Filomat 38:12 (2024), 4009–4019
https://doi.org/10.2298/FIL2412009B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On some Grüss-type inequalities via k-weighted fractional operators

Bouharket Benaissa[a], **Noureddine Azzouz**[b], **Mehmet Zeki Sarikaya**[c,*]

[a] *Laboratory of Informatics and Mathematics Faculty of Material Sciences, University of Tiaret, Algeria*
[b] *Faculty of Sciences, University, Center Nour Bachir El Bayadh, Algeria*
[c] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey*

Abstract. In this paper, we employ the concept of k-weighted fractional integration of one function with respect to another function to extend the scope of Grüss-type fractional integral inequalities. Furthermore, we establish and provide proofs for a set of inequalities that incorporate k-weighted fractional integrals.

1. Introduction

In 1935, G. Grüss [1] proved the well known inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx\right| \leq \frac{(M-m)(N-n)}{4} \tag{1}$$

provided that f, g are two integrable functions on $[a,b]$ and satisfying the conditions:

$$m \leq f(x) \leq M,\ n \leq g(x) \leq N; \quad n, m, N, M \in \mathbf{R} \text{ and } x \in [a,b].$$

In 2010, Dahmani et al. [2], proved the following fractional version inequality by using Riemann– Liouville fractional integral

$$\left|\frac{x^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(fg)(x) - I_a^\alpha f(x)\, I_a^\alpha g(x)\right| \leq \left(\frac{x^\alpha}{\Gamma(\alpha+1)}\right)^2 (M-m)(N-n). \tag{2}$$

In 2014, Tariboon et al. [3], replaced the constants which appeared as bounds of the functions f and g by four integrable functions on $[0,\infty)$, as

$$\left|\frac{x^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(fg)(x) - I_a^\alpha f(x)\, I_a^\alpha g(x)\right| \leq \sqrt{H(f,F_1,F_2)\, H(g,G_1,G_2)}, \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 26D10; 26A33; 26D15.
Keywords. Grüss inequality, k-weighted fractional Operator, Hölder inequality.
Received: 12 September 2023; Revised: 02 November 2023; Accepted: 06 November 2023
Communicated by Dragan S. Djordjević
* Corresponding author: Mehmet Zeki Sarikaya
Email addresses: bouharket.benaissa@univ-tiaret.dz (Bouharket Benaissa), azzouz_n@live.fr (Noureddine Azzouz), sarikayamz@gmail.com (Mehmet Zeki Sarikaya)

where $F_1(x) \leq f(x) \leq F_2(x),\ G_1(x) \leq g(x) \leq G_2(x)$ and $H(f, F_1, F_2)$ defined by

$$\begin{aligned} H(f,F_1,F_2)(x) &= \left(I_a^\alpha F_2(x) - I_a^\alpha f(x)\right)\left(I_a^\alpha f(x) - I_a^\alpha F_1(x)\right) \\ &+ \tfrac{x^\alpha}{\Gamma(\alpha+1)} I_a^\alpha \left[F_2(x) f(x)\right] - I_a^\alpha F_2(x) I_a^\alpha f(x) \\ &+ \tfrac{x^\alpha}{\Gamma(\alpha+1)} I_a^\alpha \left[f(x) F_1(x)\right] - I_a^\alpha f(x) I_a^\alpha F_1(x) \\ &+ I_a^\alpha F_2(x) I_a^\alpha F_1(x) - \tfrac{x^\alpha}{\Gamma(\alpha+1)} I_a^\alpha \left[F_2(x) F_1(x)\right]. \end{aligned}$$

Throughout the past decade, numerous authors have introduced and validated various novel integral inequalities of the forms (2) and (3) by employing diverse fractional integral operators. Refer to the following sources for more details, [4]-[11], [13].

On the other hand [12], the weighted fractional integrals is defined, for an integrable function f on the interval $[a, b]$ and for a differentiable function μ such that $\mu'(t) \neq 0$ for all $t \in [a, b]$, as follows,

$${}_{a^+}I_w^\beta f(x) = \frac{1}{w(x)\Gamma(\beta)} \int_a^x \mu'(s) \left(\mu(x) - \mu(s)\right)^{\beta-1} w(s) f(s) ds, \quad x > a,$$

where w is a weighted function (positive measurable function).

2. *k*-weighted fractional Operator

In this section, we present a definition of the k-weighted fractional integrals of a function f with respect to the function ψ and we prove that they are bounded in a specified space. Let $[a, b] \subseteq [0, +\infty)$, where $a < b$.

