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On the A_α spectral radius of generalized weighted digraphs

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Abstract. Let $G = (V(G), E(G))$ be a generalized weighted digraph without loops and multiple arcs, where the weight of each arc is a nonnegative and symmetric matrix of same order p. For $v_i \in V(G)$, let $w_i^+ = \sum\limits_{v_j \in N_i^+} w_{ij}$, where w_{ij} is the weight of the arc (v_i, v_j), and N_i^+ is the set of out-neighbors of the vertex v_i. Let $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $0 \le \alpha \le 1$, $A(G)$ is the adjacency matrix of the generalized weighted digraph G, and $D(G) = diag(w_1^+, w_2^+, \ldots, w_n^+)$. The spectral radius of $A_\alpha(G)$ is called the A_α spectral radius of G. In this paper, we give some upper bounds on the A_α spectral radius of generalized weighted digraphs, and characterize the digraphs achieving the upper bounds. As application, we obtain some upper bounds on the A_α spectral radius of weighted digraphs and unweighted digraphs.

1. Introduction

Let $G = (V(G), E(G))$ be a digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. If $(v_i, v_j) \in E(G)$, then v_i is called the tail and v_j is called the head of the arc (v_i, v_j). A digraph G is strongly connected if for every order pair of vertices $v_i, v_j \in V(G)$, there exists a directed path from v_i to v_j and a directed path from v_j to v_i. For any vertex $v_i \in V(G)$, let $N_i^+ = N_{v_i}^+(G) = \{v_j : (v_i, v_j) \in E(G)\}$ denote the set of out-neighbors of the vertex v_i. Throughout this paper, we only consider the digraphs without loops and multiple arcs.

A generalized weighted digraph is a digraph in which each arc is assigned a square matrix, called the weight of the arc. All the weight matrices will be assumed to be nonnegative and symmetric matrices with the same order. A generalized weighted digraph can be view as weighted digraph if the weight of each arc is a positive number, and an unweighted digraph if each arc bearing weight 1.

Let G be a generalized weighted digraph with n vertices, denote by w_{ij} the weight of the arc (v_i, v_j), which is a nonnegative and symmetric matrix of order p. For $v_i \in V(G)$, let $w_i^+ = \sum\limits_{v_j \in N_i^+} w_{ij}$. Clearly, w_i^+ is a nonnegative and symmetric matrix, we use $\rho(w_i^+)$ to denote the spectral radius of w_i^+. A generalized weighted digraph G is a generalized weight-semiregular bipartite digraph if it is a strongly connected digraph whose vertex set can be partitioned into two disjoint nonempty subsets V_1 and V_2 such that $\rho(w_i^+)$

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is a constant for every vertex v_i in V_1, and $\rho(w_j^+)$ is a constant for every vertex v_j in V_2. If $\rho(w_k^+)$ is a constant for every vertex v_k in G, then G is called a generalized weight-regular digraph.

The adjacency matrix $A(G)$ of a generalized weighted digraph G is a block matrix, where the matrix block a_{ij} of order p is defined by

$$a_{ij} = \begin{cases} w_{ij} \text{ (or } w_{i,j}), & \text{if } (v_i, v_j) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The signless Laplacian matrix of a generalized weighted digraph G is $Q(G) = A(G) + D(G)$, where $D(G) = diag(w_1^+, w_2^+, \ldots, w_n^+)$. For any real number $\alpha \in [0, 1]$, the convex combinations $A_\alpha(G)$ of $D(G)$ and $A(G)$ defined by

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

Clearly,

$$A_0(G) = A(G), \quad A_1(G) = D(G), \quad \text{and} \quad 2A_{\frac{1}{2}}(G) = Q(G).$$

In general, $A_\alpha(G)$ is not symmetric and its eigenvalues can be complex numbers. The spectral radius of $A_\alpha(G)$ is called the A_α spectral radius of G, denoted by $\mu_\alpha(G)$. Since $A_\alpha(G)$ is a nonnegative matrix, it follows from Perron Frobenius Theorem that $\mu_\alpha(G)$ is an eigenvalue of $A_\alpha(G)$ and there is a nonnegative eigenvector corresponding to $\mu_\alpha(G)$. There has been some results on the A_α spectral radius of unweighted digraphs, see [3, 9, 16, 17].

Since graphs in the design of networks and electronic circuits are usually weighted, the spectra of weighted graphs are often used to solve problems. Fiedler [2] had introduced the following question: What is the optimal distribution of nonnegative weights (with total sum 1) among the edges of a given graph, so that the spectral radius of the resulting matrix is minimum? He himself proved that the optimum solution is achieved. It is natural for us to consider the Fiedler's problem on the largest spectral radius of matrices associated with weighted graphs. In fact, the spectral radius, Laplacian spectral radius and signless Laplacian spectral radius of weighted undirected graphs have been well treated in the literature [4, 5, 8, 10–14, 18]. However, there are a few of results for the generalized weighted digraphs. Recently, Ş.B. Bozkurt and D. Bozkurt [1] obtained some upper bounds for the spectral radius of generalized weighted digraphs and characterized the digraphs achieving the upper bounds. P. Li and Q.X. Huang [7] gave a sharp upper bound for the spectral radius of generalized weighted digraphs, and if the digraph is strongly connected, they also characterized the digraphs achieving the upper bound. W.G. Xi and L.G. Wang [15] obtained an upper bound for the signless Laplacian spectral radius of generalized weighted digraphs, and if the digraph is strongly connected, they also characterized the digraphs achieving the upper bound. In this paper, we will give some upper bounds on the A_α spectral radius of generalized weighted digraphs. As application, we also obtain some upper bounds on the A_α spectral radius of weighted digraphs and unweighted digraphs.

