Filomat 38:12 (2024), 4241–4251
https://doi.org/10.2298/FIL2412241Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The semigroups of orientation-preserving transformations with restricted range

Ping Zhao[a,b], Huabi Hu[a,*]

[a] *School of Biology and Engineering, Guizhou Medical University, Guiyang 50004, Guizhou Province, China*
[b] *School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, Guizhou Province, China*

Abstract. Let X_n be a chain with n elements ($n \in \mathcal{N}$), and let $\mathcal{OP}_n$ be the monoid of all orientation-preserving transformations of X_n. Given a non-empty subset Y of X_n, we denote by $\mathcal{OP}_n(Y)$ the subsemigroup of $\mathcal{OP}_n$ of all full orientation-preserving transformations with range contained in Y. We also denote by $\overline{\mathcal{OP}}_Y$ the semigroup of all singular orientation-preserving transformations of Y. In this article, we consider the subsemigroup $\mathcal{FOP}_n(Y) = \{\alpha \in \mathcal{OP}_n(Y) \mid \mathrm{im}(\alpha) = Y\alpha\}$ of $\mathcal{OP}_n(Y)$: We characterize the connections between the maximal (regular) subsemibands of the core $C(\mathcal{FOP}_n(Y))$ of $\mathcal{FOP}_n(Y)$ and the maximal (regular) subsemibands of $\overline{\mathcal{OP}}_Y$. Moreover, we compute the rank of the semigroup $\mathcal{FOP}_n(Y)$ and characterize the structure of the idempotent generating sets of the semigroup $C(\mathcal{FOP}_n(Y))$. We also determine the maximal subsemibands as well as the maximal regular subsemibans of the semigroup $C(\mathcal{FOP}_n(Y))$.

1. Introduction and preliminaries

Let X_n be a chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$. We denote by $\mathcal{T}_n$ the monoid of all full transformations on X_n. We say that a transformation $\alpha \in \mathcal{T}_n$ is *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$, for all $x, y \in X_n$. Denote by $\mathcal{O}_n$ the submonoid of $\mathcal{T}_n$ of all full order-preserving transformations of X_n. Let $c = (c_1, c_2, \ldots, c_t)$ be a sequence of t ($t \geq 0$) elements from the chain X_n. We say that c is cyclic if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $c_i > c_{i+1}$, where c_{t+1} denotes c_1. We say that $\alpha \in \mathcal{T}_n$ is *orientation-preserving* if the sequence of its image $(1\alpha, 2\alpha, \ldots, n\alpha)$ is cyclic. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{T}_n$ of all full orientation-preserving transformations of X_n.

The notion of an orientation-preserving transformation was introduced by McAlister in [3] and, independently, by Catarino and Higgins in [3]. Several properties of the monoid $\mathcal{OP}_n$ have been investigated in these two articles. A presentation for the monoid $\mathcal{OP}_n$, in terms of $2n-1$ generators, was given by Catarino in [2]. Another presentation for $\mathcal{OP}_n$, in terms of 2 (its rank) generators, was found by Arthur and Rušk̆uc [1]. The congruences of the monoid $\mathcal{OP}_n$ was completely described by Fernandes et al. in [5].

Let Y be a non-empty subset of X_n, we denote by $\mathcal{OP}_n(Y)$ the subsemigroup $\{\alpha \in \mathcal{OP}_n \mid \mathrm{im}(\alpha) \subseteq Y\}$ of $\mathcal{OP}_n$ of all elements with range (image) restricted to Y. We also denote by $\mathcal{OP}_Y$ the monoid of all full

2020 *Mathematics Subject Classification*. 20M20; 20M10.
Keywords. rank, maximal subsemiband, maximal regular subsemiband.
Received: 19 April 2023; Revised: 24 October 2023; Accepted: 08 November 2023
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (No.12261022) and the National Natural Science Foundation of China (No.11461014).
* Corresponding author: Huabi Hu
Email addresses: zhaoping731108@hotmail.com (Ping Zhao), huhuabi1978@163.com (Huabi Hu)

orientation-preserving transformations of Y, by C_Y the cycle group on Y and by $\overline{\mathcal{OP}}_Y$ the subsemigroup $\mathcal{OP}_Y \backslash C_Y$ of $\mathcal{OP}_Y$ of all singular orientation-preserving transformations.

In the abstract theory of semigroups, idempotents are extremely important in the structure theory of semigroups, both finite and infinite. They help classify different types of semigroups, identify subgroups, determine left or right ideals, and describe the general structure of a given semigroup. In 2016, Fernandes et al.[6] consider the subsemigroup of $\mathcal{OP}_n(Y)$ defined by

$$\mathcal{FOP}_n(Y) = \{\alpha \in \mathcal{OP}_n(Y) \mid \operatorname{im}(\alpha) = Y\alpha\}.$$

Notice that, if $Y = X_n$, then $\mathcal{FOP}_n(Y) = \mathcal{OP}_n$. They computed the rank of $\mathcal{OP}_n(Y)$ and showed that $\mathcal{FOP}_n(Y)$ is the largest regular subsemigroup of $\mathcal{OP}_n(Y)$. However, the results about algebraic and maximal properties of the semigroup $\mathcal{FOP}_n(Y)$ are very few. In view of the above work, we can consider the core $C(\mathcal{FOP}_n(Y))$ of the semigroup $\mathcal{FOP}_n(Y)$. The main aim of this paper is to study the semigroup $C(\mathcal{FOP}_n(Y))$, the structure of the idempotent generating sets of the semigroup $C(\mathcal{FOP}_n(Y))$ are characterized and complete classifications of maximal subsemigroups as well as the maximal regular subsemibans of the semigroup $C(\mathcal{FOP}_n(Y))$ are obtained.

This paper is organized as follows. We characterize the connections between the maximal (regular) subsemibands of the core $C(\mathcal{FOP}_n(Y))$ of the semigroup $\mathcal{FOP}_n(Y)$ and the maximal (regular) subsemibands of $\overline{\mathcal{OP}}_Y$ in Sec.2. In Sec.3, we compute the rank of the semigroup $\mathcal{FOP}_n(Y)$. In Sec.4, we characterize the structure of the idempotent generating sets of the core $C(\mathcal{FOP}_n(Y))$ of the semigroup $\mathcal{FOP}_n(Y)$. As applications, we compute the number of distinct minimal (idempotent) generating sets of $C(\mathcal{FOP}_n(Y))$. In Sec.5, we determine the maximal subsemibands as well as the maximal regular subsemibands of $C(\mathcal{FOP}_n(Y))$.

Remark 1 In this paper, it will always be from context when additions are taken modulo n (or modulo t where t is the number of elements of any sequence).

Let S be a semigroup. Given a subset U of a semigroup S and $\alpha \in S$, we denote by $E(U)$ the set of idempotents of S belonging to U and by L_α, R_α and H_α the $\mathscr{L}$-class, $\mathscr{R}$-class and $\mathscr{H}$-class of α, respectively. For general background on Semigroup Theory, we refer the reader to Howie's book [8].

2. Preliminary results

In this section, we present several structural properties of the core $C(\mathcal{FOP}_n(Y))$ of the semigroup $\mathcal{FOP}_n(Y)$.

Let Y be a non-empty subset of X_n with $|Y| = r$. Throughout this paper we always assume that $Y = \{y_1 < y_2 < \cdots < y_r\}$.

If $\alpha \in \mathcal{T}_n$, we will write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_m \\ a_1 & \dots & a_m \end{pmatrix}$$

to indicate that $X_n = A_1 \cup \cdots \cup A_m$, $\operatorname{im}(\alpha) = \{a_1, \dots, a_m\}$ and $A_i\alpha = a_i$ for each $i \in \{1, \dots, m\}$. As usual, we denote the *kernel* of $\alpha \in \mathcal{T}_n$ by

$$\ker(\alpha) = \{(x, y) \in X_n \times X_n \mid x\alpha = y\alpha\}.$$

We will sometimes write $\ker(\alpha) = (A_1|\dots|A_m)$ to indicate that $\ker(\alpha)$ has equivalence classes $A_1, \dots, A_m$, and this notation will always imply that A_i are pairwise disjoint and non-empty.