Definition 2.1. *The k-weighted fractional integral operators are defined as follows: Let $\alpha > 0$, $k > 0$ and ψ be a positive, strictly increasing differentiable function such that $\psi'(s) \neq 0$ for all $s \in [a, b]$. The left and right sided k-weighted fractional integral of a function f with respect to the function ψ on $[a, b]$ are defined respectively as follows.*

$${}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) = \frac{1}{w(x) k \Gamma_k(\alpha)} \int_a^x \psi'(t) \left(\psi(x) - \psi(t)\right)^{\frac{\alpha}{k}-1} w(t) f(t) dt, \quad a < x \leq b. \tag{4}$$

$${}_{b^-}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) = \frac{1}{w(x) k \Gamma_k(\alpha)} \int_x^b \psi'(t) \left(\psi(t) - \psi(x)\right)^{\frac{\alpha}{k}-1} w(t) f(t) dt, \quad a \leq x < b, \tag{5}$$

where w is a weighted non-decreasing function and Γ_k is the k-gamma function defined by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt.$$

The space $L_p^W[a, b]$ of all real-valued Lebesgue measurable functions f on $[a, b]$ with norm conditions:

$$\| f \|_p^W = \left(\int_a^b | f(x) |^p \, W(x) dx \right)^{\frac{1}{p}} < \infty, \ 1 \leq p < +\infty$$

is known as weighted Lebesgue space, where W be a weight function (measurable and positive).

1. Put $W \equiv 1$, the space $L_p^W[a, b]$ reduces to the classical space $L_p[a, b]$.
2. Choose $W(x) = w^p(x)\,\psi'(x)$ and $p = 1$, we get

$$L_{X_w}[a, b] = \left\{ f : \| f \|_{X_w} = \int_a^b | w(x) f(x) | \, \psi'(x) dx < \infty \right\}. \tag{6}$$

In the next theorem, we show that the the k-weighted fractional operators are bounded.

Theorem 2.2. *The fractional integrals (4), (5) are defined for functions $f \in L_{X_{\psi'}}[a,b]$, existing almost everywhere and*

$$ {}_{a^+}\mathrm{I}_w^{\psi} f(x) \in L_{X_w}[a,b], \qquad {}_{b^-}\mathrm{I}_w^{\psi} f(x) \in L_{X_w}[a,b]. \tag{7}$$

Moreover

$$ \left\| {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) \right\|_{X_w} \leq C \left\| f(x) \right\|_{X_w}, \qquad \left\| {}_{b^-}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) \right\|_{X_w} \leq C \left\| f(x) \right\|_{X_w}, \tag{8}$$

where

$$ C = \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}. $$

Proof. Let $f \in L_{X_w}[a,b]$. Appling Fubini's Theorem, we get

$$ \left\| {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f \right\|_{X_w} = \int_a^b \mid w(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) \mid \psi'(x)\,dx $$

$$ \leq \frac{1}{k\,\Gamma_k(\alpha)} \int_a^b \int_a^x \left| w(s) f(s) \right| \psi'(s) \left(\psi(x) - \psi(s)\right)^{\frac{\alpha}{k}-1} \psi'(x)\,ds\,dx $$

$$ = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^b \left| w(s) f(s) \right| \left(\int_s^b (\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} \psi'(x)\,dx \right) \psi'(s)\,ds $$

$$ = \frac{1}{\alpha\,\Gamma_k(\alpha)} \int_a^b \left| w(s) f(s) \right| (\psi(b)-\psi(s))^{\frac{\alpha}{k}} \psi'(s)\,ds $$

$$ \leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \int_a^b \left| w(s) f(s) \right| \psi'(s)\,ds $$

$$ = C \left\| f \right\|_{X_w}. $$

Similarly

$$ \int_a^b \mid w(x)\, {}_{b^-}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) \mid \psi'(x)dx \leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \left\| f \right\|_{X_w}. $$

This gives us our desired formulas (7) and (8). □

One important feature of the left side k-weighted fractional operators is that they depend on the choice of the functions ψ give rise to certain types of left side k-weighted fractional integrals.

1. Taking $w(\tau) = 1$, we get the left side k-Hilfer operator (generalized k-fractional integral) of order $\alpha > 0$

$$ {}_{a^+}\mathcal{J}_k^{\alpha,\psi} f(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^x (\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} \psi'(s) f(s) ds, \quad x > a, $$

2. Taking $\psi(\tau) = \tau$, we get the left side k-weighted Riemann-Liouville fractional operator of order $\alpha > 0$

$$ {}_{a^+}\mathcal{RL}_{k,w}^{\alpha} f(x) = \frac{1}{w(x) k\,\Gamma_k(\alpha)} \int_a^x (x-s)^{\frac{\alpha}{k}-1} w(s) f(s) ds, \quad x > a, $$

