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Hermitian elements and solutions of related equations in a ring with involution

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Abstract. In this paper, we mainly give some new equivalence portrayals of Hermitian elements in a ring with involution. Firstly, we discuss some properties of Hermitian elements by means of Moore-Penrose inverses, invertible elements and EP elements. Next, the Hermitian element is deconstructed by constructing equations on the ring so that it has a solution on a specific set. Finally, we characterize Hermitian elements by constructing the group inverses and MP inverses.

1. Introduction

Let R be a ring and $a \in R$. If there exists $b \in R$ such that

$$aba = a,\ bab = a,\ ab = ba,$$

then a is called a group invertible element of R and b is called a group inverse of a [4, 8, 9], and it is unique, usually we write it by $a^{\#}$. We write $R^{\#}$ to denote the set of all group invertible elements of R.

If a map $* : R \to R$ satisfies

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^* \text{ for a, b} \in \text{R},$$

then R is said to be an involution ring or a $*-$ring.

Let R be a $*$-ring and $a \in R$. If there exists $b \in R$ such that

$$a = aba,\ b = bab,\ (ab)^* = ab,\ (ba)^* = ba,$$

then a is called a Moore Penrose invertible element, and b is called the Moore Penrese inverse of a [3, 6], and it is unique, usually we record it as a^+. Let R^+ denote the set of all Moore Penrese invertible elements of R.

If $a \in R^{\#} \cap R^+$ and $a^{\#} = a^+$, then a is called an EP element. On the studies of EP, the readers can refer to [2, 3, 5, 7, 10–14].

If $a \in R$ and $a = a^*$, then a is called Hermitian element. We write R^{Her} to denote the set of all Hermitian elements of R. Clearly, if $a \in R^+$ is Hermitian, then $a^{\#} = a^+$. [11].

The research hotspots of Hermitian elements are mainly matrix directions, and this paper gives some new portrayals of Hermitian elements from the perspective of ring theory. In [11], many characterizations of Hermitian elements are given. Motivated by these references, this paper mainly study the ways to characterize Hermitian elements.

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2. Some characterizations of Hermitian elements

Lemma 2.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^*a^+a^{\#} = a^{\#}aa^+$.*

Proof. " $\Rightarrow$ " Assume that $a \in R^{Her}$. Then $a = a^*$ and $a^{\#} = a^+$. It follows that

$$a^*a^+a^{\#} = aa^+a^{\#} = a^{\#} = a^{\#}aa^+.$$

" $\Leftarrow$ " Since $a^*a^+a^{\#} = a^{\#}aa^+$, $a^*a^+a = (a^*a^+a^{\#})a^2 = (a^{\#}aa^+)a^2 = a$.
Then, multiplying $a^*a^+a = a$ by a^+ from the right, we have

$$a^*a^+ = aa^+.$$

Hence $a \in R^{Her}$ by [11, Theorem 1.4.1]. □

Theorem 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $aa^+a^+a^{\#} = (a^+)^*a^+$.*

Proof. " $\Rightarrow$ " Since $a \in R^{Her}$, $a^*a^+a^{\#} = a^{\#}aa^+$ by Lemma 2.1. Multiplying the equality on the left by $(a^+)^*$, one has

$$aa^+a^+a^{\#} = (a^+)^*a^{\#}aa^+.$$

Noting that $(a^+)^*a^{\#}a = (a^+)^*$. Then $aa^+a^+a^{\#} = (a^+)^*a^+$.
" $\Leftarrow$ " From the equality $aa^+a^+a^{\#} = (a^+)^*a^+$, we obtain

$$a^*a^+a^{\#} = a^*(aa^+a^+a^{\#}) = a^*(a^+)^*a^+ = a^+.$$

Hence $a \in R^{Her}$ by [11, Theorem 1.4.2]. □

Noting that $aa^+(a^+)^* = (a^+)^*$. Then Theorem 2.2 leads to the following corollary.

Corollary 2.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^+a^+a^{\#} = a^+(a^+)^*a^+$.*

Multiplying the equality of Corollary 2.3 on the left by $(a^{\#}a)^*a$, one has $a^+a^{\#} = (a^{\#})^*a^+$, then we have:

Corollary 2.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^+a^{\#} = (a^{\#})^*a^+$.*

Proof. " $\Rightarrow$ " Since $a \in R^{Her}$, $a^* = a$ and $a^+ = a^{\#}$. This infers

$$(a^{\#})^*a^+ = (a^*)^{\#}a^+ = a^{\#}a^+ = a^+a^{\#}.$$

" $\Leftarrow$ " From the condition $a^+a^{\#} = (a^{\#})^*a^+$, one yields

$$a^+a^{\#} = ((a^{\#})^*a^+)aa^+ = a^+a^{\#}aa^+.$$

Hence $a \in R^{EP}$, this gives

$$a^{\#} = a^{\#}a^{\#}a = a^+a^{\#}a = (a^{\#})^*a^+a = (a^+)^*a^+a = (a^+)^* = (a^{\#})^*.$$

Thus $a \in R^{Her}$. □

Noting that $a^+a(a^{\#})^* = (a^{\#})^*$ and $a^{\#} = aa^+a^{\#}$. Then Corollary 2.4 infers the following corollary.