Green's relations on $\mathcal{FOP}_n(Y)$ are characterized by

$$\alpha \mathscr{L} \beta \text{ if and only if } \operatorname{im}(\alpha) = \operatorname{im}(\beta),$$

$$\alpha \mathscr{R} \beta \text{ if and only if } \ker(\alpha) = \ker(\beta),$$

$$\alpha \mathscr{J} \beta \text{ if and only if } |\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|.$$

Regarding Green's relation $\mathcal{H}$, if α is an element of $\mathcal{FOP}_n(Y)$ of rank k, for $1 \le k \le r$, then the $\mathcal{H}$-class in $\mathcal{FOP}_n(Y)$ of α is a cycle group of order k (see [6, Theorem 2.2]). Notice that the $\mathcal{J}$-class J_r has exactly one $\mathcal{L}$-class. Thus each $\mathcal{R}$-class in J_r is an $\mathcal{H}$-class.

From the above fact, we know that the semigroup $\mathcal{FOP}_n(Y)$ has r $\mathcal{J}$-classes, namely $J_1, \dots, J_r$, where $J_k = \{\alpha \in \mathcal{FOP}_n(Y) \mid |\mathrm{im}(\alpha)| = k\}$. For $1 \le k \le r$, let

$$Q_n(k) = \{\alpha \in \mathcal{FOP}_n(Y) \mid |\mathrm{im}(\alpha)| \le k\}.$$

Then the sets $Q_n(k)$ are the two-sided ideals of $\mathcal{FOP}_n(Y)$ and $Q_n(k) = J_1 \cup J_1 \cdots \cup J_k$.

Lemma 2.1. *Let $\alpha \in \mathcal{FOP}_n(Y)$. Then there exists $\lambda_\alpha \in E(J_r)$ such that $\alpha = \lambda_\alpha \alpha = \lambda_\alpha(\alpha|_Y)$.*

Proof. Suppose that $\ker(\alpha) = (A_1|\dots|A_k)$. Put

$$\varepsilon = \begin{pmatrix} A_1 & \dots & A_k \\ minA_1 & \dots & minA_k \end{pmatrix}.$$

Then $\varepsilon^2 = \varepsilon$ and $\alpha = \varepsilon\alpha$. By refining adequately the kernel of ε we can get an idempotent λ_α of rank r which is $\mathscr{R}$-above ε, that is, satisfies $\lambda_\alpha\varepsilon = \varepsilon$. Then

$$\lambda_\alpha\alpha = \lambda_\alpha(\varepsilon\alpha) = (\lambda_\alpha\varepsilon)\alpha = \varepsilon\alpha = \alpha.$$

Thus clearly $\alpha = \lambda_\alpha\alpha = \lambda_\alpha(\alpha|_Y)$. □

Let $\lambda \in E(J_r)$ and $\alpha \in \mathcal{OP}_Y$. Notice that $\mathrm{im}(\lambda) = \mathrm{dom}(\alpha) = Y$. Then clearly $\lambda\alpha \in \mathcal{FOP}_n(X, Y)$. Thus $E(J_r)\mathcal{OP}_Y \subseteq \mathcal{FOP}_n(Y)$. From Lemma 2.1, we easily obtain the following result:

Lemma 2.2. *$\mathcal{FOP}_n(Y) = E(J_r)\mathcal{OP}_Y$.*

For $1 \le k \le r$, let

$$\mathcal{I}_{\mathbf{r}}(k) = \{\alpha \in \mathcal{OP}_Y \mid |\mathrm{im}(\alpha)| \le k\}.$$

Then the sets $\mathcal{I}_{\mathbf{r}}(k)$ are the two-sided ideals of $\mathcal{OP}_Y$. Clearly $\mathcal{I}_{\mathbf{r}}(r) = \mathcal{OP}_Y$ and $\mathcal{I}_{\mathbf{r}}(r-1) = \overline{\mathcal{OP}}_Y$.

Notice that $Q_n(r) = \mathcal{FOP}_n(Y) = E(J_r)\mathcal{I}_{\mathbf{r}}(r)$ (by Lemma 2.2). In fact, we have the following lemma:

Lemma 2.3. *Let $1 \le k \le r - 1$. Then $Q_n(k) = E(J_r)\mathcal{I}_{\mathbf{r}}(k)$.*

Proof. Let $\alpha \in Q_n(k)$ be arbitrary. Then, by Lemma 2.1, there exists $\lambda_\alpha \in E(J_r)$ such that $\alpha = \lambda_\alpha(\alpha|_Y)$. Notice that $|\mathrm{im}(\alpha|_Y)| = |\mathrm{im}(\alpha)| \le k$ and $\alpha|_Y \in \mathcal{OP}_Y$. Then $\alpha = \lambda_\alpha(\alpha|_Y) \in E(J_r)\mathcal{I}_{\mathbf{r}}(k)$. Thus $Q_n(k) \subseteq E(J_r)\mathcal{I}_{\mathbf{r}}(k)$.

Conversely, let $\lambda \in E(J_r)$ and $\gamma \in \mathcal{I}_{\mathbf{r}}(k)$ be arbitrary. Then $|\mathrm{im}(\lambda\gamma)| = |\mathrm{im}(\gamma)| \le k$ and $\lambda\gamma \in E(J_r)\mathcal{I}_{\mathbf{r}}(k) \subseteq E(J_r)\mathcal{OP}_Y = \mathcal{FOP}_n(Y)$ (by Lemma 2.2). Thus $\lambda\gamma \in Q_n(k)$. Hence $E(J_r)\mathcal{I}_{\mathbf{r}}(k) \subseteq Q_n(k)$. □

Recall that a right identity of a semigroup S is an element $u \in S$ such that $x = xu$ for all $x \in S$. It is obvious that the elements of $E(J_r)$ are right identities of $\mathcal{FOP}_n(Y)$.

Lemma 2.4. *Let $1 \le k \le r - 1$. Then $Q_n(k) = \langle E(J_k)\rangle$.*

Proof. Let $\alpha \in Q_n(k)$ be arbitrary. Then, by Lemma 2.1, there exist $\lambda \in E(J_r)$ and $\beta \in \mathcal{I}_{\mathbf{r}}(k)$ such that $\alpha = \lambda\beta$. Notice that $\alpha \in \mathcal{OP}_Y$ and $|\mathrm{im}(\alpha)| \le k$. Since $k \le r - 1$, then, by [12, Lemma 2.3], there exist idempotents $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m \in \mathcal{OP}_Y$ each of which has rank k such that $\beta = \varepsilon_1\varepsilon_2\dots\varepsilon_m$. Notice that the elements of $E(J_r)$ are right identities. It follows that $(\lambda\varepsilon_i)^2 = (\lambda\varepsilon_i)(\lambda\varepsilon_i) = \lambda(\varepsilon_i)^2 = \lambda\varepsilon_i$ and so $\alpha = \lambda\beta = \lambda\varepsilon_1\varepsilon_2\dots\varepsilon_m = (\lambda\varepsilon_1)(\lambda\varepsilon_2)\dots(\lambda\varepsilon_m) \in \langle E(J_k)\rangle$. □

Recall that the subsemigroup $\langle E(S)\rangle$ of a semigroup S is called the *core* of S, and S is said to be a *semiband* if T equals its own core. We denote by $C(S)$ the core of S.

