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Generalized metric properties at a subset on the Vietoris hyperspace $\mathcal{F}(X)$

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Abstract. In this paper, we study generalized metric properties at a subset on the hyperspace $\mathcal{F}(X)$ of finite subsets of a space X endowed with the Vietoris topology. We prove that X has the covering property γ at A if and only if $\mathcal{F}(X)$ has the covering property γ at $\langle A\rangle_{\mathcal{F}(X)}$ for each $A \subset X$ and some γ. By these results, we obtain some results related to the images of metric spaces under some kinds of continuous mappings at a subset on the Vietoris hyperspace $\mathcal{F}(X)$.

1. Introduction and preliminaries

Recently, the generalized metric properties on hyperspaces with the Vietoris topology have been studied by many authors ([3, 4, 9–11, 13–18]).

In 2020 and 2022, covering concepts such as external bases, *so*-networks, *sn*-networks, *cs*-networks, cs^*-networks, point-regular covers, point-finite covers, point-countable covers at a subset A for a space X were introduced and studied by S. Lin, X.W. Ling, Y. Ge and W. He ([5, 6]). They obtained some good results. In this paper, we also introduce some more covering concepts like *cn*-networks, *ck*-networks, compact-finite covers, compact-countable covers, locally finite covers, locally countable covers at a subset A for a space X and study them on the Vietoris hyperspace $\mathcal{F}(X)$ at a subset $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$. Throughout this paper, (P) is assumed to be one of the following properties: point-finite, point-countable, compact-finite, compact-countable, locally finite, locally countable. Moreover, all spaces are assumed to be T_1 and regular, $\mathbb{N}$ denotes the set of all positive integers. For $A \subset X$, we prove that

1. X has a sequence of open covers (resp., *so*-covers, *cs*-covers, cs^*-covers) at A which is a point-star network at A for X if and only if $\mathcal{F}(X)$ has a sequence of open covers (resp., *so*-covers, *cs*-covers, cs^*-covers) at $\langle A\rangle_{\mathcal{F}(X)}$ which is a point-star network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$;
2. X has a sequence of open covers (resp., *so*-covers, *cs*-covers, cs^*-covers) with property (P) at A which is a point-star network at A for X if and only if $\mathcal{F}(X)$ has a sequence of open covers (resp., *so*-covers, *cs*-covers, cs^*-covers) with property (P) at $\langle A\rangle_{\mathcal{F}(X)}$ which is a point-star network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$;
3. X has an external base (resp., an *so*-network, an *sn*-network, a *cs*-network, a cs^*-network, a *cn*-network, a *ck*-network) with property σ-(P) at A for X if and only if $\mathcal{F}(X)$ has an external base (resp., an *so*-network, an *sn*-network, a *cs*-network, a cs^*-network, a *cn*-network, a *ck*-network) with property σ-(P) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

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By these results, we obtain that

1. X has a point-regular external base (resp., *so*-network, *sn*-network, *cs*-network, cs^*-network) at A for X if and only if $\mathcal{F}(X)$ has a point-regular external base (resp., *so*-network, *sn*-network, *cs*-network, cs^*-network) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$;
2. X has a point-countable external base (resp., *so*-network, *sn*-network, *cs*-network, cs^*-network, *cn*-network, *ck*-network) at A for X if and only if $\mathcal{F}(X)$ has a point-countable external base (resp., *so*-network, *sn*-network, *cs*-network, cs^*-network, *cn*-network, *ck*-network) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

On the other hand, we also get some results about the images of metric spaces under some kinds of continuous mappings at a subset on the Vietoris hyperspace $\mathcal{F}(X)$. Moreover, if $A = X$, then $\langle A\rangle_{\mathcal{F}(X)} = \mathcal{F}(X)$, we get some new results and get back some known results (for example, [4, Theorem 4.7], [14, Theorems 37, 41], [18, Theorem 2.6], [18, Corollaries 2.7, 2.8]) on the Vietoris hyperspace $\mathcal{F}(X)$.

For a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, we say that $\{x_n\}_{n\in\mathbb{N}}$ is *eventually* in P if $\{x\} \cup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}_{n\in\mathbb{N}}$ is *frequently* in P if some subsequence of $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P. Furthermore, if $\mathcal{P}$ is a family of subsets of a space X and $A \subset X$, then

$$\mathtt{St}(A,\mathcal{P}) = \bigcup\{P \in \mathcal{P} : P \cap A \neq \emptyset\};$$
$$(\mathcal{P})_A = \{P \in \mathcal{P} : P \cap A \neq \emptyset\}.$$

For $x \in X$, we use the notation $\mathtt{St}(x,\mathcal{P})$ instead of $\mathtt{St}(\{x\},\mathcal{P})$.

Given a space X, we define its *hyperspaces* as the following sets:

1. $CL(X) = \{A \subset X : A \text{ is closed and nonempty}\}$;
2. $\mathbb{K}(X) = \{A \in CL(X) : A \text{ is compact}\}$;
3. $\mathcal{F}_n(X) = \{A \in CL(X) : |A| \leq n\}$, where $n \in \mathbb{N}$;
4. $\mathcal{F}(X) = \{A \in CL(X) : A \text{ is finite}\}$.

The set $CL(X)$ is topologized by the *Vietoris topology* defined as the topology generated by

$$\mathcal{B} = \{\langle U_1, \ldots, U_k\rangle : U_1, \ldots, U_k \text{ are open subsets of } X,\ k \in \mathbb{N}\},$$

where

$$\langle U_1, \ldots, U_k\rangle = \Big\{A \in CL(X) : A \subset \textstyle\bigcup_{i\leq k} U_i,\ A \cap U_i \neq \emptyset \text{ for each } i \leq k\Big\}.$$

Note that, by definition, $\mathbb{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subspaces of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

1. $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X;
2. $\mathbb{K}(X)$ is called the *hyperspace of nonempty compact subsets of* X;
3. $\mathcal{F}_n(X)$ is called the *n-fold symmetric product of* X;
4. $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of* X.

On the other hand, it is obvious that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty}\mathcal{F}_n(X)$ and $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in \mathbb{N}$.

Remark 1.1. ([15]) Let X be a space and let $n \in \mathbb{N}$.

1. $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$.
2. $f_1 : X \twoheadrightarrow \mathcal{F}_1(X)$ given by $f_1(x) = \{x\}$ is a homeomorphism.
3. Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m, n \in \mathbb{N}$, $m < n$.

Notation 1.2. ([14]) *If $U_1, \ldots, U_s$ are open subsets of a space X, then $\langle U_1, \ldots, U_s\rangle_{\mathcal{F}(X)}$ denotes the intersection of the open set $\langle U_1, \ldots, U_s\rangle$ of the Vietoris topology, with $\mathcal{F}(X)$.*

Notation 1.3. ([17]) *Let X be a space. If $\{x_1, \ldots, x_r\}$ is a point of $\mathcal{F}(X)$ and $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_{\mathcal{F}(X)}$, then for each $j \leq r$, we let*

$$U_{x_j} = \bigcap\{U \in \{U_1, \dots, U_s\} : x_j \in U\}.$$

Observe that $\langle U_{x_1}, \dots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \langle U_1, \dots, U_s\rangle_{\mathcal{F}(X)}$.