Corollary 2.5. *Let $a \in R^{\#} \cap R^{+}$. Then the followings are equivalent:*
(1) $a \in R^{Her}$;
(2) $a^{\#} = a(a^{\#})^*a^+$;
(3) $a^+ = a(a^{\#})^*a^+$.

Proof. (1) $\Leftrightarrow$ (2) It follows from Corollary 2.4.
(2) $\Rightarrow$ (3) Since $a^{\#} = a(a^{\#})^*a^+$, we have

$$aa^+ = aa^{\#}aa^+ = a^2(a^{\#})^*a^+aa^+ = a^2(a^{\#})^*a^+ = aa^{\#}.$$

Hence $a \in R^{EP}$, this implies $a^+ = a^{\#} = a(a^{\#})^*a^+$.
(3) $\Rightarrow$ (2) Suppose that $a^+ = a(a^{\#})^*a^+$. Then we have

$$a^+a = a((a^{\#})^*a^+a) = a^{\#}a(a(a^{\#})^*a^+a) = a^{\#}aa^+a = a^{\#}a.$$

Hence $a \in R^{EP}$. Then $a^{\#} = a^+ = a(a^{\#})^*a^+$. □

3. Construct Moore-Penrose inverses to characterize Hermitian elements

Lemma 2.1 inspires us to give the following lemma.

Lemma 3.1. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $(a^*a^+a^{\#})^+ = a^+a^3(a^{\#})^*a^+a$;
(2) $(a^{\#}aa^+)^+ = a^2a^+$.

Proof. It is routine. □

Theorem 3.2. *Let $a \in R^{\#} \cap R^{+}$. Then the followings are equivalent:*
(1) $a \in R^{Her}$;
(2) $a^+a^3(a^{\#})^*a^+a = a^2a^+$;
(3) $(a^*a^+a^{\#})^+ = a(a^{\#})^*a^*$;
(4) $(a^{\#}aa^+)^+ = a^*aa^{\#}$.

Proof. (1) ⇔ (2) It follows from Lemma 2.1 and Lemma 3.1.
(1) ⇒ (3) Assume that $a \in R^{Her}$. Then $a = a^*$, $a^+ = a^{\#}$ and

$$(a^*a^+a^{\#})^+ = (aa^+a^{\#})^+ = (a^{\#})^+ = (a^+)^+ = a = aa^{\#}a = a(a^{\#}a)^* = a(a^{\#})^*a^*.$$

(3) ⇒ (1) From the assumption, we have

$$a(a^{\#})^*a^* = (a^*a^+a^{\#})^+ = a^+a^3(a^{\#})^*a^+a = a^+a(a^+a^3(a^{\#})^*a^+a) = a^+a^2(a^{\#})^*a^*.$$

Multiplying the equality on the right by a^+, one has $aa^+ = a^+a^2a^+$. Hence $a \in R^{EP}$, this leads to

$$a = a(a^{\#})^*a^* = a^+a^3(a^{\#})^*a^+a = a^2(a^{\#})^*$$

and

$$aa^* = a^2(a^{\#})^*a^* = a^2.$$

Thus $a \in R^{Her}$ by [11, Theorem 1.4.1].
(1) ⇒ (4) Suppose that $a \in R^{Her}$. Then $a = a^*$ and $a^+ = a^{\#}$. By Lemma 3.1, we have $(a^{\#}aa^+)^+ = a^2a^+ = a^*aa^{\#}$.
(4) ⇒ (1) From the assumption and Lemma 3.1, we get

$$a^2a^+ = a^*aa^{\#} = (a^*aa^{\#})a^+a = a^2a^+a^+a.$$

It follows that $aa^+ = a^{\#}(a^2a^+) = a^{\#}(a^2a^+a^+a) = aa^+a^+a$. Hence $a \in R^{EP}$. This induces $a = a^2a^+ = a^*aa^{\#} = a^*aa^+ = a^*$. Thus $a \in R^{Her}$. □

Lemma 3.3. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $(a(a^{\#})^*a)^+ = a^+a^*a^+$;
(2) $(a(a^{\#})^*a)^{\#} = a^{\#}aa^+a^*a^+aa^{\#}$;
(3) $(a^*aa^{\#})^+ = a^+a(a^+)^*$.

Proof. It is routine. □

Theorem 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then the followings are equivalent:*
(1) $a \in R^{Her}$;
(2) $a^*a^+a^{\#} = a^+a^*a^+$;
(3) $a^*a^+a^{\#} = a^{\#}aa^+a^*a^+aa^{\#}$;
(4) $a^{\#}aa^+ = a^+a(a^+)^*$.