2. Lemmas and Results

Lemma 2.1. *([6]) Let B be an $n \times n$ real nonnegative symmetric matrix, $\rho(B)$ be the largest eigenvalue. Then for any $x \in R^n (x \neq 0)$, $y \in R^n (y \neq 0)$,*

$$|x^T B y| \leq \rho(B) \sqrt{x^T x} \sqrt{y^T y}. \tag{1}$$

The equality holds in (1) if and only if x is an eigenvector of B corresponding to $\rho(B)$ and $y = \alpha x$ for some $\alpha \in R$.

Lemma 2.2. *([6]) Let M be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then for any $x \in R^n (x \neq 0)$,*

$$x^T M x \geq \lambda_n x^T x. \tag{2}$$

The equality holds if and only if x is an eigenvector of M corresponding to the eigenvalue λ_n.

Lemma 2.3. *([7]) Let $G = (V(G), E(G))$ be a generalized weighted digraph and w_{ij} be a nonnegative symmetric matrix of order p of the arc $(v_i, v_j) \in E(G)$ and $w_i^+ = \sum_{v_j \in N_i^+} w_{ij}$. Also let x be an eigenvector of w_{ij} corresponding to the largest eigenvalue $\rho(w_{ij})$ for all i, j. Then x is also an eigenvector of w_i^+ corresponding to the largest eigenvalue $\rho(w_i^+)$ for all i, and $\rho(w_i^+) = \sum_{v_j \in N_i^+} \rho(w_{ij})$.*

In the following, we give some upper bounds for the A_α spectral radius of generalized weighted digraphs.

Theorem 2.4. *Let $G = (V(G), E(G))$ be a strongly connected generalized weighted digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. Then*

$$\mu_\alpha(G) \le \max_{(v_i,v_j)\in E(G)} \left\{ \frac{\alpha(\rho(w_i^+) + \rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+) - \rho(w_j^+))^2 + 4(1-\alpha)^2 \sum_{v_k \in N_i^+} \rho(w_{ik}) \sum_{v_k \in N_j^+} \rho(w_{jk})}}{2} \right\}, \tag{3}$$

where w_{ij} is the nonzero nonnegative symmetric matrix of order p of the arc $(v_i, v_j) \in E(G)$. Moreover, the equality holds in (3) *if and only if the following two conditions hold:*

(i) w_{ij} has a common eigenvector corresponding to the largest eigenvalue $\rho(w_{ij})$ for all v_i, v_j;

(ii) G is an weight-regular digraph or G is an weight-semiregular bipartite digraph.

Proof. Let $\mathbf{X} = (x_1^T, x_2^T, \cdots, x_n^T)^T$ be a nonnegative eigenvector of $A_\alpha(G)$ corresponding to the eigenvalue $\mu_\alpha(G) > 0$, where $x_i \ge 0$ is a column vector in R^p corresponding to the vertex v_i of G. Since $A_\alpha(G)\mathbf{X} = \mu_\alpha(G)\mathbf{X}$,

$$\mu_\alpha(G)x_i = \alpha w_i^+ x_i + (1-\alpha) \sum_{v_k \in N_i^+} w_{ik} x_k,$$

that is

$$(\mu_\alpha(G) I_p - \alpha w_i^+) x_i = (1-\alpha) \sum_{v_k \in N_i^+} w_{ik} x_k. \tag{4}$$

By multiplying x_i^T to (4), we get

$$\begin{aligned} x_i^T(\mu_\alpha(G) I_p - \alpha w_i^+) x_i &= (1-\alpha) \sum_{v_k \in N_i^+} x_i^T w_{ik} x_k \\ &\le (1-\alpha) \sum_{v_k \in N_i^+} |x_i^T w_{ik} x_k| \\ &\le (1-\alpha) \sum_{v_k \in N_i^+} \rho(w_{ik}) \sqrt{x_i^T x_i} \sqrt{x_k^T x_k} \ \text{(using (1))}. \end{aligned}$$

From (2), we have

$$(\mu_\alpha(G) - \alpha\rho(w_i^+)) x_i^T x_i \le x_i^T(\mu_\alpha(G) I_p - \alpha w_i^+) x_i.$$

Therefore

$$\begin{aligned} (\mu_\alpha(G) - \alpha\rho(w_i^+)) x_i^T x_i &\le x_i^T(\mu_\alpha(G) I_p - \alpha w_i^+) x_i \\ &= (1-\alpha) \sum_{v_k \in N_i^+} x_i^T w_{ik} x_k \\ &\le (1-\alpha) \sum_{v_k \in N_i^+} |x_i^T w_{ik} x_k| \\ &\le (1-\alpha) \sum_{v_k \in N_i^+} \rho(w_{ik}) \sqrt{x_i^T x_i} \sqrt{x_k^T x_k}. \end{aligned} \tag{5}$$

