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# $M$-fuzzifying rough sets via $M$-fuzzifying algebraic relations

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**Abstract.** The aim of this paper is to study $M$-fuzzifying algebraic rough sets in a constructive approach. For this purpose, the notion of $M$-fuzzifying algebraic relation is introduced and a pair of lower and upper $M$-fuzzifying approximation operators are presented. Several conditions of $M$-fuzzifying algebraic relations such as seriality, (resp., primitive, weak) symmetry, reflexivity and (resp., strong) transitivity are characterized by $M$-fuzzifying algebraic approximation operators. Then relationships among $M$-fuzzifying algebraic rough sets, $M$-fuzzifying convex structures and $M$-fuzzifying rough sets are investigated. Specifically, the category of reflexive and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of $M$-fuzzifying convex spaces. The category of reflexive, symmetric and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of reflexive, symmetric and transitive $M$-fuzzifying rough spaces. In particular, the category of reflexive, weakly symmetric and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of $M$-fuzzifying convex matroids.

## 1. Introduction

The notion of rough sets was originally proposed by Pawlak [19]. It is an effective mathematical tool for handling uncertainty, imprecision, and vagueness presented by incomplete and insufficient information. Its theory has been applied in many fields such as decision making [21, 25], machine learning [4, 5], pattern recognition [11, 28], data mining [6, 9] and medical diagnosis [18].

In rough set theory, approximation operators are primary concepts. Generally, there are mainly two approaches to develop rough set theory, namely the constructive approach and the axiomatic approach. In the constructive approach, a primitive structure of rough set theory is proposed. It is usually an approximation space consisting of a universe of discourse and the primitive notions imposed on it, such as binary relations [39], coverings [10, 33], neighborhood systems [36, 37] and algebras [22]. Based on the approximation space, a pair of concrete lower and upper approximation operators are constructed and some special properties of the primitive notion are characterized. In the axiomatic approach, a pair of abstract lower and upper approximation operators are proposed as the primitive notions. They are set-theoretic operators mainly focus on the algebraic system for the theory of rough sets [8, 35, 36]. Based

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on the abstract lower and upper approximation operators, a notion similar to the primitive notion in the constructive approach is induced. In turn, the lower and upper approximation operators induced by this notion are the same as the abstract lower and upper approximation operators [8]. Rough set theory has been applied in many fields, such as decision making [21, 25], machine learning [4, 5], data mining [6, 9], pattern recognition [11, 28].

Since Zadeh introduced the concept of fuzzy sets [40], many concepts and theories have been extended into fuzzy setting [1, 14–17, 23, 34]. The concept of rough sets has been extended into fuzzy settings. In the framework of fuzzy rough set theory, based on some basic fuzzy notions including fuzzy binary relations, fuzzy coverings and fuzzy neighborhood systems, lower and upper approximation operators have been generalized into various forms via both the constructive approach and the axiomatic approach [2, 3, 7, 12–14, 27, 31, 38, 39]. As for fuzzy generalization of rough sets constructed by binary relations, there are mainly three kinds of fuzzy rough sets, the most common kind is the one obtained by directly replacing crisp relations and subsets with fuzzy corresponding ones. The second kind is composed by using crisp relation and fuzzy approximation subsets. The third kind is composed by using fuzzy relation and crisp approximation subsets. The common feature of these three kinds of fuzzy rough sets are based on a crisp relation or a fuzzy relation which imposes on a pair of points or fuzzy points. In addition, rough sets constructed by relations or fuzzy relations are usually point-to-point relations. As a result, the corresponding approximation spaces are usually connected topological spaces or Alexander spaces [14, 20].

It is sometimes required to consider relation between a point and a subset and to consider connections between approximations spaces and convex spaces. Motivated by this, we present this paper. The arrangement of this paper is as follows. In Section 2, we present some basic notions and results. In Section 3, we introduce the notion of $M$-fuzzifying algebraic relations and present notions of lower $M$-fuzzifying algebraic approximation operators and upper $M$-fuzzifying algebraic approximation operators. We study $M$-fuzzifying algebraic rough sets constructed by one or more conditions of $M$-fuzzifying algebraic relations such as seriality, reflexivity, symmetry and transitivity. In Section 4, we prove that the category of $M$-fuzzifying convex spaces is isomorphic to the category of reflexive and transitive $M$-fuzzifying algebraic rough sets and $M$-fuzzifying convex spaces. In particular, we prove that the category of reflexive, weakly symmetric and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of $M$-fuzzifying convex matroids. In Section 5, we prove that there is a Galois's connection between the category of $M$-fuzzifying rough spaces and the category of $M$-fuzzifying algebraic rough spaces. In particular, we prove that the category of reflexive, symmetric and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of reflexive, symmetric and transitive $M$-fuzzifying rough spaces.

## 2. Preliminaries

In this paper, $U$ and $V$ are nonempty sets. The power set of $U$ is denoted by $\mathcal{P}(U)$. The set of all finite subsets of $U$ is denoted by $\mathcal{P}_{fin}(U)$. The set of all nonempty finite subsets of $U$ is denoted by $\mathcal{P}^*_{fin}(U)$. For any $A \in \mathcal{P}(U)$, the compliment of $A$ is denoted by $A^c$. For any $x \in U$, we denote $A\backslash\{x\} = \{y \in A : y \neq x\}$. A subset $\{A_i\}_{i\in I} \subseteq \mathcal{P}(U)$ is said to be directed, denoted by $\{A_i\}^{dir}_{i\in I} \subseteq \mathcal{P}(U)$, if any pair of index $i, j \in I$ yields an index $k \in I$ such that $A_i \cup A_j \subseteq A_k$. In this case, $\bigcup_{i\in I} A_i$ will be denoted by $\bigcup^{dir}_{i\in I} A_i$. A subset $\{B_i\}_{i\in I} \subseteq \mathcal{P}(U)$ is said to be co-directed, denoted by $\{B_i\}^{cdir}_{i\in I} \subseteq \mathcal{P}(U)$, if any pair of index $i, j \in I$ yields an index $k \in I$ such that $B_k \subseteq B_i \cap B_j$. In this case, $\bigcap_{i\in I} B_i$ will be denoted by $\bigcap^{cdir}_{i\in I} B_i$.

$M$ is a distributive lattice with an inverse involution $'$. The smallest (resp. largest) element in $M$ is denoted by $\perp$ (resp. $\top$). An element $a \in M$ is called a co-prime element, if for all $b, c \in M$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all co-prime elements in $M\backslash\{\perp\}$ is denoted by $J(M)$. For any $a \in M$, there is $M_1 \subseteq J(M)$ such that $a = \bigvee_{b\in M_1} b$. A binary relation $\prec$ on $M$ is defined by $a \prec b$ iff for each $M_1 \subseteq M$, $b \leq \bigvee M_1$ implies a $d \in M_1$ with $a \leq d$. The mapping $\beta : M \longrightarrow \mathcal{P}(M)$, defined by $\beta(a) = \{b \in M : b \prec a\}$ for any $a \in M$, satisfies $\beta(\bigvee_{i\in I} a_i) = \bigcup_{i\in I} \beta(a_i)$ for any $\{a_i\}_{i\in I} \subseteq M$. We denote $\beta^*(a) = \beta(a) \cap J(M)$ for any $a \in M$. It is proved that $a = \bigvee \beta(a) = \bigvee \beta^*(a)$, $\beta(a) = \bigcup_{b\in\beta^*(a)} \beta(b)$ and $\beta^*(a) = \bigcup_{b\in\beta^*(a)} \beta^*(b)$. A binary relation $\prec^{op}$ on $M$ is defined by $a \prec^{op} b$ iff for each $M_2 \subseteq M$, $\bigwedge M_2 \leq b$ implies a $d \in M_2$ with $d \leq a$. The mapping $\alpha : M \longrightarrow \mathcal{P}(M)$, defined by $\alpha(a) = \{b \in M : b \prec^{op} a\}$ for any $a \in M$, satisfies $\alpha(\bigwedge_{i\in I} a_i) = \bigcup_{i\in I} \alpha(a_i)$ for any $\{a_i\}_{i\in I} \subseteq M$. For any $a \in M$, we denote $\alpha^*(a) = \alpha(a) \cap P(M)$. It is proved that $a = \bigwedge \alpha(a) = \bigwedge \alpha^*(a)$, $\alpha(a) = \bigcap_{a \in \alpha^*(b)} \alpha(b)$ and $\alpha^*(a) = \bigcap_{a \in \alpha^*(b)} \alpha^*(b)$. For any $a, b \in M$, it is clear that $a \prec B$ iff $b' \prec^{op} a'$ [23].

An $M$-fuzzy set on $U$ is a mapping $A : U \longrightarrow M$. The set of all $M$-fuzzy sets is denoted by $M^U$. For any subset $W \in \mathcal{P}(U)$, the characterization function of $W$ is an $M$-fuzzy set $\chi_W$ defined by $\chi_W(x) = \top$ for any $x \in W$ and $\chi_W(x) = \bot$ for any $x \notin W$ [23].

**Definition 2.1.** ([14]) An $M$-fuzzifying relation from $U$ to $V$ is a mapping $\hat{\mathcal{R}} : U \times V \longrightarrow M$. The triple $(U, V, \hat{\mathcal{R}})$ is called an $M$-fuzzifying approximation space. In Particular, if $U = V$ then $\hat{\mathcal{R}}$ is simply called an $M$-fuzzifying relation on $U$ and $(U, V, \hat{\mathcal{R}})$ is simply denoted by $(U, \hat{\mathcal{R}})$.

**Definition 2.2.** ([14]) Let $(U, V, \hat{\mathcal{R}})$ be an $M$-fuzzifying approximation space. $\hat{\mathcal{R}}$ is said to be

(1) serial, if $\bigvee_{y \in V} \hat{\mathcal{R}}(x, y) = \top$ for any $x \in U$.

If $U = V$ then $\hat{\mathcal{R}}$ is said to be

(2) reflexive, if $\hat{\mathcal{R}}(x, x) = \top$ for any $x \in U$;

(3) symmetric, if $\hat{\mathcal{R}}(x, y) = \hat{\mathcal{R}}(y, x)$ for all $x, y \in U$;

(4) transitive, if $\hat{\mathcal{R}}(x, z) \wedge \hat{\mathcal{R}}(z, y) \leq \hat{\mathcal{R}}(x, y)$ for all $x, y, z \in U$.

**Definition 2.3.** ([14]) Let $(U, V, \hat{\mathcal{R}})$ be an $M$-fuzzifying approximation space. The lower approximation operator and the upper approximation operator $\underline{\mathcal{R}} : \mathcal{P}(V) \longrightarrow M^U$ and $\overline{\mathcal{R}} : \mathcal{P}(V) \longrightarrow M^U$, defined by

$$\forall A \in \mathcal{P}(V), \forall x \in U, \quad \underline{\hat{\mathcal{R}}}(A)(x) = \bigwedge_{y \notin A} [\hat{\mathcal{R}}(x, y)]' \quad \text{and} \quad \overline{\hat{\mathcal{R}}}(A)(x) = \bigvee_{y \in A} \hat{\mathcal{R}}(x, y).$$

**Proposition 2.4.** ([14]) *Let $(U, V, \hat{\mathcal{R}})$ be an $M$-fuzzifying approximation space. The following statements are valid.*

*(1) $\hat{\mathcal{R}}$ is serial iff $\overline{\hat{\mathcal{R}}}(V) = \chi_U$ iff $\underline{\hat{\mathcal{R}}}(\emptyset) = \chi_\emptyset$.*

*(2) $\hat{\mathcal{R}}$ is reflexive iff $\overline{\hat{\mathcal{R}}}(A)(x) = \top$ for any $x \in A \in \mathcal{P}(V)$ iff $\underline{\hat{\mathcal{R}}}(A)(x) = \bot$ for any $A \in \mathcal{P}(U)$ and any $x \notin A$.*

*(3) $\hat{\mathcal{R}}$ is symmetric iff $\overline{\hat{\mathcal{R}}}(\{x\})(y) = \overline{\hat{\mathcal{R}}}(\{y\})(x)$ for all $x, y \in U$ iff $\underline{\hat{\mathcal{R}}}(\{x\}^c)(y) = \underline{\hat{\mathcal{R}}}(\{y\}^c)(x)$ for all $x, y \in U$.*

*(4) $\hat{\mathcal{R}}$ is transitive iff $\overline{\hat{\mathcal{R}}}(A)(x) = \bigwedge_{A \subseteq C} \overline{\hat{\mathcal{R}}}(C)(x) \vee \bigvee_{y \notin C} \overline{\hat{\mathcal{R}}}(C)(y)$ for all $A \in \mathcal{P}(V)$ and $x \in U$ iff $\underline{\hat{\mathcal{R}}}(A)(x) = \bigvee_{B \subseteq A} \underline{\hat{\mathcal{R}}}(x) \wedge \bigwedge_{y \in B} \underline{\hat{\mathcal{R}}}(B)(y)$ for all $A \in \mathcal{P}(U)$ and $x \in U$.*

**Definition 2.5.** ([23]) A mapping $\mathcal{C} : \mathcal{P}(U) \longrightarrow M$ is called an $M$-fuzzifying convex structure on $U$ and the pair $(U, \mathcal{C})$ is called an $M$-fuzzifying convex space, if $\mathcal{C}$ satisfies

(MC1) $\mathcal{C}(U) = \mathcal{C}(\emptyset) = \top$;

(MC2) $\forall \{A_i\}_{i \in i} \subseteq \mathcal{P}(U)$, $\mathcal{C}(\bigcap_{i \in I} A_i) \geq \bigwedge_{i \in I} \mathcal{C}(A_i)$;

(MC3) $\forall \{A_i\}_{i \in I}^{dir} \subseteq \mathcal{P}(U)$, $\mathcal{C}(\bigcup_{i \in I}^{dir} A_i) \geq \bigwedge_{i \in I} \mathcal{C}(A_i)$.

Let $(U_1, \mathcal{C}_1)$ and $(U_2, \mathcal{C}_2)$ be $M$-fuzzifying convex spaces. A mapping $f : U_1 \longrightarrow U_2$ is called an $M$-fuzzifying convexity preserving mapping, if $\mathcal{C}_2(B) \leq \mathcal{C}_1(f^{-1}(B))$ for any $B \in \mathcal{P}(U)$ [23]. The category consisting of $M$-fuzzifying convex spaces as objects and $M$-fuzzifying convexity preserving mappings as morphisms is denoted by $M$-**CS**.

**Definition 2.6.** ([23])An operator $co : \mathcal{P}(U) \longrightarrow M^U$ is called an $M$-fuzzifying convex hull operator on $U$ and the pair $(U, co)$ is an $M$-fuzzifying convex hull space, if for all $A \in \mathcal{P}(U)$, $\{A_i\}_{i \in I}^{dir} \subseteq \mathcal{P}(U)$ and $x \in U$,

(MCO1) $co(\emptyset)(x) = \bot$;

(MCO2) $co(A)(x) = \top$ whenever $x \in A$;

(MCO3) $co(A)(x) = \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} co(B)(y)$;

(MCO4) $co(\bigcup_{i \in I}^{dir} A_i)(x) = \bigvee_{i \in I} co(A_i)(x)$.

**Theorem 2.7.** ([23]) *(1) Let $(U, C)$ be an M-fuzzifying convex space. Define an operator $co_C : \mathcal{P}(U) \longrightarrow M^U$ by*

$$\forall A \in \mathcal{P}(U), \forall x \in U, \quad co_C(A)(x) = \bigwedge_{x \notin B \supseteq A} [C(B)]'.$$

*Then $co_C$ is an M-fuzzifying convex hull operator on $U$.*

*(2) Let $(U, co)$ be an M-fuzzifying convex hull space. Define a mapping $C_{co} : \mathcal{P}(U) \longrightarrow M$ by*

$$\forall A \in \mathcal{P}(U), \quad C_{co}(A) = \bigwedge_{x \notin A} [co(A)(x)]'.$$

*Then $C_{co}$ is an M-fuzzifying convex structure on $U$.*

*(3) If $(U, C)$ is an M-fuzzifying convex space then $C_{co_C} = C$.*

*(4) If $(U, co)$ is an M-fuzzifying convex hull space then $co_{C_{co}} = co$.*

**Definition 2.8.** ([23]) An $M$-fuzzifying convex structure $C$ on $U$ is called an $n$-arity convex structure ($n \in \mathbb{N}$), if $co_C(A) = \bigvee_{F \in \mathcal{P}_{fin}(A), |F| \leq n} co_C(F)$ for any $A \in \mathcal{P}(U)$ where $|F|$ is the cardinality of $F$.

As the dual concept of $M$-fuzzifying convex hull operator, $M$-fuzzifying concave hull operator and its relationship with $M$-fuzzifying convex structure are presented as follows.