Definition 1.4. Let $A \subset X$ and $\mathcal{P}$ be a family of subsets of a space X.

1. A is called a *sequential neighborhood* of $x \in X$ [5], if each sequence L converging to x is eventually in A.
2. A is called *sequentially open* [5], if A is a sequential neighborhood of each point in A.
3. $\mathcal{P}$ is called a *network* at $x \in X$ [5], if $x \in \bigcap \mathcal{P}$, and for each neighborhood U of x in X, there is $P \in \mathcal{P}$ such that $P \subset U$.
4. $\mathcal{P}$ is called a *cs-cover* [5] (resp., *cs*-cover* [6]) at A for X, if every sequence L converging to $x \in A$ in X is eventually (resp., frequently) in some $P \in \mathcal{P}$.
5. $\mathcal{P}$ is called an *open cover* (resp., *so-cover*) at A for X [5], if each element of $\mathcal{P}$ is an open (resp., a sequentially open) set in X and $A \subset \bigcup \mathcal{P}$.
6. $\mathcal{P}$ is called a *cs-network* (resp., *cs*-network*) at A for X [5], if for each $x \in A$, any sequence L converging to $x \in U$ with U open in X, then L is eventually (resp., frequently) in $P \subset U$ for some $P \in \mathcal{P}$.
7. $\mathcal{P}$ is called an *sn-network* at a point $x \in X$ [5], if the following are satisfied: (i) $\mathcal{P}$ is a network at x in X; (ii) if $U, V \in \mathcal{P}$, then $W \subset U \cap V$ for some $W \in \mathcal{P}$; (iii) each element of $\mathcal{P}$ is a sequential neighborhood of x in X.
8. $\mathcal{P} = \bigcup_{x\in A} \mathcal{P}_x$ is called an *external base* (resp., *sn-network, so-network*) at A for X [5], if $\mathcal{P}_x$ is a local base (resp., an *sn*-network, an *sn*-network consisting of sequentially open sets) at x in X for each $x \in A$.
9. $\mathcal{P}$ is called a *cn-network* at $x \in X$ [2], if for each neighborhood O_x of x, the set $\bigcup\{P \in \mathcal{P} : x \in P \subset O_x\}$ is a neighborhood of x; $\mathcal{P}$ is a *cn-network* at A for X, if $\mathcal{P}$ is a *cn*-network at each point $x \in A$.
10. $\mathcal{P}$ is called a *ck-network* at $x \in X$ [2], if for any neighborhood O_x of x, there is a neighborhood $U_x \subset O_x$ of x such that for each compact subset $K \subset U_x$, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset O_x$; $\mathcal{P}$ is a *ck-network* at A for X, if $\mathcal{P}$ is a *ck*-network at each point $x \in A$.

Remark 1.5.
1. External base (at A) ⇒ *so*-network (at A) ⇒ *sn*-network (at A) ⇒ *cs*-network (at A) ⇒ *cs**-network (at A).
2. External base (at A) ⇒ *ck*-network (at A) ⇒ *cn*-network (at A).

Definition 1.6. Let $A \subset X$ and $\mathcal{P}$ be a family of subsets of a space X.

1. $\mathcal{P}$ is said to be *point-finite* (resp., *point-countable*) at A [5], if the family $(\mathcal{P})_x$ is finite (resp., countable) for each $x \in A$.
2. $\mathcal{P}$ is said to be *compact-finite* (resp., *compact-countable*) at A, if for each compact subset K in the subspace A of X, the family $(\mathcal{P})_K$ is finite (resp., countable).
3. $\mathcal{P}$ is said to be *locally finite* (resp., *locally countable*) at A, if for each $x \in A$, there exists an open neighborhood V of x such that the family $(\mathcal{P})_V$ is finite (resp., countable).
4. $\mathcal{P}$ is said to be *point-regular* at A [5], if for each $x \in A$ and $x \in U$ with U open in X, $\{P \in (\mathcal{P})_x : P \not\subset U\}$ is finite.

Definition 1.7. For a cover $\mathcal{P}$ of a subset A of a space X. We say that $\mathcal{P}$ has *property* σ-(P) at A, if $\mathcal{P}$ can be expressed as $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$, where each $\mathcal{P}_n$ has property (P) at A, and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$.

Definition 1.8. ([5]) Let X be a space and $A \subset X$. A sequence $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ of families of subsets in X is called a *point-star network* at A for X, if $\{\mathtt{St}(x, \mathcal{P}_n)\}_{n\in\mathbb{N}}$ is a network at x in X for each $x \in A$.

Remark 1.9. ([5]) Point-star networks for a space are also called *σ-strong networks*.

For some undefined or related concepts, we refer the reader to [2, 5–8, 18].

2. Main results

Let X be a space. We say that a sequence $\{A_n\}_{n\in\mathbb{N}}$ consisting of subsets of X converges to a subset $A \subset X$, if for each open set U in X with $A \subset U$, there exists $k \in \mathbb{N}$ such that $A_n \subset U$ for each $n > k$.

Notation 2.1. *Let $P_1, \ldots, P_s$ be subsets of a space X. Then, we denote*

$$\langle P_1, \ldots, P_s\rangle_{\mathcal{F}(X)} = \langle P_1, \ldots, P_s\rangle \cap \mathcal{F}(X) \text{ and } \langle P_1, \ldots, P_s\rangle_{\mathbb{K}(X)} = \langle P_1, \ldots, P_s\rangle \cap \mathbb{K}(X),$$

where

$$\langle P_1, \ldots, P_s\rangle = \Big\{A \in CL(X) : A \subset \bigcup_{i\le s} P_i,\ A \cap P_i \neq \emptyset \text{ for each } i \le s\Big\}.$$

Lemma 2.2. ([18, Lemma 2.1]) *Let X be a space and $\{F_m\}_{m\in\mathbb{N}}$ be a sequence of points of $\mathcal{F}(X)$. If $\{F_m\}_{m\in\mathbb{N}}$ converges to $F = \{x_1, \ldots, x_r\}$ in $\mathcal{F}(X)$ and $\{U_1, \ldots, U_r\}$ is a family of pairwise disjoint open subsets of X such that $x_j \in U_j$ for each $j \le r$, then $\{F_m \cap U_j\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X for each $j \le r$.*

Lemma 2.3. ([16, Lemma 2.1]) *Let $\langle U_1, \ldots, U_s\rangle$, $\langle V_1, \ldots, V_r\rangle \subset CL(X)$. If there exists $i_0 \le s$ such that $U_{i_0} \cap (\bigcup_{j\le r} V_j) = \emptyset$, then $\langle U_1, \ldots, U_s\rangle \cap \langle V_1, \ldots, V_r\rangle = \emptyset$.*

Lemma 2.4. *Let X be a space and $A \subset X$. If $\mathcal{K}$ is compact in the subspace $\langle A\rangle_{\mathbb{K}(X)}$ of $\mathbb{K}(X)$, then $\bigcup \mathcal{K}$ is compact in the subspace A of X.*