Let $(v_i, v_j) \in E(G)$. Similarly, we have

$$(\mu_\alpha(G) - \alpha\rho(w_j^+))x_j^T x_j \le x_j^T(\mu_\alpha(G)I_p - \alpha w_j^+)x_j \le (1-\alpha)\sum_{v_t \in N_j^+} \rho(w_{jt})\sqrt{x_j^T x_j}\sqrt{x_t^T x_t}. \tag{6}$$

Let $a = \max_{1\le k\le n}\{x_k^T x_k\} > 0$. We now choose x_{i_0}, the vector component of $\mathbf{X}$ such that $x_{i_0}^T x_{i_0} = a$ and there exists $v_{j_0} \in N_{i_0}^+$ satisfying $x_{j_0}^T x_{j_0} = \max_{v_k \in N_{i_0}^+}\{x_k^T x_k\} \ge \max_{v_t \in N_i^+}\{x_t^T x_t\}$ whenever $x_i^T x_i = a$. Clearly, $(v_{i_0}, v_{j_0}) \in E(G)$.

Taking $i = i_0$ in (5), we get

$$\begin{aligned}
(\mu_\alpha(G) - \alpha\rho(w_{i_0}^+))x_{i_0}^T x_{i_0} &\le x_{i_0}^T(\mu_\alpha(G)I_p - \alpha w_{i_0}^+)x_{i_0} &(7)\\
&= (1-\alpha)\sum_{v_k \in N_{i_0}^+} x_{i_0}^T w_{i_0k}x_k &(8)\\
&\le (1-\alpha)\sum_{v_k \in N_{i_0}^+} |x_{i_0}^T w_{i_0k}x_k| &(9)\\
&\le (1-\alpha)\sum_{v_k \in N_{i_0}^+} \rho(w_{i_0k})\sqrt{x_{i_0}^T x_{i_0}}\sqrt{x_k^T x_k} &(10)\\
&\le (1-\alpha)\sqrt{x_{i_0}^T x_{i_0}}\sqrt{x_{j_0}^T x_{j_0}}\sum_{v_k \in N_{i_0}^+} \rho(w_{i_0k}). &(11)
\end{aligned}$$

Taking $j = j_0$ in (6), we get

$$\begin{aligned}
(\mu_\alpha(G) - \alpha\rho(w_{j_0}^+))x_{j_0}^T x_{j_0} &\le x_{j_0}^T(\mu_\alpha(G)I_p - \alpha w_{j_0}^+)x_{j_0}\\
&= (1-\alpha)\sum_{v_k \in N_{j_0}^+} x_{j_0}^T w_{j_0k}x_k\\
&\le (1-\alpha)\sum_{v_k \in N_{j_0}^+} |x_{j_0}^T w_{j_0k}x_k| &(12)\\
&\le (1-\alpha)\sum_{v_k \in N_{j_0}^+} \rho(w_{j_0k})\sqrt{x_{j_0}^T x_{j_0}}\sqrt{x_k^T x_k} &(13)\\
&\le (1-\alpha)\sqrt{x_{i_0}^T x_{i_0}}\sqrt{x_{j_0}^T x_{j_0}}\sum_{v_k \in N_{j_0}^+} \rho(w_{j_0k}). &(14)
\end{aligned}$$

We now claim that $x_{j_0} \ne 0$. Indeed, by contradiction, we can suppose that $x_{j_0} = 0$. Then $x_k = 0$ for all k with $v_k \in N_{i_0}^+$. Furthermore, by the (8), $0 \le (\mu_\alpha(G) - \alpha\rho(w_{i_0}^+))x_{i_0}^T x_{i_0} \le 0$. That is $(\mu_\alpha(G) - \alpha\rho(w_{i_0}^+))x_{i_0}^T x_{i_0} = 0$, and so $(\mu_\alpha(G) - \alpha\rho(w_{i_0}^+)) = 0$ since $x_{i_0}^T x_{i_0} \ne 0$, which is impossible. Thus by multiplying the two sides of (11) and (14), we have

$$(\mu_\alpha(G) - \alpha\rho(w_{i_0}^+))(\mu_\alpha(G) - \alpha\rho(w_{j_0}^+)) \le (1-\alpha)^2\sum_{v_k \in N_{i_0}^+} \rho(w_{i_0k})\sum_{v_k \in N_{j_0}^+} \rho(w_{j_0k}).$$

Therefore,

$$\mu_\alpha(G) \le \frac{\alpha(\rho(w_{i_0}^+) + \rho(w_{j_0}^+)) + \sqrt{\alpha^2(\rho(w_{i_0}^+) - \rho(w_{j_0}^+))^2 + 4(1-\alpha)^2\sum_{v_k \in N_{i_0}^+} \rho(w_{i_0k})\sum_{v_k \in N_{j_0}^+} \rho(w_{j_0k})}}{2},$$

which leads to the result since $(v_{i_0}, v_{j_0}) \in E(G)$.