**Definition 2.9.** An operator $ca : \mathcal{P}(U) \longrightarrow M^U$ is called an $M$-fuzzifying concave hull operator on $U$ and the pair $(U, ca)$ is an $M$-fuzzifying concave hull space, if for all $A \in \mathcal{P}(U)$, $\{A_i\}_{i \in I}^{cdir} \subseteq \mathcal{P}(U)$ and $x \in U$,

(MCA1) $ca(U)(x) = \top$;

(MCA2) $ca(A)(x) = \bot$ whenever $x \notin A$;

(MCA3) $ca(A)(x) = \bigvee_{x \in B \subseteq A} \bigwedge_{y \in B} ca(B)(y)$;

(MCA4) $ca(\bigcap_{i \in I}^{cdir} A_i)(x) = \bigwedge_{i \in I} ca(A_i)(x)$.

**Theorem 2.10.** *(1) Let $(U, C)$ be an M-fuzzifying convex space. Define an operator $ca_C : \mathcal{P}(U) \longrightarrow M^U$ by*

$$\forall A \in \mathcal{P}(U), \forall x \in U, \quad ca_C(A)(x) = \bigvee_{x \in B \subseteq A} C(B^c).$$

*Then $ca_C$ is an M-fuzzifying concave hull operator on $U$.*

*(2) Let $(U, ca)$ be an M-fuzzifying concave hull space. Define a mapping $C_{ca} : \mathcal{P}(U) \longrightarrow M$ by*

$$\forall A \in \mathcal{P}(U), \quad C_{ca}(A) = \bigwedge_{x \in A^c} ca(A^c)(x).$$

*Then $C_{ca}$ is an M-fuzzifying concave structure on $U$.*

*(3) If $(U, C)$ is an M-fuzzifying convex space then $C_{ca_C} = C$.*

*(4) If $(U, ca)$ is an M-fuzzifying concave hull space then $ca_{C_{ca}} = ca$.*

**Definition 2.11.** ([24]) An operator $\mathcal{H} : \mathcal{P}_{fin}(U) \longrightarrow M^U$ is called an $M$-fuzzifying restricted hull operator on $U$ and the pair $(U, \mathcal{H})$ is called an $M$-fuzzifying restricted hull space, if $\mathcal{H}$ satisfies

(MRH1) $\mathcal{H}(\emptyset)(x) = \bot$ for any $x \in U$;

(MRH2) $\mathcal{H}(F)(x) = \top$ for all $F \in \mathcal{P}_{fin}(U)$ and $x \in F$;

(MRH3) $\mathcal{H}(G)(x) \wedge \bigwedge_{y \in G} \mathcal{H}(F)(y) \leq \mathcal{H}(F)(x)$ for all $F, G \in \mathcal{P}_{fin}(U)$ and $x \in U$.

**Theorem 2.12.** ([24]) *(1) Let $(U, C)$ be an M-fuzzifying convex space. Define an operator $\mathcal{H}_C : \mathcal{P}_{fin}(U) \longrightarrow M^U$ by*

$$\forall F \in \mathcal{P}_{fin}(U), \forall x \in U, \quad \mathcal{H}_C(F)(x) = co_C(F)(x).$$

*Then $\mathcal{H}_C$ is an M-fuzzifying restricted hull operator on $U$.*

*(2) Let $(U, \mathcal{H})$ be an M-fuzzifying restricted hull space. Define a mapping $C_{\mathcal{H}} : \mathcal{P}(U) \longrightarrow M$ by*

$$\forall A \in \mathcal{P}(U), \quad C_{\mathcal{H}}(A) = \bigwedge_{x \notin A} \bigwedge_{F \in \mathcal{P}_{fin}(U)} [\mathcal{H}(F)(x)]'.$$

*Then $C_{\mathcal{H}}$ is an M-fuzzifying convex structure on U.*

*(3) If $(U, C)$ is an M-fuzzifying convex space then $C_{\mathcal{H}_C} = C$.*

*(4) If $(U, \mathcal{H})$ is an M-fuzzifying restricted hull space then $\mathcal{H}_{C_{\mathcal{H}}} = \mathcal{H}$.*

**Definition 2.13.** ([29]) An $M$-fuzzifying convex space $(U, C)$ is called an $M$-fuzzifying convex matroid, if for all $u, v \in U$ and $A \in \mathcal{P}(U)$,

$$co_C(A)(v) \vee co_C(A \cup \{v\})(u) \leq co_C(A)(u) \vee co_C(A \cup \{u\})(v).$$

**Definition 2.14.** ([32]) A mapping $\mathcal{I} : U \times U \longrightarrow M^U$ is called an $M$-fuzzifying interval operator on $U$ and the pair $(U, \mathcal{I})$ is called an $M$-fuzzifying interval space, if for all $x, y \in U$,

(MI1) $\mathcal{I}(x, y)(x) = \mathcal{I}(x, y)(y) = \top$;

(MI2) $\mathcal{I}(x, y) = \mathcal{I}(y, x)$.

**Proposition 2.15.** ([32]) *Let $\mathcal{I}$ be an M-fuzzifying interval operator on U. Then the mapping $C_{\mathcal{I}} : 2^X \longrightarrow M$, defined by $C_{\mathcal{I}}(A) = \bigwedge_{z \notin A} \bigwedge_{x,y \in A} [\mathcal{I}(x, y)(z)]'$ for all $A \in \mathcal{P}(U)$, is an M-fuzzifying convex structure on U.*

**Definition 2.16.** ([23]) An $M$-fuzzifying vector space is a pair $(V, \mu)$, where $V$ is a vector space over a totally ordered field $\mathbb{K}$ and $\mu : V \longrightarrow M$ is a mapping satisfying $\mu(kx + ly) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in V$ and $k, l \in \mathbb{K}$. If $(V, \mu)$ is an $M$-fuzzifying vector space then $\mu_{[a]}$ is a subspace of $V$ for any $a \in M \backslash \{\bot\}$. Further, $(V, \mu)$ is called an $M$-fuzzifying affine vector space, if $\mu_{[a]}$ is an affine subspace of $V$ for any $a \in M \backslash \{\bot\}$ (i.e., $A \in \mu_{[a]}$ iff $kv + lw \in A$ for all $v, w \in A$ and $k, l \in \mathbb{K}$ with $k + l = 1$).

**Proposition 2.17.** ([23, 29]) *Let $(V, \mu)$ is an M-fuzzifying vector space. Define a mapping $C_\mu : \mathcal{P}(V) \longrightarrow M$ by $C_\mu(A) = \bigvee \{a \in M \backslash \{\bot\} : A \in \mu_{[a]}\}$ for any $A \in \mathcal{P}(V)$. Then $C_\mu$ is an M-fuzzifying convex structure. In particular, if $(V, \mu)$ is an M-fuzzifying affine vector space then $(V, \mu)$ is an M-fuzzifying convex matroid.*

**Definition 2.18.** ([30]) An $M$-fuzzifying binary relation $\varepsilon : \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow M$ is called an $M$-fuzzifying convex enclosed relation on $U$ and the pair $(U, \varepsilon)$ is called an $M$-fuzzifying convex enclosed relation space, if

(MCER1) $\varepsilon(\emptyset, \emptyset) = \top$;

(MCER2) $\varepsilon(A, B) \neq \bot$ implies $A \subseteq B$;

(MCER3) $\varepsilon(A, \bigcap_{i \in I} B_i) = \bigwedge_{i \in I} \varepsilon(A, B_i)$;

(MCER4) $\varepsilon(A, B) \leq \bigvee_{C \in \mathcal{P}(U)} \varepsilon(A, C) \wedge \varepsilon(C, B)$;

(MCER5) $\varepsilon(\bigcup_{i \in I}^{dir} A_i, B) = \bigwedge_{i \in I} \varepsilon(A_i, B)$.

**Proposition 2.19.** ([30]) *Let $(U, \varepsilon)$ be an M-fuzzifying convex enclosed relation space. Define $co_\varepsilon : \mathcal{P}(U) \longrightarrow M^U$ and $ca_\varepsilon : \mathcal{P}(U) \longrightarrow M^U$ by*

$$\forall A \in \mathcal{P}(U), \forall u \in U, \quad co_\varepsilon(A)(u) = \bigwedge_{u \notin B} [\varepsilon(A, B)]' \quad \text{and} \quad ca_\varepsilon(A)(u) = \bigvee_{u \in B} \varepsilon(A^c, B^c).$$

*Then $co_\varepsilon$ is an M-fuzzifying convex hull operator and $ca_\varepsilon$ is an M-fuzzifying concave hull operator.*

## 3. Approximation operators via algebraic relations

**Definition 3.1.** An $M$-fuzzifying algebraic relation from $U$ to $V$ is a mapping $\mathcal{R} : U \longrightarrow M^{\mathcal{P}_{fin}(V)}$ satisfying

$$\forall x \in U, \forall F \in \mathcal{P}_{fin}(V), \quad \mathcal{R}(x)(F) = \bigvee_{G \in \mathcal{P}^*_{fin}(F)} \mathcal{R}(x)(G).$$

In particular, if $U = V$ then $\mathcal{R}$ is called an $M$-fuzzifying algebraic relation on $U$.

For any $x \in U$ and $F \in \mathcal{P}_{fin}(V)$, the value $\mathcal{R}(x)(F)$ can be regarded as the degree that $x$ is algebraic related to $F$. It is clear that $\mathcal{R}(x)(\emptyset) = \bot$ for any $x \in U$.

**Definition 3.2.** Let $\mathcal{R}$ be an $M$-fuzzifying algebraic relation from $U$ to $V$. Define two operators $\underline{\mathcal{R}}, \overline{\mathcal{R}} : \mathcal{P}(V) \longrightarrow M^U$ by $\forall A \in \mathcal{P}(V), \forall x \in U$,

$$\underline{\mathcal{R}}(A)(x) = \bigwedge_{G \in \mathcal{P}_{fin}(A^c)} [\mathcal{R}(x)(G)]' \quad \text{and} \quad \overline{\mathcal{R}}(A)(x) = \bigvee_{G \in \mathcal{P}_{fin}(A)} \mathcal{R}(x)(G).$$

The operators $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are respectively called the lower $M$-fuzzifying algebraic approximation operator and the upper $M$-fuzzifying algebraic approximation operator with respect to the $M$-fuzzifying algebraic relation $\mathcal{R}$. The pair $(\underline{\mathcal{R}}(A), \overline{\mathcal{R}}(A))$ is called an $M$-fuzzifying algebraic rough set and the triple $(U, V, \mathcal{R})$ is called an $M$-fuzzifying algebraic approximation space. In particular, $(U, V, \mathcal{R})$ is simply denoted by $(U, \mathcal{R})$ when $U = V$.

The followings are some examples of $M$-fuzzifying algebraic relations.

**Example 3.3.** Let $(U, \mathcal{H})$ be an $M$-fuzzifying restricted hull space. Define an $M$-fuzzifying relation $\mathcal{R}_{\mathcal{H}} : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by

$$\forall x \in U, \forall F \in \mathcal{P}_{fin}(U), \quad \mathcal{R}_{\mathcal{H}}(x)(F) = \mathcal{H}(F)(x).$$

Then $\mathcal{R}_{\mathcal{H}}$ is an $M$-fuzzifying algebraic relation satisfying $\overline{\mathcal{R}}_{\mathcal{H}}(F) = \mathcal{H}(F)$ for any $F \in \mathcal{P}_{fin}(U)$.

**Example 3.4.** Let $(U, \mathcal{I})$ be an $M$-fuzzifying interval space. Define a relation $\mathcal{R}_{\mathcal{I}} : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by

$$\forall x \in U, \forall F \in \mathcal{P}_{fin}(U), \quad \mathcal{R}_{\mathcal{I}}(x)(F) = \bigwedge_{x \notin B \supseteq F} \bigvee_{z \notin B} \bigvee_{w, v \in B} \mathcal{I}(w, v)(z).$$

It is clear that $\mathcal{R}_{\mathcal{I}}(x)(F) = \mathcal{H}_{C_{\mathcal{I}}}(F)(x)$ for any $F \in \mathcal{P}_{fin}(U)$ and $x \in U$. Thus $\mathcal{R}_{\mathcal{I}}$ is an $M$-fuzzifying algebraic relation.

**Example 3.5.** Let $(U, \varepsilon)$ be an $M$-fuzzifying enclosed relation space. Define a mapping $\mathcal{R}_\varepsilon : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by $\mathcal{R}_\varepsilon(x)(F) = co_\varepsilon(F)(x)$ for any $F \in \mathcal{P}_{fin}(U)$ and $x \in U$. Then $\mathcal{R}_\varepsilon$ is an $M$-fuzzifying algebraic relation satisfying $\overline{\mathcal{R}}_\varepsilon = co_\varepsilon$ and $\underline{\mathcal{R}}_\varepsilon = ca_\varepsilon$.

**Example 3.6.** Let $(V, \mu)$ be an $M$-fuzzifying vector space over a totally ordered field $\mathbb{K}$. Define $\mathcal{R}_V : V \longrightarrow M^{\mathcal{P}_{fin}(V)}$ by

$$\forall F \in \mathcal{P}_{fin}(V), \forall x \in V, \quad \mathcal{R}_V(x)(F) = \mathcal{H}_{C_\mu}(F)(x).$$

Then $\mathcal{R}_V$ is an $M$-fuzzifying algebraic relation satisfying $\overline{\mathcal{R}_V} = co_{C_\mu}$.

As some further examples of $M$-fuzzifying algebraic relations, we will investigate relationships among $M$-fuzzifying algebraic relation and $M$-fuzzifying convex structure in Sections 4 and 5.

Next, we investigate some basic properties of lower $M$-fuzzifying algebraic approximation operators and upper $M$-fuzzifying algebraic approximation operators.

**Proposition 3.7.** *Let $(U, V, \mathcal{R})$ be an $M$-fuzzifying algebraic approximation space. For all $A, B \in \mathcal{P}(V)$ and $x \in U$,*
*(1) $A \subseteq B$ implies $\underline{\mathcal{R}}(A) \leq \underline{\mathcal{R}}(B)$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$;*
*(2) $\underline{\mathcal{R}}(V) = \chi_U$ and $\overline{\mathcal{R}}(\emptyset) = \chi_\emptyset$;*
*(3) $\mathcal{R}(x)(F) = [\underline{\mathcal{R}}(F^c)(x)]' = \overline{\mathcal{R}}(F)(x)$.*

*Proof.* (1) and (2) are direct.
(3) It is clear that

$$\mathcal{R}(x)(F) = \bigvee_{H \in \mathcal{P}_{fin}(F)} \mathcal{R}(x)(H) = [\bigwedge_{H \in \mathcal{P}_{fin}((F^c)^c)} [\mathcal{R}(x)(H)]']' = [\underline{\mathcal{R}}(F^c)(x)]'$$

and $\mathcal{R}(x)(F) = \bigvee_{H \in \mathcal{P}_{fin}(F)} \mathcal{R}(x)(H) = \overline{\mathcal{R}}(F)(x)$. □

**Proposition 3.8.** *Let $(U, V, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. For any $\{A_i\}_{i\in I}^{cdir}, \{B_i\}_{i\in I}^{dir} \subseteq \mathcal{P}(V)$ and $A \in \mathcal{P}(V)$,*

*(MRL1)* $\underline{\mathcal{R}}(A) = [\overline{\mathcal{R}}(A^c)]'$;

*(MRU1)* $\overline{\mathcal{R}}(A) = [\underline{\mathcal{R}}(A^c)]'$;

*(MRL2)* $\underline{\mathcal{R}}(\bigcap_{i\in I}^{cdir} A_i)(x) = \bigwedge_{i\in I} \underline{\mathcal{R}}(A_i)(x)$;

*(MRU2)* $\overline{\mathcal{R}}(\bigcup_{i\in I}^{dir} B_i)(x) = \bigvee_{i\in I} \overline{\mathcal{R}}(B_i)(x)$.

*Proof.* (MRL1) For any $x \in U$ and $A \in \mathcal{P}(V)$,

$$[\overline{\mathcal{R}}(A^c)]'(x) = [\overline{\mathcal{R}}(A^c)(x)]' = [\bigvee_{G\in\mathcal{P}_{fin}(A^c)} \mathcal{R}(x)(G)]' = \bigwedge_{G\in\mathcal{P}_{fin}(A^c)} [\mathcal{R}(x)(G)]' = \underline{\mathcal{R}}(A)(x).$$

This shows that $\underline{\mathcal{R}}(A) = [\overline{\mathcal{R}}(A^c)]'$.