Proof. Let $\mathcal{U}$ be an open cover of $\bigcup \mathcal{K}$ in the subspace A of X. Then, for each $U \in \mathcal{U}$, there exists an open subset V_U in X such that $U = V_U \cap A$. Take any $E \in \mathcal{K}$, we have that $E \subset \bigcup \mathcal{K} \subset \bigcup_{U\in\mathcal{U}} V_U$. Since E is a compact subset of X, there exists a finite subcover $\{V_{U_1}, \ldots, V_{U_{n(E)}}\}$ of E such that $E \cap V_{U_i} \neq \emptyset$ for each $i \le n(E)$. Thus, $E \in \langle V_{U_1}, \ldots, V_{U_{n(E)}}\rangle_{\mathbb{K}(X)}$. Now, if we put

$$\mathfrak{U} = \Big\{\langle V_{U_1}, \ldots, V_{U_{n(E)}}\rangle_{\mathbb{K}(X)} : E \in \mathcal{K}\Big\},$$

then $\mathfrak{U}$ is an open cover of $\mathcal{K}$ in $\mathbb{K}(X)$. Hence, $\{\mathcal{H} \cap \langle A\rangle_{\mathbb{K}(X)} : \mathcal{H} \in \mathfrak{U}\}$ is an open cover of $\mathcal{K}$ in the subspace $\langle A\rangle_{\mathbb{K}(X)}$ of $\mathbb{K}(X)$. Since $\mathcal{K}$ is compact in the subspace $\langle A\rangle_{\mathbb{K}(X)}$ of $\mathbb{K}(X)$, there exists a finite subfamily $\mathfrak{U}_0$ of $\mathfrak{U}$ such that $\mathcal{K} \subset \bigcup_{\mathcal{H}\in\mathfrak{U}_0}(\mathcal{H} \cap \langle A\rangle_{\mathbb{K}(X)})$. Put

$$\mathfrak{U}_0 = \Big\{\mathcal{O}_{E_j} = \langle V_{U_{1(E_j)}}, \ldots, V_{U_{n(E_j)}}\rangle_{\mathbb{K}(X)} : j \le m\Big\},$$
$$\mathcal{V} = \{U_{1(E_j)}, \ldots, U_{n(E_j)} : j \le m\}.$$

Then, $\mathcal{V}$ is a finite subfamily of $\mathcal{U}$. Moreover, $\bigcup \mathcal{K} \subset \bigcup \mathcal{V}$. Indeed, let $z \in \bigcup \mathcal{K}$. Then, there exists $E \in \mathcal{K}$ such that $z \in E$. Since $E \in \mathcal{K}$, there exists $j \le m$ such that

$$E \in \mathcal{O}_{E_j} \cap \langle A\rangle_{\mathbb{K}(X)} = \langle V_{U_{1(E_j)}}, \ldots, V_{U_{n(E_j)}}\rangle_{\mathbb{K}(X)} \cap \langle A\rangle_{\mathbb{K}(X)}.$$

This implies that there exists $1 \le i \le n$ such that

$$z \in V_{U_{i(E_j)}} \cap A = U_{i(E_j)} \subset \bigcup \mathcal{V}.$$

Therefore, $\bigcup \mathcal{K}$ is compact in the subspace A of X. □

Lemma 2.5. *Let X be a space, $A \subset X$ and $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ be a sequence of families of subsets in X. For each $n \in \mathbb{N}$, put*

$$\mathfrak{P}_n = \{\langle P_1^{(n)}, \ldots, P_s^{(n)}\rangle_{\mathcal{F}(X)} : P_1^{(n)}, \ldots, P_s^{(n)} \in \mathcal{P}_n, s \in \mathbb{N}\}.$$

1. *If $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ is a point-star network at A for X, then $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ is a point-star network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*
2. *For each $n \in \mathbb{N}$, if $\mathcal{P}_n$ is an open cover (resp., an so-cover, a cs-cover, a cs*-cover) at A for X, then $\mathfrak{P}_n$ is an open cover (resp., an so-cover, a cs-cover, a cs*-cover) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

Proof. We can assume that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$. Let $F = \{x_1, \ldots, x_r\} \in \langle A \rangle_{\mathcal{F}(X)}$ and $\mathcal{U}$ be an open neighborhood of F in $\mathcal{F}(X)$. Then, $F \subset A$ and there exist open subsets $U_1, \ldots, U_s$ of X such that

$$F \in \langle U_1, \ldots, U_s \rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Because X is Hausdorff, it follows from Notation 1.3 that we can find pairwise disjoint open subsets $U_{x_1}, \ldots, U_{x_r}$ of X such that $x_j \in U_{x_j}$ for each $j \leq r$ and

$$F \in \langle U_{x_1}, \ldots, U_{x_r} \rangle_{\mathcal{F}(X)} \subset \langle U_1, \ldots, U_s \rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

(1) For each $j \leq r$, since $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ is a point-star network at A for X, $\{\mathtt{St}(x_j, \mathcal{P}_n)\}_{n \in \mathbb{N}}$ is a network at x_j in X for each $x_j \in A$. Thus, there exists $m_j \in \mathbb{N}$ such that $x_j \in \mathtt{St}(x_j, \mathcal{P}_n) \subset U_{x_j}$ whenever $n \geq m_j$. If we put $m = \max\{m_j : j \leq r\}$, then

$$F \in \langle \mathtt{St}(x_1, \mathcal{P}_n), \ldots, \mathtt{St}(x_r, \mathcal{P}_n) \rangle_{\mathcal{F}(X)} \subset \langle U_{x_1}, \ldots, U_{x_r} \rangle_{\mathcal{F}(X)}$$

for every $n \geq m$. Moreover, it is easy to see that

$$\mathtt{St}(F, \mathfrak{P}_n) \subset \langle \mathtt{St}(x_1, \mathcal{P}_n), \ldots, \mathtt{St}(x_r, \mathcal{P}_n) \rangle_{\mathcal{F}(X)}.$$

Hence, $F \in \mathtt{St}(F, \mathfrak{P}_n) \subset \mathcal{U}$ for every $n \geq m$. Therefore, $\{\mathtt{St}(F, \mathfrak{P}_n)\}_{n \in \mathbb{N}}$ is a network at F in $\mathcal{F}(X)$ for each $F \in \langle A \rangle_{\mathcal{F}(X)}$. This shows that $\{\mathfrak{P}_n\}_{n \in \mathbb{N}}$ is a point-star network at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

(2) *Case 1.* $\mathcal{P}_n$ is an open cover (resp., *so*-cover) at A for X. Let $A \subset \bigcup \mathcal{P}_n$. Then, for each $j \leq r$, since $x_j \in A$, there exists $P_j^{(n)} \in \mathcal{P}_n$ such that $x_j \in P_j^{(n)}$. This implies that $F \in \langle P_1^{(n)}, \ldots, P_r^{(n)} \rangle_{\mathcal{F}(X)}$. Thus, $\langle A \rangle_{\mathcal{F}(X)} \subset \bigcup \mathfrak{P}_n$.