Now, we suppose that the equality holds in (3). Then all inequalities in the above argument must be equalities. Then by the equality (7), x_{i_0} is an eigenvector of matrix $w_{i_0}^+$ corresponding to the eigenvalue $\rho(w_{i_0}^+)$. By the equalities (10) and (11), for any $v_k \in N_{i_0}^+$, x_{i_0} is a common eigenvector of w_{i_0k} corresponding to the eigenvalue $\rho(w_{i_0k})$, $x_k = b_{i_0k}x_{i_0}$ and $x_k^T x_k = x_{j_0}^T x_{j_0}$. Hence, for $(v_{i_0}, v_z) \in E(G)$ and $(v_{i_0}, v_s) \in E(G)$, $b_{i_0z}^2 x_{i_0}^T x_{i_0} = x_z^T x_z = x_s^T x_s = b_{i_0s}^2 x_{i_0}^T x_{i_0}$, which implies $b_{i_0z}^2 = b_{i_0s}^2$. Moreover, for any $v_k \in N_{i_0}^+$, $x_{i_0}^T w_{i_0k} x_{i_0} > 0$ since $\rho(w_{i_0k}) > 0$. Furthermore, by the equality (9), $b_{i_0k} x_{i_0}^T w_{i_0k} x_{i_0} = x_{i_0}^T w_{i_0k} x_k = |x_{i_0}^T w_{i_0k} x_k| > 0$. Hence $b_{i_0k} > 0$ for any $v_k \in N_{i_0}^+$. So $b_{i_0z} = b_{i_0s} = b$. Thus, for any $v_k \in N_{i_0}^+$, $x_k = bx_{i_0}$, and x_{i_0} is a common eigenvector of w_{i_0k} corresponding to the eigenvalue $\rho(w_{i_0k})$.

For any $v_j \in N_{i_0}^+$, $x_j = bx_{i_0} = x_{j_0}$. By replacing j_0 with j in the equality (14), for any x_l with $v_l \in N_j^+$,

$$\begin{aligned}
(\mu_\alpha(G) - \alpha\rho(w_j^+))x_j^T x_j &= x_j^T(\mu_\alpha(G)I_p - \alpha w_j^+)x_j \\
&= (1-\alpha)\sum_{v_l \in N_j^+} |x_j^T w_{jl} x_l| \\
&= (1-\alpha)\sum_{v_l \in N_j^+} \rho(w_{jl})\sqrt{x_j^T x_j}\sqrt{x_l^T x_l} \\
&= (1-\alpha)\sqrt{x_{i_0}^T x_{i_0}}\sqrt{x_j^T x_j}\sum_{v_l \in N_j^+} \rho(w_{jl}).
\end{aligned}$$

Hence, we have x_j is an eigenvector of matrix w_j^+ corresponding to the eigenvalue $\rho(w_j^+)$, and for any $v_l \in N_j^+$, x_j is a common eigenvector of w_{jl} corresponding to the eigenvalue $\rho(w_{jl})$, $x_l = c_{jl}x_j$ and $x_l^T x_l = x_{i_0}^T x_{i_0}$. Then $(c_{jl}b)^2 x_{i_0}^T x_{i_0} = x_{i_0}^T x_{i_0}$, which implies $(c_{jl}b)^2 = 1$. Moreover, $c_{jl} x_j^T w_{jl} x_j = x_j^T w_{jl} x_l = |x_j^T w_{jl} x_l| > 0$. Noting that $x_j^T w_{jl} x_j > 0$ since $\rho(w_{jl}) > 0$, we have $c_{jl} > 0$ and thus $c_{jl} = \frac{1}{b}$. Therefore, for any $v_l \in N_j^+$, $x_l = \frac{1}{b}x_j = \frac{1}{b}bx_{i_0} = x_{i_0}$.

Now let $p = v_{i_0}v_{i_1}\ldots v_{i_r}$ be a directed path in G. We have the following claim.

Claim: If $0 \le t \le r$ is even, then $x_{i_t} = x_{i_0}$, if $1 \le t \le r$ is odd, then $x_{i_t} = bx_{i_0}$.

Firstly, we know that v_{i_0} corresponding to x_{i_0}. Then the claim holds for $t = 0, 1, 2$ by the above proof. Now let $t = 2$, we know that $x_{i_2} = x_{i_0}$. Let $x_h^T x_h = \max_{v_q \in N_{i_2}^+}\{x_q^T x_q\} \le x_{j_0}^T x_{j_0}$. Since $(v_{i_2}, v_h) \in E(G)$ and $x_{i_2} = x_{i_0}$. As similar as the inequality (11), we have

$$\begin{aligned}
(\mu_\alpha(G) - \alpha\rho(w_{i_2}^+))x_{i_2}^T x_{i_2} &\le x_{i_2}^T(\mu_\alpha(G)I_p - \alpha w_{i_2}^+)x_{i_2} \\
&= (1-\alpha)\sum_{v_q \in N_{i_2}^+} x_{i_2}^T w_{i_2q} x_q \\
&\le (1-\alpha)\sum_{v_q \in N_{i_2}^+} |x_{i_2}^T w_{i_2q} x_q| \\
&\le (1-\alpha)\sum_{v_q \in N_{i_2}^+} \rho(w_{i_2q})\sqrt{x_{i_2}^T x_{i_2}}\sqrt{x_q^T x_q} \quad \text{(using (1))} \\
&\le (1-\alpha)\sum_{v_q \in N_{i_2}^+} \rho(w_{i_2q})\sqrt{x_{i_2}^T x_{i_2}}\sqrt{x_h^T x_h} \\
&\le (1-\alpha)\sum_{v_q \in N_{i_2}^+} \rho(w_{i_2q})\sqrt{x_{i_2}^T x_{i_2}}\sqrt{x_{j_0}^T x_{j_0}}. \qquad (15)
\end{aligned}$$