3. Using $\psi(\tau)=\ln\tau$, we obtain the left side k-weighted Hadamard fractional operator of order $\alpha>0$

$$ {}_{a^+}\mathcal{H}^{\alpha}_{k,w}f(x)=\frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x\left(\ln\frac{x}{s}\right)^{\frac{\alpha}{k}-1}w(s)f(s)\frac{ds}{s},\quad x>a>1, $$

4. Putting $\psi(\tau)=\frac{\tau^{\rho+1}}{\rho+1}$ where $\rho>0$, we deduce the left side k-weighted Katugompola fractional operators((k,s)–weighted fractional) of order $\alpha>0$,

$$ {}_{a^+}\mathcal{K}^{\alpha}_{k,w}f(x)=\frac{(\rho+1)^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)}\int_a^x\left(x^{\rho+1}-s^{\rho+1}\right)^{\frac{\alpha}{k}-1}w(s)f(s)s^{\rho}ds,\ x>a, $$

5. Setting $\psi(\tau)=\frac{(\tau-a)^{\theta}}{\theta}$ where $\theta>0$, we result the left sided k-weighted fractional conformable operator of order $\alpha>0$ [13].

$$ {}_{a^+}C^{\alpha}_{k,w}f(x)=\frac{\theta^{1-\frac{\alpha}{k}}}{w(x)k\,\Gamma_k(\alpha)}\int_a^x\left((x-a)^{\theta}-(s-a)^{\theta}\right)^{\frac{\alpha}{k}-1}\frac{w(s)\,f(s)}{(s-a)^{1-\theta}}ds,\ x>a, $$

Remark 2.3. *Since w is non-decreasing on $[a,b]$, apply the notion of k-weighted fractional integral* (4)*, we get*

$$ \begin{aligned} {}_{a^+}J^{\alpha,\psi}_{k,w}1(x) &= \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x\psi'(s)\left(\psi(x)-\psi(s)\right)^{\frac{\alpha}{k}-1}w(s)\,ds\\ &\le \frac{1}{w(a)k\,\Gamma_k(\alpha)}\int_a^x\psi'(s)\left(\psi(x)-\psi(s)\right)^{\frac{\alpha}{k}-1}w(b)\,ds\\ &= \frac{w(b)}{w(a)\,k\,\Gamma_k(\alpha)}\int_a^x\psi'(s)\left(\psi(x)-\psi(s)\right)^{\frac{\alpha}{k}-1}\,ds, \end{aligned} $$

another part

$$ \begin{aligned} {}_{a^+}J^{\alpha,\psi}_{k,w}1(x) &\le \frac{1}{w(x)k\,\Gamma_k(\alpha)}\int_a^x\psi'(s)\left(\psi(x)-\psi(s)\right)^{\frac{\alpha}{k}-1}w(x)\,ds\\ &= \frac{1}{k\,\Gamma_k(\alpha)}\int_a^x\psi'(s)\left(\psi(x)-\psi(s)\right)^{\frac{\alpha}{k}-1}\,ds. \end{aligned} $$

Therefore

$$ {}_{a^+}J^{\alpha,\psi}_{k,w}1(x)\le\frac{(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\le\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}}{w(a)\,\Gamma_k(\alpha+k)}. \tag{9} $$

If $w=1$, we get equality in (9).

The following Hölder inequality can be deduced from Corollary 2 [14] for $m=2$ and $q>1$:

$$ \begin{aligned} &\int_a^x\int_a^x|\,\mu(t,s)\Phi(t,s)\Psi(t,s)\,|\,dt\,ds\\ &\le\left(\int_a^x\int_a^x|\,\mu(t,s)\,||\,\Phi(t,s)\,|^{q}\,dt\,ds\right)^{\frac{1}{q}}\left(\int_a^x\int_a^x|\,\mu(t,s)||\Psi(t,s)\,|^{q'}\,dt\,ds\right)^{\frac{1}{q'}}. \end{aligned} \tag{10} $$

By $q=2$, then the above inequality (10) can be rewritten as follows:

$$ \begin{aligned} &\left(\int_a^x\int_a^x\mu(t,s)\Phi(t,s)\Psi(t,s)dt\,ds\right)^2\\ &\le\left(\int_a^x\int_a^x|\,\mu(t,s)\,|\,\Phi^2(t,s)dt\,ds\right)\left(\int_a^x\int_a^x|\,\mu(t,s)\,|\,\Psi^2(t,s)dt\,ds\right). \end{aligned} \tag{11} $$