If each element of $\mathcal{P}_n$ is open in X, then it is obvious that each element of $\mathfrak{P}_n$ is open in $\mathcal{F}(X)$. Therefore, $\mathfrak{P}_n$ is an open cover at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

If each element of $\mathcal{P}_n$ is a sequentially open set in X, then each element of $\mathfrak{P}_n$ is a sequentially open set in $\mathcal{F}(X)$. Indeed, take any $\mathcal{W} = \langle P_1^{(n)}, \ldots, P_s^{(n)} \rangle_{\mathcal{F}(X)} \in \mathfrak{P}_n$, we only need to prove that $\mathcal{W}$ is a sequentially open set in $\mathcal{F}(X)$. Assume that $B = \{y_1, \ldots, y_m\} \in \mathcal{W}$ and the sequence $\{B_k\}_{k \in \mathbb{N}}$ converges to B in $\mathcal{F}(X)$.

Claim. *There exists $N \in \mathbb{N}$ such that $B_k \subset \bigcup_{i \leq s} P_i^{(n)}$ for each $k > N$.*

Otherwise, then there exists a subsequence $\{B_{k_l}\}_{l \in \mathbb{N}}$ such that $B_{k_l} \not\subset \bigcup_{i \leq s} P_i^{(n)}$ for each $l \in \mathbb{N}$ and $\{k_l\}_{l \in \mathbb{N}}$ is strictly increasing. For each $l \in \mathbb{N}$, take $z_l \in B_{k_l} \setminus \bigcup_{i \leq s} P_i^{(n)}$. Since the sequence $\{B_k\}_{k \in \mathbb{N}}$ converges to B in $\mathcal{F}(X)$, it is obvious that $\{B\} \cup \{B_k : k \in \mathbb{N}\}$ is a compact subset in $\mathcal{F}(X) = \langle X \rangle_{\mathcal{F}(X)}$. This implies that $\{B\} \cup \{B_k : k \in \mathbb{N}\}$ is a compact subset in $\langle X \rangle_{\mathbb{K}(X)} = \mathbb{K}(X)$. It follows from Lemma 2.4 that $B \cup \bigcup_{k \in \mathbb{N}} B_k$ is a compact subset in X. Moreover, since $B \cup \bigcup_{k \in \mathbb{N}} B_k$ is countable, $B \cup \bigcup_{k \in \mathbb{N}} B_k$ has a countable network. This implies that $B \cup \bigcup_{k \in \mathbb{N}} B_k$ is metrizable since a Hausdorff compact space (i.e., compactum) with a countable network is metrizable [1]. Thus, $B \cup \bigcup_{k \in \mathbb{N}} B_k$ is a compact metrizable subspace of X. Therefore, $\{z_l\}_{l \in \mathbb{N}}$ must have a subsequence $\{z_{l_p}\}_{p \in \mathbb{N}}$ converges to z. Then, $z_{l_p} \in B_{k_{l_p}}$ for each $p \in \mathbb{N}$. Now, we prove that $z \in B$. If not, since X is Hausdorff, there exist an open neighborhood U of z and an open neighborhood V of B in X such that $U \cap V = \emptyset$. Because the sequence $\{B_k\}_{k \in \mathbb{N}}$ converges to B in $\mathcal{F}(X)$, the subsequence $\{B_{k_{l_p}}\}_{p \in \mathbb{N}}$ converges to B in $\mathcal{F}(X)$. Moreover, since U is an open neighborhood of z in X and $\langle V \rangle_{\mathcal{F}(X)}$ is an open neighborhood of B in $\mathcal{F}(X)$, there exist $p_0, p_1 \in \mathbb{N}$ such that $z_{l_p} \in U$ for each $p > p_0$ and $B_{k_{l_p}} \subset V$ for each $p > p_1$. If we put $p_2 = \max\{p_0, p_1\}$, then $z_{l_p} \in U$ and $z_{l_p} \in B_{k_{l_p}} \subset V$ for each $p > p_2$. This implies that $U \cap V \neq \emptyset$, which is a contradiction. Since $z \in B$, there exist $q \leq m$ and $i \leq s$ such that $z = y_q \in P_i^{(n)}$. Since $P_i^{(n)}$ is a sequentially open set in X, there exists $N \in \mathbb{N}$ such that $z_{l_p} \in P_i^{(n)}$ for each $p > N$. This is a contradiction.

By Claim, without loss of generality, we may assume that $B_k \subset \bigcup_{i \leq s} P_i^{(n)}$ for each $k \in \mathbb{N}$. Next, we prove that there exists $p \in \mathbb{N}$ such that $B_k \in \mathcal{W}$ for each $k > p$. Suppose not, there exist a subsequence $\{B_{k_l}\}_{l \in \mathbb{N}}$ and $i \leq s$ such that $B_{k_l} \notin \mathcal{W}$ and $B_{k_l} \cap P_i^{(n)} = \emptyset$ for each $l \in \mathbb{N}$. Moreover, since $B \in \mathcal{W}$, there exists $t \leq m$ such that $y_t \in P_i^{(n)}$. Since the sequence $\{B_k\}_{k \in \mathbb{N}}$ converges to B in $\mathcal{F}(X)$, the subsequence $\{B_{k_l}\}_{l \in \mathbb{N}}$ converges to B in $\mathcal{F}(X)$. Let $\mathcal{O}$ be an open neighborhood of B in $\mathcal{F}(X)$. Similar to the above proof, there exist pairwise disjoint open subsets $U_{y_1}, \ldots, U_{y_m}$ of X such that $y_t \in U_{y_t}$ for each $t \leq m$, and

$$B \in \langle U_{y_1}, \ldots, U_{y_m} \rangle_{\mathcal{F}(X)} \subset \mathcal{O}.$$

By Lemma 2.2, the sequence $\{B_{k_l} \cap U_{y_t}\}_{l\in\mathbb{N}}$ converges to $\{y_t\}$ in X. Moreover, since $P_i^{(n)}$ is a sequentially open set in X and $y_t \in P_i^{(n)}$, there exists $N \in \mathbb{N}$ such that $B_{k_l} \cap U_{y_t} \subset P_i^{(n)}$ for each $l > N$, which is a contradiction. Hence, $\mathcal{W}$ is a sequentially open set in $\mathcal{F}(X)$. This implies that $\mathfrak{P}_n$ is an *so*-cover at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

Case 2. $\mathcal{P}_n$ is a *cs*-cover (resp., *cs**-cover) at A for X. Let $\{F_m\}_{m\in\mathbb{N}}$ be a sequence converging to F in $\mathcal{F}(X)$. Then, for each $j \le r$, the sequence $\{F_m \cap U_{x_j}\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X by Lemma 2.2.