Noting that $x_{i_1} = x_{j_0} = bx_{i_0}$, then we have

$$(\mu_\alpha(G) - \alpha\rho(w_{i_1}^+))x_{i_1}^T x_{i_1} \le (1-\alpha)\sum_{v_k \in N_{i_1}^+} \rho(w_{i_1k})\sqrt{x_{i_1}^T x_{i_1}}\sqrt{x_{i_0}^T x_{i_0}}.$$

Therefore,

$$\begin{aligned}
\mu_\alpha(G) &\le \frac{\alpha(\rho(w_{i_1}^+)+\rho(w_{i_2}^+)) + \sqrt{\alpha^2(\rho(w_{i_1}^+)-\rho(w_{i_2}^+))^2 + 4(1-\alpha)^2 \sum\limits_{v_k\in N_{i_1}^+}\rho(w_{i_1k}) \sum\limits_{v_k\in N_{i_2}^+}\rho(w_{i_2k})}}{2} \\
&\le \max_{(v_i,v_j)\in E(G)} \left\{ \frac{\alpha(\rho(w_i^+)+\rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+)-\rho(w_j^+))^2 + 4(1-\alpha)^2 \sum\limits_{v_k\in N_i^+}\rho(w_{ik}) \sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2} \right\} \\
&= \mu_\alpha(G),
\end{aligned}$$

which means

$$\mu_\alpha(G) = \frac{\alpha(\rho(w_{i_1}^+)+\rho(w_{i_2}^+)) + \sqrt{\alpha^2(\rho(w_{i_1}^+)-\rho(w_{i_2}^+))^2 + 4(1-\alpha)^2 \sum\limits_{v_k\in N_{i_1}^+}\rho(w_{i_1k}) \sum\limits_{v_k\in N_{i_2}^+}\rho(w_{i_2k})}}{2}.$$

Furthermore, we have (15) must be equality. Thus, we get $x_{i_2} = x_{i_0}$ is an eigenvector of matrix $w_{i_2}^+$ corresponding to the eigenvalue $\rho(w_{i_2}^+)$, and for any $v_q \in N_{i_2}^+$, $x_q^T x_q = x_{j_0}^T x_{j_0} = x_{i_1}^T x_{i_1}$. Again the equality (15) means $x_{i_2} = x_{i_0}$ is a common eigenvector of w_{i_2q} corresponding to the eigenvalue $\rho(w_{i_2q})$, $x_q = b_q x_{i_2} = b_q x_{i_0}$ where $b_q > 0$ similar as the proof of $b_{i_0k} > 0$. Thus $b_q^2 x_{i_0}^T x_{i_0} = x_{j_0}^T x_{j_0} = b^2 x_{i_0}^T x_{i_0}$, and so $b_q = b$. Hence, $x_{i_3} = bx_{i_0} = x_{j_0}$. Noting that $x_{i_2} = x_{i_0}$ and $x_{i_3} = x_{j_0}$. Regarding v_{i_2} as v_{i_0} and repeating the above process, we will get the claim by induction since G is strongly connected. At last, x_{i_0} is a common eigenvector of w_{ij} corresponding to the eigenvalue $\rho(w_{ij})$ for every arc (v_i, v_j) of G.

Now let $V_1 = \{v_i \mid x_i = x_{i_0}\}$ and $V_2 = \{v_i \mid x_i = bx_{i_0}\}$. Since G is strongly connected digraph, then $V(G) = V_1 \cup V_2$ and the subdigraphs induced by V_1 and V_2 respectively are empty digraphs if $b \neq 1$, and $V(G) = V_1$ if $b = 1$. In the following, we consider the following two cases.

Case 1: $b = 1$.

In this case, for any $1 \le k \le n$, $x_k = x_{i_0}$, and x_{i_0} is a common eigenvector of w_{ij} corresponding to the eigenvalue $\rho(w_{ij})$ for every arc (v_i, v_j) of G. By Lemma 2.3, x_{i_0} is also an eigenvector of w_k^+ corresponding to the largest eigenvalue $\rho(w_k^+)$ for all k, and $\rho(w_k^+) = \sum\limits_{v_j\in N_k^+}\rho(w_{kj})$. Then, for any $v_k \in V(G)$,

$$\mu_\alpha(G)x_{i_0} = \alpha w_k^+ x_{i_0} + (1-\alpha)\sum_{v_i\in N_k^+} w_{ki}x_{i_0} = \alpha\rho(w_k^+)x_{i_0} + (1-\alpha)\rho(w_k^+)x_{i_0} = \rho(w_k^+)x_{i_0},$$

which implies that $\rho(w_k^+) = \mu_\alpha(G)$. Therefore, G is a generalized weight-regular digraph.