Lemma 2.5. *Let $n \geq 3$. Then $E(J_r)$ is a left zero subsemigroup of $\mathcal{FOP}_n(Y)$ and*

$$C(\mathcal{FOP}_n(Y)) = E(J_r) \cup Q_n(r-1).$$

Proof. Let $\lambda, \mu \in E(J_r)$ be arbitrary. Then $\lambda|_Y = \mu|_Y = 1_Y$ and $\mathrm{im}(\lambda) = \mathrm{im}(\mu) = Y$. Then $\lambda\mu = \lambda 1_Y = \lambda$. Then $E(J_r)$ is a left zero subsemigroup of $\mathcal{FOP}_n(Y)$. Notice that $Q_n(r-1) = \langle E(Q_n(r-1))\rangle$ (by Lemma 2.4). It follows that

$$E(J_r) \cup Q_n(r-1) = \langle E(J_r) \cup Q_n(r-1)\rangle = \langle E(J_r) \cup E(Q_n(r-1))\rangle = \langle E(\mathcal{FOP}_n(Y))\rangle.$$

Thus $C(\mathcal{FOP}_n(Y)) = E(J_r) \cup Q_n(r-1)$. □

Notice that $\overline{\mathcal{OP}}_Y = \mathcal{OP}_Y \backslash C_Y$. Let S be a subsemigroup of $\overline{\mathcal{OP}}_Y$. We define

$$S^{\Delta} = E(J_r)S^{1_Y}.$$

Notice that the elements of $E(J_r)$ are right identities of $\mathcal{FOP}_n(X, Y)$. In fact, it is obvious that $\alpha\lambda = \alpha$, for $\alpha \in \mathcal{OP}_Y$, $\lambda \in E(J_r)$. Then S^{Δ} is a semigroup. Now, we define a mapping $\phi : S^{\Delta} \longrightarrow S^{1_Y}$ by the rule that, for any $\lambda\alpha \in S^{\Delta}$ with $\lambda \in E(J_r)$ and $\alpha \in S^{1_Y}$,

$$(\lambda\alpha)\phi = \alpha.$$

Notice again that the elements of $E(J_r)$ are right identities. The following lemma is immediate by the definition of the mapping ϕ:

Lemma 2.6. *Let S be a subsemigroup of $\overline{\mathcal{OP}}_Y$. Then the map ϕ is an epimorphism.*

Let $\lambda \in E(J_r)$ and $\alpha \in \mathcal{OP}_Y$. Notice that $\lambda|_Y = 1_Y$. Then clearly $(\lambda\alpha)|_Y = \alpha$. With this fact, we can prove the following result:

Lemma 2.7. *Let S and T be subsemigroups of $\overline{\mathcal{OP}}_Y$. Then $S^{\Delta} = T^{\Delta}$ if and only if $S = T$.*

Proof. If $S = T$, then clearly $S^{\Delta} = T^{\Delta}$. Conversely, suppose that $S^{\Delta} = T^{\Delta}$, i.e., $E(J_r)S^{1_Y} = E(J_r)T^{1_Y}$. Let $\alpha \in S$ be arbitrary, and let $\lambda \in E(J_r)$. Then $\lambda\alpha \in E(J_r)S^{1_Y} = E(J_r)T^{1_Y}$. Thus there exist $\mu \in E(J_r)$ and $\beta \in T^{1_Y}$ such that $\lambda\alpha = \mu\beta$. It follows that $\alpha = (\lambda\alpha)|_Y = (\mu\beta)|_Y = \beta \in T^{1_Y}$. Since $\alpha \in S \subseteq \overline{\mathcal{OP}}_Y$, we have $\alpha \neq 1_Y$ and so $\alpha = \beta \in T$. Thus $S \subseteq T$. Similarly, we can prove that $T \subseteq S$. Hence $S = T$. □

Notice also that the elements of $E(J_r)$ are right identities. This observation allows us to easily deduce the following properties:

Lemma 2.8. *Let S be a subsemigroup of $\overline{\mathcal{OP}}_Y$. Then*

(1) *S is regular if and only if S^{Δ} is regular.*
(2) *S is a semiband if and only if S^{Δ} is a semiband.*
(3) *$S = \overline{\mathcal{OP}}_Y$ if and only if $S^{\Delta} = C(\mathcal{FOP}_n(Y))$.*

Proof. (1) Suppose that S^{Δ} is regular. Then, by Lemma 2.6, S^{1_Y} is regular and so S is regular. Conversely, suppose that S is regular. Let $\alpha \in S^{\Delta}$ be arbitrary. Then there exist $\lambda \in E(J_r)$ and $\beta \in S^{1_Y}$ such that $\alpha = \lambda\beta$. Then $\lambda|_Y = 1_Y$, and there exists $\beta^* \in S^{1_Y}$ such that $\beta = \beta\beta^*\beta$. Notice that the elements of $E(J_r)$ are right identities. Let $\alpha^* = \lambda\beta^*$. Then $\alpha^* \in E(J_r)S^{1_Y} = S^{\Delta}$ and $\alpha\alpha^*\alpha = (\lambda\beta)(\lambda\beta^*)(\lambda\beta) = \lambda(\beta\beta^*\beta) = \lambda\beta = \alpha$. Thus S^{Δ} is regular.

(2) Suppose that S^Δ is a semiband. Then, by Lemma 2.6, S^{1_Y} is a semiband and so S is a semiband. Conversely, suppose that S is a semiband. Then $S = \langle E(S)\rangle$ and so $S^\Delta = E(J_r)\langle E(S)\rangle^{1_Y}$. Let $\alpha \in S^\Delta$ be arbitrary. Then there exist $\lambda \in E(J_r)$ and $\varepsilon_1, \ldots, \varepsilon_m \in E(S)^{1_Y}$ such that $\alpha = \lambda\varepsilon_1 \ldots \varepsilon_m$. Notice that the elements of $E(J_r)$ are right identities. Since $(\lambda\varepsilon_i)^2 = (\lambda\varepsilon_i)(\lambda\varepsilon_i) = \lambda\varepsilon_i\varepsilon_i = \lambda\varepsilon_i$, we have $\lambda\varepsilon_i \in E(S^\Delta)$, for $1 \le i \le m$. Then $\alpha = \lambda\varepsilon_1 \ldots \varepsilon_m = (\lambda\varepsilon_1)(\lambda\varepsilon_2)\ldots(\lambda\varepsilon_m) \in \langle E(S^\Delta)\rangle$. Thus S^Δ is a semiband.

(3) Suppose that $S = \overline{\mathcal{OP}}_Y$. Then, by Lemmas 2.3 and 2.5, $S^\Delta = E(J_r)\overline{\mathcal{OP}}_Y^{1_Y} = E(J_r) \cup E(J_r)\mathcal{I}_{\mathbf{r}}(r-1) = E(J_r) \cup Q_n(r-1) = C(\mathcal{FOP}_n(Y))$. Conversely, suppose that $S^\Delta = C(\mathcal{FOP}_n(Y))$. Then, by Lemmas 2.3 and 2.5, $S^\Delta = C(\mathcal{FOP}_n(Y)) = E(J_r) \cup Q_n(r-1) = E(J_r) \cup E(J_r)\mathcal{I}_{\mathbf{r}}(r-1) = E(J_r)\overline{\mathcal{OP}}_Y^{1_Y} = \overline{\mathcal{OP}}_Y^{\Delta}$. Thus, by Lemma 2.7, $S = \overline{\mathcal{OP}}_Y$. □

Lemma 2.9. *Let S be a subsemigroup of $C(\mathcal{FOP}_n(Y))$. Let $M_S = \{\alpha|_Y \mid \alpha \in S\}\backslash\{1_Y\}$. If $E(J_r) \subseteq S$, then $S = M_S^\Delta$.*

Proof. Let $\alpha \in S$ be arbitrary. Then, by Lemma 2.1, there exists $\lambda_\alpha \in E(J_r)$ such that $\alpha = \lambda_\alpha(\alpha|_Y)$. Then $\alpha = \lambda_\alpha(\alpha|_Y) \in E(J_r)M_S^{1_Y} = M_S^\Delta$. Thus $S \subseteq M_S^\Delta$. Conversely, let $\alpha \in M_S^\Delta = E(J_r)M_S^{1_Y}$ be arbitrary. Then there exist $\varepsilon \in E(J_r)$ and $\beta \in S$ such that $\alpha = \varepsilon(\beta|_Y)$ (notice that, if $\beta \in E(J_r) \subseteq S$, then $\beta|_Y = 1_Y$). By Lemma 2.1, there exists $\lambda_\beta \in E(J_r)$ such that $\beta = \lambda_\beta(\beta|_Y)$. Notice that the elements of $E(J_r)$ are right identities. Then $\alpha = \varepsilon(\beta|_Y) = \varepsilon\lambda_\beta(\beta|_Y) = \varepsilon[\lambda_\beta(\beta|_Y)] = \varepsilon\beta$. It follows from $E(J_r) \subseteq S$ that $\alpha = \varepsilon\beta \in S$. Then $M_S^\Delta \subseteq S$. □