If $\mathcal{P}_n$ is a *cs*-cover at A for X, then there exist $P_j^{(n)} \in \mathcal{P}_n$ and $k_j \in \mathbb{N}$ such that

$$\{x_j\} \cup \Big(\bigcup\{F_m \cap U_j : m \ge k_j\}\Big) \subset P_j^{(n)}.$$

If we put $k = \max\{k_j : j \le r\}$, then $\langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)} \in \mathfrak{P}_n$ and

$$\{F\} \cup \{F_m : m > k\} \subset \langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)}.$$

This shows that $\mathfrak{P}_n$ is a *cs*-cover at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

If $\mathcal{P}_n$ is a *cs**-cover at A for X, by induction on r, then there exist $P_j^{(n)} \in \mathcal{P}_n$ and a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\{x_j\} \cup \Big(\bigcup\{F_{m_k} \cap U_j : k \in \mathbb{N}\}\Big) \subset P_j^{(n)}.$$

This implies that $\langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)} \in \mathfrak{P}_n$ and

$$\{F\} \cup \{F_{m_k} : k \in \mathbb{N}\} \subset \langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)}.$$

Therefore, $\mathfrak{P}_n$ is a *cs**-cover at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$. □

Let $\mathcal{P}$ be a family of subsets of a space X. If we put

$$\mathfrak{P} = \Big\{\langle P_1, \dots, P_s\rangle_{\mathcal{F}(X)} : P_1, \dots, P_s \in \mathcal{P}, s \in \mathbb{N}\Big\},$$

then observe that $\mathfrak{P}$ is a family of subsets of $\mathcal{F}(X)$.

Lemma 2.6. *Let X be a space and $A \subset X$. If $\mathcal{P}$ has property (P) at A for X, then $\mathfrak{P}$ has property (P) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

Proof. Let $F = \{x_1, \dots, x_r\} \in \langle A\rangle_{\mathcal{F}(X)}$. Then, $F \subset A$.

Case 1. (P) is point-finite (resp., point-countable). Then, for each $j \le r$, since $\mathcal{P}$ is point-finite (resp., point-countable) at A for X, $\mathcal{P}_j = \{P \in \mathcal{P} : x_j \in P\}$ is finite (resp., countable). If we put $\mathcal{P}_0 = \bigcup_{j\le r} \mathcal{P}_j$, then $\mathcal{P}_0$ is finite (resp., countable). Therefore, to prove that $\mathfrak{P}$ is point-finite (resp., point-countable) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$, we only need to show that

$$\Big\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W}\Big\} \subset \Big\{\langle P_1, \dots, P_s\rangle_{\mathcal{F}(X)} : P_1, \dots, P_s \in \mathcal{P}_0, s \in \mathbb{N}\Big\}.$$

In fact, take any $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \notin \Big\{\langle P_1, \dots, P_s\rangle_{\mathcal{F}(X)} : P_1, \dots, P_s \in \mathcal{P}_0, s \in \mathbb{N}\Big\}$ with $k \in \mathbb{N}$. Then, $E_{i_0} \notin \mathcal{P}_0$ for some $i_0 \le k$. This implies that $x_j \notin E_{i_0}$ for every $j \le r$. Thus, $F \notin \langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)}$. Hence, $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \notin \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W}\}$.

Case 2. (P) is compact-finite (resp., compact-countable). Let $\mathcal{K}$ be compact in the subspace $\langle A\rangle_{\mathcal{F}(X)}$ of $\mathcal{F}(X)$. Then, $\mathcal{K}$ is compact in the subspace $\langle A\rangle_{\mathbb{K}(X)}$ of $\mathbb{K}(X)$. It follows from Lemma 2.4 that $K = \bigcup \mathcal{K}$ is compact in the subspace A of X. Moreover, since $\mathcal{K} \subset \langle K\rangle_{\mathcal{F}(X)}$, we claim that

$$\{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \mathcal{K} \ne \emptyset\} \subset \{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \langle K\rangle_{\mathcal{F}(X)} \ne \emptyset\}.$$

Since $\mathcal{P}$ is compact-finite (resp., compact-countable) at A for X, $\mathcal{P}_0 = \{P \in \mathcal{P} : P \cap K \ne \emptyset\}$ is finite (resp., countable). On the other hand,

$$\{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \langle K\rangle_{\mathcal{F}(X)} \ne \emptyset\} \subset \{\langle P_1, \dots, P_s\rangle_{\mathcal{F}(X)} : P_1, \dots, P_s \in \mathcal{P}_0,\ s \in \mathbb{N}\}.$$

In fact, let $k \in \mathbb{N}$ and $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \notin \{\langle P_1, \dots, P_s\rangle_{\mathcal{F}(X)} : P_1, \dots, P_s \in \mathcal{P}_0,\ s \in \mathbb{N}\}$. Then, there exists $i_0 \leq k$ such that $E_{i_0} \notin \mathcal{P}_0$. This implies that $E_{i_0} \cap K = \emptyset$. By Lemma 2.3, $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \cap \langle K\rangle_{\mathcal{F}(X)} = \emptyset$. Thus, $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \notin \{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \langle K\rangle_{\mathcal{F}(X)} \neq \emptyset\}$.

Hence, $\{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \mathcal{K} \neq \emptyset\}$ is finite (resp., countable). This shows that $\mathfrak{P}$ is compact-finite (resp., compact-countable) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

Case 3. (P) is locally finite (resp., locally countable). Then, for each $i \leq r$, there exists an open neighborhood W_i of x_i such that $(\mathcal{P})_{W_i}$ is finite (resp., countable). If we put

$$V_i = W_i \setminus \{x_j : j \leq r, j \neq i\},$$

then V_i is open in X for every $i \leq r$, and $\langle V_1, \dots, V_r\rangle_{\mathcal{F}(X)}$ is an open neighborhood of F in $\mathcal{F}(X)$. On the other hand, $\{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \langle V_1, \dots, V_r\rangle_{\mathcal{F}(X)} \neq \emptyset\}$ is finite (resp., countable). In fact, for each $i \leq r$, since $\mathcal{P}$ is locally finite (resp., locally countable) at A in X, $\mathcal{P}_i = \{P \in \mathcal{P} : P \cap V_i \neq \emptyset\}$ is finite (resp., countable). If we put $\mathcal{P}_0 = \bigcup_{i\leq r}\mathcal{P}_i$, then $\mathcal{P}_0$ is finite (resp., countable). Now, take any $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \notin \left\{\langle P_1, \dots, P_s\rangle_{\mathcal{F}(X)} : P_1, \dots, P_s \in \mathcal{P}_0, s \in \mathbb{N}\right\}$ with $k \in \mathbb{N}$. Then, there exists $i_0 \leq k$ such that $E_{i_0} \notin \mathcal{P}_0$. Thus, $E_{i_0} \cap V_i = \emptyset$ for every $i \leq r$. It follows from Lemma 2.3 that $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \cap \langle V_1, \dots, V_r\rangle_{\mathcal{F}(X)} = \emptyset$. Hence, $\langle E_1, \dots, E_k\rangle_{\mathcal{F}(X)} \notin \{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \langle V_1, \dots, V_r\rangle_{\mathcal{F}(X)} \neq \emptyset\}$. This implies that

$$\{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \langle V_1, \dots, V_r\rangle_{\mathcal{F}(X)} \neq \emptyset\} \subset \{\langle P_1, \dots, P_s\rangle_{\mathcal{F}(X)} : P_1, \dots, P_s \in \mathcal{P}_0,\ s \in \mathbb{N}\}.$$