(MRU1) For any $x \in U$, it follows that

$$[\underline{\mathcal{R}}(A^c)]'(x) = [\underline{\mathcal{R}}(A^c)(x)]' = [\bigwedge_{G\in\mathcal{P}_{fin}((A^c)^c)} [\mathcal{R}(x)(G)]']' = \bigvee_{G\in\mathcal{P}_{fin}(A)} \mathcal{R}(x)(G) = \overline{\mathcal{R}}(A)(x).$$

This shows that $\overline{\mathcal{R}}(A) = [\underline{\mathcal{R}}(A^c)]'$.

(MRL2) For any $x \in U$, it follows that

$$\underline{\mathcal{R}}(\bigcap_{i\in I}^{cdir} A_i)(x) = \bigwedge_{F\in\mathcal{P}_{fin}((\bigcap_{i\in I}^{cdir} A_i)^c)} [\mathcal{R}(x)(F)]' = \bigwedge_{F\in\mathcal{P}_{fin}(\bigcup_{i\in I}^{dir} A_i^c)} [\mathcal{R}(x)(F)]' = \bigwedge_{i\in I} \bigwedge_{F\in\mathcal{P}_{fin}(A_i^c)} [\mathcal{R}(x)(F)]' = \bigwedge_{i\in I} \underline{\mathcal{R}}(A_i)(x).$$

Thus $\underline{\mathcal{R}}(\bigcap_{i\in I}^{cdir} A_i) = \bigwedge_{i\in I} \underline{\mathcal{R}}(A_i)$.

(MRU2) For any $u \in U$, it follows (MRU1) and (MRL2) that

$$\overline{\mathcal{R}}(\bigcup_{i\in I}^{dir} A_i)(x) = [\underline{\mathcal{R}}(\bigcap_{i\in I}^{cdir} A_i^c)(x)]' = [\bigwedge_{i\in I} \underline{\mathcal{R}}(A_i^c)(x)]' = \bigvee_{i\in I} [\underline{\mathcal{R}}(A_i^c)(x)]' = \bigvee_{i\in I} \overline{\mathcal{R}}(A_i)(x).$$

Thus $\overline{\mathcal{R}}(\bigcup_{i\in I}^{dir} A_i) = \bigvee_{i\in I} \overline{\mathcal{R}}(A_i)$. □

In the following subsections, we present notions of serial (resp. reflexive, transitive) $M$-fuzzifying algebraic relations and study some characterizations of them.

### 3.1. Serial M-fuzzifying algebraic relations

**Definition 3.9.** An $M$-fuzzifying algebraic relation $\mathcal{R}$ from $U$ to $V$ is called serial, if $\bigvee_{F\in\mathcal{P}_{fin}(V)} \mathcal{R}(x)(F) = \top$ for any $x \in U$.

**Proposition 3.10.** *Let $(U, V, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. $\mathcal{R}$ is serial iff (MRL3) holds iff (MRU3) holds.*

*(MRL3)* $\underline{\mathcal{R}}(\emptyset) = \chi_\emptyset$.

*(MRU3)* $\overline{\mathcal{R}}(V) = \chi_U$.

*Proof.* It is clear that $\mathcal{R}$ is serial iff $\overline{\mathcal{R}}(V)(x) = \bigvee_{G\in\mathcal{P}_{fin}(V)} \mathcal{R}(x)(G) = \top$ for any $x \in U$. Thus $\mathcal{R}$ is serial iff $\overline{\mathcal{R}}(V) = \chi_U$. Also, the equivalence between (MRL3) and (MRU3) follows from (MRL1) and (MRU1). □

### 3.2. Reflexive M-fuzzifying algebraic relations

**Definition 3.11.** An $M$-fuzzifying algebraic relation $\mathcal{R}$ on $U$ is called reflexive, if $\mathcal{R}(x)(F) = \top$ for all $x \in F \in \mathcal{P}_{fin}(U)$.

**Proposition 3.12.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. $\mathcal{R}$ is reflexive iff one of the following conditions holds.*

*(MARL4) $\underline{\mathcal{R}}(F^c) \leq \chi_{F^c}$ for any $F \in \mathcal{P}_{fin}(U)$.*

*(MARU4) $\chi_F \leq \overline{\mathcal{R}}(F)$ for any $F \in \mathcal{P}_{fin}(U)$.*

*(MARL4') $\underline{\mathcal{R}}(A) \leq \chi_A$ for any $A \in \mathcal{P}(U)$.*

*(MARU4') $\chi_A \leq \overline{\mathcal{R}}(A)$ for any $A \in \mathcal{P}(U)$.*

*Proof.* We prove that $\mathcal{R}$ is reflexive iff (MARL4) holds for $\underline{\mathcal{R}}$.

($\Longrightarrow$) If $x \in F \in \mathcal{P}_{fin}(U)$ then $\mathcal{R}(x)(F) = \top$. Thus $\underline{\mathcal{R}}(F^c)(x) = [\mathcal{R}(x)(F)]' = \bot$. Therefore $\underline{\mathcal{R}}(F^c) \leq \chi_{F^c}$.

($\Longleftarrow$) For all $x \in F \in \mathcal{P}_{fin}(U)$, (MARL4) implies $\mathcal{R}(x)(F) = [\underline{\mathcal{R}}(F^c)(x)]' = \top$ by Proposition 3.7(3). Therefore $\mathcal{R}$ is reflexive.

Proofs of (MARL4) $\Longleftrightarrow$ (MARU4) and (MARL4') $\Longleftrightarrow$ (MARU4') directly follow from (MARL1) and (MARU1). It is trivial that (MARL4') implies (MARL4). Conversely, assume that (MARL4) holds and let $A \in \mathcal{P}(U)$. Since the set $\varphi_A = \{F \in \mathcal{P}_{fin}(U) : F \cap A = \emptyset\}$ is a directed set satisfying $A = \bigcap^{cdir}_{F \in \varphi_A} F^c$, (MARL4) and (MARL2) yield that

$$\underline{\mathcal{R}}(A) = \underline{\mathcal{R}}(\bigcap^{cdir}_{F \in \varphi_A} F^c) = \bigwedge^{cdir}_{F \in \varphi_A} \underline{\mathcal{R}}(F^c) \leq \bigwedge^{cdir}_{F \in \varphi_A} \chi_{F^c} = \chi_A.$$

This shows that (MARL4') holds. □

### 3.3. Symmetric M-fuzzifying algebraic relations

**Definition 3.13.** Let $(U, \mathcal{R})$ be an $M$-fuzzifying algebraic approximation space. $\mathcal{R}$ is called

(1) symmetric, if $\mathcal{R}(x)(F) = \bigvee_{y \in F} \mathcal{R}(y)(\{x\})$ for all $x \in U$ and $F \in \mathcal{P}_{fin}(U)$;

(2) weakly symmetric, if $\mathcal{R}(x)(F \cup \{y\}) \leq \mathcal{R}(x)(F) \vee \mathcal{R}(y)(F \cup \{x\})$ for all $x, y \in U$ and $F \in \mathcal{P}_{fin}(U)$;

(3) primitively symmetric, if $\mathcal{R}(x)(\{y\}) \leq \mathcal{R}(y)(\{x\})$ for all $x, y \in U$.

**Proposition 3.14.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. The following statements are valid.*

*(1) Symmetry $\Longrightarrow$ weak symmetry $\Longrightarrow$ primitive symmetry.*

*(2) If $\mathcal{R}(x)(F) = \bigvee_{y \in F} \mathcal{R}(x)(\{y\})$ for all $x \in U$ and $F \in \mathcal{P}_{fin}(U)$ then symmetry, weak symmetry and primitive symmetry are equivalent.*

*(3) If $\mathcal{R}$ is symmetric then $\mathcal{R}(x)(F) = \bigvee_{y \in F} \mathcal{R}(x)(\{y\})$ for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$.*

*Proof.* (1) Let $\mathcal{R}$ be symmetric. We check that $\mathcal{R}$ is both weakly symmetric and primitively symmetric.

(i) For all $x, y \in U$, the symmetry of $\mathcal{R}$ directly yields that $\mathcal{R}(x)(\{y\}) = \bigvee_{z \in \{y\}} \mathcal{R}(z)(\{x\}) = \mathcal{R}(y)(\{x\})$. So $\mathcal{R}$ is primitively symmetric.

(ii) Let $x, y \in U$ and let $F \in \mathcal{P}_{fin}(U)$. Let $a \in J(M)$ with $a \leq \mathcal{R}(y)(F) \vee \mathcal{R}(x)(F \cup \{y\})$. If $a \leq \mathcal{R}(y)(F)$ or $a \leq \mathcal{R}(x)(F)$, then it is trivial that $a \leq \mathcal{R}(x)(F) \vee \mathcal{R}(y)(F \cup \{x\})$. Thus we assume that $a \nleq \mathcal{R}(y)(F) \vee \mathcal{R}(x)(F)$. Then $a \leq \mathcal{R}(x)(F \cup \{y\}) = \bigvee_{z \in F \cup \{y\}} \mathcal{R}(z)(\{x\})$. Since $F \cup \{y\}$ is finite, there is $z \in F \cup \{y\}$ such that $a \leq \mathcal{R}(z)(\{x\})$. We say that $z = y$. Otherwise, $z \in F$ and $a \leq \mathcal{R}(z)(\{x\}) = \mathcal{R}(x)(\{z\}) \leq \mathcal{R}(x)(F)$ by (i). It is a contradiction. Hence $z = y$ and $a \leq \mathcal{R}(y)(\{x\}) \leq \mathcal{R}(x)(F) \vee \mathcal{R}(y)(F \cup \{x\})$. It follows from the arbitrariness of $a \in J(M)$ that $\mathcal{R}(y)(F) \vee \mathcal{R}(x)(F \cup \{y\}) \mathcal{R}(x)(F) \vee \mathcal{R}(y)(F \cup \{x\})$. Therefore $\mathcal{R}$ is weakly symmetric.

(iii) Let $\mathcal{R}$ be weakly symmetric. For all $x, y \in U$, the weak symmetry of $\mathcal{R}$ yields that

$$\mathcal{R}(x)(\{y\}) = \mathcal{R}(y)(\emptyset) \vee \mathcal{R}(x)(\emptyset \cup \{y\}) \leq \mathcal{R}(x)(\emptyset) \vee \mathcal{R}(y)(\emptyset \cup \{x\}) = \mathcal{R}(y)(\{x\}).$$

So $\mathcal{R}$ is primitively symmetric.

(2) By (1), it is sufficient to prove that the primitive symmetry implies symmetry upon the assumed hypothesis. For all $x \in U$ and $F \in \mathcal{P}_{fin}(U)$, the assumed hypothesis and the primitive symmetry of $\mathcal{R}$ yield that $\mathcal{R}(x)(F) = \bigvee_{y \in F} \mathcal{R}(x)(\{y\}) = \bigvee_{y \in F} \mathcal{R}(y)(\{x\})$. This shows that $\mathcal{R}$ is symmetric.

(3) If $\mathcal{R}$ is symmetric then $\mathcal{R}$ is primitively symmetric by (1). For all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$, it is clear that $\mathcal{R}(x)(F) = \bigvee_{y \in F} \mathcal{R}(y)(\{x\}) = \bigvee_{y \in F} \mathcal{R}(x)(\{y\})$. □

**Remark 3.15.** In general, primitive symmetry $\not\Longleftarrow$ weak symmetry $\not\Longleftarrow$ symmetry.

(i) Let $U = \{x, y, z\}$ and $M = [0, 1]$ with $a' = 1 - a$ for any $a \in M$. Define a mapping $\mathcal{R} : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by $\mathcal{R}(x)(\{y\}) = \mathcal{R}(y)(\{x\}) = \frac{1}{4}$, $\mathcal{R}(x)(\{x, y\}) = \mathcal{R}(y)(\{x, y\}) = \mathcal{R}(z)(\{x, z\}) = \frac{1}{2}$, $\mathcal{R}(x)(U) = \mathcal{R}(y)(U) = \mathcal{R}(z)(U) = 1$ and other values are 0. Then $\mathcal{R}$ is a primitively weakly symmetric $M$-fuzzifying algebraic relation on $U$. Let $F = \{z\}$. We have $\mathcal{R}(z)(F \cup \{x\}) = \frac{1}{2} \not\leq 0 = \mathcal{R}(z)(F) \vee \mathcal{R}(x)(F \cup \{z\})$. So $\mathcal{R}$ is not weakly symmetric.

(ii) Let $U = \{x, y, z\}$ and $M = [0, 1]$ with $a' = 1 - a$ for any $a \in M$. Define a mapping $\mathcal{R} : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by

$$\forall u \in U, \forall F \in \mathcal{P}_{fin}(U), \quad \mathcal{R}(u)(F) = \begin{cases} 1, & F = U; \\ \frac{1}{2}, & u \in \{x, y\} \ \& \ F \notin \{\emptyset, \{z\}, U\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{R}$ is a weakly symmetric $M$-fuzzifying algebraic relation on $U$. But $\mathcal{R}(z)(U) = 1 \neq 0 = \bigvee_{v \in U} \mathcal{R}(v)(\{z\})$. So $\mathcal{R}$ is not symmetric.

**Proposition 3.16.** *Let $(U, \mathcal{R})$ be an $M$-fuzzifying algebraic approximation space. Then the following conditions are equivalent.*

*(1) $\mathcal{R}$ is primitively symmetric.*

*(2) $\forall x, y \in U$, $\underline{\mathcal{R}}(\{y\}^c)(x) = \underline{\mathcal{R}}(\{x\}^c)(y)$;*

*(3) $\forall x, y \in U$, $\overline{\mathcal{R}}(\{y\})(x) = \overline{\mathcal{R}}(\{x\})(y)$.*

*Proof.* (1) $\Longrightarrow$ (2) Let $x, y \in U$. Then the primitive symmetry of $\mathcal{R}$ and Proposition 3.7(3) yield that $\underline{\mathcal{R}}(\{y\}^c)(x) = [\mathcal{R}(x)(\{y\})]' = [\mathcal{R}(y)(\{x\})]' = \underline{\mathcal{R}}(\{x\}^c)(y)$. Thus (2) holds.

(2) $\Longrightarrow$ (3) It directly follows from (MARU1).

(3) $\Longrightarrow$ (1) Let $x, y \in U$. Proposition 3.7(3) yields $\mathcal{R}(x)(\{y\}) = \overline{\mathcal{R}}(\{y\})(x) = \overline{\mathcal{R}}(\{x\})(y) = \mathcal{R}(y)(\{x\})$. Therefore $\mathcal{R}$ is primitively symmetric. □

**Proposition 3.17.** *Let $(U, \mathcal{R})$ be an algebraic approximation space. Then the following conditions are equivalent.*

*(1) $\mathcal{R}$ is weakly symmetric.*

*(2) $\forall x \in U$, $\forall F \in \mathcal{P}_{fin}(U)$, $\overline{\mathcal{R}}(F \cup \{y\})(x) \leq \overline{\mathcal{R}}(F)(x) \vee \overline{\mathcal{R}}(F \cup \{x\})(y)$.*

*(3) $\forall x \in U$, $\forall F \in \mathcal{P}_{fin}(U)$, $\underline{\mathcal{R}}(F^c)(x) \wedge \underline{\mathcal{R}}(F^c \backslash \{x\})(y) \leq \underline{\mathcal{R}}(F^c \backslash \{y\})(x)$.*

*(4) $\forall x \in U$, $\forall A \in \mathcal{P}(U)$, $\overline{\mathcal{R}}(A \cup \{y\})(x) \leq \overline{\mathcal{R}}(A)(x) \vee \overline{\mathcal{R}}(A \cup \{x\})(y)$.*

*(5) $\forall x \in U$, $\forall A \in \mathcal{P}(U)$, $\underline{\mathcal{R}}(A)(x) \wedge \underline{\mathcal{R}}(A \backslash \{x\})(y) \leq \underline{\mathcal{R}}(A \backslash \{y\})(x)$.*

*Proof.* (1) $\Longleftrightarrow$ (2) It directly follows from Proposition 3.7(3). (2) $\Longleftrightarrow$ (3) and (4) $\Longleftrightarrow$ (5) are clear. (4) $\Longrightarrow$ (2) is trivial. Thus we only need to prove (2) $\Longrightarrow$ (4). For all $A \in \mathcal{P}(U)$ and $x \in U$, (MARU2) implies that

$$\begin{aligned} \overline{\mathcal{R}}(A \cup \{y\})(x) &\leq \overline{\mathcal{R}}\Big(\bigcup_{F \in \mathcal{P}_{fin}(A)} F\Big)(y) \vee \overline{\mathcal{R}}\Big(\bigcup_{F \in \mathcal{P}_{fin}(A)} F \cup \{y\}\Big)(x) \\ &= \bigvee_{F \in \mathcal{P}_{fin}(A)} \overline{\mathcal{R}}(F)(y) \vee \overline{\mathcal{R}}(F \cup \{y\})(x) \\ &\leq \bigvee_{F \in \mathcal{P}_{fin}(A)} \overline{\mathcal{R}}(F)(x) \vee \overline{\mathcal{R}}(F \cup \{x\})(y) \\ &= \overline{\mathcal{R}}(A)(x) \vee \overline{\mathcal{R}}(A \cup \{x\})(y). \end{aligned}$$

Thus (4) holds. □

**Proposition 3.18.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. Then $\mathcal{R}$ is symmetric iff one of the following conditions holds.*

*(MARL5)* $\forall x \in U$, $\forall F \in \mathcal{P}_{fin}(U)$, $\bigwedge_{y \in F} \underline{\mathcal{R}}(\{x\}^c)(y) \leq \underline{\mathcal{R}}(F^c)(x)$.