Case 2: $b \neq 1$.

Therefore, G is a bipartite digraph. For any $v_s \in V_1$, $x_s = x_{i_0}$, and for any $v_t \in V_2$, $x_t = bx_{i_0}$. And x_{i_0} is a common eigenvector of w_{ij} corresponding to the eigenvalue $\rho(w_{ij})$ for every arc (v_i, v_j) of G. By Lemma 2.3, x_{i_0} is also an eigenvector of w_k^+ corresponding to the largest eigenvalue $\rho(w_k^+)$ for all k, and $\rho(w_k^+) = \sum\limits_{v_j\in N_k^+}\rho(w_{kj})$.

If $v_s \in V_1$,

$$\mu_\alpha(G)x_{i_0} = \alpha w_s^+ x_{i_0} + (1-\alpha)\sum_{v_k\in N_s^+} w_{sk}bx_{i_0} = \alpha\rho(w_s^+)x_{i_0} + (1-\alpha)\rho(w_s^+)bx_{i_0},$$

which implies that $\mu_\alpha(G) = \alpha\rho(w_s^+) + (1-\alpha)\rho(w_s^+)b$, that is $\rho(w_s^+) = \frac{\mu_\alpha(G)}{\alpha+(1-\alpha)b}$.

If $v_t \in V_2$,

$$\mu_\alpha(G)bx_{i_0} = \alpha w_t^+ bx_{i_0} + (1-\alpha)\sum_{v_k\in N_t^+} w_{tk}x_{i_0} = \alpha\rho(w_t^+)bx_{i_0} + (1-\alpha)\rho(w_t^+)x_{i_0},$$

which implies that $\mu_\alpha(G)b = \alpha\rho(w_t^+)b + (1-\alpha)\rho(w_t^+)$, that is $\rho(w_t^+) = \frac{\mu_\alpha(G)b}{\alpha b+(1-\alpha)}$. Therefore, G is a generalized weight-semiregular bipartite digraph.

For converse, suppose that the conditions $(i)-(ii)$ shown in the second part of the theorem hold for the digraph G. Then we must prove that

$$\mu_\alpha(G) = \max_{(v_i,v_j)\in E(G)} \left\{ \frac{\alpha(\rho(w_i^+)+\rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+)-\rho(w_j^+))^2 + 4(1-\alpha)^2 \sum\limits_{v_k\in N_i^+}\rho(w_{ik}) \sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2} \right\}.$$

Let x be a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\rho(w_{ij})$ for all v_i, v_j. By Lemma 2.3, x is also an eigenvector of w_i^+ corresponding to the largest eigenvalue $\rho(w_i^+)$ for all i, and $\rho(w_i^+) = \sum\limits_{v_j\in N_i^+}\rho(w_{ij})$. Firstly, we suppose that G is an weight regular digraph. Let $\rho\left(w_i^+\right) = \gamma$ for all $v_i \in V(G)$. Let $\mathbf{X} = \{x^T, x^T, \ldots, x^T\}^T$. Then the following equation can be easily seen $A_\alpha(G)X = \gamma X$. Therefore γ is an eigenvalue of $A_\alpha(G)$. So $\gamma \le \mu_\alpha(G)$. On the other hand,

$$\max_{(v_i,v_j)\in E(G)} \left\{ \frac{\alpha(\rho(w_i^+)+\rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+)-\rho(w_j^+))^2 + 4(1-\alpha)^2 \sum\limits_{v_k\in N_i^+}\rho(w_{ik}) \sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2} \right\} = \gamma.$$

Thus

$$\mu_\alpha(G) \le \max_{(v_i,v_j)\in E(G)} \left\{ \frac{\alpha(\rho(w_i^+)+\rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+)-\rho(w_j^+))^2 + 4(1-\alpha)^2 \sum\limits_{v_k\in N_i^+}\rho(w_{ik}) \sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2} \right\} = \gamma.$$

Hence,

$$\mu_\alpha(G) = \gamma = \max_{(v_i,v_j)\in E(G)} \left\{ \frac{\alpha(\rho(w_i^+)+\rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+)-\rho(w_j^+))^2 + 4(1-\alpha)^2 \sum\limits_{v_k\in N_i^+}\rho(w_{ik}) \sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2} \right\}.$$

In the following, suppose G is an weight-semiregular bipartite digraph. Let U, W be the partite sets of G. Let $\rho(w_i^+) = \gamma$ for any $v_i \in U$, and $\rho(w_j^+) = \beta$ for any $v_j \in W$. Without loss of generality, let $U = \{v_1, v_2, \ldots, v_k\}$ and $W = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$. Therefore,

$$A_\alpha(G) = \left(\begin{array}{ccc|ccc} \alpha w_1^+ & \cdots & 0 & (1-\alpha)w_{1,k+1} & \cdots & (1-\alpha)w_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \alpha w_k^+ & (1-\alpha)w_{k,k+1} & \cdots & (1-\alpha)w_{k,n} \\ \hline (1-\alpha)w_{k+1,1} & \cdots & (1-\alpha)w_{k+1,k} & \alpha w_{k+1}^+ & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ (1-\alpha)w_{n,1} & \cdots & (1-\alpha)w_{n,k} & 0 & \cdots & \alpha w_n^+ \end{array}\right).$$