Therefore, we claim that $\{\mathcal{W} \in \mathfrak{P} : \mathcal{W} \cap \langle V_1, \dots, V_r\rangle_{\mathcal{F}(X)} \neq \emptyset\}$ is finite (resp., countable). Hence, $\mathfrak{P}$ is locally finite (resp., locally countable) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$. □

Theorem 2.7. *Let X be a space and $A \subset X$.*

1. *X has a sequence of open covers (resp., so-covers, cs-covers, cs*-covers) at A which is a point-star network at A for X if and only if $\mathcal{F}(X)$ has a sequence of open covers (resp., so-covers, cs-covers, cs*-covers) at $\langle A\rangle_{\mathcal{F}(X)}$ which is a point-star network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*
2. *X has a sequence of open covers (resp., so-covers, cs-covers, cs*-covers) with property (P) at A which is a point-star network at A for X if and only if $\mathcal{F}(X)$ has a sequence of open covers (resp., so-covers, cs-covers, cs*-covers) with property (P) at $\langle A\rangle_{\mathcal{F}(X)}$ which is a point-star network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

Proof. Necessity. By Lemmas 2.5 and 2.6.

Sufficiency. Assume that $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ is a sequence of open covers (resp., *so*-covers, *cs*-covers, *cs**-covers) at $\langle A\rangle_{\mathcal{F}(X)}$, and a point-star network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$. For each $n \in \mathbb{N}$, we put

$$\mathfrak{Q}_n = \{\mathcal{W} \cap \mathcal{F}_1(X) : \mathcal{W} \in \mathfrak{P}_n\}.$$

Then, $\{\mathfrak{Q}_n\}_{n\in\mathbb{N}}$ is a sequence of open covers (resp., *so*-covers, *cs*-covers, *cs**-covers) at $\langle A\rangle_{\mathcal{F}_1(X)} = \langle A\rangle \cap \mathcal{F}_1(X)$, and a point-star network at $\langle A\rangle_{\mathcal{F}_1(X)} = \langle A\rangle \cap \mathcal{F}_1(X)$ for $\mathcal{F}_1(X)$. On the other hand, for each $n \in \mathbb{N}$, if $\mathfrak{P}_n$ has property (P) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$, then $\mathfrak{Q}_n$ has property (P) at $\langle A\rangle_{\mathcal{F}_1(X)} = \langle A\rangle \cap \mathcal{F}_1(X)$ for $\mathcal{F}_1(X)$. By Remark 1.1, the proof of sufficiency is completed. □

In Theorem 2.7, if (P) is point-finite, then by [5, Theorems 3.3, 3.4] and [6, Theorem 3.4], we obtain the following corollary.

Corollary 2.8. *Let X be a space and $A \subset X$. Then, X has a point-regular external base (resp., so-network, sn-network, cs-network, cs*-network) at A for X if and only if $\mathcal{F}(X)$ has a point-regular external base (resp., so-network, sn-network, cs-network, cs*-network) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

By Corollary 2.8, [6, Theorems 3.4, 3.10] and [8, Theorem 5.3], we obtain the following corollary.

Corollary 2.9. *Let X be a space and $A \subset X$. Then, X is the image of a metric space under an almost-open (resp., a strictly countably bi-quotient, a sequence-covering, a 1-sequence-covering, sequentially quotient) and compact mapping at A for X if and only if $\mathcal{F}(X)$ is the image of a metric space under an almost-open (resp., a strictly countably bi-quotient, a sequence-covering, a 1-sequence-covering, sequentially quotient) and compact mapping at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

Lemma 2.10. *Let X be a space and $A \subset X$. Then, $\mathcal{P}$ is an external base (resp., an so-network, an sn-network, a cs-network, a cs*-network, a cn-network, a ck-network) at A for X, then $\mathfrak{P}$ is an external base (resp., an so-network, an sn-network, a cs-network, a cs*-network, a cn-network, a ck-network) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

Proof. Suppose that $F = \{x_1, \ldots, x_r\} \in \langle A\rangle_{\mathcal{F}(X)}$ and $\mathcal{U}$ is an open neighborhood of F in $\mathcal{F}(X)$. Then, $F \subset A$ and by the proof of Lemma 2.5, there exist pairwise disjoint open subsets $U_{x_1}, \ldots, U_{x_r}$ of X such that $x_j \in U_{x_j}$ for each $j \leq r$, and

$$F \in \langle U_{x_1}, \ldots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Case 1. Assume that $\mathcal{P} = \bigcup_{x\in A} \mathcal{P}_x$ is an external base (resp., *so*-network, *sn*-network) at A for X, where each $\mathcal{P}_x$ is a local base (resp., an *sn*-network consisting of sequentially open sets, an *sn*-network) at x in X for each $x \in A$. Put

$$\mathfrak{P}_F = \Big\{\langle P_{x_1}, \ldots, P_{x_r}\rangle_{\mathcal{F}(X)} : P_{x_j} \in \mathcal{P}_{x_j}, P_{x_i} \cap P_{x_j} \neq \emptyset, i \neq j, i, j \leq r\Big\}.$$

If P_x is a local base at x in X for each $x \in A$, then it is easy to check that $\mathfrak{P}_F$ is a local base at F in $\mathcal{F}(X)$.

If $\mathcal{P}_x$ is an *sn*-network (resp., an *sn*-network consisting of sequentially open sets) at x in X for each $x \in A$, then it is obvious that $\mathfrak{P}_F$ is a network at F in $\mathcal{F}(X)$ and if $\mathcal{W}_1, \mathcal{W}_2 \in \mathfrak{P}_F$, then $\mathcal{W} \subset \mathcal{W}_1 \cap \mathcal{W}_2$ for some $\mathcal{W} \in \mathfrak{P}_F$. Now, take any $\mathcal{W} \in \mathfrak{P}_F$. Then, $\mathcal{W} = \langle P_{x_1}, \ldots, P_{x_r}\rangle_{\mathcal{F}(X)}$, where each $P_{x_j} \in \mathcal{P}_{x_j}$ and $P_{x_i} \cap P_{x_j} \neq \emptyset$ if $i \neq j$.

If each element of $\mathcal{P}_x$ is a sequentially open set in X, then by the proof of Case 1 of Lemma 2.5(2), we claim that $\mathcal{W}$ is a sequentially open set in $\mathcal{F}(X)$. This implies that each element of $\mathfrak{P}_F$ is a sequentially open set in $\mathcal{F}(X)$.

If each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X, then $\mathcal{W}$ is a sequential neighborhood of F in $\mathcal{F}(X)$. In fact, let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence converging to F in $\mathcal{F}(X)$. By the proof of Case 1 of Lemma 2.5(2), we claim that there exists $m \in \mathbb{N}$ such that $F_n \in \mathcal{W}$ for each $n > m$. This shows that $\mathcal{W}$ is a sequential neighborhood of F in $\mathcal{F}(X)$. Therefore, each element of $\mathfrak{P}_F$ is a sequential neighborhood of F in $\mathcal{F}(X)$.