*(MARU5)* $\forall x \in U$, $\forall F \in \mathcal{P}_{fin}(U)$, $\overline{\mathcal{R}}(F)(x) \leq \bigvee_{y \in F} \overline{\mathcal{R}}(\{x\})(y)$.

*(MARL5′)* $\forall x \in U$, $\forall A \in \mathcal{P}(U)$, $\bigwedge_{y \notin A} \underline{\mathcal{R}}(\{x\}^c)(y) \leq \underline{\mathcal{R}}(A)(x)$.

*(MARU5′)* $\forall x \in U$, $\forall A \in \mathcal{P}(U)$, $\overline{\mathcal{R}}(A)(x) \leq \bigvee_{y \in A} \overline{\mathcal{R}}(\{x\})(y)$.

*Proof.* (MARL5) $\Longleftrightarrow$ (MARU5) and (MARL5′) $\Longleftrightarrow$ (MARU5′) directly follow from (MARL1) and (MARU1). It directly follows from the symmetry of $\mathcal{R}$ and Proposition 3.7(3) that $\mathcal{R}$ is symmetric iff (MARU5) holds. Also, (MARU5′) $\Longrightarrow$ (MARU5) is clear. Thus it is sufficient to prove (MARU5) $\Longrightarrow$ (MARU5′).

Let $A \in \mathcal{P}(U)$ and $x \in U$. (MARU2) and (MARU5) imply

$$\overline{\mathcal{R}}(A)(x) = \overline{\mathcal{R}}(\bigcup_{F \in \mathcal{P}_{fin}(A)}^{dir} F)(x) = \bigvee_{F \in \mathcal{P}_{fin}(A)} \overline{\mathcal{R}}(F)(x) \leq \bigvee_{F \in \mathcal{P}_{fin}(A)} \bigvee_{y \in F} \overline{\mathcal{R}}(\{x\})(y) = \bigvee_{y \in A} \overline{\mathcal{R}}(\{x\})(y).$$

This shows that (MARU5) holds. □

**Proposition 3.19.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. Then the following conditions are equivalent.*

*(1) $\mathcal{R}$ is symmetric.*

*(2)* $\forall F \in \mathcal{P}_{fin}(U)$, $F \subseteq \bigcap_{a \in \alpha(b)} \underline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a']})^{(b)}$ *for any* $a \in \alpha(M)$;

*(3)* $\forall F \in \mathcal{P}_{fin}(U)$, $\bigcup_{a \in \beta(b)} \overline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)^{(a')})_{[b]} \subseteq F^c$ *for any* $a \in \beta(M)$;

*(4)* $\forall A \in \mathcal{P}(U)$, $A \subseteq \bigcap_{a \in \alpha(b)} \underline{\mathcal{R}}(\overline{\mathcal{R}}(A)_{[a']})^{(b)}$ *for any* $a \in \alpha(M)$;

*(5)* $\forall A \in \mathcal{P}(U)$, $\bigcup_{a \in \beta(b)} \overline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a')})_{[b]} \subseteq A$ *for any* $a \in \beta(M)$.

*Proof.* We first prove that (1) $\Longleftrightarrow$ (5).

($\Longrightarrow$) Let $A \in \mathcal{P}(U)$ and $a \in \beta(M)$. We check that $\bigcup_{a \in \beta(b)} \overline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a')})_{[b]} \subseteq A$.

If $x \in \bigcup_{a \in \beta(b)} \overline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a')})_{[b]}$ then there is $b \in M$ such that $a \in \beta(b)$ and $x \in \overline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a')})_{[b]}$. So $a \prec b \leq \overline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a')})(x)$. Thus there is $F \in \mathcal{P}_{fin}(\underline{\mathcal{R}}(A)^{(a')})$ such that $a \prec \mathcal{R}(x)(F)$. The symmetry of $\mathcal{R}$ yields a $y \in F$ such that $a \prec \mathcal{R}(y)(\{x\})$. Hence $y \in F \subseteq \underline{\mathcal{R}}(A)^{(a')}$ which implies $\underline{\mathcal{R}}(A)(y) \nleq a'$.

Suppose that $x \notin A$. Then $\{x\} \in \mathcal{P}_{fin}(A^c)$. So $\bigwedge_{G \in \mathcal{P}_{fin}(A^c)} [\mathcal{R}(y)(G)]' = \underline{\mathcal{R}}(A)(y) \nleq a'$ implies $[\mathcal{R}(y)(\{x\})]' \nleq a'$. This shows $a \nleq \mathcal{R}(y, \{x\})$. It is a contradiction. Hence $x \in A$ must be true. Therefore $\bigcup_{a \in \beta(b)} \overline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a')})_{[b]} \subseteq A$.

($\Longleftarrow$) Assume that (5) holds. We verify that $\mathcal{R}$ is symmetric.

Let $x \in U$ and $F \in \mathcal{P}_{fin}(U)$. Let $a \in \beta(M)$ with $a \prec \mathcal{R}(x)(F)$. For any $a \in \beta(b)$, (5) yields $\overline{\mathcal{R}}(\underline{\mathcal{R}}(\{u\}^c)^{(a')})_{[b]} \subseteq \{u\}^c$. This shows $x \notin \overline{\mathcal{R}}(\underline{\mathcal{R}}(\{x\}^c)^{(a')})_{[b]}$. In particular, let $b = \mathcal{R}(x)(F)$. Then $a \in \beta(b)$ and $x \in \overline{\mathcal{R}}(F)$. This implies $F \nsubseteq \underline{\mathcal{R}}(\{x\}^c)^{(a')}$. Thus there is a $z \in F$ such that $z \notin \underline{\mathcal{R}}(\{x\}^c)^{(a')}$. This indicates that

$$[\mathcal{R}(z)(\{x\})]' = \bigwedge_{G \in \mathcal{P}_{fin}(\{x\})} [\mathcal{R}(z)(G)]' = \underline{\mathcal{R}}(\{x\}^c)(z) \leq a'.$$

Hence $a \leq \mathcal{R}(z)(\{x\}) \leq \bigvee_{y \in F} \mathcal{R}(y)(\{x\})$. So $\mathcal{R}(x)(F) \leq \bigvee_{y \in F} \mathcal{R}(y)(\{x\})$. Therefore $\mathcal{R}$ is symmetric.

Proofs of (2) $\Longleftrightarrow$ (3) and (4) $\Longleftrightarrow$ (5) directly follow from (ARL1) and (ARU1). Also, (5) $\Longrightarrow$ (3) is trivial.

We next verify that (3) $\Longrightarrow$ (5). Let $A \in \mathcal{P}(U)$ and $a \in \beta(M)$. Since $A = \bigcap^{cdir}_{F\in\mathcal{P}_{fin}(A^c)} F^c$, (3) and (MARL2) imply

$$\begin{aligned}\bigcup_{a\in\beta(b)} \overline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a')})_{[b]} &\subseteq \bigcup_{a\in\beta(b)} \overline{\mathcal{R}}(\bigcap^{cdir}_{F\in\mathcal{P}_{fin}(A^c)} \underline{\mathcal{R}}(F^c)^{(a')})_{[b]} \\ &\subseteq \bigcap_{F\in\mathcal{P}_{fin}(A^c)} \bigcup_{a\in\beta(b)} \overline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)^{(a')})_{[b]} \\ &= \bigcap_{F\in\mathcal{P}_{fin}(A^c)} F^c = A.\end{aligned}$$

Therefore (5) holds for $\mathcal{R}$. □

*3.4. Transitive M-fuzzifying algebraic relations*

**Definition 3.20.** Let $(U, \mathcal{R})$ be an $M$-fuzzifying algebraic approximation space. $\mathcal{R}$ is called
(1) transitive, if $\mathcal{R}(x)(F) \geq \mathcal{R}(x)(G) \wedge \bigwedge_{y\in G} \mathcal{R}(y)(F)$ for all $x \in U$ and $F, G \in \mathcal{P}_{fin}(U)$;
(2) strongly transitive, if $\bigwedge_{F\subseteq A} \bigvee_{y\notin A} \bigvee_{G\in\mathcal{P}_{fin}(A)} \mathcal{R}(x)(G) \vee \mathcal{R}(y)(G) \leq \mathcal{R}(x)(F)$ for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$.

**Proposition 3.21.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. The following conditions are equivalent.*
*(1) $\mathcal{R}$ is transitive.*
*(2) $\overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})_{[a]} \subseteq \overline{\mathcal{R}}(F)_{[a]}$ for any $F \in \mathcal{P}_{fin}(U)$ and any $a \in M$.*
*(3) $\overline{\mathcal{R}}(\overline{\mathcal{R}}(A)_{[a]})_{[a]} \subseteq \overline{\mathcal{R}}(A)_{[a]}$ for any $A \in \mathcal{P}(U)$ and any $a \in M$.*
*(4) $\underline{\mathcal{R}}(F^c)^{(a)} \subseteq \underline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)^{(a)})^{(a)}$ for any $F \in \mathcal{P}_{fin}(U)$ and any $a \in M$.*
*(5) $\underline{\mathcal{R}}(A)^{(a)} \subseteq \underline{\mathcal{R}}(\underline{\mathcal{R}}(A)^{(a)})^{(a)}$ for any $A \in \mathcal{P}(U)$ and any $a \in M$.*

*Proof.* (1) $\Longrightarrow$ (2) Assume that $\mathcal{R}$ is transitive. Let $F \in \mathcal{P}_{fin}(U)$ and let $a \in M$. If $x \in \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})_{[a]}$ then $a \leq \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(x)$. For any $b \in \beta(a)$, it follows that

$$b \prec a \leq \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(x) = \bigvee_{G\in\mathcal{P}_{fin}(\overline{\mathcal{R}}(F)_{[a]})} \mathcal{R}(x)(G).$$

Then there is a finite set $G \in \mathcal{P}_{fin}(\overline{\mathcal{R}}(F)_{[a]})$ such that $b \leq \mathcal{R}(x)(G)$. Since $G \subseteq \overline{\mathcal{R}}(F)_{[a]}$, it is clear that $a \leq \overline{\mathcal{R}}(F)(y) = \mathcal{R}(y)(F)$. Thus the transitivity of $\mathcal{R}$ implies that

$$b \leq \mathcal{R}(x)(G) \wedge \bigwedge_{y\in G} \mathcal{R}(y)(F) \leq \mathcal{R}(x)(F).$$

This shows that $b \leq \mathcal{R}(x)(F) = \overline{\mathcal{R}}(F)(x)$. It follows from arbitrariness of $b \in \beta(a)$ that $a \leq \overline{\mathcal{R}}(F)(x)$. Hence $x \in \overline{\mathcal{R}}(F)_{[a]}$ which implies $\overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})_{[a]} \subseteq \overline{\mathcal{R}}(F)_{[a]}$.

(2) $\Longrightarrow$ (3) Let $x \in \overline{\mathcal{R}}(\overline{\mathcal{R}}(A)_{[a]})_{[a]}$ and $b \in \beta(a)$. Then $b \prec a \leq \overline{\mathcal{R}}(\overline{\mathcal{R}}(A)_{[a]})(x)$. Thus there is a finite set $F \in \mathcal{P}_{fin}(\overline{\mathcal{R}}(A)_{[a]})$ such that $b \leq \mathcal{R}(x)(F) = \overline{\mathcal{R}}(F)(x)$. Further, for any $y \in F$, it is clear that

$$b \prec a \leq \overline{\mathcal{R}}(A)(x) = \bigvee_{G\in\mathcal{P}_{fin}(A)} \mathcal{R}(y)(G) = \bigvee_{G\in\mathcal{P}_{fin}(A)} \overline{\mathcal{R}}(G)(y).$$

So there is a $G_y \in \mathcal{P}_{fin}(A)$ such that $b \leq \overline{\mathcal{R}}(G_y)(y)$. Let $H = \bigcup_{y\in F} G_y$. Then $H \in \mathcal{P}_{fin}(A)$ and $F \subseteq \overline{\mathcal{R}}(H)_{[b]}$. So

$$x \in \overline{\mathcal{R}}(F)_{[b]} \subseteq \overline{\mathcal{R}}(\overline{\mathcal{R}}(H)_{[b]})_{[b]} \subseteq \overline{\mathcal{R}}(H)_{[b]} \subseteq \overline{\mathcal{R}}(A)_{[b]}.$$

Hence $x \in \bigcap_{b\in\beta(a)} \overline{\mathcal{R}}(A)_{[b]} = \overline{\mathcal{R}}(A)_{[a]}$. Therefore $\overline{\mathcal{R}}(\overline{\mathcal{R}}(A)_{[a]})_{[a]} \subseteq \overline{\mathcal{R}}(A)_{[a]}$.

(3) $\Longrightarrow$ (1) Let $F, G \in \mathcal{P}_{fin}(U)$ and $x \in U$. For any $a \in M$ with $a \leq \mathcal{R}(x)(G) \wedge \bigwedge_{y\in G} \mathcal{R}(y)(F)$, it is clear that $x \in \overline{\mathcal{R}}(G)_{[a]}$ and $G \in \mathcal{P}_{fin}(\overline{\mathcal{R}}(F)_{[a]})$. Thus $x \in \overline{\mathcal{R}}(G)_{[a]} \subseteq \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})_{[a]} \subseteq \overline{\mathcal{R}}(F)_{[a]}$ which implies

$a \leq \overline{\mathcal{R}}(F)(x) = \mathcal{R}(x)(F)$. It follows from the arbitrariness of $a \in M$ that $\mathcal{R}(x)(G) \wedge \bigwedge_{y \in G} \mathcal{R}(y)(F) \leq \mathcal{R}(x)(F)$. Therefore $\mathcal{R}$ is transitive.

Proofs of (2) $\Longleftrightarrow$ (4) and (2) $\Longleftrightarrow$ (5) directly follow from (MRL1) and (MRU1). $\square$

**Proposition 3.22.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. Then $\mathcal{R}$ is transitive iff one of the following conditions holds.*

*(MARL6)* $\underline{\mathcal{R}}(F^c)(x) \leq \bigvee_{B \subseteq U}[\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{z \in B} \underline{\mathcal{R}}(F^c)(z)]$ *for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$.*

*(MARU6)* $\overline{\mathcal{R}}(F)(x) \geq \bigwedge_{C \subseteq U}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z \notin C} \overline{\mathcal{R}}(F)(z)]$ *for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$.*

*(MARL6′)* $\underline{\mathcal{R}}(A)(x) \leq \bigvee_{B \subseteq U}[\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{z \in B} \underline{\mathcal{R}}(A)(z)]$ *for all $A \in \mathcal{P}(U)$ and $x \in U$.*

*(MARU6′)* $\overline{\mathcal{R}}(A)(x) \geq \bigwedge_{C \subseteq U}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z \notin C} \overline{\mathcal{R}}(A)(z)]$ *for all $A \in \mathcal{P}(U)$ and $x \in U$.*

*Proof.* We firstly prove that $\mathcal{R}$ is transitive iff (MARL6) holds for $\underline{\mathcal{R}}$.

($\Longleftarrow$) Let $F \in \mathcal{P}_{fin}(U)$ and $x \in U$. For any $a \in M$ with $x \in \underline{\mathcal{R}}(F^c)^{(a)}$, $\underline{\mathcal{R}}(F^c)(x) \nleq a$. (MARL6) implies that

$$\bigvee_{B \in \mathcal{P}(U)} [\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{z \in B} \underline{\mathcal{R}}(F^c)(z)] \nleq a.$$

Thus there is $B \in \mathcal{P}(U)$ such that $\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{z \in B} \underline{\mathcal{R}}(F^c)(z) \nleq a$. This implies that $x \in \underline{\mathcal{R}}(B)^{(a)}$ and $z \in \underline{\mathcal{R}}(F^c)^{(a)}$ for any $z \in B$. Hence $B \subseteq \underline{\mathcal{R}}(F^c)^{(a)}$ and $x \in \underline{\mathcal{R}}(B)^{(a)} \subseteq \underline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)^{(a)})^{(a)}$. Therefore $\underline{\mathcal{R}}(F^c)^{(a)} \subseteq \underline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)^{(a)})^{(a)}$. It follows from Proposition 3.21(3) that $\mathcal{R}$ is transitive.