Let $\mathbf{X} = (\underbrace{x^T, x^T, \ldots, x^T}_{k}, \underbrace{\theta x^T, \theta x^T, \ldots, \theta x^T}_{n-k})^T$, where

$$\theta = \frac{\alpha(\beta-\gamma) + \sqrt{\alpha^2(\gamma-\beta)^2 + 4(1-\alpha)^2\gamma\beta}}{2(1-\alpha)\gamma}.$$

Let

$$T=\frac{\alpha(\gamma+\beta)+\sqrt{\alpha^2(\gamma-\beta)^2+4(1-\alpha)^2\gamma\beta}}{2}.$$

Since

$$A_\alpha(G)\mathbf{X}=\begin{pmatrix}\alpha w_1^+x+(1-\alpha)\sum\limits_{v_t\in N_1^+}w_{1,t}\theta x\\ \vdots\\ \alpha w_k^+x+(1-\alpha)\sum\limits_{v_t\in N_k^+}w_{k,t}\theta x\\ \alpha w_{k+1}^+\theta x+(1-\alpha)\sum\limits_{v_t\in N_{k+1}^+}w_{k+1,t}x\\ \vdots\\ \alpha w_n^+\theta x+(1-\alpha)\sum\limits_{v_t\in N_n^+}w_{n,t}x\end{pmatrix}$$
$$=\begin{pmatrix}\alpha\rho(w_1^+)x+(1-\alpha)\sum\limits_{v_t\in N_1^+}\rho(w_{1,t})\theta x\\ \vdots\\ \alpha\rho(w_k^+)x+(1-\alpha)\sum\limits_{v_t\in N_k^+}\rho(w_{k,t})\theta x\\ \alpha\rho(w_{k+1}^+)\theta x+(1-\alpha)\sum\limits_{v_t\in N_{k+1}^+}\rho(w_{k+1,t})x\\ \vdots\\ \alpha\rho(w_n^+)\theta x+(1-\alpha)\sum\limits_{v_t\in N_n^+}\rho(w_{n,t})x\end{pmatrix}$$

$$=\begin{pmatrix}\alpha\rho(w_1^+)x+(1-\alpha)\rho(w_1^+)\theta x\\ \vdots\\ \alpha\rho(w_k^+)x+(1-\alpha)\rho(w_k^+)\theta x\\ \alpha\rho(w_{k+1}^+)\theta x+(1-\alpha)\rho(w_{k+1}^+)x\\ \vdots\\ \alpha\rho(w_n^+)\theta x+(1-\alpha)\rho(w_n^+)x\end{pmatrix}$$
$$=\begin{pmatrix}\alpha\gamma x+(1-\alpha)\gamma\theta x\\ \vdots\\ \alpha\gamma x+(1-\alpha)\gamma\theta x\\ \alpha\beta\theta x+(1-\alpha)\beta x\\ \vdots\\ \alpha\beta\theta x+(1-\alpha)\beta x\end{pmatrix},$$

$$
\begin{aligned}
&\alpha\gamma + (1-\alpha)\gamma\theta \\
=&\alpha\gamma + (1-\alpha)\gamma\frac{\alpha(\beta-\gamma) + \sqrt{\alpha^2(\gamma-\beta)^2 + 4(1-\alpha)^2\gamma\beta}}{2(1-\alpha)\gamma} \\
=&\alpha\gamma + \frac{\alpha(\beta-\gamma) + \sqrt{\alpha^2(\gamma-\beta)^2 + 4\,(1-\alpha^2)\,\gamma\beta}}{2} \\
=&\frac{\alpha(\gamma+\beta) + \sqrt{\alpha^2(\gamma-\beta)^2 + 4(1-\alpha)^2\gamma\beta}}{2} = T,
\end{aligned}
$$

and

$$
\begin{aligned}
&T\theta - \alpha\beta\theta = (T-\alpha\beta)\theta \\
=&\left[\frac{\alpha(\gamma-\beta) + \sqrt{\alpha^2(\gamma-\beta)^2 + 4(1-\alpha)^2\gamma\beta}}{2}\right]\left[\frac{\alpha(\beta-\gamma) + \sqrt{\alpha^2(\gamma-\beta)^2 + 4(1-\alpha)^2\gamma\beta}}{2(1-\alpha)\gamma}\right] \\
=&\frac{\alpha^2(\gamma-\beta)^2 + 4(1-\alpha)^2\gamma\beta - \alpha^2(\gamma-\beta)^2}{4(1-\alpha)\gamma} \\
=&(1-\alpha)\beta\,,
\end{aligned}
$$

then

$$(1-\alpha)\beta + \alpha\beta\theta = T\theta.$$

Therefore, $A_\alpha(G)\mathbf{X} = T\mathbf{X}$. Hence, T is an eigenvalue of $A_\alpha(G)$. Thus $T \le \mu_\alpha(G)$. On the other hand, because

$$\max_{(v_i,v_j)\in E(G)} = \left\{\frac{\alpha(\rho(w_i^+) + \rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+) - \rho(w_j^+))^2 + 4(1-\alpha)^2\sum\limits_{v_k\in N_i^+}\rho(w_{ik})\sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2}\right\} = T,$$

hence

$$\mu_\alpha(G) \le \max_{(v_i,v_j)\in E(G)} \left\{\frac{\alpha(\rho(w_i^+) + \rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+) - \rho(w_j^+))^2 + 4(1-\alpha)^2\sum\limits_{v_k\in N_i^+}\rho(w_{ik})\sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2}\right\} = T.$$