We shall say that a proper subsemigroup S of $C(\mathcal{FOP}_n(Y))$ is *maximal (regular) subsemiband* if S is a (regular) subsemiband, and any (regular) subsemiband of $C(\mathcal{FOP}_n(Y))$ properly containing S must be $C(\mathcal{FOP}_n(Y))$. Using Lemmas 2.7, 2.8 and 2.9, we can prove the following result:

Theorem 2.10. *Let S be a subsemigroup of $\overline{\mathcal{OP}}_Y$. Then S is a maximal (regular) subsemiband of $\overline{\mathcal{OP}}_Y$ if and only if S^Δ is a maximal (regular) subsemiband of $C(\mathcal{FOP}_n(Y))$.*

Proof. Suppose that S is a maximal (regular) subsemiband of $\overline{\mathcal{OP}}_Y$. Then, by Lemma 2.8, S^Δ is a (regular) subsemiband of $C(\mathcal{FOP}_n(Y))$. Suppose that S^Δ is not a maximal (regular) subsemiband of $C(\mathcal{FOP}_n(Y))$. Then there exists a maximal (regular) subsemiband T of $C(\mathcal{FOP}_n(Y))$ such that $S^\Delta \subset T \subset C(\mathcal{FOP}_n(Y))$. Notice that $E(J_r) \subseteq E(J_r)S^{1_Y} = S^\Delta \subset T$. Put $M_T = \{\alpha|_Y \mid \alpha \in T\}\backslash\{1_Y\}$. Then, by Lemma 2.9, $T = M_T^\Delta$. Notice that T is a (regular) semiband. Thus, by Lemma 2.8, M_T is a (regular) semiband of $\overline{\mathcal{OP}}_Y$. Let $\alpha \in S \subseteq \overline{\mathcal{OP}}_Y$ be arbitrary, and let $\lambda \in E(J_r)$. Clearly $\alpha \neq 1_Y$. Then $\lambda\alpha \in E(J_r)S^{1_Y} = S^\Delta \subset T$. Thus $\alpha = (\lambda\alpha)|_Y \in M_T$. Thus $S \subseteq M_T$. By the maximality of S, we have $M_T = \overline{\mathcal{OP}}_Y$ or $S = M_T$. If $M_T = \overline{\mathcal{OP}}_Y$, then, by Lemma 2.8, $T = M_T^\Delta = \overline{\mathcal{OP}}_Y^{\Delta} = C(\mathcal{FOP}_n(Y))$, a contradiction. If $S = M_T$, then $S^\Delta = M_T^\Delta = T$, a contradiction.

Conversely, suppose that S^Δ is a maximal (regular) subsemiband of $C(\mathcal{FOP}_n(Y))$. Then, by Lemma 2.8, S is a (regular) subsemiband of $\overline{\mathcal{OP}}_Y$. Suppose that S is not a maximal (regular) subsemiband of $\overline{\mathcal{OP}}_Y$. Then there exists a maximal (regular) subsemiband M of $\overline{\mathcal{OP}}_Y$ such that $S \subset M \subset \overline{\mathcal{OP}}_Y$. Notice that $M^\Delta = E(J_r)M^{1_Y}$. Then, by Lemma 2.8, M^Δ is a (regular) subsemiband of $C(\mathcal{FOP}_n(Y))$. Clearly $S^\Delta \subseteq M^\Delta$. By the maximality of S, we have $M^\Delta = C(\mathcal{FOP}_n(Y))$ or $S^\Delta = M^\Delta$. If $M^\Delta = C(\mathcal{FOP}_n(Y))$, then, by Lemma 2.8, $M = \overline{\mathcal{OP}}_Y$, a contradiction. If $S^\Delta = M^\Delta$, then, by Lemma 2.7, $M = S$, a contradiction. □

3. Rank of the semigroup $\mathcal{FOP}_n(Y)$

In this section, we compute the rank of the semigroup $\mathcal{FOP}_n(Y)$.

We denote by $[i,k]$ the set $\{i, i+1, \ldots, k-1, k\}$ for $i,k \in X_n$. A subset C of X_n is said to be *convex* if C has the form $[i, i+t]$, for some $i,k \in X_n$ and $0 \le t \le n-1$. We shall refer to an equivalence π on X_n as convex if its classes are convex subsets of X_n, and we shall say that π is of weight k if $|X/\pi| = k$. An convex equivalence π on X_n is *Y-convex* if each class of π contains at least one element of Y.

It is known that every kernel $\ker(\alpha)$ of $\alpha \in \mathcal{FOP}_n(Y)(\subseteq \mathcal{OP}_n)$ is convex (see [3]). Let $\alpha \in J_r$. Considering the kernel classes of α, we obtain a type of partitions of the domain X_n of α into convex subsets:

$\mathrm{dom}(\alpha) = X_n = \cup_{i=1}^{r} P_i$ with $\alpha = \left(\begin{array}{c|c|c} P_1 & \cdots & P_r \\ a_1 & \cdots & a_r \end{array}\right)$, and $P_i \cap Y \neq \emptyset$, for $1 \leq i \leq r$.

Notice that $P_1, \ldots, P_r$ are precisely the kernel classes of α. We can associate to α a Y-convex relation of weight r (with classes $P_1, \ldots, P_r$). Therefore the number of $\mathcal{R}$-classes of $\mathcal{FOP}_n(Y)$ of rank r is equal to the number of Y-convex equivalences of weight r on X_n.

Notice that $Y = \{y_1 < y_2 < \cdots < y_r\}$. Put

$$m_i = |\{x \in X_n \mid y_i < x < y_{i+1}\}|,\ 1 \leq i \leq r-1,\ \ \textit{and}\ m_r = |\{x \in X_n \mid x < y_1\} \cup \{x \in X_n \mid x > y_r\}|.$$

Then $m_i + 1 = y_{i+1} - y_i$, for $1 \leq i \leq r-1$, and $m_r = n - (y_r - y_1 + 1)$. We denote by p_r the number of Y-convex equivalences of weight r on X_n. Then clearly

$$p_r = (m_1 + 1)(m_2 + 1) \ldots (m_{r-1} + 1)(m_r + 1) = \prod_{i=1}^{r-1}(y_{i+1} - y_i)[n - (y_r - y_1)].$$

Thus the number of $\mathcal{R}$-classes of $\mathcal{FOP}_n(Y)$ of rank r is p_r. Notice also that each $\mathcal{H}$-class in J_r is a cycle group of order r (see [6, Theorem 2.2]) and each $\mathcal{R}$-class in J_r is an $\mathcal{H}$-class. Then the $\mathcal{J}$-class J_r is a union of p_r groups, each of which is a cycle group of order r.

As usual, the rank of a semigroup S is defined by $\mathrm{rank}\, S = \min\{|A| \mid A \subseteq S, \langle A \rangle = S\}$. If S is generated by its set E of idempotents, then the idempotent rank of S is defined by $\mathrm{idrank}\, S = \min\{|A| \mid A \subseteq E, \langle A \rangle = S\}$. Clearly, $\mathrm{rank}\, S \leq \mathrm{idrank}\, S$.

Let $\alpha, \beta \in J_r$ be arbitrary. Then $\mathrm{im}(\alpha) = \mathrm{im}(\beta) = Y$. Thus clearly $\mathrm{im}(\alpha\beta) = \mathrm{im}(\beta)$ and so $\alpha\beta \in J_r$. Hence J_r is a semigroup. Now, notice that, if α is an element of $\mathcal{FOP}_n(Y)$ of rank r and β and γ are two elements of $\mathcal{FOP}_n(Y)$ such that $\alpha = \beta\gamma$, then $\ker(\alpha) = \ker(\beta)$ and $\mathrm{im}(\alpha) = \mathrm{im}(\gamma)$. Then any generating set of the semigroup J_r contains at least one element of rank r from each $\mathscr{R}$-classes of $\mathcal{FOP}_n(Y)$ of rank r. Thus $\mathrm{rank}\, J_r \geq p_r$.