Proof. It is an immediate result of Theorem 3.2 and Lemma 3.3. □

4. Using EP elements to characterize Hermitian elements

Theorem 4.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a \in R^{EP}$ and $a^*aa^{\#} \in R^{Her}$.*

Proof. " ⇒ " Suppose that $a \in R^{Her}$. Then, certainly, $a \in R^{EP}$ and $a^* = a$. This gives

$$a^*aa^{\#} = a^2a^{\#} = a \in R^{Her}.$$

" ⇐ " Since $a^*aa^{\#} \in R^{Her}, a^*aa^{\#} = (a^*aa^{\#})^* = a^*(a^{\#})^*a$. Multiplying the equality on the left by $(a^+)^*$, one gets

$$aa^{\#} = aa^+(a^{\#})^*a.$$

Noting that $a \in R^{EP}$. Then $aa^+(a^{\#})^* = (a^{\#})^*$. It follows $aa^{\#} = (a^{\#})^*a$.
Hence

$$a^{\#} = aa^{\#}a^{\#} = (a^{\#})^*aa^{\#} = (a^{\#})^*aa^+ = (a^{\#})^*.$$

Thus $a \in R^{Her}$. □

It is well known that $a \in R^{Her}$ if and only if $a^+ \in R^{Her}$. From Lemma 3.3 and Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a \in R^{EP}$ and $a^+a(a^+)^* \in R^{Her}$.*

Noting that $a \in R^{Her}$ if and only if $a^* \in R^{Her}$. Then Theorem 4.1 implies the following corollary.

Corollary 4.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a \in R^{EP}$ and $(aa^{\#})^*a \in R^{Her}$.*

Theorem 4.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a \in R^{EP}$ and $aa^+(a^{\#})^* \in R^{Her}$.*

Proof. " ⇒ " Since $a \in R^{Her}$, $a^* = a$ and $(a^{\#})^* = a^{\#} = a^+$. It follows that

$$aa^+(a^{\#})^* = aa^+a^{\#} = a^{\#} = a^{\#}aa^+ = (aa^+(a^{\#})^*)^*.$$

Hence $aa^+(a^{\#})^* \in R^{Her}$.
" ⇐ " Suppose that $aa^+(a^{\#})^* \in R^{Her}$ and $a \in R^{EP}$. Then

$$(a^{\#})^* = aa^+(a^{\#})^* = a^{\#}aa^+ = a^{\#}.$$

Hence $a \in R^{Her}$. □

Noting that $(aa^+(a^{\#})^*)^+ = aa^+a^*$. Then Theorem 4.4 induces the following corollary.

Corollary 4.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a \in R^{EP}$ and $aa^+a^* \in R^{Her}$.*

5. Using invertible elements to characterize Hermitian elements

It is well known that if $a \in R^{\#}$, then $a + 1 - aa^{\#} \in R^{-1}$ and $(a + 1 - aa^{\#})^{-1} = a^{\#} + 1 - aa^{\#}$. This implies us to give the following lemma by Lemma 3.3.

Lemma 5.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a^*aa^{\#} + 1 - a^+a \in R^{-1}$ and $(a^*aa^{\#} + 1 - a^+a)^{-1} = a^+a(a^+)^* + 1 - a^+a$.*

From Lemma 5.1 and Theorem 3.4, we have the following theorem.

Theorem 5.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $(a^*aa^{\#} + 1 - a^+a)^{-1} = a^{\#}aa^+ + 1 - a^+a$.*

Theorem 5.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $aa^*aa^{\#}a^+ + 1 - aa^+ \in R^{-1}$ and $(aa^*aa^{\#}a^+ + 1 - aa^+)^{-1} = a^{\#} + 1 - aa^+$.*

Proof. " ⇒ " Assume that $a \in R^{Her}$. Then by Theorem 5.2, we have

$$(a^*aa^\# + 1 - a^+a)^{-1} = a^\#aa^+ + 1 - a^+a.$$

Since

$$a^*aa^\# + 1 - a^+a = 1 - a^+(a - aa^*aa^\#),$$

and

$$(1 - a^+(a - aa^*aa^\#))^{-1} = a^\#aa^+ + 1 - a^+a,$$

it follows that

$$\begin{aligned}(1 - (a - aa^*aa^\#)a^+)^{-1} &= 1 + (a - aa^*aa^\#)(1 - a^+(a - aa^*aa^\#))^{-1}a^+ \\ &= 1 + (a - aa^*aa^\#)(a^\#aa^+ + 1 - a^+a)a^+ \\ &= 1 + (a - aa^*aa^\#)(a^\#aa^+a^+) \\ &= aa^+a^+ + 1 - aa^*aa^\#a^+a^+.\end{aligned}$$

That is

$$(aa^*aa^\#a^+ + 1 - aa^+)^{-1} = aa^+a^+ + 1 - aa^*aa^\#a^+a^+.$$

Since $a^* = a$ and $a^\# = a^+$, one has $aa^+a^+ = aa^\#a^\# = a^\#$ and $aa^*aa^\#a^+a^+ = a^3a^\#a^\#a^+ = aa^+$. Hence, $(aa^*aa^\#a^+ + 1 - aa^+)^{-1} = a^\# + 1 - aa^+$.