3. Weight Grüss-type inequalities

Let f, F_1, F_2 be functions defined on $L_{X_w^p}[a,b]$ verified the condition

$$F_1(x) \le f(x) \le F_2(x), \quad \text{for all } x \in [a,b]. \tag{12}$$

Consider $T_\alpha^\beta(f, F_1, F_2)$ to be an operator defined as follows:

$$\begin{aligned} T_\alpha^\beta(f,F_1,F_2)(x) &= \left({}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}F_2(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\right)\left({}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_1(x)\right) \\ &+ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}\left[F_2(x)f(x)\right] - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}F_2(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) \\ &+ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}\left[f(x)F_1(x)\right] - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_1(x) \\ &+ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}F_2(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_1(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}\left[F_1(x)F_2(x)\right]. \end{aligned} \tag{13}$$

Taking $F_2(x) = M$ and $F_1(x) = m$, we get

$$\begin{aligned} T_\alpha^\beta(f,m,M)(x) &= \left(M\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\right)\left({}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) - m\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}1(x)\right) \\ &+ m\left({}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\right). \end{aligned} \tag{14}$$

Denote $T =: T_\alpha^\beta$ for $\alpha = \beta$, we have

$$T(f,m,M)(x) = \left(M\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}1(x) - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x)\right)\left({}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) - m\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}1(x)\right). \tag{15}$$

Appling the elementary inequality $AB \le (\frac{A+B}{2})^2$, we deduce

$$T(f,m,M)(x) \le \left(\frac{{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}1(x)\,(M-m)}{2}\right)^2.$$

By using inequality (9), we get

$$T(f,m,M)(x) \le \left(\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}\,(M-m)}{2\,w(a)\,\Gamma_k(\alpha+k)}\right)^2. \tag{16}$$

The first theorem is now presented.

Theorem 3.1. *Let f, F_1, F_2 be functions defined on $L_{X_w^p}[a,b]$ that satisfy the condition* (12) *and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$, $x > a$ and α, β, $k > 0$. Then the following inequalities hold:*

$$\begin{aligned} &{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}F_2(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) + {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_1(x) \\ &\ge {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}F_2(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_1(x) + {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_2(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) + {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_1(x) \\ &\ge {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_2(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}F_1(x) + \left({}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x)\right)^2. \end{aligned} \tag{18}$$

Proof. Given the hypothesis (12), then for all $t, s \in [a, b]$ we get

$$(f(t) - F_1(t))(F_2(s) - f(s)) \geq 0,$$

then

$$f(t)F_2(s) + F_1(t)f(s) \geq F_1(t)F_2(s) + f(t)f(s). \tag{19}$$

Multiplying the inequality (19) by $\frac{\psi'^{\frac{\alpha}{k}-1}w(t)}{w(x)k\Gamma_k(\alpha)}$ and integrating with respect to t over (a, x), we get

$$F_2(s)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) + f(s)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} F_1(t) \geq F_2(s)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} F_1(t) + f(s)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(t). \tag{20}$$

Now, Multiplying the above inequality (20) by $\frac{\psi'^{\frac{\beta}{k}-1}w(s)}{w(x)k\Gamma_k(\beta)}$ and integrating with respect to s over (a, x), we get

$$\begin{aligned} &{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} F_2(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) + {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} F_1(x) \\ \geq\ &{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} F_2(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} F_1(x) + {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x). \end{aligned}$$

Thus, we get the acquired inequality (17).
By putting $\beta = \alpha$ through the inequality (17), we obtain the inequality (18). □

Corollary 3.2. *Let $f \in L_{X_w^p}[a, b]$ and ψ be an increasing and positive function on $[a, b]$, having a continuous derivative ψ' on $[a, b]$, $x > a$ and $\alpha, \beta, k > 0$. Suppose that $m \leq f(x) \leq M$, for all $x \in [a, b]$, then the follows inequalities hold:*

$$\begin{aligned} &M\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) + m\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} f(x) \\ \geq\ &M\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} 1(x)\, m\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x) + {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) \end{aligned} \tag{21}$$

and

$$(M + m)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x) \geq M\, m \left({}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\right)^2 + \left({}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\right)^2. \tag{22}$$

Proof. Put $F_2(x) = M$ and $F_1(x) = m$, hence

$${}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} F_2(x) = M\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x), \quad \text{and} \quad {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} F_1(x) = m\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x).$$

So, applying Theorem 3.1, we can deduce our results. □

Remark 3.3.
1. *Put $w = \psi = 1$ and $k = 1$, we get Theorem 2 in [3].*
2. *Put $w = 1$ and $k = 1$, we get Theorem 2.11 and Corollary 2.14 in [4].*
3. *Put $w = 1$ and $\psi(x) = \frac{x^{r+1}}{r+1}$, we get Theorem 2.1 and Lemma 2.4 in [10].*
4. *Put $w = 1$, $\psi(x) = \frac{x^{r+1}}{r+1}$ where $r > 0$, we get Theorem 2.1 and Lemma 2.4 in [10], and for $k = 1$ we get Theorem 5 and Corollary 2 in [5].*
5. *Put $w = 1$, $k = 1$ and $\psi(x) = \frac{(x-a)^s}{s}$ where $s > 0$, we get Theorem 2.1 and Theorem 2.3 in [7] and Theorem 2.1 in [8].*
6. *Put $w = 1$, $k = 1$, $a = 0$ and $\psi(x) = \frac{(x-a)^s}{s}$ where $s > 0$, we get Theorem 2.1 and Corollary 2.1 in [9].*

The following Lemma is required to prove the second basic Theorem 3.6.