Lemma 3.1. *Let* $2 \leq r \leq n$. *Then* $\mathrm{rank}\, J_r = \prod_{i=1}^{r-1}(y_{i+1} - y_i)[n - (y_r - y_1)]$.

Proof. Notice that the $\mathcal{J}$-class J_r is a union of p_r groups, each of which is a cycle group of order r. We can suppose that

$$J_r = \bigcup_{i=1}^{p_r} H_i = \bigcup_{i=1}^{p_r} \langle \lambda_i \rangle,$$

where H_i is a cycle group of order r and λ_i is a generator of cyclic group H_i, for $1 \leq i \leq r$. Let $G = \{\lambda_1, \lambda_2, \ldots, \lambda_{p_r}\}$. Then $J_r = \bigcup_{i=1}^{p_r} H_i \subseteq \langle G \rangle$. Notice that $G \subseteq J_r$. Then $\langle G \rangle \subseteq \langle J_r \rangle = J_r$. Thus $J_r = \langle G \rangle$. Since the set G has cardinality p_r, which equals the number of $\mathscr{R}$-classes of $\mathcal{FOP}_n(Y)$ of rank r, it follows immediately that $\mathrm{rank}\, J_r = p_r = \prod_{i=1}^{r-1}(y_{i+1} - y_i)[n - (y_r - y_1)]$. □

Since J_r is a subsemigroup of $\mathcal{FOP}_n(Y)$, then any element of $Q_n(r-1)$ cannot be generated by elements of J_r. And it is clear that if $\alpha \in J_r$ and $\alpha = \beta\gamma$, then $\beta, \gamma \in J_r$. Then $\mathrm{rank}\, \mathcal{FOP}_n(Y) \geq \mathrm{rank}\, J_r + 1$.

Theorem 3.2. *Let* $2 \leq r \leq n$. *Then*

$$\mathrm{rank}\, \mathcal{FOP}_n(Y) = \prod_{i=1}^{r-1}(y_{i+1} - y_i)[n - (y_r - y_1)] + 1.$$

Proof. Put

$$g = \begin{pmatrix} y_1 & \cdots & y_{r-1} & y_r \\ y_2 & \cdots & y_r & y_1 \end{pmatrix} \textit{ and } \epsilon = \begin{pmatrix} y_1 & y_2 & \cdots & y_r \\ y_2 & y_2 & \cdots & y_r \end{pmatrix}.$$

Then $\mathcal{OP}_Y = \langle \epsilon, g \rangle$ (see [4, Proposition 1.3] and $C_Y = \langle g \rangle$. Now, take $\lambda \in E(J_r)$ and let $\eta = \lambda\epsilon$. Notice that the elements of $E(J_r)$ are right identities and $E(J_r)g \subseteq J_r$ and $\alpha\lambda = \alpha$, for $\alpha \in \mathcal{OP}_Y$, $\lambda \in E(J_r)$. It follows that

$$E(J_r)\langle \epsilon, g \rangle = \langle E(J_r)\epsilon, E(J_r)g \rangle \subseteq \langle E(J_r)\epsilon, J_r \rangle$$

and

$$E(J_r)\epsilon = E(J_r)\lambda\epsilon = E(J_r)\eta \subseteq J_r\eta \subseteq \langle J_r \cup \{\eta\}\rangle.$$

Then, by Lemma 2.2,

$$\mathcal{FOP}_n(Y) = E(J_r)\mathcal{OP}_Y = E(J_r)\langle\epsilon, g\rangle = \langle E(J_r)\epsilon, E(J_r)g\rangle \subseteq \langle J_r \cup \{\eta\}\rangle.$$

Thus $\mathcal{FOP}_n(Y) = \langle J_r \cup \{\eta\}\rangle$. Let A be a generating set of J_r with $|A| = \operatorname{rank} J_r$. Then $\mathcal{FOP}_n(Y) = \langle A \cup \{\eta\}\rangle$ and so $\operatorname{rank}\mathcal{FOP}_n(Y) \leq |A| + 1 = \operatorname{rank} J_r + 1$. Since $\operatorname{rank}\mathcal{FOP}_n(Y) \geq \operatorname{rank} J_r + 1$, it follows that $\operatorname{rank}\mathcal{FOP}_n(Y) = \operatorname{rank} J_r + 1$. Thus, by Lemma 3.1, $\operatorname{rank}\mathcal{FOP}_n(Y) = p_r + 1 = \prod_{i=1}^{r-1}(y_{i+1} - y_i)[n - (y_r - y_1)] + 1$. □

4. The idempotent-generated sets of $C(\mathcal{FOP}_n(Y))$

In this section, we characterize the structure of the idempotent generating sets of the core $C(\mathcal{FOP}_n(Y))$ of $\mathcal{FOP}_n(Y)$. As applications, we compute the number of distinct minimal idempotent generating sets of $C(\mathcal{FOP}_n(Y))$.

Notice that $\overline{\mathcal{OP}}_Y = \mathcal{OP}_Y \backslash \mathcal{C}_Y$. For $1 \leq i \leq r$, let

$$\tau_i = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i & y_{i+1} & y_{i+2} & \cdots & y_r \\ y_1 & \cdots & y_{i-1} & y_{i+1} & y_{i+1} & y_{i+2} & \cdots & y_r \end{pmatrix}$$

and

$$\varsigma_i = \begin{pmatrix} y_1 & \cdots & y_{i-1} & y_i & y_{i+1} & y_{i+2} & \cdots & y_r \\ y_1 & \cdots & y_{i-1} & y_i & y_i & y_{i+2} & \cdots & y_r \end{pmatrix}.$$

Then τ_i and ς_i are idempotents in $\overline{\mathcal{OP}}_Y$ of rank $r-1$. Notice that $\tau_r = \begin{pmatrix} y_1 & \cdots & y_{r-1} & y_r \\ y_1 & \cdots & y_{r-1} & y_1 \end{pmatrix}$ and $\varsigma_r = \begin{pmatrix} y_2 & \cdots & y_r & y_1 \\ y_2 & \cdots & y_r & y_r \end{pmatrix}$. We denoted by $E_{r-1}^{\overline{\mathcal{OP}}_Y}$ the set of all idempotents in $\overline{\mathcal{OP}}_Y$ of rank $r-1$. For $1 \leq i \leq r$, let

$$E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)} = \{\tau_i \mid 1 \leq i \leq r\} \text{ and } E_{n-1}^{(\overline{\mathcal{OP}}_Y,-)} = \{\varsigma_i \mid 1 \leq i \leq r\}.$$

Then $E_{r-1}^{\overline{\mathcal{OP}}_Y} = E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)} \cup E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)}$. Recall that Zhao, Xu and Yang [13, Theorem 2.1] proved the following lemma:

Lemma 4.1. *Let G be subset of $E(\overline{\mathcal{OP}}_Y)$. Then*

$$\langle G\rangle = \overline{\mathcal{OP}}_Y \text{ if and only if } E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)} \subseteq G \text{ or } E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)} \subseteq G.$$

For $\alpha \in \mathcal{OP}_Y$, we define

$$\Delta_\alpha = \{\beta \in \mathcal{FOP}_n(Y) \mid \beta|_Y = \alpha\}.$$

Notice that if $\beta \in \Delta_\alpha$, then $\operatorname{im}(\beta) = \operatorname{im}(\alpha)$.