" ⇐ " From the assumption, we get

$$\begin{aligned}1 &= (aa^*aa^\#a^+ + 1 - aa^+)(a^\# + 1 - aa^+) \\ &= aa^*a^\#a^\# + 1 - aa^+.\end{aligned}$$

This gives $aa^*a^\#a^\# = aa^+$. So

$$a^+ = a^+aa^+ = a^+aa^*a^\#a^\# = a^*a^\#a^\# = (a^*a^\#a^\#)a^+a = a^+a^+a.$$

Hence $a \in R^{EP}$ and $a^+ = a^*a^\#a^\# = a^*a^+a^\#$. By [11, Theorem 1.4.2], $a \in R^{Her}$. □

Noting that $(aa^*aa^\#a^+ + 1 - aa^+)^{-1} = a(a^+)^*a^+ + 1 - aa^+$. Then Theorem 5.3 leads to the following corollary.

Corollary 5.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^\# = a(a^+)^*a^+$.*

Corollary 5.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^\#a^\# = (a^+)^*a^+$.*

Proof. It is an immediate result of Corollary 5.4. □

Noting that $(aa^*)^+ = (a^+)^*a^+$. Then Corollary 5.5 implies the following corollary.

Corollary 5.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^\#a^\# = (aa^*)^+$.*

Since $(a^\#a^\#)^+ = a^+a^4a^+$, we have the following corollary by Corollary 5.6.

Corollary 5.7. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^+a^4a^+ = aa^*$.*

Theorem 5.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^+a^*a^*a^+a = a^*$.*

Proof. $\Longrightarrow$ Since $a \in R^{Her}$, $a^+a^4a^+ = aa^*$ by Corollary 5.7. Applying the involution on the equality, one has

$$aa^+a^*a^*a^+a = aa^*.$$

Multiplying the last equality on the left by a^+, one yields $a^+a^*a^*a^+a = a^*$.

$\Longleftarrow$ Assume that $a^+a^*a^*a^+a = a^*$. Multiplying the equality on the right by $(a^\#)^*$, one obtains

$$a^+a^* = (aa^\#)^*,$$

this gives

$$a^+ = a^+a^*(a^\#)^* = (aa^\#)^*(a^\#)^* = (a^\#)^*.$$

Hence

$$a^* = a^+a^*a^*a^+a = a^+a^*a^*(a^\#)^*a = a^+a^*a = (aa^\#)^*a.$$

Applying the involution on the last equality, we have

$$a = a^*aa^\#.$$

Thus $a \in R^{Her}$ by [11, Theorem 1.4.2]. $\square$

Corollary 5.9. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if $a^+a^3(a^+)^*a^\# = aa^+$.*

Proof. $\Longrightarrow$ Since $a \in R^{Her}$, $a^+a^4a^+ = aa^*$ by Corollary 5.7. Multiplying the equality on the right by $(a^+)^*a^+$, one has

$$a^+a^3(a^+)^*a^+ = aa^+.$$

Noting that $a \in R^{EP}$, Then one yields $a^+a^3(a^+)^*a^\# = aa^+$.

$\Longleftarrow$ Assume that $a^+a^3(a^+)^*a^\# = aa^+$. Multiplying the equality on the right by a^2, one obtains

$$a^+a^3(a^+)^*a = a^2.$$

Applying the involution on the last equality, one has

$$a^*a^* = a^*a^+a^*a^*a^+a.$$

This gives

$$a^* = (a^\#)^*a^*a^* = (a^\#)^*a^*a^+a^*a^*a^+a = a^+a^*a^*a^+a.$$

Thus $a \in R^{Her}$ by Theorem 5.8. $\square$

6. Characterizing Hermitian elements by the solution of univariate equations in a given set

Observing Lemma 2.1, we can establish the following equation:

$$a^*xa^\# = a^\#ax. \tag{1}$$

Theorem 6.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(6.1) has at least one solution in $\rho_a = \{a, a^\#, a^+, a^*, (a^+)^*, (a^\#)^*, (a^+)^\#, (a^\#)^+\}$.*

Proof. " $\Rightarrow$ " If $a \in R^{Her}$, then $x = a^+$ is a solution by Lemma 2.1.

" $\Leftarrow$ " (1) If $x = a$ is a solution, then $a^*aa^\# = a^\#aa = a$. Multiplying the equality by a from the right side, we have

$$a^*a = a^2.$$

Hence $a \in R^{Her}$ by [11, Theorem 1.4.1];

(2) If $x = a^\#$ is a solution, then $a^*a^\#a^\# = a^\#aa^\# = a^\#$. Multiplying the equality by a from the right side, we have

$$a^*a^\# = a^\#a.$$

Hence $a \in R^{Her}$ by [11, Theorem 1.4.2];

(3) If $x = a^+$ is a solution, then $a^*a^+a^\# = a^\#aa^+$, it follows from Lemma 2.1 that $a \in R^{Her}$;