Lemma 3.4. *Let f, F_1, F_2 be a functions defined on $L_{X_w^p}[a,b]$ and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$, $x > a$ and α, $k > 0$. Suppose that the condition* (12) *holds. Then*

$$\begin{aligned} & {}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f^2(x) - {}_{a^+}J^{\beta,\psi}_{k,w}f(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f(x) \\ & = T^{\beta}_{\alpha}(f, F_1, F_2)(x) - {}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}\left[(F_2(x) - f(x))(f(x) - F_1(x))\right], \end{aligned} \tag{23}$$

and

$${}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f^2(x) - {}_{a^+}J^{\beta,\psi}_{k,w}f(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f(x) \leq T^{\beta}_{\alpha}(f, F_1, F_2)(x). \tag{24}$$

Proof. For all $t, s \in [a,b]$, we have

$$\begin{aligned} & (F_2(s) - f(s))(f(t) - F_1(t)) - (F_2(t) - f(t))(f(t) - F_1(t)) \\ & = f^2(t) - f(t)f(s) \\ & -F_2(t)f(t) + F_2(s)f(t) - f(t)F_1(t) + f(s)F_1(t) - F_2(s)F_1(t) + F_2(t)F_1(t). \end{aligned} \tag{25}$$

Multiplying the inequality (25) by $\frac{\psi'^{\frac{\alpha}{k}-1}w(t)}{w(x)\,k\,\Gamma_k(\alpha)}$ and integrating with respect to t over (a,x), we get

$$\begin{aligned} & (F_2(s) - f(s))({}_{a^+}J^{\alpha,\psi}_{k,w}f(x) - {}_{a^+}J^{\alpha,\psi}_{k,w}F_1(x)) - {}_{a^+}J^{\alpha,\psi}_{k,w}\left[(F_2(x) - f(x))(f(x) - F_1(x))\right] \\ & = {}_{a^+}J^{\alpha,\psi}_{k,w}f^2(x) - f(s)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f(x) - {}_{a^+}J^{\alpha,\psi}_{k,w}\left[F_2(x)f(x)\right] + F_2(s)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f(x) \\ & - {}_{a^+}J^{\alpha,\psi}_{k,w}\left[F_1(x)f(x)\right] + f(s)\, {}_{a^+}J^{\alpha,\psi}_{k,w}F_1(x) - F_2(s)\, {}_{a^+}J^{\alpha,\psi}_{k,w}F_1(x) + {}_{a^+}J^{\alpha,\psi}_{k,w}\left[F_2(x)F_1(x)\right]. \end{aligned} \tag{26}$$

Now, Multiplying the above inequality (26) by $\frac{\psi'^{\frac{\beta}{k}-1}w(s)}{w(x)k\,\Gamma_k(\beta)}$ and integrating with respect to s over (a,x), we get

$$\begin{aligned} & ({}_{a^+}J^{\beta,\psi}_{k,w}F_2(x) - {}_{a^+}J^{\beta,\psi}_{k,w}f(x))({}_{a^+}J^{\alpha,\psi}_{k,w}f(x) - {}_{a^+}J^{\alpha,\psi}_{k,w}F_1(x)) \\ & - {}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}\left[(F_2(x) - f(x))(f(x) - F_1(x))\right] \\ & = {}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f^2(x) - {}_{a^+}J^{\beta,\psi}_{k,w}f(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f(x) \\ & - {}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}\left[F_2(x)f(x)\right] + {}_{a^+}J^{\beta,\psi}_{k,w}F_2(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}f(x) \\ & - {}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}\left[F_1(x)f(x)\right] + {}_{a^+}J^{\beta,\psi}_{k,w}f(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}F_1(x) \\ & - {}_{a^+}J^{\beta,\psi}_{k,w}F_2(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}F_1(x) + {}_{a^+}J^{\beta,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}\left[F_2(x)F_1(x)\right]. \end{aligned}$$

Thus, we get the acquired inequality (23).
Since

$$\left[(F_2(x) - f(x))(f(x) - F_1(x))\right] \geq 0,$$

then

$${}_{a^+}J^{\alpha,\psi}_{k,w}1(x)\, {}_{a^+}J^{\alpha,\psi}_{k,w}\left[(F_2(x) - f(x))(f(x) - F_1(x))\right] \geq 0.$$

This gives us the inequality (24).
□

If F_1 and F_2 are some constants, then from the above Lemma we have the following Corollary.