($\Longrightarrow$) Let $\underline{\mathcal{R}}(F^c)(x) \nleq a$. Then there is $b \in \alpha(a)$ such that $\underline{\mathcal{R}}(F^c)(x) \nleq b$. So $x \in \underline{\mathcal{R}}(F^c)^{(b)} \subseteq \underline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)^{(b)})^{(b)}$. Thus $\underline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)(b))(x) \nleq b$. For any $z \in \underline{\mathcal{R}}(F^c)^{(b)}$, it is clear that $\underline{\mathcal{R}}(F^c)(z) \nleq b$. Thus

$$\underline{\mathcal{R}}(\underline{\mathcal{R}}(F^c)^{(b)})(x) \wedge \bigwedge_{z \in \underline{\mathcal{R}}(F^c)^{(b)}} \underline{\mathcal{R}}(F^c)(z) \nleq a.$$

Hence $\bigvee_{B \in \mathcal{P}(U)}[\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{z \in B} \underline{\mathcal{R}}(F^c)(z)] \nleq a$. From the arbitrariness of $a \in M$,

$$\underline{\mathcal{R}}(F^c)(x) \leq \bigvee_{B \in \mathcal{P}(U)} [\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{z \in B} \underline{\mathcal{R}}(F^c)(z)].$$

Therefore (MARL6) holds for $\underline{\mathcal{R}}$.

By replacing the set $F^c$ by $A$ in the above proof, we can directly prove that $\mathcal{R}$ is transitive iff (MARL6′) holds for $\underline{\mathcal{R}}$. Thus (MARL6) and (MARL6′) are equivalent. Finally, proofs of (MARL6) $\Longleftrightarrow$ (MARU6) and (MARL6′) $\Longleftrightarrow$ (MARU6′) directly follow from (MARL1) and (MARU1). $\square$

**Proposition 3.23.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. Then $\mathcal{R}$ is strongly transitive iff one of the following conditions holds.*

*(MARL6*)* $\underline{\mathcal{R}}(F^c)(x) = \bigvee_{B \subseteq F^c}[\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{y \in B} \underline{\mathcal{R}}(B)(y)]$ *for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$.*

*(MARU6*)* $\overline{\mathcal{R}}(F)(x) = \bigwedge_{F \subseteq C}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{y \notin C} \overline{\mathcal{R}}(C)(y)]$ *for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$.*

*(MARL6*′)* $\underline{\mathcal{R}}(A)(x) = \bigvee_{B \subseteq A}[\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{y \in B} \underline{\mathcal{R}}(B)(y)]$ *for all $A \in \mathcal{P}(U)$ and $x \in U$.*

*(MARU6*′)* $\overline{\mathcal{R}}(A)(x) = \bigwedge_{A \subseteq C}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{y \notin C} \overline{\mathcal{R}}(C)(y)]$ *for all $A \in \mathcal{P}(U)$ and $x \in U$.*

*Proof.* We verify that $\mathcal{R}$ is strongly transitive iff (MARU6*) holds. Indeed, for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$, it follows from (MARU2) that

$$\bigwedge_{F \subseteq C} \bigvee_{y \notin C} \bigvee_{G \in \mathcal{P}_{fin}(C)} \mathcal{R}(x)(G) \vee \mathcal{R}(y)(G) = \bigwedge_{F \subseteq C} \bigvee_{y \notin C} \overline{\mathcal{R}}(C)(x) \vee \overline{\mathcal{R}}(C)(y) = \bigwedge_{F \subseteq C} \overline{\mathcal{R}}(C)(x) \vee \bigvee_{y \notin C} \overline{\mathcal{R}}(C)(y).$$

Thus $\mathcal{R}$ is strongly transitive iff (MARU6*) holds.

Proofs of (MARL6*) $\Longleftrightarrow$ (MARU6*) and (MARL6*′) $\Longleftrightarrow$ (MARU6*′) follow from (MARL1) and (MARU1). Also, it is clear that (MARU6*′) implies (MARU6*). We next prove (MARU6*) implies (MARU6*′).

Let $A \in \mathcal{P}(U)$ and $x \in U$. It is clear that

$$\overline{\mathcal{R}}(A)(x) \leq \bigwedge_{A\subseteq C} \overline{\mathcal{R}}(C)(x) \leq \bigwedge_{A\subseteq C} [\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z)].$$

Conversely, let $a \in \beta(M)$ with $a \prec \bigwedge_{A\subseteq C}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z)]$. Suppose that $a \not\leq \overline{\mathcal{R}}(A)(x)$. Then $b \not\leq \overline{\mathcal{R}}(A)(x)$ for some $b \in \beta(a)$. Again, there is $d \in \beta(b)$ such that $d \not\leq \overline{\mathcal{R}}(A)(x) = \bigvee_{F\in\mathcal{P}_{fin}(A)} \overline{\mathcal{R}}(F)(x)$. Let $F \in \mathcal{P}_{fin}(A)$. Since

$$d \not\leq \overline{\mathcal{R}}(F)(x) = \bigwedge_{F\subseteq C} [\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z),$$

there is a $C \supseteq F$ such that $d \not\leq \overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z)$. Let $C_F = \bigcap \psi_F^d$, where

$$\psi_F^d = \{D \in \mathcal{P}(U) : F \subseteq D,\ d \not\leq \overline{\mathcal{R}}(D)(x) \vee \bigvee_{z\notin D} \overline{\mathcal{R}}(D)(z)\}.$$

Since $\mathcal{P}_{fin}(A)$ is directed, the set $\{C_F : F \in \mathcal{P}_{fin}(A)\}$ is also directed. Thus $A = \bigvee^{dir}_{F\in\mathcal{P}_{fin}(A)} F \subseteq \bigvee^{dir}_{F\in\mathcal{P}_{fin}(A)} C_F$. Let $E = \bigvee_{F\in\mathcal{P}_{fin}(A)} C_F$. For any $z \notin E$, it is clear that $z \notin C_F$ for any $F \in \mathcal{P}_{fin}(A)$. Hence there is a $D \in \psi_F^b$ such that $z \notin D$. So

$$d \not\leq \overline{\mathcal{R}}(D)(x) \vee \overline{\mathcal{R}}(D)(z) \geq \overline{\mathcal{R}}(C_F)(x) \vee \overline{\mathcal{R}}(C_F)(z).$$

This implies that $d \not\leq \overline{\mathcal{R}}(C_F)(x) \vee \overline{\mathcal{R}}(C_F)(z)$. From the arbitrariness of $z \notin C_F$, we conclude that $b \not\leq \overline{\mathcal{R}}(C_F)(x) \vee \bigvee_{z\notin E} \overline{\mathcal{R}}(C_F)(z)$. Hence

$$\begin{aligned} a &\not\leq \bigvee^{dir}_{F\in\mathcal{P}_{fin}(A)} [\overline{\mathcal{R}}(C_F)(x) \vee \bigvee_{z\notin E} \overline{\mathcal{R}}(C_F)(z)] \\ &= \overline{\mathcal{R}}(E)(x) \vee \bigvee_{z\notin E} \overline{\mathcal{R}}(E)(z) \\ &\geq \bigwedge_{A\subseteq C} [\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z)]. \end{aligned}$$

This contradicts $a \prec \bigwedge_{A\subseteq C}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z)]$. Therefore $a \leq \overline{\mathcal{R}}(A)(x)$. So

$$\bigwedge_{A\subseteq C} [\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z)] \leq \overline{\mathcal{R}}(A)(x).$$

This shows that (MARU6*′) holds for $\mathcal{R}$. □

**Proposition 3.24.** *Let $(U, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. Then the following results holds.*
*(1) Strong transitivity implies transitivity.*
*(2) If $\mathcal{R}(x)(F) \geq \mathcal{R}(y)(F)$ for all $x, y \in U$ and $F \in \mathcal{P}_{fin}(U)$ with $x \in F$ and $y \notin F$, then $\mathcal{R}$ is transitive iff $\mathcal{R}$ is strongly transitive.*
*(3) If $\mathcal{R}$ is reflexive then $\mathcal{R}$ is transitive iff it is strongly transitive.*

*Proof.* (1) Let $\mathcal{R}$ be strongly transitive. For all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$, it follows from (MARU6*) that

$$\bigwedge_{C\subseteq U} [\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(F)(z)] \leq \bigwedge_{F\subseteq C} [\overline{\mathcal{R}}(C)(x) \vee \bigvee_{y\notin C} \overline{\mathcal{R}}(C)(y)] \leq \overline{\mathcal{R}}(F)(x).$$

This shows that (MARU6) holds for $\mathcal{R}$. That is, $\mathcal{R}$ is transitive.

(2) It is sufficient to prove that if $\mathcal{R}$ is transitive then (MARU6*) holds for $\mathcal{R}$.

Let $\mathcal{R}$ is transitive. Let $F \in \mathcal{P}_{fin}(U)$ and $x \in U$. It is clear that

$$\overline{\mathcal{R}}(F)(x) \leq \bigwedge_{F\subseteq C} [\overline{\mathcal{R}}(C)(x) \leq \bigwedge_{F\subseteq C} [\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C} \overline{\mathcal{R}}(C)(z)].$$

Conversely, let $a \in \beta(M)$ with $a \prec \bigwedge_{F\subseteq C}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C}\overline{\mathcal{R}}(C)(z)]$. Suppose that $a \nleq \overline{\mathcal{R}}(F)(x)$. Since $a \prec \overline{\mathcal{R}}(F)(x) \vee \bigvee_{z\notin F}\overline{\mathcal{R}}(F)(z)$, there is a $z \notin F$ such that $a \leq \overline{\mathcal{R}}(F)(z)$. For any $w \in F$, it follows from the hypothesis that $a \leq \overline{\mathcal{R}}(F)(z) = \mathcal{R}(z, F) \leq \mathcal{R}(w, F) = \overline{\mathcal{R}}(F)(w)$. This shows that $w \in \overline{\mathcal{R}}(F)_{[a]}$. So $F \subseteq \overline{\mathcal{R}}(F)_{[a]}$ which implies that

$$a \prec \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(x) \vee \bigvee_{z\notin\overline{\mathcal{R}}(F)_{[a]}} \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(z).$$

Thus $a \prec \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(x)$ or $a \prec \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(z)$ for some $z \notin \overline{\mathcal{R}}(F)_{[a]}$. If

$$a \prec \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(x) = \overline{\mathcal{R}}(\bigcup^{dir}_{G\in\mathcal{P}_{fin}(\overline{\mathcal{R}}(F)_{[a]})} G)(x) = \bigvee_{G\in\mathcal{P}_{fin}(\overline{\mathcal{R}}(F)_{[a]})} \overline{\mathcal{R}}(G)(x),$$

there is a $G \in \mathcal{P}_{fin}(\overline{\mathcal{R}}(F)_{[a]})$ such that $a \leq \overline{\mathcal{R}}(G)(x)$. The transitivity of $\mathcal{R}$ implies $a \leq \overline{\mathcal{R}}(G)(x) \wedge \bigwedge_{y\in G}\overline{\mathcal{R}}(F)(y) \leq \overline{\mathcal{R}}(F)(x)$. This shows that $a \leq \overline{\mathcal{R}}(F)(x)$. It is a contradiction. If $a \prec \overline{\mathcal{R}}(\overline{\mathcal{R}}(F)_{[a]})(z)$ for $z \notin \overline{\mathcal{R}}(F)_{[a]}$, there is $H \in \mathcal{P}_{fin}(\overline{\mathcal{R}}(F)_{[a]})$ such that $a \leq \overline{\mathcal{R}}(H)(z)$. The transitivity of $\mathcal{R}$ implies $a \leq \overline{\mathcal{R}}(H)(z) \wedge \bigwedge_{y\in H}\overline{\mathcal{R}}(F)(y) \leq \overline{\mathcal{R}}(F)(z)$. It contradicts $z \notin \overline{\mathcal{R}}(F)_{[a]}$. Thus $a \leq \overline{\mathcal{R}}(F)(x)$ must hold. Hence

$$\bigwedge_{F\subseteq C}[\overline{\mathcal{R}}(C)(x) \vee \bigvee_{z\notin C}\overline{\mathcal{R}}(C)(z)] \leq \overline{\mathcal{R}}(F)(x).$$

Therefore (MARU6*) holds for $\mathcal{R}$.

(3) If $\mathcal{R}$ is reflexive then the hypothesis in (2) holds. Thus $\mathcal{R}$ is transitive iff it is strongly transitive. □

**Remark 3.25.** (1) Clearly, the reflexivity implies the hypothesis in Proposition 3.24 (i.e., $\mathcal{R}(x)(F) \geq \mathcal{R}(y)(F)$ for all $x, y \in U$ and $F \in \mathcal{P}_{fin}(U)$ with $x \in F$ and $y \notin F$). But the inverse implication is not true.

(2) Transitivity fails to imply strong transitivity without the hypothesis in Proposition 3.24(2).

Let $U = \{x, y, z\}$ and let $M = \{\bot, a, b, \top\}$ be a diamond lattice with two incomparable elements $a$ and $b$. The inverse involution $'$ on $M$ is defined by $\bot' = \top$ and $a' = b$. Define a relation $\mathcal{R} : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by

$$\forall w \in U, \forall F \in \mathcal{P}_{fin}(U), \quad \mathcal{R}(w)(F) = \begin{cases} \bot, & F = \emptyset, \\ a, & w \in F \neq U, \\ b, & w \notin F \neq \emptyset, \\ \top, & F = U. \end{cases}$$

It is clear that $\mathcal{R}$ is an $M$-fuzzifying algebraic relation. We check the following results.

(i) $\mathcal{R}$ has the transitive property.

Let $w \in U$ and $F, G \in \mathcal{P}_{fin}(U)$. We verify that $\mathcal{R}(w)(G) \wedge \bigwedge_{v\in G}\mathcal{R}(v)(F) \leq \mathcal{R}(w)(F)$.

If one of the cases $F = U$, $F = \emptyset$, $G = \emptyset$ or $w \in G$ holds, then the desired result is clear. Assume that $F \neq \emptyset$, $F \neq U$, $G \neq \emptyset$ and $w \notin G$. If $w \notin F$ then the desired result is also clear since $w \notin G$. Thus we further assume that $w \in F$. If $G \subseteq F$, then $\mathcal{R}(w)(G) = \mathcal{R}(w)(F)$. If $G \nsubseteq F$, then there is a point $v \in G$ such that $v \notin F$. Thus $\mathcal{R}(v)(F) = \mathcal{R}(w)(F)$. In either case, the desired result is true. Therefore $\mathcal{R}$ has the transitive property.

(ii) The presupposition fails. In fact, $\mathcal{R}(x)(\{x, y\}) = a \ngeq b = \mathcal{R}(z)(\{x, y\})$.

(iii) $\mathcal{R}$ fails to satisfy (MARL6*). That is, $\mathcal{R}$ is not strongly transitive.

In fact, let $F = \{x\}$. Then $\bigvee_{B\subseteq F^c}[\underline{\mathcal{R}}(B)(x) \wedge \bigwedge_{v\in B}\underline{\mathcal{R}}(B)(v)] = \bot$ and $\underline{\mathcal{R}}(F^c)(x) = b$. So (MARL6*) fails for $\mathcal{R}$.

(3) The hypothesis is independent with transitivity and strong transitivity.

Let $U = \{x, y\}$ and let $M = \{\bot, a, b, \top\}$ be a diamond lattice with two incomparable elements $a$ and $b$. The inverse involution $'$ on $M$ is defined by $\bot' = \top$ and $a' = b$. Define a relation $\mathcal{R} : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by $\mathcal{R}(x)(\emptyset) = \mathcal{R}(x)(\{y\}) = \bot$, $\mathcal{R}(x)(\{x\}) = \mathcal{R}(x)(U) = a$ and $\mathcal{R}(y)(\{x\}) = \mathcal{R}(y)(\{y\}) = \mathcal{R}(y)(U) = b$. Then $\mathcal{R}$ satisfies (MARU6*). This is, $\mathcal{R}$ is strongly transitive. But $\mathcal{R}(x, x) = a \ngeq b = \mathcal{R}(y, x)$. Therefore $\mathcal{R}$ fails for the hypothesis.

**Proposition 3.26.** *Let $(U,\mathcal{R})$ be an M-fuzzifying algebraic approximation space. Then $\mathcal{R}$ is reflexive and transitive iff one of the following conditions holds.*

*(MRL7)* $\underline{\mathcal{R}}(F^c)(x) = \bigvee_{x\in B\subseteq F^c} \bigwedge_{y\in B} \underline{\mathcal{R}}(B)(y)$ *for all* $F \in \mathcal{P}_{fin}(U)$ *and* $x \in U$.