(4) If $x = a^*$ is a solution, then $a^*a^*a^\# = a^\#aa^*$. Multiplying the equality on the left by $(a^+)^*$, one yields

$$(a^2a^+)^*a^\# = aa^+.$$

Now,

$$aa^\# = (aa^+)aa^\# = (a^2a^+)^*a^\#aa^\# = (a^2a^+)^*a^\# = aa^+.$$

Hence $a \in R^{EP}$ and $a^\# = a^+$.
Therefore

$$a^*a^*a^\# = a^\#aa^* = a^+aa^* = a^*.$$

Thus $a \in R^{Her}$ by [11, Theorem 1.4.2];

(5) If $x = (a^+)^*$ is a solution, then $a^*(a^+)^*a^\# = a^\#a(a^+)^* = (a^+)^*$. Multiplying the equality on the right by a, one gets

$$a^+a = (a^+)^*a.$$

Now,

$$a^\#a = a^\#aa^+a = a^\#a(a^+)^*a = (a^+)^*a = a^+a.$$

Then $a \in R^{EP}$ and $a^+ = a^*(a^+)^*a^\# = (a^+)^*$. Hence $a \in R^{Her}$.

(6) If $x = (a^\#)^*$ is a solution, then $a^*(a^\#)^*a^\# = a^\#a(a^\#)^*$. Multiplying the equality on the left by a^2a^+, one yields

$$aa^\# = a(a^\#)^*.$$

Therefore

$$aa^+ = aa^\#aa^+ = a(a^\#)^*aa^+ = a(a^\#)^* = aa^\#.$$

Then $a \in R^{EP}$ and $a^+ = a^\#$, we have $(a^\#)^* = (a^+)^*$. Thus $a \in R^{Her}$ by (5).

(7) If $x = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$ is a solution, then $a^*(aa^\#)^*a(aa^\#)^*a^\# = a^\#a(aa^\#)^*a(aa^\#)^*$. Multiplying the equality on the left by a^+a, one obtains

$$a^*a(aa^\#)^*a^\# = (aa^\#)^*a(aa^\#)^*.$$

It follows that

$$a^\#a(aa^\#)^*a(aa^\#)^* = (aa^\#)^*a(aa^\#)^*.$$

Multiplying the equality on the right by a^+, one gets

$$(a^\#a)(aa^\#)^* = (aa^\#)^*.$$

Hence $aa^\#$ is Hermitian, this infers $a \in R^{EP}$ by [11, Theorem 1.1.3]. Thus $x = (a^+)^\# = (a^\#)^\# = a$, one obtains $a \in R^{Her}$ by (1).

(8) If $x = (a^\#)^+ = a^+a^3a^+$ is a solution, then $a^*a^+a^3a^+a^\# = a^\#aa^+a^3a^+$.
e.g.,

$$a^*a^+a = a^2a^+.$$

Multiplying the equality on the left by $(a^\#)^*$, one has

$$a^+a = (a^\#)^*a^2a^+ = ((a^\#)^*a^2a^+)(aa^\#)^* = a^+a(aa^\#)^* = (aa^\#)^*.$$

It follows that $aa^+ = aa^\#$. Hence $a \in R^{EP}$, this infers $x = (a^\#)^+ = (a^+)^+ = a$.
Thus $a \in R^{Her}$ by (1). □

Since $a \in R^{Her}$ if and only if $a^* \in R^{Her}$. Replacing a in Eq.(6.1) by a^* , one gets

$$ax(a^{\#})^* = (a^{\#})^* a^* x. \tag{2}$$

Corollary 6.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if Eq.(6.2) has at least one solution in ρ_a.*

Now we construct the following equation

$$(a^+)^* xa = (aa^{\#})^* x. \tag{3}$$

Theorem 6.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if Eq.(6.3) has at least one solution in ρ_a.*

Proof. " $\Rightarrow$ " If $a \in R^{Her}$, then $a = a^*$. It is easy to check that $x = a^*$ is a solution.

" $\Leftarrow$ " (1) If $x = a$, then $(a^+)^* aa = (aa^{\#})^* a$. Multiplying the equality on the left by $aa^{\#}$, we have

$$(a^+)^* aa = (aa^{\#})(aa^{\#})^* a.$$

It follows that

$$(aa^{\#})^* a = (aa^{\#})(aa^{\#})^* a.$$

Again multiplying the last equality by a^+ from the right side, we get

$$(aa^{\#})^* = (aa^{\#})(aa^{\#})^*.$$

This gives $aa^{\#}$ is Hermitian. Then $a \in R^{EP}$, one has

$$(a^{\#})^* a^2 = (a^+)^* aa = (aa^{\#})^* a = (aa^+)^* a = a.$$

Applying the involution to the above equality, we obtain $a^* a^* a^{\#} = a^*$. Thus $a \in R^{Her}$ by [11, Theorem 1.4.2];

(2) If $x = a^{\#}$, then $(a^+)^* a^{\#} a = (aa^{\#})^* a^{\#}$. Multiplying the equality by a^2 from the right side, one gets

$$(a^+)^* a^2 = (aa^{\#})^* a.$$

Then $a \in R^{Her}$ by (1);