Now, it is easy to prove the main result of this section:

Theorem 4.2. *Let E be an idempotent set of $C(\mathcal{FOP}_n(Y))$. Then E is an idempotent generating set of $C(\mathcal{FOP}_n(Y))$ if and only if $E(J_r) \subseteq E$ and $E \cap \Delta_\varepsilon \neq \emptyset$, for all $\varepsilon \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)}$ or $E \cap \Delta_\varepsilon \neq \emptyset$, for all $\varepsilon \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)}$.*

Proof. Notice that $\overline{\mathcal{OP}}_Y = \mathcal{I}_{\mathbf{r}}(r-1)$. By Lemmas 2.3 and 2.5, we have

$$C(\mathcal{FOP}_n(Y)) = E(J_r) \cup E(J_r)\overline{\mathcal{OP}}_Y = E(J_r)(\overline{\mathcal{OP}}_Y)^{1_Y}.$$

Let E be an idempotent generating set of $C(\mathcal{FOP}_n(Y))$. Let $\lambda \in E(J_r)$ be arbitrary. Then there exist $\varepsilon_1, \ldots, \varepsilon_m \in E$ such that $\lambda = \varepsilon_1 \ldots \varepsilon_m$. Notice that the elements of $E(J_r)$ are right identities. Then

$$\varepsilon_1 = \varepsilon_1\lambda = \varepsilon_1^2\varepsilon_2 \ldots \varepsilon_m = \varepsilon_1 \ldots \varepsilon_m = \lambda$$

and so $\lambda = \varepsilon_1 \in E$. Thus $E(J_r) \subseteq E$. Suppose that there exist $\sigma \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)}$ and $\rho \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)}$ such that $E \cap \Delta_\sigma = \emptyset$ and $E \cap \Delta_\rho = \emptyset$. Notice that $E_{r-1}^{\overline{\mathcal{OP}}_Y} = E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)} \cup E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)}$ and $C(\mathcal{FOP}_n(Y)) = E(J_r) \cup Q_n(r-1)$. Let $\delta \in E_{r-1}^{\overline{\mathcal{OP}}_Y}$ be arbitrary. Take $\widehat{\delta} \in \Delta_\delta \subseteq Q_n(r-1) \subseteq C(\mathcal{FOP}_n(Y))$. Since E is an idempotent generating set of $C(\mathcal{FOP}_n(Y))$, there exist $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m \in E$ such that $\widehat{\delta} = \varepsilon_1\varepsilon_2 \cdots \varepsilon_m$. Since $|\mathrm{im}(\widehat{\delta})| = |\mathrm{im}(\delta)| = r-1$, we have $|\mathrm{im}(\varepsilon_j)| \geq r-1$, for $1 \leq j \leq m$ (otherwise $|\mathrm{im}(\widehat{\delta})| = |\mathrm{im}(\varepsilon_1\varepsilon_2 \cdot \cdots \cdot \varepsilon_m)| \leq r-2$, a contradiction). Notice again that $E \cap \Delta_\sigma = \emptyset$ and $E \cap \Delta_\rho = \emptyset$. If $|\mathrm{im}(\varepsilon_j)| = r-1$, then $\varepsilon_j|_Y \in E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\sigma, \rho\}$; if $|\mathrm{im}(\varepsilon_j)| = r$, then $\varepsilon_j|_Y = 1_Y$. Thus

$$\delta = \widehat{\delta}|_Y = \varepsilon_1|_Y\varepsilon_2|_Y \cdots \varepsilon_m|_Y \in \langle E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\sigma, \rho\} \cup \{1_Y\}\rangle = \langle E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\sigma, \rho\}\rangle \cup \{1_Y\}.$$

It follows from $\delta \in E_{r-1}^{\overline{\mathcal{OP}}_Y}$ that $\delta \in \langle E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\sigma, \rho\}\rangle$. Hence $E_{r-1}^{\overline{\mathcal{OP}}_Y} \subseteq \langle E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\sigma, \rho\}\rangle$. By Lemma 4.1, we have $\langle E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\sigma, \rho\}\rangle = \langle E_{r-1}^{\overline{\mathcal{OP}}_Y}\rangle = \overline{\mathcal{OP}}_Y$. Then $E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\sigma, \rho\}$ is an idempotent-generating set of $\overline{\mathcal{OP}}_Y$, a contradition (by Lemma 4.1 again).

Conversely, notice that $E(J_r) \subseteq E \subseteq \langle E\rangle$ and each $\alpha \in Q_n(r-1)$ has the form $\alpha = \lambda\beta$ with $\lambda \in E(J_r)$ and $\beta \in \overline{\mathcal{OP}}_Y$, where $\beta = \varepsilon_1\varepsilon_2 \ldots \varepsilon_m$ with $\varepsilon_1, \ldots, \varepsilon_m \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)}$ or $\varepsilon_1, \ldots, \varepsilon_m \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)}$ (by Lemmas 2.3 and 4.1). Since $\lambda \in E(J_r) \subseteq E$, and there exist $\widehat{\varepsilon_i} \in \Delta_{\varepsilon_i} \cap E$, for $1 \leq i \leq m$, we have

$$\alpha = \lambda\beta = \lambda\varepsilon_1\varepsilon_2 \ldots \varepsilon_m = \lambda(\widehat{\varepsilon_1}|_Y)(\widehat{\varepsilon_2}|_Y) \ldots (\widehat{\varepsilon_m}|_Y) = \lambda\widehat{\varepsilon_1}\widehat{\varepsilon_2} \ldots \widehat{\varepsilon_m} \in \langle E\rangle.$$

Then $Q_n(r-1) \subseteq \langle E\rangle$. Thus $C(\mathcal{FOP}_n(Y)) = E(J_r) \cup Q_n(r-1) = \langle E\rangle$. □

Let E be an idempotent generating set of $C(\mathcal{FOP}_n(Y))$. Then, by Theorem 4.2, E is a minimal idempotent generating set of $C(\mathcal{FOP}_n(Y))$ if and only if $E(J_r) \subseteq E$ and E be the subset having exactly one element from each Δ_ε, for all $\varepsilon \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)}$ or $\varepsilon \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)}$. Notice that $\prod_{\varepsilon \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)}} |\Delta_\varepsilon| = \prod_{\varepsilon \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,-)}} |\Delta_\varepsilon|$. Thus, we immediately deduce:

Corollary 4.3. *Let E be a minimal idempotent generating set of $C(\mathcal{FOP}_n(Y))$. Then the number of distinct sets E is*

$$\prod_{\varepsilon \in E_{r-1}^{(\overline{\mathcal{OP}}_Y,+)}} |\Delta_\varepsilon| + 1.$$

5. The subsemigroups of $C(\mathcal{FOP}_n(Y))$

In this section, we determine the maximal subsemibands as well as the maximal regular subsemibands of $C(\mathcal{FOP}_n(Y))$.

As in [10], let $k \in \{0, 1, 2, \cdots, n-1\}$, define a total order $\leq_k$ on X_n by

$$k+1 \leq_k k+2 \leq_k \cdots \leq_k n \leq_k 1 \leq_k \cdots \leq_k k.$$

We also use ${}_k \geq$ to denote $\leq_k$. Notice that $Y = \{y_1 < y_2 < \cdots < y_r\}$. We denote by $\mathcal{O}_Y$ the subsemigroup of $\mathcal{T}(Y)$ of all order-preserving transformations of Y. Let $\overline{\mathcal{O}}_Y = \mathcal{O}(Y) \backslash \{1_Y\}$. For $1 \leq k \leq r$, let

$$\overline{\mathcal{O}}_Y^k = \{a^{-k}fa^k : f \in \overline{\mathcal{O}}_Y\},$$

where $a = (y_1 y_2 \dots y_r)$ is the fixed generator of cyclic group C_Y. For $1 \le i, j \le r$, let

$$M_{\mathbf{r},(y_i,y_j)} = \langle E_{r-1}^{\overline{\mathcal{OP}}_Y} \backslash \{\tau_i, \varsigma_{j-1}\} \rangle.$$

The following lemma was proved by Zhao, Xu and Yang [13, Lemma 2.2]:

Lemma 5.1. *Let $1 \le i \le r$. Then $\overline{\mathcal{O}}_Y^i = M_{\mathbf{r},(y_i,y_{i+1})}$.*