Finally, since $\mathfrak{P} = \bigcup_{F\in\langle A\rangle_{\mathcal{F}(X)}} \mathfrak{P}_F$, $\mathfrak{P}$ is an external base (resp., *so*-network, *sn*-network) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

Case 2. $\mathcal{P}$ is a *cs*-network (resp., *cs**-network) at A for X. Suppose that $\{F_m\}_{m\in\mathbb{N}}$ is a sequence converging to F in $\mathcal{F}(X)$. For each $j \leq r$, it follows from Lemma 2.2 that the sequence $\{F_m \cap U_{x_j}\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X.

If $\mathcal{P}$ is a *cs*-network at A for X, then there exist $P_j \in \mathcal{P}$ and $k_j \in \mathbb{N}$ such that

$$\{x_j\} \cup \Big(\bigcup\{F_m \cap U_{x_j} : m \geq k_j\}\Big) \subset P_j \subset U_{x_j}.$$

Put $k = \max\{k_j : j \leq r\}$. Then, $\langle P_1, \ldots, P_r\rangle_{\mathcal{F}(X)} \in \mathfrak{P}$ and

$$\{F\} \cup \{F_m : m > k\} \subset \langle P_1, \ldots, P_r\rangle_{\mathcal{F}(X)} \subset \langle U_{x_1}, \ldots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Therefore, $\mathfrak{P}$ is a *cs*-network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

If $\mathcal{P}$ is a *cs**-network at A for X, by induction on r, then there exist $P_j \in \mathcal{P}$ and a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\{x_j\} \cup \Big(\bigcup\{F_{m_k} \cap U_{x_j} : k \in \mathbb{N}\}\Big) \subset P_j \subset U_{x_j}.$$

This implies that $\langle P_1, \ldots, P_r\rangle_{\mathcal{F}(X)} \in \mathfrak{P}$ and

$$\{F\} \cup \{F_{m_k} : k \in \mathbb{N}\} \subset \langle P_1, \ldots, P_r\rangle_{\mathcal{F}(X)} \subset \langle U_{x_1}, \ldots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Hence, $\mathfrak{P}$ is a *cs**-network at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

Case 3. $\mathcal{P}$ is a *cn*-network at A for X. For each $j \leq r$, if we put $\mathcal{P}_j = \{P \in \mathcal{P} : x_j \in P \subset U_{x_j}\}$, then $\bigcup \mathcal{P}_j$ is a neighborhood of x_j in X for each $j \leq r$. This implies that for each $j \leq r$, there exists an open subset V_j in X such that $x_j \in V_j \subset \bigcup \mathcal{P}_j$. Moreover, if we put $\mathcal{R} = \bigcup_{j\leq r} \mathcal{P}_j$, then

$$
\begin{aligned}
F \in \langle V_1, \ldots, V_r \rangle_{\mathcal{F}(X)} &\subset \Big\langle \bigcup \mathcal{P}_1, \ldots, \bigcup \mathcal{P}_r \Big\rangle_{\mathcal{F}(X)} \\
&\subset \bigcup \{\langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)} : F \in \langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)}, P_1, \ldots, P_s \in \mathcal{R},\ s \in \mathbb{N}\} \\
&\subset \bigcup \{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}.
\end{aligned}
$$

On the other hand, since $\langle V_1, \ldots, V_r \rangle_{\mathcal{F}(X)}$ is open in $\mathcal{F}(X)$, we claim that $\bigcup\{\mathcal{W} \in \mathfrak{P} : F \in \mathcal{W} \subset \mathcal{U}\}$ is a neighborhood of F in $\mathcal{F}(X)$. This shows that $\mathfrak{P}$ is a *cn*-network at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

Case 4. $\mathcal{P}$ is a *ck*-network at A for X. For each $j \le r$, there is a neighborhood V_{x_j} of x_j in X such that $V_{x_j} \subset U_{x_j}$ and for each compact subset $A_j \subset V_{x_j}$, there exists a finite subfamily $\mathcal{A}_j$ of $\mathcal{P}$ satisfying $x_j \in \bigcap \mathcal{A}_j$ and $A_j \subset \bigcup \mathcal{A}_j \subset U_{x_j}$. Next, for each $j \le r$, since X is regular, there exists an open subset W_{x_j} in X such that $x_j \in W_{x_j} \subset \overline{W}_{x_j} \subset V_{x_j}$. Now, if we put $\mathcal{V}_F = \langle W_{x_1}, \ldots, W_{x_r} \rangle_{\mathcal{F}(X)}$, then for each compact subset $\mathcal{K} \subset \mathcal{V}_F$, $\bigcup \mathcal{K} \subset \bigcup_{j \le r} \overline{W}_{x_j}$. Since $\mathcal{K}$ is compact in $\mathcal{F}(X)$, $\mathcal{K}$ is compact in $\langle X \rangle_{\mathbb{K}(X)} = \mathbb{K}(X)$. It follow from Lemma 2.4 that $\bigcup \mathcal{K}$ is compact in X. Hence, we claim that $K_j = (\bigcup \mathcal{K}) \cap \overline{W}_{x_j}$ is compact in X and $K_j \subset V_{x_j}$. Thus, there exists a finite subfamily $\mathcal{F}_j \subset \mathcal{P}$ such that $x_j \in \bigcap \mathcal{F}_j$ and $K_j \subset \bigcup \mathcal{F}_j \subset U_{x_j}$. Lastly, if we put $\mathcal{R} = \bigcup_{j \le r} \mathcal{F}_j$ and

$$\mathcal{F} = \{\langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)} : F \in \langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)}, P_1, \ldots, P_s \in \mathcal{R},\ s \in \mathbb{N}\},$$

then $\mathcal{F}$ is finite, $F \in \bigcap \mathcal{F}$ and $\bigcup \mathcal{F} \subset \langle U_{x_1}, \ldots, U_{x_r} \rangle_{\mathcal{F}(X)}$. Furthermore, $\mathcal{K} \subset \bigcup \mathcal{F}$. In fact, take any $\{y_1, \ldots, y_p\} \in \mathcal{K}$. Then, $\{y_1, \ldots, y_p\} \subset \bigcup \mathcal{K}$. For each $k \le p$, since $\bigcup \mathcal{K} = \bigcup_{j \le r} K_j$, there exists $j_0 \le r$ such that $y_k \in K_{j_0} \subset \bigcup \mathcal{F}_{j_0}$. This implies that $\{y_1, \ldots, y_p\} \in \bigcup \mathcal{F}$. Thus, $\mathcal{K} \subset \bigcup \mathcal{F} \subset \langle U_{x_1}, \ldots, U_{x_r} \rangle_{\mathcal{F}(X)}$. Therefore, $\mathfrak{P}$ is a *ck*-network at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$. □

Theorem 2.11. *Let X be a space and $A \subset X$. Then, X has an external base (resp., an so-network, an sn-network, a cs-network, a cs*-network, a cn-network, a ck-network) with property σ-(P) at A for X if and only if $\mathcal{F}(X)$ has an external base (resp., an so-network, an sn-network, a cs-network, a cs*-network, a cn-network, a ck-network) with property σ-(P) at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