(3) If $x = a^+$, then $(a^+)^* a^+ a = (aa^{\#})^* a^+$. Therefore $(a^+)^* = a^+$. Hence $a \in R^{Her}$;

(4) If $x = a^*$, then $(a^+)^* a^* a = (aa^{\#})^* a^*$. Therefore $a = a^*$. Thus $a \in R^{Her}$;

(5) If $x = (a^+)^*$, then $(a^+)^*(a^+)^* a = (aa^{\#})^*(a^+)^*$. Applying the involution on the equality, we get

$$a^* a^+ a^+ = a^+ aa^{\#}.$$

Multiplying the equality on the right by aa^+, one yields

$$a^+ aa^{\#} = a^+.$$

Then $a \in R^{EP}$, it follows that $a^* a^+ a^+ = a^{\#}$. Thus $a \in R^{Her}$ by [11, Theorem 1.4.2];

(6) If $x = (a^{\#})^*$, then $(a^+)^*(a^{\#})^* a = (aa^{\#})^*(a^{\#})^*$. Taking involution of the equality, we have

$$a^* a^{\#} a^+ = a^{\#}.$$

Multiplying the equality by a from the right side, one gets

$$a^* a^{\#} = a^{\#} a.$$

Thus $a \in R^{Her}$ by [11, Theorem 1.4.2];

(7) If $x = (a^+)^{\#} = (aa^{\#})^* a (aa^{\#})^*$, then $(a^+)^*(aa^{\#})^* a (aa^{\#})^* a = (aa^{\#})^*(aa^{\#})^* a (aa^{\#})^*$. Multiplying the equality on the right by $a^+ a^+ a^*$, we obtain

$$aa^+ = a^+ a^*.$$

Taking the involution of this equality, we get $aa^+ = a(a^+)^*$. Therefore,

$$aa^\# = aa^+aa^\# = a(a^+)^*aa^\# = a(a^+)^* = aa^+.$$

Then $a \in R^{EP}$, it follows that $x = (a^+)^\# = (a^\#)^\# = a$. Thus $a \in R^{Her}$ by (1);

(8) If $x = (a^\#)^+ = a^+a^3a^+$, then $(a^+)^*a^+a^3a^+a = (aa^\#)^*a^+a^3a^+$. Multiplying the equality on the right by $a^\#$, one has

$$(a^+)^*a = a^+a.$$

Therefore,

$$a^\#a = a^\#aa^+a = a^\#a(a^+)^*a = (a^+)^*a = a^+a.$$

Then $a \in R^{EP}$, one gets $x = (a^\#)^+ = (a^+)^+ = a$. Thus $a \in R^{Her}$ by (1). □

Applying the involution on Eq.(6.3) , we get

$$a^*xa^+ = xaa^\#. \tag{4}$$

Corollary 6.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(6.4) has at least one solution in ρ_a.*

Multiplying Eq.(6.4) on the right by a, and then revise as follows:

$$a^*xaa^+ = xa. \tag{5}$$

Theorem 6.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(6.5) has at least one solution in ρ_a.*

7. The general solution of bivariate equations

Now we generalize Eq.(6.1) as follows

$$a^*xa^\# = a^\#ay. \tag{6}$$

Theorem 7.1. *Let $a \in R^\# \cap R^+$. Then the general solution of Eq.(7.1) is given by*

$$\begin{cases} x = (a^\#)^*a^+p + u - aa^+uaa^+ \\ y = a^+pa^\# + v - a^+av \end{cases}, \text{ where } p,\ u,\ v \in R \text{ with } a^+p = aa^+a^+p. \tag{7}$$

Proof. First

$$\begin{aligned} a^*((a^\#)^*a^+p + u - aa^+uaa^+)a^\# &= a^+pa^\# = aa^+a^+pa^\# \\ &= a^\#aaa^+a^+pa^\# = a^\#aa^+pa^\# \\ &= a^\#a(a^+pa^\# + v - a^+av). \end{aligned}$$

It follows that the formula (7.2) is the solution of Eq.(7.1).

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(7.1). Then

$$a^*x_0a^\# = a^\#ay_0.$$

Choose $p = a(aa^\#)^*a^\#ay_0a$, $u = x_0 - (a^\#)^*a^+p$ and $v = y_0 - a^+pa^\#$.