Corollary 3.5. *Let $f \in L_{X_w^p}[a,b]$ and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$ and also $x > a$, α, β, $k > 0$. Suppose that $m \leq f(x) \leq M$ for all $x \in [a,b]$, then the following inequalities hold:*

$$\begin{aligned} &{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f^2(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) \\ &= T_\alpha^\beta(f,m,M)(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}\left[(M-f(x))(f(x)-m)\right], \end{aligned} \tag{27}$$

and

$$ {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f^2(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x) \leq T_\alpha^\beta(f,m,M)(x). \tag{28}$$

If $\alpha = \beta$, then from inequalities (28) and (16) we get

$$ {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f^2(x) - \left({}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x)\right)^2 \leq \left(\frac{w(b)\,(\psi(x)-\psi(a))^{\frac{\alpha}{k}}\,(M-m)}{2\,w(a)\,\Gamma_k(\alpha+k)}\right)^2. \tag{29}$$

Theorem 3.6. *Let $f, g, F_1, F_2, G_1, G_2 \in L_{X_w^p}[a,b]$ and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$, $x > a$ and α, β, $k > 0$. Suppose that*

$$F_1(x) \leq f(x) \leq F_2(x) \text{ and } G_1(x) \leq g(x) \leq G_2(x), \quad \text{for all } x \in [a,b], \tag{30}$$

then the following inequality holds:

$$\begin{aligned} &\left|A_\beta^\alpha(f,g)(x) + A_\alpha^\beta(f,g)(x)\right| \leq \\ &\sqrt{\left(T_\alpha^\beta(f,F_1,F_2)(x) + T_\beta^\alpha(f,F_1,F_2)(x)\right)\left(T_\alpha^\beta(g,G_1,G_2)(x)+,T_\beta^\alpha(g,G_1,G_2)(x)\right)}, \end{aligned} \tag{31}$$

where

$$A_\beta^\alpha(f,g)(x) = {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}\left[f(x)g(x)\right] - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}g(x).$$

Proof. Let

$$K(t,s) = (f(t)-f(s))(g(t)-g(s)), \text{ for all } t,s \in [a,b], \tag{32}$$

then

$$K(t,s) = f(t)g(t) + f(s)g(s) - f(t)g(s) - f(s)g(t).$$

Multiplying the inequality (32) by $\frac{\psi'^{\frac{\alpha}{k}-1}w(t)}{w(x)\,k\,\Gamma_k(\alpha)}\frac{\psi'^{\frac{\beta}{k}-1}w(s)}{w(x)k\Gamma_k(\beta)}$ and integrating the resulting identity with respect to t and s over (a,x), we get

$$\begin{aligned} &\int_a^x\int_a^x \frac{\psi'^{\frac{\alpha}{k}-1}w(t)\psi'^{\frac{\beta}{k}-1}w(s)}{(w(x)\,k)^2\,\Gamma_k(\alpha)\,\Gamma_k(\beta)}K(t,s)\,dt\,ds \\ &= {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}\left[f(x)g(x)\right] + {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}1(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}\left[f(x)g(x)\right] \\ &- {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}g(x) - {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi}f(x)\,{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi}g(x) \\ &= A_\alpha^\beta(f,g)(x) + A_\beta^\alpha(f,g)(x). \end{aligned} \tag{33}$$

Since $K(t,s) = (f(t) - f(s))(g(t) - g(s))$, apply Hölder inequality (11), we get

$$\begin{aligned}
&\left(\int_a^x \int_a^x \frac{\psi'^{\frac{\alpha}{k}-1} w(t) \psi'^{\frac{\beta}{k}-1} w(s)}{(w(x)\,k)^2\, \Gamma_k(\alpha)\, \Gamma_k(\beta)} K(t,s)\, dt\, ds\right)^2 \\
&\leq \left(\int_a^x \int_a^x \frac{\psi'^{\frac{\alpha}{k}-1} w(t) \psi'^{\frac{\beta}{k}-1} w(s)}{(w(x)\,k)^2\, \Gamma_k(\alpha)\, \Gamma_k(\beta)} (f(t) - f(s))^2 dt\, ds\right) \\
&\times \left(\int_a^x \int_a^x \frac{\psi'^{\frac{\alpha}{k}-1} w(t) \psi'^{\frac{\beta}{k}-1} w(s)}{(w(x)\,k)^2\, \Gamma_k(\alpha)\, \Gamma_k(\beta)} (g(t) - g(s))^2 dt\, ds\right).
\end{aligned} \tag{34}$$