*(MRU7)* $\overline{\mathcal{R}}(F)(x) = \bigwedge_{x\notin C\supseteq F} \bigvee_{y\notin C} \overline{\mathcal{R}}(C)(y)$ *for all* $F \in \mathcal{P}_{fin}(U)$ *and* $x \in U$.

*(MRL7′)* $\underline{\mathcal{R}}(A)(x) = \bigvee_{x\in B\subseteq A} \bigwedge_{y\in B} \underline{\mathcal{R}}(B)(y)$ *for all* $A \in \mathcal{P}(U)$ *and* $x \in U$.

*(MRU7′)* $\overline{\mathcal{R}}(A)(x) = \bigwedge_{x\notin C\supseteq A} \bigvee_{y\notin C} \overline{\mathcal{R}}(C)(y)$ *for all* $A \in \mathcal{P}(U)$ *and* $x \in U$.

*Proof.* It directly follows from (1) of Remark 3.25 and Propositions 3.12 and 3.23. □

**Remark 3.27.** (1) If an $M$-fuzzifying restricted hull operator $\mathcal{H} : \mathcal{P}_{fin}(U) \longrightarrow M^{\mathcal{P}(U)}$ satisfies $\mathcal{H}(\emptyset)(x) = \bot$ and $\mathcal{H}(F)(x) = \top$ for any $x \in U$ and $F \in \mathcal{P}^*_{fin}(U)$ (or, $\mathcal{H}(F)(x) = \top$ whenever $x \in F$, and $\mathcal{H}(F)(x) = \bot$ otherwise), then $\mathcal{R}_{\mathcal{H}}$ is serial, symmetric, reflexive and transitive.

(2) $M$-fuzzifying algebraic relations in Examples 3.3-3.5 are serial, reflexive and transitive. $M$-fuzzifying algebraic relations in Examples 3.6 and 3.5 are primitively symmetric. The $M$-fuzzifying algebraic relation in Example 3.5 is weakly symmetric.

## 4. Relationships between $M$-fuzzifying algebraic rough spaces and $M$-fuzzifying convex spaces

In this section, we discuss relationships between $M$-fuzzifying algebraic approximation spaces and $M$-fuzzifying convex structures.

**Proposition 4.1.** *Let $(U,\mathcal{R})$ be an M-fuzzifying algebraic approximation space. Then the following conditions are equivalent:*

*(1) $\mathcal{R}$ is reflexive and transitive;*

*(2) the operator $\mathcal{H}_{\mathcal{R}} : \mathcal{P}_{fin}(U) \longrightarrow M^U$, defined by $\mathcal{H}_{\mathcal{R}}(F)(x) = \mathcal{R}(x)(F)$ for all $F \in \mathcal{P}_{fin}(U)$ and $x \in X$, is an M-fuzzifying restricted hull operator.*

*(3) $\overline{\mathcal{R}}$ is an M-fuzzifying convex hull operator;*

*(4) $\underline{\mathcal{R}}$ is an M-fuzzifying concave hull operator.*

*Proof.* (1) $\Longrightarrow$ (2) We check that $\mathcal{H}_{\mathcal{R}}$ is an $M$-fuzzifying restricted hull operator.

(MRH1) It is clear that $\mathcal{H}_{\mathcal{R}}(\emptyset)(x) = \mathcal{R}(x)(\emptyset) = \bot$.

(MRH2) For all $F \in \mathcal{P}_{fin}(U)$ and $x \in F$, the reflexivity of $\mathcal{R}$ directly implies that $\mathcal{H}_{\mathcal{R}}(F)(x) = \mathcal{R}(x)(F) = \top$.

(MRH3) For all $F, G \in \mathcal{P}_{fin}(U)$ and $x \in U$, the transitivity of $\mathcal{R}$ directly implies

$$\mathcal{H}_{\mathcal{R}}(G)(x) \wedge \bigwedge_{y\in G} \mathcal{H}_{\mathcal{R}}(F)(y) \leq \mathcal{H}_{\mathcal{R}}(F)(x).$$

Therefore $\mathcal{H}_{\mathcal{R}}$ is an $M$-fuzzifying restricted hull operator.

(2) $\Longrightarrow$ (3) For any $A \in \mathcal{P}(U)$ and any $x \in X$, it is clear that

$$\overline{\mathcal{R}}(A)(x) = \bigvee_{F\in\mathcal{P}_{fin}(A)} \mathcal{R}(x)(F) = \bigvee_{F\in\mathcal{P}_{fin}(A)} \mathcal{H}_{\mathcal{R}}(F)(x).$$

Thus $\overline{\mathcal{R}}$ is an $M$-fuzzifying convex hull operator.

(3) $\Longrightarrow$ (1) Since $\overline{\mathcal{R}}$ is an $M$-fuzzifying convex hull operator, (MCO3) directly implies that (MARU7) holds for $\overline{\mathcal{R}}$. Thus $\mathcal{R}$ is reflexive and transitive.

(3) $\Longleftrightarrow$ (4) Since $M$-fuzzifying convex hull operator and $M$-fuzzifying concave hull operator are dual concepts are dual concepts, the equivalence is clear. □

**Proposition 4.2.** *Let $(U,C)$ be an M-fuzzifying convex space. Define a mapping $\mathcal{R}_C : U \longrightarrow M^{\mathcal{P}_{fin}(U)}$ by $\mathcal{R}_C(x)(F) = \mathcal{H}_C(F)(x)$ for all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$. Then $\mathcal{R}_C$ is a reflexive and transitive M-fuzzifying algebraic relation satisfying $\underline{\mathcal{R}_C} = ca_C$ and $\overline{\mathcal{R}_C} = co_C$. In addition, if $(U,C)$ is an M-fuzzifying convex matroid then $\mathcal{R}_C$ is weakly symmetric.*

*Proof.* It is clear that $\mathcal{R}_C$ is an $M$-fuzzifying algebraic relation. For all $A \in \mathcal{P}(U)$ and $x \in U$, (MCO4) yields

$$\overline{\mathcal{R}_C}(A)(x) = \bigvee_{F \in \mathcal{P}_{fin}(A)} \mathcal{H}_C(F)(x) = \bigvee_{F \in \mathcal{P}_{fin}(A)} co_C(F)(x) = co_C(A)(x).$$

Thus $\underline{\mathcal{R}_C}(A)(x) = [\overline{\mathcal{R}_C}(A^c)(x)]' = [co_C(A^c)(x)]' = ca_C(A)(x)$. So $\underline{\mathcal{R}_C} = ca_C$ and $\overline{\mathcal{R}_C} = co_C$.

For all $\{A_i\}_{i\in I}^{cdir} \subseteq \mathcal{P}(U)$ and $x \in U$, (MCA3) implies that

$$\begin{aligned}
\underline{\mathcal{R}_C}(\overset{cdir}{\bigcap_{i\in I}} A_i)(x) &= \bigwedge_{i\in I} ca_C(A_i)(x) \\
&= \bigwedge_{i\in I} \bigvee_{x\in B_i\subseteq A_i} \bigwedge_{y\in B_i} ca_C(B_i)(y) \\
&= \bigwedge_{i\in I} \bigvee_{x\in B_i\subseteq A_i} \bigwedge_{y\in B_i} \underline{\mathcal{R}_C}(B_i)(y).
\end{aligned}$$

Hence (MARL7) holds for $\underline{\mathcal{R}_C}$. Therefore $\mathcal{R}_C$ is reflexive and transitive.

Also, if $(U, C)$ is an $M$-fuzzifying convex matroid then it directly follows from Definition 2.13 and Proposition 3.7(3) that $\mathcal{R}_C$ is weakly symmetric. □

**Proposition 4.3.** *Let $(U, \mathcal{R})$ be a reflexive and transitive M-fuzzifying algebraic approximation space. Define a mapping $C_{\mathcal{R}} : \mathcal{P}(U) \longrightarrow M$ by*

$$\forall A \in \mathcal{P}(U), \quad C_{\mathcal{R}}(A) = \bigwedge_{x\in A^c} \underline{\mathcal{R}}(A^c)(x) = \bigwedge_{x\notin A} [\overline{\mathcal{R}}(A)(x)]'.$$

*Then $C_{\mathcal{R}}$ is an M-fuzzifying convex structure on U satisfying $ca_{C_{\mathcal{R}}} = \underline{\mathcal{R}}$ and $co_{C_{\mathcal{R}}} = \overline{\mathcal{R}}$. In addition, if $\mathcal{R}$ is also weakly symmetric then $(U.C_{\mathcal{R}})$ is an M-fuzzifying convex matroid.*

*Proof.* We check that $C_{\mathcal{R}}$ satisfies (MC1)–(MC3).

(MC1) Clearly, $C_{\mathcal{R}}(U) = \bigwedge \emptyset = \top$. Also, $C_{\mathcal{R}}(\emptyset) = \bigwedge_{x\in U} \chi_U(x) = \top$ by Proposition 3.7(2).

(MC2) For any $\{A_i\}_{i\in I} \subseteq \mathcal{P}(U)$, Proposition 3.7(1) yields that

$$C_{\mathcal{R}}(\bigcap_{i\in I} A_i) = \bigwedge_{x\in\bigcup_{i\in I} A_i^c} \underline{\mathcal{R}}(\bigcup_{i\in I} A_i^c)(x) \geq \bigwedge_{i\in I} \bigwedge_{x\in A_i^c} \underline{\mathcal{R}}(A_i^c)(x) = \bigwedge_{i\in I} C_{\mathcal{R}}(A_i).$$

(MC3) For any $\{A_i\}_{i\in I}^{dir} \subseteq \mathcal{P}(U)$, the set $\{A_i^c : i \in I\}$ is co-directed. Thus

$$C_{\mathcal{R}}(\overset{dir}{\bigcup_{i\in I}} A_i) = \bigwedge_{x\in\bigcap_{i\in I}^{cdir} A_i^c} \underline{\mathcal{R}}(\overset{cdir}{\bigcap_{i\in I}} A_i^c)(x) \geq \bigwedge_{i\in I} \bigwedge_{x\in A_i^c} \underline{\mathcal{R}}(A_i^c)(x) = \bigwedge_{i\in I} C_{\mathcal{R}}(A_i).$$

Therefore $C_{\mathcal{R}}$ is an $M$-fuzzifying convex structure.

Further, for all $A \in \mathcal{P}(U)$ and $x \in U$, (MCA3) and (MARL7′) implies that

$$ca_{C_{\mathcal{R}}}(A)(x) = \bigvee_{x\in B\subseteq A} \bigwedge_{y\in B} \underline{\mathcal{R}}(B)(y) = \underline{\mathcal{R}}(A)(x).$$

This implies that $co_{C_{\mathcal{R}}}(A) = [\underline{\mathcal{R}}(A^c)]' = \overline{\mathcal{R}}(A)$. Therefore $ca_{C_{\mathcal{R}}} = \underline{\mathcal{R}}$ and $co_{C_{\mathcal{R}}} = \overline{\mathcal{R}}$.

If $\mathcal{R}$ is weakly symmetric then directly follows from Definition 2.13 and Proposition 3.7(3) and 3.17(3) that $C_{\mathcal{R}}$ is an $M$-fuzzifying convex matroid. □

**Proposition 4.4.** *Let $(U, \mathcal{R})$ be a reflexive and transitive M-fuzzifying algebraic approximation space. Then $\mathcal{R}_{C_{\mathcal{R}}} = \mathcal{R}$.*

*Proof.* For all $x \in U$ and $F \in \mathcal{P}_{fin}(U)$, Proposition 4.3 yields that

$$\mathcal{R}_{C_{\mathcal{R}}}(x)(F) = \mathcal{H}_{C_{\mathcal{R}}}(F)(x) = co_{C_{\mathcal{R}}}(F)(x) = [ca_{C_{\mathcal{R}}}(F^c)(x)]' = [\underline{\mathcal{R}}(F^c)(x)]' = \mathcal{R}(x)(F).$$

Thus $\mathcal{R}_{C_{\mathcal{R}}} = \mathcal{R}$. □

**Theorem 4.5.** *Let $(U, C)$ be an M-fuzzifying convex space. Then $C_{\mathcal{R}_C} = C$.*

*Proof.* For any $A \in \mathcal{P}(U)$, Proposition 4.3 and Theorem 2.10(4) imply that

$$C_{\mathcal{R}_C}(A) = \bigwedge_{x \in A^c} \underline{\mathcal{R}_C}(A^c)(x) = \bigwedge_{x \in A^c} ca_C(A^c)(x) = C_{ca_C}(A) = C(A).$$

This shows that $C_{\mathcal{R}_C} = C$. □

In order to discuss categorical relationships between $M$-fuzzifying convex structures and reflexive and transitive $M$-fuzzifying algebraic relations, we present the following notion.

**Definition 4.6.** Let $(U_1, V_1, \mathcal{R}_1)$ and $(U_2, V_2, \mathcal{R}_2)$ be $M$-fuzzifying algebraic approximation spaces. Let $f : U_1 \longrightarrow U_2$ and $g : V_1 \longrightarrow V_2$ be mappings. The pairwise mapping $(f, g) : U_1 \times V_1 \longrightarrow U_2 \times V_2$ is called a pairwise $M$-fuzzifying algebraic relation preserving mapping, if $\mathcal{R}_1(x)(F) \leq \mathcal{R}_2(f(x))(g(F))$ for any $x \in U_1$ and $F \in \mathcal{P}_{fin}(V_1)$.

In particular, if $U_i = V_i$ $(i = 1, 2)$ and $f = g$ then $(f, g)$ reduces to a single mapping $f$ which is simply called an $M$-fuzzifying algebraic relation preserving mapping.

**Proposition 4.7.** *Let $(U_1, V_1, \mathcal{R}_1)$ and $(U_2, V_2, \mathcal{R}_2)$ be M-fuzzifying algebraic approximation spaces. For a pairwise mapping $(f, g) : U_1 \times V_1 \longrightarrow U_2 \times V_2$, the following conditions are equivalent.*

*(1) $(f, g)$ is a pairwise M-fuzzifying algebraic relation preserving mapping;*

*(2) $\overline{\mathcal{R}_1}(A)(x) \leq \overline{\mathcal{R}_2}(g(A))(f(x))$ for all $A \in \mathcal{P}(V_1)$ and $x \in U_1$;*

*(3) $\overline{\mathcal{R}_1}(F)(x) \leq \overline{\mathcal{R}_2}(g(F))(f(x))$ for any $F \in \mathcal{P}_{fin}(V_1)$ and $x \in U_1$;*

*(4) $\underline{\mathcal{R}_2}(g(F)^c)(f(x)) \leq \underline{\mathcal{R}_1}(F^c)(x)$ for any $F \in \mathcal{P}_{fin}(V_1)$ and $x \in U_1$.*

*(5) $\underline{\mathcal{R}_2}(g(A)^c)(f(x)) \leq \underline{\mathcal{R}_1}(A^c)(x)$ for any $A \in \mathcal{P}(V_1)$ and $x \in U_1$.*

*Proof.* (1) $\Longrightarrow$ (2) For all $A \in \mathcal{P}(V)$ and $x \in U$,

$$\overline{\mathcal{R}_1}(A)(x) = \bigvee_{F \in \mathcal{P}_{fin}(A)} \mathcal{R}_1(x)(F) \leq \bigvee_{F \in \mathcal{P}_{fin}(A)} \mathcal{R}_2(f(x))(g(F)) \leq \overline{\mathcal{R}_2}(g(A))(f(x)).$$

(2) $\Longrightarrow$ (3) It is clear.

(3) $\Longrightarrow$ (4) Let $F \in \mathcal{P}_{fin}(V)$ and $x \in U$. (MARL1) implies that

$$\underline{\mathcal{R}_2}(g(F)^c)(f(x)) = [\overline{\mathcal{R}_2}(g(F))(f(x))]' \leq [\overline{\mathcal{R}_1}(F)(x)]' = \underline{\mathcal{R}_1}(F^c)(x).$$

Hence (4) holds.