Then

$$\begin{aligned} a^+p &= a^+a(aa^\#)^*a^\#ay_0a = (aa^\#)^*a^\#ay_0a \\ &= (aa^\#)^*(a^*x_0a^\#)a = a^*x_0a^\#a = a^\#ay_0a, \end{aligned}$$

and

$$aa^+a^+p = aa^+(a^\# ay_0a) = a^\# ay_0a = a^+p.$$

Since

$$\begin{aligned} aa^+uaa^+ &= aa^+(x_0 - (a^\#)^*a^+p)aa^+ = aa^+x_0aa^+ - aa^+(a^\#)^*a^+paa^+ \\ &= aa^+x_0aa^+ - aa^+(a^\#)^*a^\# ay_0a^2a^+ = aa^+x_0aa^+ - aa^+(a^\#)^*a^*x_0a^\# a^2a^+ \\ &= aa^+x_0aa^+ - aa^+x_0aa^+ = 0. \end{aligned}$$

It follows that

$$x_0 = (a^\#)^*a^+p + u - aa^+uaa^+,$$

and

$$\begin{aligned} a^+av &= a^+a(y_0 - a^+pa^\#) = a^+ay_0 - a^+pa^\# \\ &= a^+ay_0 - a^\# ay_0aa^\# = a^+ay_0 - a^*x_0a^\# aa^\# \\ &= a^+ay_0 - a^+a(a^*x_0a^\#) = a^+ay_0 - a^+a(a^\# ay_0) \\ &= a^+ay_0 - a^+ay_0 = 0. \end{aligned}$$

Then

$$y_0 = a^+pa^\# + v - a^+av.$$

Hence the general solution of Eq.(7.1) is given by the formula (7.2). □

Theorem 7.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if the general solution of Eq.(7.1) is given by*

$$\begin{cases} x = (a^\#)^*a^+p + u - aa^+uaa^+ \\ y = a^+p(a^\#)^* + v - a^+av \end{cases}, \text{where } p, u, v \in R. \tag{8}$$

Proof. " ⇒ " Since $a \in R^{Her}$, $a \in R^{EP}$ and $a^\# = (a^\#)^*$. It follows that $a^+ = aa^+a^+$. Hence the formula (7.3) is the same as the formula (7.2). By Theorem 7.1, we are done.

" ⇐ " From the assumption, we have

$$a^*((a^\#)^*a^+p + u - aa^+uaa^+)a^\# = a^\# a(a^+p(a^\#)^* + v - a^+av),$$

e.g.

$$a^+pa^\# = a^\# aa^+p(a^\#)^* \text{ forall p} \in \text{R}.$$

Especially, choose $p = a^2$, one yields

$$a^+a = a(a^\#)^*.$$

So $a^+ = a(a^\#)^*a^+$. By Corollary 2.5, $a \in R^{Her}$. □

We establish the following equation

$$aa^+xaa^+(a^\#)^* = (a^+)^*y. \tag{9}$$

Theorem 7.3. *Let $a \in R^\# \cap R^+$. Then the general solution of Eq.(7.4) is given by*

$$\begin{cases} x = (a^\#)^*a^+p + u - aa^+uaa^+ \\ y = a^+p(a^\#)^* + v - a^+av \end{cases}, \text{ where } p,\ u,\ v \in R \text{ with } pa^+ = paa^+a^+. \tag{10}$$

Proof. First, we have

$$aa^+((a^\#)^*a^+p + u - aa^+uaa^+)aa^+(a^\#)^* = aa^+(a^\#)^*a^+paa^+(a^\#)^*$$
$$= (aa^+(a^\#)^*a^+a)a^+paa^+(a^\#)^* = (a^+)^*a^+paa^+(a^\#)^* = (a^+)^*a^+paa^+a^+a(a^\#)^*$$
$$= (a^+)^*a^+pa^+a(a^\#)^* = (a^+)^*a^+p(a^\#)^* = (a^+)^*(a^+p(a^\#)^* + v - a^+av).$$

Hence the formula (7.5) is the solution of Eq.(7.4).

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(7.4). Then we have

$$aa^+x_0aa^+(a^\#)^* = (a^+)^*y_0.$$

Choose $p = ay_0a^*, u = x_0 - (a^\#)^*a^+p, v = y_0$. Then

$$paa^+a^+ = ay_0a^*aa^+a^+ = ay_0a^*a^+ = pa^+,$$

and

$$aa^+uaa^+ = aa^+(x_0 - (a^\#)^*a^+p)aa^+ = aa^+x_0aa^+ - aa^+(a^\#)^*a^+paa^+$$
$$= aa^+x_0aa^+ - aa^+(a^\#)^*a^+ay_0a^*aa^+ = aa^+x_0aa^+ - (a^+)^*y_0a^*$$
$$= aa^+x_0aa^+ - aa^+x_0aa^+(a^\#)^*a^* = aa^+x_0aa^+ - aa^+x_0aa^+ = 0.$$

One gets

$$x_0 = (a^\#)^*a^+p + u - aa^+uaa^+.$$

Also,

$$a^+p(a^\#)^* = a^+ay_0(a^\#a)^* = a^*((a^+)^*y_0)(aa^\#)^*$$
$$= a^*(aa^+x_0aa^+(a^\#)^*)(aa^\#)^* = a^*(aa^+x_0aa^+(a^\#)^*)$$
$$= a^*(a^+)^*y_0 = a^+ay_0 = a^+av.$$

This infers

$$y_0 = a^+p(a^\#)^* + v - a^+av.$$

Hence every solution of Eq.(7.4) has the form of the formula (7.5). Thus the general solution of Eq.(7.4) is given by formula (7.5). □

Theorem 7.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Her}$ if and only if Eq.(7.1) has the same solution as Eq.(7.4).*

Proof. " ⇒ " If $a \in R^{Her}$, then $a \in R^{EP}$ and so $a^+ = aa^+a^+$. It follows that the formula (7.3) is the same as the formula (7.5). Hence by Theorem 7.2 and Theorem 7.3, we are done.