For $1 \le i, j, k \le r$, let

$$S_{\mathbf{r},j}^k = \{\alpha \in \overline{\mathcal{O}}_Y^k \mid (\forall x \in Y)\ x\ {}_{y_k}\!\ge y_j \Longrightarrow x\alpha\ {}_{y_k}\!\ge y_j\},\ j \ne k+1\ (mod(r)),$$

$$T_{\mathbf{r},j}^k = \{\ \alpha \in \overline{\mathcal{O}}_Y^k \mid (\forall x \in Y)\ x \le_{y_k} y_j \Longrightarrow x\alpha \le_{y_k} y_j\},\ j \ne k\ (mod(r)).$$

$$S_j^k = \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid (\forall x \in X_n)\ x\ {}_{y_k}\!\ge y_j \Longrightarrow x\alpha\ {}_{y_k}\!\ge y_j \text{ and } \alpha|_Y \in \overline{\mathcal{O}}_Y^k\},\ j \ne k+1\ (mod(r)),$$

$$T_j^k = \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid (\forall x \in X_n)\ x \le_{y_k} y_j \Longrightarrow x\alpha \le_{y_k} y_j \text{ and } \alpha|_Y \in \overline{\mathcal{O}}_Y^k\},\ j \ne k\ (mod(r)).$$

Lemma 5.2. *Let $1 \le j, k \le r$. Then $S_j^k = E(J_r)S_{\mathbf{r},j}^k$ and $T_j^k = E(J_r)T_{\mathbf{r},j}^k$.*

Proof. Let $\alpha \in S_j^k$ be arbitrary. Then $\alpha|_Y \in \overline{\mathcal{O}}_Y^k$ and $(\forall\ x \in Y \subseteq X_n)\ x\ {}_{y_k}\!\ge y_j \Longrightarrow x\alpha\ {}_{y_k}\!\ge y_j$. Thus $\alpha|_Y \in S_{\mathbf{r},j}^k$. By Lemma 2.1, there exists $\lambda_\alpha \in E(J_r)$ such that $\alpha = \lambda_\alpha(\alpha|_Y)$. Then $\alpha = \lambda_\alpha(\alpha|_Y) \in E(J_r)S_{\mathbf{r},j}^k$. Thus $S_j^k \subseteq E(J_r)S_{\mathbf{r},j}^k$.

Conversely, let $\lambda \in E(J_r)$ and $\alpha \in S_{\mathbf{r},j}^k$ be arbitrary. Clearly $\lambda\alpha \in \mathcal{FOP}_n(Y)$. Since $\alpha \in S_{\mathbf{r},j}^k$ and $\lambda|_Y = 1_Y$, we have

$$(\lambda\alpha)|_Y = \alpha \in S_{\mathbf{r},j}^k \subseteq \overline{\mathcal{O}}_Y^k$$

and

$$x\ {}_{y_k}\!\ge y_j \Longrightarrow x\lambda\alpha = x\alpha\ {}_{y_k}\!\ge y_j\ (\forall x \in Y).$$

Then $\lambda\alpha \in S_j^k$. Thus $E(J_r)S_{\mathbf{r},j}^k \subseteq S_j^k$. Hence $S_j^k = E(J_r)S_{\mathbf{r},j}^k$. Similarly, we can prove that $T_j^k = E(J_r)T_{\mathbf{r},j}^k$. □

Lemma 5.3. *Let $1 \le i \le r$. Then*

$$E(J_r)M_{\mathbf{r},(y_i,y_{i+1})} = \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid \alpha|_Y \in \overline{\mathcal{O}}_Y^i\}.$$

Proof. Let $S = \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid \alpha|_Y \in \overline{\mathcal{O}}_Y^i\}$. Let $\alpha \in S$ be arbitrary. Then, Lemma 5.1, $\alpha|_Y \in \overline{\mathcal{O}}_Y^i = M_{\mathbf{r},(y_i,y_{i+1})}$. By Lemma 2.1, there exists $\lambda_\alpha \in E(J_r)$ such that $\alpha = \lambda_\alpha(\alpha|_Y)$. Then $\alpha = \lambda_\alpha(\alpha|_Y) \in E(J_r)M_{\mathbf{r},(y_i,y_{i+1})}$. Thus $S \subseteq E(J_r)M_{\mathbf{r},(y_i,y_{i+1})}$. Conversely, let $\lambda \in E(J_r)$ and $\alpha \in M_{\mathbf{r},(y_i,y_{i+1})}$ be arbitrary. Clearly $\lambda\alpha \in \mathcal{FOP}_n(Y)$. Then, by Lemma 5.1, $(\lambda\alpha)|_Y = \alpha \in M_{\mathbf{r},(y_i,y_{i+1})} = \overline{\mathcal{O}}_Y^i$ and so $\lambda\alpha \in S$. Thus $E(J_r)M_{\mathbf{r},(y_i,y_{i+1})} \subseteq S$. □

Recall that Zhao [10, Lemma 3.2 and Theorem 2.7] proved:

Lemma 5.4. *Each maximal subsemiband S of $\overline{\mathcal{OP}}_Y$ must be one of the following forms:*
(A) $S = \mathcal{I}_\mathbf{r}(r-2) \cup M_{r,(y_i,y_{i+1})}$, $1 \le i \le r$.
(B) $S = \mathcal{I}_\mathbf{r}(r-2) \cup S_{r,j}^i \cup T_{r,i}^{j-1}$, $j \ne i+1\ (mod(r))$, $1 \le i \le r$.

Now, it is easy to prove the following result:

Theorem 5.5. *Each maximal subsemiband S of $C(\mathcal{FOP}_n(Y))$ must be one of the following forms:*
(A) $S = C(\mathcal{FOP}_n(Y))\backslash\{\varepsilon\}$, $\varepsilon \in E(J_r)$.
(B) $S = Q_n(r-2) \cup E(J_r) \cup \{\alpha \in J_{r-1} \mid \alpha|_Y \in \overline{O}^i_Y\}$, $1 \leq i \leq r$.
(C) $S = Q_n(r-2) \cup E(J_r) \cup S^i_j \cup T^{j-1}_i$, $j \neq i+1\ (mod(r))$, $1 \leq i \leq r$.

Proof. Notice that $E(J_r)$ is a zero subsemigroup of $\mathcal{FOP}_n(Y)$ (by Lemma 2.5). It is obvious that, for $\varepsilon \in E(J_r)$, $C(\mathcal{FOP}_n(Y))\backslash\{\varepsilon\}$ is a maximal subsemiband of $C(\mathcal{FOP}_n(Y))$. Let S be a maximal subsemiband of $C(\mathcal{FOP}_n(Y))$. If $E(J_r)\backslash S \neq \emptyset$, then $S = C(\mathcal{FOP}_n(Y))\backslash\{\varepsilon\}$, for some $\varepsilon \in J_r$. Let $M_S = \{\alpha|_Y \mid \alpha \in S\}\backslash\{1_Y\}$. If $E(J_r) \subseteq S$, then, by Lemma 2.9, $S = M_S^{\Delta} = E(J_r)M_S^{1_Y}$. Thus, by the maximality of S and Theorem 2.10, M_S is a maximal subsemiband of $\overline{\mathcal{OP}}_Y$. Hence, by Lemmas 2.3, 5.2, 5.3, 5.4 and Theorem 2.10, S must be one of the following forms:

$$\begin{aligned} S &= E(J_r)[\mathcal{I}_{\mathbf{r}}(r-2) \cup M_{\mathbf{r},(y_i,y_{i+1})}]^{1_Y} \\ &= Q(r-2) \cup E(J_r) \cup E(J_r)M_{\mathbf{r},(y_i,y_{i+1})} \\ &= Q_n(r-2) \cup E(J_r) \cup \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid \alpha|_Y \in \overline{O}^i_Y\} \\ &= Q_n(r-2) \cup E(J_r) \cup \{\alpha \in J_{r-1} \mid \alpha|_Y \in \overline{O}^i_Y\} \end{aligned}$$

or

$$\begin{aligned} S &= E(J_r)[\mathcal{I}_{\mathbf{r}}(r-2) \cup (S^i_{\mathbf{r},j} \cup T^{j-1}_{\mathbf{r},i})]^{1_Y} \\ &= Q(r-2) \cup E(J_r) \cup E(J_r)(S^i_{\mathbf{r},j} \cup T^{j-1}_{\mathbf{r},i}) \\ &= Q_n(r-2) \cup E(J_r) \cup S^i_j \cup T^{j-1}_i. \end{aligned}$$

□

The following lemma was proved by Zhao [11, Lemma 2.4].