(4) $\Longrightarrow$ (5) Let $A \in \mathcal{P}(V)$ and $x \in U$. Since $g(A) = \bigcup^{dir}_{F \in \mathcal{P}_{fin}(g(A))} F$, (MARL2) yields

$$\underline{\mathcal{R}_2}(g(A)^c)(f(x)) = \bigwedge_{F \in \mathcal{P}_{fin}(g(A))} \underline{\mathcal{R}_2}(F^c)(f(x)) \leq \bigwedge_{G \in \mathcal{P}_{fin}(A)} \underline{\mathcal{R}_1}(G^c)(x) = \underline{\mathcal{R}_1}(A^c)(x).$$

(5) $\Longrightarrow$ (1) For all $F \in \mathcal{P}_{fin}(V)$ and $x \in U$,

$$\mathcal{R}_1(x)(F) = [\underline{\mathcal{R}_1}(F^c)(x)]' \leq [\underline{\mathcal{R}_2}(g(F)^c)(f(x))]' = \mathcal{R}_2(f(x))(g(F)).$$

Therefore $(f, g)$ is a pairwise $M$-fuzzifying algebraic relation preserving mappings. □

**Proposition 4.8.** *Let $(U, C_U)$ and $(V, C_V)$ be M-fuzzifying convex spaces. If $f : U \longrightarrow V$ is an M-fuzzifying convexity preserving mapping with respect to $(U, C_U)$ and $(V, C_V)$, then $f : (U, \mathcal{R}_{C_U}) \longrightarrow (V, \mathcal{R}_{C_V})$ is an M-fuzzifying algebraic relation preserving mapping.*

*Proof.* For all $F \in \mathcal{P}_{fin}(U)$ and $x \in U$, Proposition 4.2 implies

$$\mathcal{R}_U(x)(F) = \overline{\mathcal{R}_{C_U}}(F)(x) = co_{C_U}(F)(x) \le co_{C_V}(f(F))(f(x)) = \overline{\mathcal{R}_{C_V}}(f(F))(f(x)) = \mathcal{R}_V(f(x))(f(F)).$$

Thus $f$ is an $M$-fuzzifying algebraic relation preserving mapping. □

**Proposition 4.9.** *Let $(U, \mathcal{R}_U)$ and $(V, \mathcal{R}_V)$ be M-fuzzifying algebraic approximation spaces. If $f : U \longrightarrow V$ is an M-fuzzifying algebraic relation preserving mapping with respect to $(U, \mathcal{R}_U)$ and $(V, \mathcal{R}_V)$, then $f : (U, C_{\mathcal{R}_U}) \longrightarrow (V, C_{\mathcal{R}_V})$ is an M-fuzzifying convexity preserving mapping.*

*Proof.* For all $A \in \mathcal{P}(U)$ and $x \in U$, Proposition 4.7(2) implies that

$$co_{C_{\mathcal{R}_U}}(A)(x) = \overline{\mathcal{R}_U}(A)(x) \le \overline{\mathcal{R}_V}(f(A))(f(x)) = co_{C_{\mathcal{R}_V}}(f(A))(f(x)).$$

Thus $f$ is an $M$-fuzzifying convexity preserving mapping. □

The category consisting of reflexive and transitive $M$-fuzzifying algebraic approximation spaces $(U, \mathcal{R})$ as objects and $M$-fuzzifying algebraic relation preserving mappings as morphisms is denoted by $M$-**RTAAS**.
Based on Propositions 4.2 and 4.8, we get a functor $\mathbb{F}$ :$M$-**CS**$\longrightarrow$ $M$-**RTAAS** by

$$\mathbb{F}((U, C)) = (U, \mathcal{R}_C) \quad \text{and} \quad \mathbb{F}(f) = f.$$

From Propositions 4.2–4.9, we find that $\mathbb{F}$ is isomorphic. Thus we have the following result.

**Theorem 4.10.** *The category M-*__RTAAS__ *is isomorphic to the category M-*__CS__.

The category of reflexive, weakly symmetric and transitive algebraic approximation spaces with the form $(U, \mathcal{R})$ as objects and algebraic relation preserving mappings as morphisms is denoted by **RWSTAAS**. The category of convex matroids as objects and convexity preserving mappings as morphisms is denoted by $M$-**CMS**. Based on Propositions 4.2 and 4.3 and Theorem 4.10, we have the following result.

**Theorem 4.11.** $M$-**RWSTAAS** *is isomorphic to* $M$-**CMS**.

**Proposition 4.12.** *If $(U, C)$ is a 1-arity M-fuzzifying convex matroid then $\mathcal{R}_C$ is symmetric.*

*Proof.* By Proposition 4.2, $\mathcal{R}_C$ is weakly symmetric. In addition, for any $A \in \mathcal{P}(U)$, $\overline{\mathcal{R}_C}(A) = co_C(A) = \bigvee_{v \in A} co_C(\{v\}) = \bigvee_{v \in A} \overline{\mathcal{R}_C}(\{v\})$. It follows from Proposition 3.14(2) that $\mathcal{R}_C$ is symmetric. □

**Proposition 4.13.** *If $(U, \mathcal{R})$ is a symmetric M-fuzzifying algebraic approximation space then $(U, C_{\mathcal{R}})$ is a 1-arity M-fuzzifying convex matriod.*

*Proof.* For any $A \in \mathcal{P}(U)$, it follows from Propositions 3.14(3) and 4.2 that $co_{C_{\mathcal{R}}}(A) = \overline{\mathcal{R}}(A) = \bigvee_{v \in A} \overline{\mathcal{R}}(\{v\}) = \bigvee_{v \in A} co_{C_{\mathcal{R}}}(\{v\})$. So $C_{\mathcal{R}}$ is 1-arity. It follows from Propositions 3.14(1) and 4.3 that $(U, C_{\mathcal{R}})$ is an $M$-fuzzifying convex matroid. □

The category of reflexive, symmetric and transitive algebraic approximation spaces with the form $(U, \mathcal{R})$ as objects and algebraic relation preserving mappings as morphisms is denoted by $M$-**RSTAAS**. The category 1-arity convex matroids as objects and convexity preserving mappings as morphisms is denoted by $M$-$\mathbf{C}_1$**MS**.
Based on Propositions 4.12 and 4.13 and Theorem 4.10, we have the following result.

**Theorem 4.14.** $M$-**RSTAAS** *is isomorphic to* $M$-$\mathbf{C}_1$**MS**.

Next, we present some examples to show relationships among $M$-fuzzifying convex structure, $M$-fuzzifying convex matroid and $M$-fuzzifying algebraic relation.

**Example 4.15.** Let $U = \{x, y, z\}$ and let $M = [0, 1]$ with an involution $'$ defined by $a' = 1 - a$ for any $a \in M$.

(1) Let $C : \mathcal{P}(U) \longrightarrow M$ be shown in Table 1. Then $C$ is an $M$-fuzzifying convex structure. In addition, $\mathcal{R}_C : U \longrightarrow M^{\mathcal{P}_{fin}}$ is a reflexive and transitive $M$-fuzzifying algebraic relation shown in Table 2.

| $A$ | $\emptyset$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x,y\}$ | $\{x,z\}$ | $\{y,z\}$ | U |
|---|---|---|---|---|---|---|---|---|
| $C$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{3}$ | $\frac{1}{2}$ | 0 | 0 | 1 |

Table 1: The definition of $C$

| $\mathcal{R}_C$ | $\emptyset$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x,y\}$ | $\{x,z\}$ | $\{y,z\}$ | U |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | $\frac{1}{3}$ | $\frac{1}{2}$ | 1 | 1 | 1 | 1 |
| $y$ | 0 | $\frac{1}{3}$ | 1 | $\frac{1}{2}$ | 1 | 1 | 1 | 1 |
| $z$ | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | 1 | $\frac{1}{3}$ | 1 | 1 | 1 |

Table 2: The calculation of $\mathcal{R}_C$

Since $co_C(\{x\})(z) = \frac{1}{3}$ and $co_C(\{z\})(x) = \frac{1}{2}$, $(U, C)$ is not an $M$-fuzzifying convex matroid. As a result, $\mathcal{R}_C$ is not weakly symmetric. since $\mathcal{R}_C(\{x\})(z) = \frac{1}{3}$ and $\mathcal{R}_C(\{z\})(x) = \frac{1}{2}$, it is clear that $\mathcal{R}_C$ is not primitive symmetric.

(2) Let $C : \mathcal{P}(U) \longrightarrow M$ be shown in Table 3. Then $(U, C)$ is an $M$-fuzzifying convex matroid. $\mathcal{R}_C : U \longrightarrow M^{\mathcal{P}_{fin}}$ is a reflexive, weakly symmetric and transitive $M$-fuzzifying algebraic relation shown in Table 4.

| $A$ | $\emptyset$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x,y\}$ | $\{x,z\}$ | $\{y,z\}$ | U |
|---|---|---|---|---|---|---|---|---|
| $C$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 1 |

Table 3: The definition of $C$

| $\mathcal{R}_C$ | $\emptyset$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x,y\}$ | $\{x,z\}$ | $\{y,z\}$ | U |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{2}{3}$ | 1 |
| $y$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | $\frac{2}{3}$ | 1 | 1 |
| $z$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{2}{3}$ | 1 | 1 | 1 |

Table 4: The calculation of $\mathcal{R}_C$

Since $\mathcal{R}_C(\{y, z\})(x) \neq \mathcal{R}_C(\{y\})(x) \vee \mathcal{R}_C(\{z\})(x)$, $\mathcal{R}_C$ is not symmetric.

(3) Let $C : \mathcal{P}(U) \longrightarrow M$ be shown in Table 5. Then $(U, C)$ is a 1-arity $M$-fuzzifying convex matroid. $\mathcal{R}_C : U \longrightarrow M^{\mathcal{P}_{fin}}$ is a reflexive, symmetric and transitive $M$-fuzzifying algebraic relation shown in Table 6.

| $A$ | $\emptyset$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x,y\}$ | $\{x,z\}$ | $\{y,z\}$ | U |
|---|---|---|---|---|---|---|---|---|
| $C$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

Table 5: The definition of $C$

| $\mathcal{R}_C$ | $\emptyset$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x,y\}$ | $\{x,z\}$ | $\{y,z\}$ | U |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | 1 |
| $y$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | 1 |
| $z$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | 1 | 1 |

Table 6: The calculation of $\mathcal{R}_C$

## 5. Relationships between $M$-fuzzifying approximation spaces and $M$-fuzzifying algebraic approximation spaces

In this section, we discuss relationships between $M$-fuzzifying approximation spaces and $M$-fuzzifying algebraic approximation spaces.

**Proposition 5.1.** *Let $(U, V, \hat{\mathcal{R}})$ be an M-fuzzifying approximation space. Define a mapping $\mathcal{R}_{\hat{\mathcal{R}}} : U \longrightarrow M^{\mathcal{P}_{fin}(V)}$ by*

$$\forall x \in U, \forall F \in \mathcal{P}_{fin}(U), \quad \mathcal{R}_{\hat{\mathcal{R}}}(x)(F) = \bigvee_{y \in F} \hat{\mathcal{R}}(x, y).$$

*Then $\mathcal{R}_{\hat{\mathcal{R}}}$ is an M-fuzzifying algebraic relation satisfying $\underline{\mathcal{R}_{\hat{\mathcal{R}}}} = \underline{\hat{\mathcal{R}}}$ and $\overline{\mathcal{R}_{\hat{\mathcal{R}}}} = \overline{\hat{\mathcal{R}}}$.*

*Proof.* It is clear that $\mathcal{R}_{\hat{\mathcal{R}}}$ is an $M$-fuzzifying algebraic relation from $U$ to $V$. Let $A \in \mathcal{P}(V)$ and $x \in U$. Then

$$\overline{\mathcal{R}_{\hat{\mathcal{R}}}}(A)(x) = \bigvee_{F \in \mathcal{P}_{fin}(A)} \mathcal{R}_{\hat{\mathcal{R}}}(x)(F) = \bigvee_{F \in \mathcal{P}_{fin}(A)} \bigvee_{y \in F} \hat{\mathcal{R}}(x, y) = \bigvee_{y \in A} \hat{\mathcal{R}}(x, y) = \overline{\hat{\mathcal{R}}}(A)(x)$$

and $\underline{\mathcal{R}_{\hat{\mathcal{R}}}}(A) = [\overline{\mathcal{R}_{\hat{\mathcal{R}}}}(A^c)]^c = [\overline{\hat{\mathcal{R}}}(A^c)]^c = \underline{\hat{\mathcal{R}}}(A)$. Therefore $\underline{\mathcal{R}_{\hat{\mathcal{R}}}} = \underline{\hat{\mathcal{R}}}$ and $\overline{\mathcal{R}_{\hat{\mathcal{R}}}} = \overline{\hat{\mathcal{R}}}$. $\square$

**Proposition 5.2.** *Let $(U, V, \hat{\mathcal{R}})$ be an M-fuzzifying approximation space. Then the following statements are valid.*

*(1) $\mathcal{R}_{\hat{\mathcal{R}}}$ is serial $\Longleftrightarrow \hat{\mathcal{R}}$ is serial.*

*(2) $\mathcal{R}_{\hat{\mathcal{R}}}$ is reflexive $\Longleftrightarrow \hat{\mathcal{R}}$ is reflexive.*

*(3) $\hat{\mathcal{R}}$ is symmetric $\Longleftrightarrow \mathcal{R}_{\hat{\mathcal{R}}}$ is symmetric $\Longleftrightarrow \mathcal{R}_{\hat{\mathcal{R}}}$ is weakly symmetric.*

*(4) $\hat{\mathcal{R}}$ is transitive $\Longleftrightarrow \mathcal{R}_{\hat{\mathcal{R}}}$ is transitive $\Longleftrightarrow \mathcal{R}_{\hat{\mathcal{R}}}$ is strongly transitive.*

*Proof.* It follows from Propositions 2.4 and 5.1 that $\hat{\mathcal{R}}$ is serial (resp. reflexive, symmetric, transitive) iff $\mathcal{R}_{\hat{\mathcal{R}}}$ is serial (resp. reflexive, weakly symmetric, strongly transitive).

We prove that $\hat{\mathcal{R}}$ is symmetric iff $\mathcal{R}_{\hat{\mathcal{R}}}$ is symmetric.

If $\mathcal{R}_{\hat{\mathcal{R}}}$ is symmetric then it is weakly symmetric. Thus $\hat{\mathcal{R}}$ is symmetric. Conversely, assume that $\hat{\mathcal{R}}$ is symmetric. Let $x \in U$ and $F \in \mathcal{P}_{fin}(U)$. Then

$$\overline{\mathcal{R}_{\hat{\mathcal{R}}}}(F)(x) = \mathcal{R}_{\hat{\mathcal{R}}}(x)(F) = \bigvee_{y \in F} \hat{\mathcal{R}}(x, y) = \bigvee_{y \in F} \overline{\hat{\mathcal{R}}}(\{y\})(x) = \bigvee_{y \in F} \overline{\hat{\mathcal{R}}}(\{x\})(y) = \bigvee_{y \in F} \overline{\mathcal{R}_{\hat{\mathcal{R}}}}(\{x\})(y).$$

This shows that (MARU5) holds for $\mathcal{R}_{\hat{\mathcal{R}}}$. Therefore $\mathcal{R}_{\hat{\mathcal{R}}}$ is symmetric.

We verify that $\hat{\mathcal{R}}$ is transitive iff $\mathcal{R}_{\hat{\mathcal{R}}}$ is transitive.

If $\hat{\mathcal{R}}$ is transitive then $\mathcal{R}_{\hat{\mathcal{R}}}$ is strongly transitive. Thus $\mathcal{R}_{\hat{\mathcal{R}}}$ is transitive. Conversely, we assume that $\mathcal{R}_{\hat{\mathcal{R}}}$ is transitive. Let $x, y, z \in U$. Then the transitivity of $\mathcal{R}_{\hat{\mathcal{R}}}$ yields that

$$\hat{\mathcal{R}}(x, y) \wedge \hat{\mathcal{R}}(y, z) = \mathcal{R}_{\hat{\mathcal{R}}}(x)(\{y\}) \wedge \mathcal{R}_{\hat{\mathcal{R}}}(y)(\{z\}) \le \mathcal{R}_{\hat{\mathcal{R}}}(x)(\{z\}) = \hat{\mathcal{R}}(x, z).$$

Thus $\hat{\mathcal{R}}$ is transitive. □

**Proposition 5.3.** *Let $(U, V, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. Define a mapping $\hat{\mathcal{R}}_{\mathcal{R}} : U \times V \longrightarrow M$ by*

$$\forall (x, y) \in U \times V, \quad \hat{\mathcal{R}}_{\mathcal{R}}(x, y) = \mathcal{R}(x)(\{y\}).$$

*Then $\hat{\mathcal{R}}_{\mathcal{R}}$ is an M-fuzzifying relation from U to V satisfying $\underline{\mathcal{R}} \le \underline{\hat{\mathcal{R}}_{\mathcal{R}}}$ and $\overline{\hat{\mathcal{R}}_{\mathcal{R}}} \le \overline{\mathcal{R}}$.*

*Proof.* It is clear that $\hat{\mathcal{R}}_{\mathcal{R}}$ is a relation. Let $A \in \mathcal{P}(U)$ and $x \in U$. Then

$$\overline{\hat{\mathcal{R}}_{\mathcal{R}}}(A)(x) = \bigvee_{y \in A} \hat{\mathcal{R}}_{\mathcal{R}}(x, y) = \bigvee_{y \in A} \mathcal{R}(x)(\{y\}) \le \bigvee_{F \in \mathcal{P}_{fin}(A)} \mathcal{R}(x)(F) = \overline{\mathcal{R}}(A)(x)$$

and $\underline{\mathcal{R}}(A) = [\overline{\mathcal{R}}(A^c)]^c \le [\overline{\hat{\mathcal{R}}_{\mathcal{R}}}(A^c)]^c = \underline{\hat{\mathcal{R}}_{\mathcal{R}}}(A)$. Therefore $\underline{\mathcal{R}} \le \underline{\hat{\mathcal{R}}_{\mathcal{R}}}$ and $\overline{\hat{\mathcal{R}}_{\mathcal{R}}} \le \overline{\mathcal{R}}$. □

**Proposition 5.4.** *Let $(U, V, \mathcal{R})$ be an M-fuzzifying algebraic approximation space. The following statements are valid.*

*(1) $\hat{\mathcal{R}}_{\mathcal{R}}$ is serial $\Longrightarrow \mathcal{R}$ is serial.*

*(2) $\hat{\mathcal{R}}_{\mathcal{R}}$ is reflexive $\Longleftrightarrow \mathcal{R}$ is reflexive.*

*(3) $\hat{\mathcal{R}}_{\mathcal{R}}$ is symmetric $\Longleftrightarrow \mathcal{R}$ is weakly symmetric.*

*(4) $\mathcal{R}$ is transitive $\Longrightarrow \hat{\mathcal{R}}_{\mathcal{R}}$ is transitive.*

*Proof.* (1) It directly follows from (ARU3), Propositions 2.4 and 5.3.