" ⇐ " From the assumption, we know that the general solution of Eq.(7.1) is given by formula (7.5). Hence

$$a^*((a^\#)^*a^+p + u - aa^+uaa^+)a^\# = a^\#a(a^+p(a^\#)^* + v - a^+av).$$

That is, $a^+pa^\# = a^\#aa^+p(a^\#)^*$ for $p \in R$ satisfying $pa^+ = paa^+a^+$. Choose $p = a^*$. Then

$$a^+a^*a^\# = a^\#aa^+a^*(a^\#)^* = a^\#aa^+.$$

Multiplying the equality on the right by a^2a^+, one has

$$a^+a^* = aa^+.$$

It follows

$$a^\# = aa^+a^\# = a^+a^*a^\# = a^\#aa^+.$$

Hence $a \in R^{EP}$, this implies

$$aa^\# = aa^+ = a^+a^*.$$

Thus $a \in R^{Her}$ by [11, Theorem 1.4.2]. □

8. Constructions of group invertible elements and Moore Penrose invertible elements

Theorem 8.1. *Let $a \in R^{\#} \cap R^{+}$. Then*

(1) $(a^*xa^{\#})^+ = (a^*xa^{\#})^{\#} = a^+a^3a^+x^{\#}(a^+)^*$, *where* $x \in \rho_a$;

(2) $(a^{\#}ax)^+ = x^+aa^+$, *where* $x \in \rho_a$;

(3) $(a^{\#}ax)^{\#} = x^{\#}a^+a$, *where* $x \in \{a, a^{\#}, (a^+)^*\} = \tau_a$;

(4) $(a^{\#}ax)^{\#} = x^+aa^+$, *where* $x \in \{a^+, a^*, (a^{\#})^*, (a^+)^{\#}, (a^{\#})^+\} = \gamma_a$.

Proof. Noting that

$$xx^{\#} = x^{\#}x = x^{\#}aa^+x = xaa^+x^{\#} = \begin{cases} aa^{\#}, x \in \tau_a \\ (aa^{\#})^*, x \in \gamma_a \end{cases} = x^{\#}a^+ax.$$

$$xx^+ = \begin{cases} aa^+, x \in \tau_a \\ a^+a, x \in \gamma_a \end{cases}.$$

$$x^+aa^{\#}x = \begin{cases} a^+a, x \in \tau_a \\ aa^+, x \in \gamma_a \end{cases}.$$

Then we can complete the proof by a routine vertification. □

The following theorem is a direct corollary of Theorem 6.1.

Theorem 8.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^+a^3a^+x^{\#}(a^+)^* = x^+aa^+$ for some $x \in \rho_a$.*

Also, we have:

Theorem 8.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $a^+a^3a^+x^{\#}(a^+)^* = x^{\#}a^+a$ for some $x \in \tau_a$.*

Theorem 8.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{Her}$ if and only if $(a^{\#}ax)^+ = x^+(a^{\#})^*a$ for some $x \in \rho_a$.*

Proof. " ⇒ " Assume that $a \in R^{Her}$. Then $a^+ = a^{\#} = (a^{\#})^*$. Hence

$$aa^+ = aa^{\#} = a^{\#}a = (a^{\#})^*a.$$

By Theorem 8.1, we have

$$(a^{\#}ax)^+ = x^+(a^{\#})^*a,$$

for all $x \in \rho_a$.

" ⇐ " By the hypothesis and Theorem 8.1, we have $x^+aa^+ = x^+(a^{\#})^*a$, for all $x \in \rho_a$. This gives

$$xx^+aa^+ = xx^+(a^{\#})^*a.$$

Noting that

$$xx^+ = \begin{cases} aa^+, x \in \tau_a \\ a^+a, x \in \gamma_a \end{cases}.$$

Thus we have $aa^+aa^+ = aa^+(a^{\#})^*a$ or $a^+aaa^+ = a^+a(a^{\#})^*a$.
If $aa^+aa^+ = aa^+(a^{\#})^*a$, then $a^+ = a^+(a^{\#})^*a$. Hence $a \in R^{Her}$ by Corollary 2.5;
If $a^+aaa^+ = a^+a(a^{\#})^*a$, then $a^2a^+ = a(a^{\#})^*a = (a(a^{\#})^*a)a^+a = a^2a^+a^+a$, it follows that

$$aa^+ = aa^+a^+a.$$

Hence $a \in R^{EP}$, one gets

$$a = a^2a^+ = a(a^{\#})^*a$$

and

$$a^{\#} = aa^{\#}a^{\#} = a(a^{\#})^*aa^{\#}a^{\#} = a(a^{\#})^*a^{\#} = a(a^{\#})^*a^+.$$

Thus $a \in R^{Her}$ by Corollary 2.5. □

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