By the equality $(f(t) - f(s))^2 = f^2(t) + f^2(s) - 2f(t)f(s)$ and using the inequalities (24), (33) and (34), we deduce

$$\begin{aligned}
&\left(A_\alpha^\beta(f,g)(x) + A_\beta^\alpha(f,g)(x))\right)^2 \\
&\leq \left({}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f^2(x) + {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} f^2(x) - 2\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} f(x)\right) \\
&\times \left({}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g^2(x) + {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} g^2(x) - 2\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\beta,\psi} g(x)\right) \\
&\leq \left(T_\alpha^\beta(f, F_1, F_2)(x) + T_\beta^\alpha(f, F_1, F_2)(x)\right)\left(T_\alpha^\beta(g, G_1, G_2)(x) +, T_\beta^\alpha(g, G_1, G_2)(x)\right).
\end{aligned}$$

This gives us the desired inequality (31). □

Setting $\alpha = \beta$, we obtain the fundamental Corollaries given below.

Corollary 3.7. *Let $f, g, F_1, F_2, G_1, G_2 \in L_{X_w^p}[a,b]$ verified* (30) *and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$, $x > a$ and α, $k > 0$. Then the following inequality holds:*

$$\begin{aligned}
&\left|{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} [f(x)g(x)] - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x)\right| \\
&\leq \sqrt{T(f, F_1, F_2)(x)\, T(g, G_1, G_2)(x)}.
\end{aligned} \tag{35}$$

If F_1, F_2, G_1 and G_2 are constants, we have the following Corollary.

Corollary 3.8. *Let $f, g \in L_{X_w^p}[a,b]$ and ψ be an increasing and positive function on $[a,b]$, having a continuous derivative ψ' on $[a,b]$, $x > a$ and α, β, $k > 0$. Suppose that*

$$m \leq f(x) \leq M \text{ and } n \leq g(x) \leq N\,, \text{ for all } x \in [a,b],$$

then

$$\begin{aligned}
&\left|{}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} [f(x)g(x)] - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x)\right| \\
&\quad \leq \left(\frac{w(b)\,(\psi(x) - \psi(a))^{\frac{\alpha}{k}}}{2\, w(a)\, \Gamma_k(\alpha + k)}\right)^2 (M - m)\,(N - n).
\end{aligned} \tag{36}$$

4. Some special cases of Weight Grüss inequality

The Grüss inequality is presented using a particular weighted operator when the function ψ is chosen.

4.1. k-Hilfer operator.

If $w(x) = 1$, $k = 1$ and $a = 0$, then Corollary 3.7 and Corollary 3.8 reduce to to Theorem 2.23 and Corollary 2.24 in [4].

4.2. k-weighted Riemann-Liouville operator.

Let $\psi(x) = x$, then

$$\left| {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} [f(x)g(x)] - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x) \right| \leq \left(\frac{w(b)\,(x-a)^{\frac{\alpha}{k}}}{2\,w(a)\,\Gamma_k(\alpha+k)} \right)^2 (M-m)\,(N-n).$$

If $w = 1$ and $a = 0$, Corollary 3.8 reduce to Theorem 3.1 in [2], and Corollary 3.7 reduce to Theorem 9 in [3].

4.3. k-weighted Hadamard operator.

Let $\psi(x) = \ln x$ and $[a, b] \subseteq [1, +\infty[$, then

$$\left| {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} [f(x)g(x)] - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x) \right| \leq \left(\frac{w(b)\,(\ln(\frac{x}{a})^{\frac{\alpha}{k}}}{2\,w(a)\,\Gamma_k(\alpha+k)} \right)^2 (M-m)\,(N-n).$$

If $w = 1$, we get a new result to Grüss inequality involving k-Hadamard operator.

4.4. k-weighted Katugompola operator.

Let $\psi(x) = \frac{x^{r+1}}{r+1}$ where $r > 0$, then

$$\left| {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} [f(x)g(x)] - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x) \right| \leq \left(\frac{w(b)\,(x^{r+1} - a^{r+1})^{\frac{\alpha}{k}}}{2\,(r+1)^{\frac{\alpha}{k}}\, w(a)\,\Gamma_k(\alpha+k)} \right)^2 (M-m)\,(N-n).$$

If $w = 1$, Corollary 3.7 reduce to Theorem 2.10 in [10].
If $w = 1$ and $k = 1$, Corollary 3.7 and Corollary 3.8 reduce to Theorem 7 and Remark 1 in [5].

4.5. k-weighted fractional conformable operator.