(2) If $\mathcal{R}$ is reflexive then $\hat{\mathcal{R}}_{\mathcal{R}}(x, x) = \mathcal{R}(x)(\{x\}) = \top$ for any $x \in U$. Thus $\hat{\mathcal{R}}_{\mathcal{R}}$ is reflexive. Conversely, we assume that $\hat{\mathcal{R}}_{\mathcal{R}}$ is reflexive. For any $x \in F \in \mathcal{P}_{fin}(U)$, it is clear that $\mathcal{R}(x)(F) \ge \mathcal{R}(x)(\{x\}) = \hat{\mathcal{R}}_{\mathcal{R}}(x, x) = \top$. Thus $\mathcal{R}$ is reflexive.

(3) The result is clear since $\hat{\mathcal{R}}_{\mathcal{R}}(x, y) = \mathcal{R}(x)(\{y\})$ for any $x, y \in U$.

(4) Let $\mathcal{R}$ be transitive and let $x, y, z \in U$. Then

$$\hat{\mathcal{R}}_{\mathcal{R}}(x, y) \wedge \hat{\mathcal{R}}_{\mathcal{R}}(y, z) = \mathcal{R}(x)(\{y\}) \wedge \mathcal{R}(y)(\{z\}) \le \mathcal{R}(x)(\{z\}) = \hat{\mathcal{R}}_{\mathcal{R}}(x, z).$$

Therefore $\hat{\mathcal{R}}_{\mathcal{R}}$ is transitive. □

**Proposition 5.5.** *If $(U, \hat{\mathcal{R}})$ is an M-fuzzifying approximation space then $\hat{\mathcal{R}}_{\mathcal{R}_{\hat{\mathcal{R}}}} = \hat{\mathcal{R}}$.*

*Proof.* Let $x, y \in U$. Then $\hat{\mathcal{R}}_{\mathcal{R}_{\hat{\mathcal{R}}}}(x, y) = \mathcal{R}_{\hat{\mathcal{R}}}(x)(\{y\}) = \hat{\mathcal{R}}(x, y)$. Thus $\hat{\mathcal{R}}_{\mathcal{R}_{\hat{\mathcal{R}}}} = \hat{\mathcal{R}}$. □

**Proposition 5.6.** *If $(U, \mathcal{R})$ is an M-fuzzifying algebraic approximation space then $\mathcal{R}_{\hat{\mathcal{R}}_{\mathcal{R}}} \leq \mathcal{R}$.*

*Proof.* Let $F \in \mathcal{P}_{fin}(U)$ and $x \in U$. Then

$$\mathcal{R}_{\hat{\mathcal{R}}_{\mathcal{R}}}(x)(F) = \bigvee_{y \in F} \hat{\mathcal{R}}_{\mathcal{R}}(x, y) = \bigvee_{y \in F} \mathcal{R}(x)(\{y\}) \leq \mathcal{R}(x)(F).$$

Therefore $\mathcal{R}_{\hat{\mathcal{R}}_{\mathcal{R}}} \leq \mathcal{R}$. □

In order to discuss categorical relationships between convex structures and reflexive and transitive algebraic relations, we present the following notions.

**Definition 5.7.** Let $(U_1, V_1, \hat{\mathcal{R}}_1)$ and $(U_2, V_2, \hat{\mathcal{R}}_2)$ be $M$-fuzzifying approximation spaces. Let $f : U_1 \longrightarrow U_2$ and $g : V_1 \longrightarrow V_2$. The pairwise mapping $(f, g) : U_1 \times V_1 \longrightarrow U_2 \times V_2$ is called a pairwise relation preserving mapping, if $\hat{\mathcal{R}}_1(x, y) \leq \hat{\mathcal{R}}_2(f(x), g(y))$ for all $x \in U_1$ and $y \in V_1$.

**Proposition 5.8.** *Let $(U_1, V_1, \hat{\mathcal{R}}_1)$ and $(U_2, V_2, \hat{\mathcal{R}}_2)$ be M-fuzzifying approximation spaces. For a pairwise mapping $(f, g) : U_1 \times V_1 \longrightarrow U_2 \times V_2$, the following conditions are equivalent.*

*(1) $(f, g)$ is a pairwise relation preserving mappings;*

*(2) $\overline{\hat{\mathcal{R}}_1}(A)(u) \leq \overline{\hat{\mathcal{R}}_2}(g(A))(f(u))$ for all $A \in \mathcal{P}(V_1)$ and $u \in U_1$;*

*(3) $\overline{\hat{\mathcal{R}}_1}(\{v\})(u) \leq \overline{\hat{\mathcal{R}}_2}(g(\{v\}))(f(u))$ for any $v \in V_1$ and $u \in U_1$;*

*(4) $\underline{\hat{\mathcal{R}}_2}(g(\{v\})^c)(f(u)) \leq \underline{\hat{\mathcal{R}}_1}(\{v\}^c)(u)$ for any $v \in V_1$ and $u \in U_1$;*

*(5) $\underline{\hat{\mathcal{R}}_2}(g(A)^c)(f(u)) \leq \underline{\hat{\mathcal{R}}_1}(A^c)(u)$ for any $A \in \mathcal{P}(V_1)$ and $u \in U_1$.*

*Proof.* It is similar to Proposition 4.7. □

**Proposition 5.9.** *Let $(U_1, V_1, \hat{\mathcal{R}}_1)$ and $(U_2, V_2, \hat{\mathcal{R}}_2)$ be M-fuzzifying approximation spaces. If $(f, g)$ is an M-fuzzifying approximation preserving mapping from $(U_1, V_1, \hat{\mathcal{R}}_1)$ to $(U_2, V_2, \hat{\mathcal{R}}_2)$, then $(f, g)$ is an M-fuzzifying algebraic relation preserving mapping with respect to $(U_1, V_1, \mathcal{R}_{\hat{\mathcal{R}}_1})$ and $(U_2, V_2, \mathcal{R}_{\hat{\mathcal{R}}_2})$.*

*Proof.* Let $x \in U_1$ and $F \in \mathcal{P}_{fin}(V_1)$. Then

$$\mathcal{R}_{\hat{\mathcal{R}}_1}(x)(F) = \bigvee_{y \in F} \hat{\mathcal{R}}_1(x, y) \leq \bigvee_{y \in F} \hat{\mathcal{R}}_2(f(u), g(v)) = \mathcal{R}_{\hat{\mathcal{R}}_2}(f(u))(g(F)).$$

Thus $(f, g)$ is an $M$-fuzzifying algebraic relation preserving mapping. □

**Proposition 5.10.** *Let $(U_1, V_1, \mathcal{R}_1)$ and $(U_2, V_2, \mathcal{R}_2)$ be M-fuzzifying algebraic approximation spaces. If $(f, g)$ is an M-fuzzifying algebraic approximation preserving mapping from $(U_1, V_1, \mathcal{R}_1)$ to $(U_2, V_2, \mathcal{R}_2)$, then $(f, g)$ is an M-fuzzifying relation preserving mapping with respect to $(U_1, V_1, \hat{\mathcal{R}}_{\mathcal{R}_1})$ and $(U_2, V_2, \hat{\mathcal{R}}_{\mathcal{R}_2})$.*

*Proof.* Let $x \in U_1$ and $y \in V_1$. Then

$$\hat{\mathcal{R}}_{\mathcal{R}_1}(u, v) = \mathcal{R}_1(u)(\{v\}) \leq \mathcal{R}_2(f(u))(\{g(v)\}) = \hat{\mathcal{R}}_{\mathcal{R}_2}(f(u), g(v)).$$

Thus $(f, g)$ is an $M$-fuzzifying relation preserving mapping. □

The category consisting of $M$-fuzzifying approximation spaces with the form $(U, V, \hat{\mathcal{R}})$ as objects and pairwise $M$-fuzzifying relation preserving mappings as morphisms is denoted by $M$-**PAS**. The category consisting of $M$-fuzzifying algebraic approximation spaces with the form $(U, V, \mathcal{R})$ as objects and pairwise $M$-fuzzifying algebraic relation preserving mappings as morphisms is denoted by $M$-**PAAS**.

Based on Propositions 5.1 and 5.9, we obtain a functor $\mathbb{F}$ :$M$-**PAAS**⟶$M$-**PAS** defined by

$$\mathbb{F}((U, \mathcal{R})) = (U, \hat{\mathcal{R}}_{\mathcal{R}}), \quad \mathbb{F}(f) = f.$$

Similarly, based on Propositions 5.3 and 5.12, we obtain a functor $\mathbb{G}$ :$M$-**PAS**⟶$M$-**PAAS** defined by

$$\mathbb{G}((U, \hat{\mathcal{R}})) = (U, \mathcal{R}_{\hat{\mathcal{R}}}), \quad \mathbb{G}(f) = f.$$

From Propositions 5.1, 5.3, 5.5, 5.6, 5.9 and 5.12, we have the following result.

**Theorem 5.11.** $(\mathbb{F}, \mathbb{G})$ *is a Galois's connection, where* $\mathbb{F}$ *is a left inverse of* $\mathbb{G}$.

**Proposition 5.12.** *If* $(U, \mathcal{R})$ *is a symmetric M-fuzzifying algebraic approximation space then* $\mathcal{R}_{\hat{\mathcal{R}}_{\mathcal{R}}} = \mathcal{R}$. *Thus* $\overline{\hat{\mathcal{R}}_{\mathcal{R}}} = \overline{\mathcal{R}}$ *and* $\underline{\hat{\mathcal{R}}_{\mathcal{R}}} = \underline{\mathcal{R}}$.

*Proof.* Let $F \in \mathcal{P}_{fin}(U)$ and $x \in U$. Since $\mathcal{R}$ is symmetric, Proposition 3.14(3) yields that

$$\mathcal{R}_{\hat{\mathcal{R}}_{\mathcal{R}}}(x)(F) = \bigvee_{y \in F} \hat{\mathcal{R}}_{\mathcal{R}}(x)(y) = \bigvee_{y \in F} \mathcal{R}(x)(\{v\}) = \mathcal{R}(x)(F).$$

So $\mathcal{R}_{\hat{\mathcal{R}}_{\mathcal{R}}} = \mathcal{R}$. In addition,

$$\overline{\mathcal{R}}(F)(x) = \mathcal{R}(x)(F) = \bigvee_{y \in F} \mathcal{R}(x)(\{y\}) = \bigvee_{y \in F} \hat{\mathcal{R}}_{\mathcal{R}}(x, y) = \overline{\hat{\mathcal{R}}_{\mathcal{R}}}(F)(x).$$

Thus $\overline{\hat{\mathcal{R}}_{\mathcal{R}}}(F) = \overline{\mathcal{R}}(F)$. Therefore $\overline{\hat{\mathcal{R}}_{\mathcal{R}}} = \overline{\mathcal{R}}$ and $\underline{\hat{\mathcal{R}}_{\mathcal{R}}} = \underline{\mathcal{R}}$. □

Based on Proposition 5.12, results in Proposition 5.2 and Theorem 5.11 can be enhanced as follows.

**Corollary 5.13.** *Let* $(U, \mathcal{R})$ *be a symmetric M-fuzzifying algebraic approximation space. Then* $\hat{\mathcal{R}}_{\mathcal{R}}$ *is symmetric. In addition,* $\hat{\mathcal{R}}_{\mathcal{R}}$ *is serial (resp., reflexive, transitive) iff* $\mathcal{R}$ *is serial (resp., reflexive, transitive).*

The category consisting of reflexive, symmetric and transitive $M$-fuzzifying approximation spaces as objects and $M$-fuzzifying relation preserving mappings as morphisms is denoted by $M$-**RSTAS**. The category consisting of reflexive, symmetric and transitive $M$-fuzzifying algebraic approximation spaces as objects and p$M$-fuzzifying algebraic relation preserving mappings as morphisms is denoted by $M$-**RSTAAS**.

**Corollary 5.14.** $M$-**RSTAS** *is isomorphic to* $M$-**RSTAAS**.

Next, we present a specific example to show relationship between reflexive, symmetric and transitive $M$-fuzzifying relations and reflexive, symmetric and transitive $M$-fuzzifying algebraic relations.

**Example 5.15.** Let $U = \{x, y, z\}$ and let $M = [0, 1]$ with an involution $'$ defined by $a' = 1 - a$ for any $a \in M$. Let $\mathcal{R} : U \longrightarrow M^{\mathcal{P}_{fin}}$ be defined in Table 7. It is a reflexive, symmetric and transitive $M$-fuzzifying algebraic relation. Thus $\hat{\mathcal{R}}_{\mathcal{R}}$ is a reflexive, symmetric and transitive relation.

| $\mathcal{R}$ | $\emptyset$ | $\{x\}$ | $\{y\}$ | $\{z\}$ | $\{x, y\}$ | $\{x, z\}$ | $\{y, z\}$ | U |
|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | 1 |
| $y$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | 1 |
| $z$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | 1 | 1 |

Table 7: The definition of $\mathcal{R}$

| $\hat{\mathcal{R}}_{\mathcal{R}}$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $y$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ |
| $z$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

Table 8: The calculation of $\hat{\mathcal{R}}_{\mathcal{R}}$

## 6. Conclusions

We developed a framework of $M$-fuzzifying algebraic rough sets in a constructive approach. For this purpose, the notion of $M$-fuzzifying algebraic relation is introduced and lower and upper $M$-fuzzifying approximation operators are constructed. In the viewpoint of category aspect, relationships among $M$-fuzzifying algebraic approximation space, $M$-fuzzifying approximation space and $M$-fuzzifying convex structures are discussed. It is proved that the category of reflexive and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of $M$-fuzzifying convex spaces. In particular, it is proved that the category of reflexive, weakly symmetric and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of $M$-fuzzifying convex matroids. It is also proved that the category of reflexive, symmetric and transitive $M$-fuzzifying algebraic rough spaces is isomorphic to the category of reflexive, symmetric and transitive $M$-fuzzifying rough spaces. Unlike classic relation or $M$-fuzzifying relation, $M$-fuzzifying algebraic relation focuses on establishing relationships between a point and a finite subset. In addition, conditions of$M$-fuzzifying algebraic relations such as (resp., weak, primitive) symmetry, transitivity and (resp., strong) transitivity can be characterized by $M$-fuzzifying algebraic approximation operators either via finite subsets or arbitrary subsets. In [26], the condition that turns an $M$-fuzzifying convex space to an $M$-fuzzifying convex matroid is the exchange law. We find in Section 4 that the exchange law is actual a kind of new symmetry of $M$-fuzzifying algebraic rough sets (i.e., weak symmetry) which is weaker than the symmetry of $M$-fuzzifying algebraic rough sets and is stronger than the primitive symmetry. In turn, the symmetry and the primitive symmetry of $M$-fuzzifying algebraic rough sets can be abstracted into $M$-fuzzifying convex structures to construct strong and weak forms of $M$-fuzzifying convex matroids. The strong form of $M$-fuzzifying convex matroids may enhance some results of $M$-fuzzifying convex invariants while the weak form of $M$-fuzzifying convex matroids is more likely to reveal essential properties of $M$-fuzzifying convex invariants.

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