Proof. Necessity. Assume that $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ is an external base (resp., an *so*-network, an *sn*-network, a *cs*-network, a *cs**-network, a *cn*-network, a *ck*-network) at A for X, where each $\mathcal{P}_n$ has property (*P*) at A for X and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$. It follows from Lemma 2.6 that

$$\mathfrak{P}_n = \{\langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)} : P_1, \ldots, P_s \in \mathcal{P}_n,\ s \in \mathbb{N}\}$$

has property (*P*) at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$, and $\mathfrak{P}_n \subset \mathfrak{P}_{n+1}$ for each $n \in \mathbb{N}$. If we put $\mathfrak{P} = \bigcup_{n \in \mathbb{N}} \mathfrak{P}_n$ then it is easy to see that

$$\mathfrak{P} \subset \{\langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)} : P_1, \ldots, P_s \in \mathcal{P},\ s \in \mathbb{N}\}.$$

Now, let $\mathcal{W} \in \{\langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)} : P_1, \ldots, P_s \in \mathcal{P},\ s \in \mathbb{N}\}$. Then, there exist $P_1, \ldots, P_s \in \mathcal{P}$ such that $\mathcal{W} = \langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)}$. Since $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$, there exists $n_i \in \mathbb{N}$ such that $P_i \in \mathcal{P}_{n_i}$ for each $i \le s$. If we put $m = \max\{n_i : i \le s\}$, then $P_1, \ldots, P_s \in \mathcal{P}_m$ and $m \in \mathbb{N}$. This implies that $\mathcal{W} \in \mathfrak{P}_m \subset \mathfrak{P}$. Thus, we claim that

$$\mathfrak{P} = \{\langle P_1, \ldots, P_s \rangle_{\mathcal{F}(X)} : P_1, \ldots, P_s \in \mathcal{P},\ s \in \mathbb{N}\}.$$

It follows from Lemma 2.10 that $\mathfrak{P}$ is an external base (resp., an *so*-network, an *sn*-network, a *cs*-network, a *cs**-network, a *cn*-network, a *ck*-network) at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.

Sufficiency. Let $\mathfrak{P} = \bigcup_{n \in \mathbb{N}} \mathfrak{P}_n$ be an external base (resp., an *so*-network, an *sn*-network, a *cs*-network, a *cs**-network, a *cn*-network, a *ck*-network) with property σ-(*P*) at $\langle A \rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$. For each $n \in \mathbb{N}$, we put

$$\mathfrak{Q}_n = \{\mathcal{W} \cap \mathcal{F}_1(X) : \mathcal{W} \in \mathfrak{P}_n\}.$$

Then, $\mathfrak{Q} = \bigcup_{n \in \mathbb{N}} \mathfrak{Q}_n$ is an external base (resp., an *so*-network, an *sn*-network, a *cs*-network, a *cs**-network, a *cn*-network, a *ck*-network) with property σ-(*P*) at $\langle A \rangle_{\mathcal{F}_1(X)} = \langle A \rangle \cap \mathcal{F}_1(X)$ for $\mathcal{F}_1(X)$. Thus, X has an external base (resp., an *so*-network, an *sn*-network, a *cs*-network, a *cs**-network, a *cn*-network, a *ck*-network) with property σ-(*P*) at A for X by Remark 1.1. □

In Theorem 2.11, if (P) is point-countable, then we obtain the following corollary.

Corollary 2.12. *Let X be a space and $A \subset X$. Then, X has a point-countable external base (resp., so-network, sn-network, cs-network, cs*-network, cn-network, ck-network) at A for X if and only if $\mathcal{F}(X)$ has a point-countable external base (resp., so-network, sn-network, cs-network, cs*-network, cn-network, ck-network) at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

By Corollary 2.12, [6, Theorem 4.6] and [8, Theorems 3.2, 3.6, 3.9], we obtain the following corollary.

Corollary 2.13. *Let X be a space and $A \subset X$. Then, X is the image of a metric space under an open (resp., a strictly countably bi-quotient, a pseudo-sequence-covering, a sequentially quotient, a sequence-covering, a 1-sequence-covering) and s-mapping at A for X if and only if $\mathcal{F}(X)$ is the image of a metric space under an open (resp., a strictly countably bi-quotient, a pseudo-sequence-covering, a sequentially quotient, a sequence-covering, a 1-sequence-covering) and s-mapping at $\langle A\rangle_{\mathcal{F}(X)}$ for $\mathcal{F}(X)$.*

In Theorems 2.7, 2.11 and Corollaries 2.8, 2.12, if $A = X$, then $\langle A\rangle_{\mathcal{F}(X)} = \mathcal{F}(X)$, we obtain the following corollary.

Corollary 2.14. *Let X be a space.*

1. *X has a sequence of open covers (resp., so-covers, cs-covers, cs*-covers) which is a point-star network for X if and only if so does $\mathcal{F}(X)$;*
2. *X has a sequence of open covers (resp., so-covers, cs-covers, cs*-covers) with property (P) which is a point-star network for X if and only if so does $\mathcal{F}(X)$;*
3. *X has a point-regular base (resp., so-network, sn-network, cs-network, cs*-network) if and only if so does $\mathcal{F}(X)$;*
4. *X has a base (resp., an so-network, an sn-network, a cs-network, a cs*-network, a cn-network, a ck-network) with property σ-(P) if and only if so does $\mathcal{F}(X)$;*
5. *X has a point-countable base (resp., so-network, sn-network, cs-network, cs*-network, cn-network, ck-network) if and only if so does $\mathcal{F}(X)$.*

In Corollary 2.14(2), if (P) is locally finite, then we get the following corollary.

Corollary 2.15. *Let X be a space. Then, X is so-metrizable (resp., strict σ-space, strict $\aleph$-space) if and only if so does $\mathcal{F}(X)$.*

Remark 2.16. By Corollary 2.14 and Remark 1.9, we get back the following known results in [4, Theorem 4.7], [14, Theorems 37, 41], [18, Theorem 2.6] and [18, Corollaries 2.7, 2.8].

1. X has a σ-strong network consisting of cs^*-covers (cs-covers) if and only if so does $\mathcal{F}(X)$;
2. X has a σ-(P)-strong network consisting of cs^*-covers (cs-covers) if and only if so does $\mathcal{F}(X)$;
3. X has a point-regular base (resp., sn-network, cs-network, cs^*-network) if and only if so does $\mathcal{F}(X)$;
4. X has a point-countable base (resp., cs-network) if and only if so does $\mathcal{F}(X)$;
5. X is an sn-metrizable space (resp., an sn-developable space, a strongly sn-developable space) if and only if so is $\mathcal{F}(X)$;
6. X is a weak Cauchy sn-symmetric space (resp., Cauchy sn-symmetric space) if and only if so is $\mathcal{F}(X)$;
7. X is a Cauchy sn-symmetric space with a σ-(P)-property cs^*-network (resp., cs-network, sn-network) if and only if so is $\mathcal{F}(X)$;
8. X has property γ if and only if so does $\mathcal{F}(X)$, where property γ is one of the images of metric spaces under some kinds of continuous mappings, which is determined in [18, Corollary 2.8].

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