Let $\psi(x) = \frac{(x-a)^s}{s}$ where $s > 0$, then

$$\left| {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} 1(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} [f(x)g(x)] - {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} f(x)\, {}_{a^+}\mathrm{J}_{k,w}^{\alpha,\psi} g(x) \right| \leq \left(\frac{w(b)\,((x-a)^s)^{\frac{\alpha}{k}}}{2\,w(a)\,s^{\frac{\alpha}{k}}\,\Gamma_k(\alpha+k)} \right)^2 (M-m)\,(N-n).$$

If $w = 1$ and $k = 1$, then Corollary 3.7 reduce to Theorem 2.10 [7] and Theorem 2.10 in [8].
If $w = 1$, $k = 1$ and $a = 0$, then Corollary 3.7 reduce to Theorem 2.4 in [9] and Corollary 3.8 reduce to Theorem 1 in [11].

5. Conclusion

This paper introduces a fresh perspective on Grüss-type inequalities by incorporating k-weighted fractional operators. Additionally, it explores a range of related weight inequalities that hinge upon specific operators reliant on the functions w and ψ. These newly introduced weight operators have the potential to extend the scope of certain existing works in future research endeavors.

References

[1] D. Grüss, Uber das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math.Z. 39 (1935), 215–226.

[2] Z. Dahmani, L. Tabharit, S. Taf, New generalisations of Grüss inequality using Riemann-Liouville fractional integrals, Bull. Math. Anal. Appl. vol. 2, no. 3,(2010), 93–99.

[3] J. Tariboon, S. K. Ntouyas, W. Sudsutad, Some new Riemann-Liouville fractional integral inequalities, Int. J. Math. Math. Sci., 2014 (2014) , Article ID 869434. https://doi.org/10.1155/2014/869434

[4] E. Kaçar, Z. Kacar, H. Yildirim, Integral inequalities for Riemann-Liouville fractional integrals of a function with respect to another function, Iran. J. Math. Sci. Inform. vol. 3, no. 3, (2018), 1–13. https://doi.org/10.7508/ijmsi.2018.1.001

[5] E. Kaçar, H. Yildirim, Grüss type integral inequalities for generalized Riemann-Liouville fractional integrals, Int. J. Pur. App. Mat. vol. 101, no. 1, (2015), 55–70. http://dx.doi.org/10.12732/ijpam.v101i1.6

[6] T. A. Aljaaidi, D. B. Pachpatte, Some Grüss-type inequalities using generalized Katugampola fractional integral, AIMS Mathematics, 2020, 5(2),1011–1024. http://dx.doi.org/10.3934/math.2020070

[7] S.Habib, G. Farid, S. Mubeen, Grüss type integral inequalities for a new class of *k*-fractional integrals, Int. J. Nonlinear Anal. Appl. 12 (2021) No. 1, 541–554. http://dx.doi.org/10.22075/ijnaa.2021.4836

[8] S.K. Yildirim, H.Yildirim, Grüss Type Integral Inequalities For Generalized η- Conformable Fractional Integrals, Turk. J. Math. Comput. Sci. 14(1),(2022), 201–211. http://dx.doi.org/10.47000/tjmcs.816174

[9] G. Rahman, S. Nisar, F. Qi, Some new inequalities of the Grüss type for conformable fractional integrals Inequalities for conformable fractional integrals. AIMS Mathematics, 2018, 3 (4), 575–583. http://dx.doi.org/10.3934/Math.2018.4.575

[10] S. Mubeen, S. Iqbal, Grüss type integral inequalities for generalized Riemann-Liouville *k*-fractional integrals. J. Inequal. Appl. 2016, 109 (2016). https://doi.org/10.1186/s13660-016-1052-x

[11] E. Set, I. Mumcu, M.E. Ozdemir, Grüss type inequalities involving new conformable fractional integral operators, AIP Conference Proceedings 1991, 020020 (2018); https://doi.org/10.1063/1.5047893

[12] F. Jarad, T. Abdeljawad , K. Shah, On the weighted fractional operators of a function with respect to another function, Fractals, vol. 28, no. 8 (2020) 2040011 (12 pages). https://doi.org/10.1142/S0218348X20400113

[13] F. Qi, S. Habib, S. Mubeen, M. N. Naeem, Generalized *k*-fractional conformable integrals and related inequalities, AIMS Mathematics, **4 (3)** (2019), 343-358.

[14] B. Benaissa, M. Z. Sarikaya, Diamond-alpha inequalities with two parameters on time scales, Bull. Transilv. Univ. Brasov Ser. III. Math. Comput. Sci. vol.3(65), no. 1, (2023), 41-56. https://doi.org/10.31926/but.mif.2023.3.65.1.4.