Lemma 5.6. *Let $1 \leq i \leq r$. Then $M_{\mathbf{r},(y_i,y_i)} = \{\alpha \in \overline{\mathcal{OP}}_Y \mid y_i\alpha = y_i\}$.*

With above lemma, we can prove the following lemma:

Lemma 5.7. *Let $1 \leq i \leq r$. Then*

$$E(J_r)M_{\mathbf{r},(y_i,y_i)} = \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid y_i\alpha = y_i\}.$$

Proof. Let $S = \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid y_i\alpha = y_i\}$. Let $\alpha \in S$ be arbitrary. Then, Lemma 5.6, $\alpha|_Y \in \{\alpha \in \overline{\mathcal{OP}}_Y \mid y_i\alpha = y_i\} = M_{\mathbf{r},(y_i,y_i)}$. By Lemma 2.1, there exists $\lambda_\alpha \in E(J_r)$ such that $\alpha = \lambda_\alpha(\alpha|_Y)$. Then $\alpha = \lambda_\alpha(\alpha|_Y) \in E(J_r)M_{\mathbf{r},(y_i,y_i)}$. Thus $S \subseteq E(J_r)M_{\mathbf{r},(y_i,y_i)}$. Conversely, let $\lambda \in E(J_r)$ and $\alpha \in M_{\mathbf{r},(y_i,y_i)}$ be arbitrary. Then, by Lemma 5.6, $y_i\alpha = y_i$ and so

$$y_i(\lambda\alpha) = (y_i\lambda)\alpha = y_i\alpha = y_i.$$

Thus $\lambda\alpha \in S$. Hence $E(J_r)M_{\mathbf{r},(y_i,y_i)} \subseteq S$. □

Recall again that Zhao [11, Lemmas 2.4, 2.12 and Theorem 2.2] proved:

Lemma 5.8. *Each maximal regular subsemiband S of $\overline{\mathcal{OP}}_Y$ must be one of the following forms:*
(A) $S = \mathcal{I}_{\mathbf{r}}(r-2) \cup M_{\mathbf{r},(y_i,y_{i+1})}$, $1 \leq i \leq r$.
(B) $S = \mathcal{I}_{\mathbf{r}}(r-2) \cup M_{\mathbf{r},(y_i,y_i)}$, $1 \leq i \leq r$.

Finally, we can prove the following theorem:

Theorem 5.9. *Each maximal regular subsemiband S of $C(\mathcal{FOP}_n(Y))$ must be one of the following forms:*
(*A*) $S = C(\mathcal{FOP}_n(Y))\backslash\{\varepsilon\}$, $\varepsilon \in E(J_r)$.
(*B*) $S = \mathcal{Q}_n(r-2) \cup E(J_r) \cup \{\alpha \in J_{r-1} \mid \alpha|_Y \in \overline{O}_Y^i\}$, $1 \le i \le r$.
(*C*) $S = \mathcal{Q}_n(r-2) \cup E(J_r) \cup \{\alpha \in J_{r-1} \mid y_i\alpha = y_i\}$, $1 \le i \le r$.

Proof. Notice that $E(J_r)$ is a zero subsemigroup of $\mathcal{FOP}_n(Y)$ (by Lemma 2.5). It is obvious that, for $\varepsilon \in E(J_r)$, $C(\mathcal{FOP}_n(Y))\backslash\{\varepsilon\}$ is a maximal regular subsemiband of $C(\mathcal{FOP}_n(Y))$. Let S be a maximal regular subsemiband of $C(\mathcal{FOP}_n(Y))$. If $E(J_r)\backslash S \neq \emptyset$, then $S = C(\mathcal{FOP}_n(Y))\backslash\{\varepsilon\}$, for some $\varepsilon \in E(J_r)$. Let $M_S = \{\alpha|_Y \mid \alpha \in S\}\backslash\{1_Y\}$. If $E(J_r) \subseteq S$, then, by Lemma 2.9, $S = M_S^{\Delta} = E(J_r)M_S^{1_Y}$. Thus, by the maximality of S and Theorem 2.10, M_S is a maximal regular subsemiband of $\overline{\mathcal{OP}}_Y$. Hence, by Lemmas 2.3, 5.3, 5.7, 5.8 and Theorem 2.10, S must be one of the following forms:

$$\begin{aligned} S &= E(J_r)[\mathcal{I}_{\mathbf{r}}(r-2) \cup M_{\mathbf{r},(y_i,y_{i+1})}]^{1_Y} \\ &= \mathcal{Q}(r-2) \cup E(J_r) \cup E(J_r)M_{\mathbf{r},(y_i,y_{i+1})} \\ &= \mathcal{Q}_n(r-2) \cup E(J_r) \cup \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid \alpha|_Y \in \overline{O}_Y^i\} \\ &= \mathcal{Q}_n(r-2) \cup E(J_r) \cup \{\alpha \in J_{r-1} \mid \alpha|_Y \in \overline{O}_Y^i\} \end{aligned}$$

or

$$\begin{aligned} S &= E(J_r)[\mathcal{I}_{\mathbf{r}}(r-2) \cup M_{\mathbf{r},(y_i,y_i)}]^{1_Y} \\ &= \mathcal{Q}(r-2) \cup E(J_r) \cup E(J_r)M_{\mathbf{r},(y_i,y_i)} \\ &= \mathcal{Q}_n(r-2) \cup E(J_r) \cup \{\alpha \in C(\mathcal{FOP}_n(Y)) \mid y_i\alpha = y_i\} \\ &= \mathcal{Q}_n(r-2) \cup E(J_r) \cup \{\alpha \in J_{r-1} \mid y_i\alpha = y_i\}. \end{aligned}$$

□

Acknowledgments We wish to thank the anonymous referee for her/his valuable suggestions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] R.E. Arthur, N. Ruškuc, Presentations for two extensions of the monoid of order-preserving mappings on a finite chain, Southeast Asian Bull. Math., 24(1)(2000), 1–7.
[2] P.M. Catarino, Monoids of orientation-preserving transformations of a finite chain and their presentation, Semigroups and Applications, J.M. Howie and N. Ruškuc, eds., World Scientific, Singapore, 1998, pp. 39–46.
[3] P.M. Catarino, P.M. Higgins, The monoid of orientation-preserving mappings on a chain, Semigroup Forum, 58(2)(1999),190–206.
[4] I. Dimitrova, V.H. Fernandes, J. Koppitz, The maximal subsemigroups of semigroups of transformations preserving or reversing the orientation on a finite chain, Publicationes Mathematicae Debrecen, 81(1-2)(2012), 11–29.
[5] V.H. Fernandes, G.M.S. Gomes, M.M. Jesus, Congruences on monoids of transformation preserving the orientation on a finite chain, J. Algebra, 321(3)(2009), 743–757.
[6] V.H. Fernandes, P. Honyam, T.M. Quinteiro, B. Singha, On Semigroups of Orientation-preserving Transformations with Restricted Range, Comm. Algebra, 44(1)(2016), 253–264.
[7] G.U. Garba, On the idempotent ranks of certain semigroups of order-preserving transformations, Portugaliae Mathematica, 51(2)(1994), 185–204.
[8] J.M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, London, 2003.
[9] D.B. McAlister, Semigroups generated by a group and an idempotent, Comm. Algebra, 26(2)(1998), 515–547.
[10] P. Zhao, A classifification of maximal idempotent-generated subsemigroups of singular orientation-preserving transformation semigroups, Semigroup Forum, 79(2)(2009), 377–384.
[11] P. Zhao, Locally maximal subsemibands of SOP_n, Bull. Malays. Math. Sci. Soc., 37(4)(2014), 881–891.
[12] P. Zhao, V.H. Fernandes, The ranks of ideals in various transformation monoids, Comm. Algebra, 43(2)(2015), 674–692.
[13] P. Zhao, B. Xu, Mei, Y., Locally maximal idempotent-generated subsemigroups of singular orientation-preserving transformation semigroups, Semigroup Forum, 77(2)(2008), 187–195.