Thus

$$\mu_\alpha(G) = T = \max_{(v_i,v_j)\in E(G)} \left\{\frac{\alpha(\rho(w_i^+) + \rho(w_j^+)) + \sqrt{\alpha^2(\rho(w_i^+) - \rho(w_j^+))^2 + 4(1-\alpha)^2\sum\limits_{v_k\in N_i^+}\rho(w_{ik})\sum\limits_{v_k\in N_j^+}\rho(w_{jk})}}{2}\right\}.$$

This completes the proof. □

From Theorem 2.4, we have the following two corollaries.

Corollary 2.5. *Let $G = (V(G), E(G))$ be a strongly connected weighted digraph, where the weight w_{ij} of each arc $(v_i, v_j) \in E(G)$ is a positive number. Then*

$$\mu_\alpha(G) \le \max_{(v_i,v_j)\in E(G)} \left\{\frac{\alpha(w_i^+ + w_j^+) + \sqrt{\alpha^2(w_i^+ + w_j^+)^2 + 4(1-2\alpha)w_i^+w_j^+}}{2}\right\},$$

where $w_i^+ = \sum\limits_{v_k\in N_i^+} w_{ik}$.

Proof. For weighted digraph where the weight w_{ij} of each arc is a positive number, we have $\rho(w_i^+) = w_i^+$, $\rho(w_{ij}) = w_{ij}$ for all v_i, v_j. Then by Theorem 2.4, we get the required result. □

Corollary 2.6. *Let $G = (V(G), E(G))$ be a strongly connected unweighted digraph. Then*

$$\mu_\alpha(G) \leq \max_{(v_i,v_j)\in E(G)} \left\{ \frac{\alpha(d_i^+ + d_j^+) + \sqrt{\alpha^2(d_i^+ + d_j^+)^2 + 4(1-2\alpha)d_i^+ d_j^+}}{2} \right\},$$

where d_i^+ is the outdegree of vertex v_i in the unweighted digraph G.

Proof. For an unweighted digraph, we have $w_{ij} = 1$ for $(v_i, v_j) \in E(G)$, $w_i^+ = d_i^+$. By Corollary 2.5, the result follows. □

Theorem 2.7. *Let $G = (V(G), E(G))$ be a strongly connected generalized weighted digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. Then*

$$\mu_\alpha(G) \leq \max_{(v_i,v_j)\in E(G)} \left\{ \alpha\rho(\sum_{v_k \in N_i^+} w_{ik}) + (1-\alpha) \sum_{v_k \in N_j^+} \rho(w_{jk}) \right\}, \tag{16}$$

where w_{ij} is the nonzero nonnegative symmetric matrix of order p of the arc $(v_i, v_j) \in E(G)$. Moreover, if G is an weight-regular digraph and w_{ij} has a common eigenvector corresponding to the largest eigenvalue $\rho(w_{ij})$ for all v_i, v_j, then the equality holds.

Proof. Let M be a block diagonal matrix $diag(\beta_1 I_p, \beta_2 I_p, \ldots, \beta_n I_p)$, where I_p is the $p \times p$ identity matrix, $\beta_i = \sum_{v_k \in N_i^+} \rho(w_{ik})$. Let $\mathbf{X} = (x_1^T, x_2^T, \cdots, x_n^T)^T$ be a nonnegative eigenvector of $M^{-1}A_\alpha(G)M$ corresponding to the eigenvalue $\mu_\alpha(G) > 0$, where $x_i \geq 0$ is a column vector in R^p corresponding to the vertex v_i of G. Let x_s is the vector component of $\mathbf{X}$ such that $x_s^T x_s = \max_{1\leq k\leq n}\{x_k^T x_k\} > 0$. Since $\mathbf{X}$ is nonzero, so is x_s. The (i, j)-th block of $M^{-1}A_\alpha(G)M$ is

$$\begin{cases} \alpha w_i^+, & \text{if } i = j, \\ (1-\alpha)\frac{\beta_j}{\beta_i} w_{ij}, & \text{if } (v_i, v_j) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $M^{-1}A_\alpha(G)M\mathbf{X} = \mu_\alpha(G)\mathbf{X}$. Furthermore, we get

$$\mu_\alpha(G)x_s = \alpha w_s^+ x_s + (1-\alpha) \sum_{v_k \in N_s^+} w_{sk} \frac{\beta_k}{\beta_s} x_k,$$

i.e.,

$$(\mu_\alpha(G)I_p - \alpha w_s^+)x_s = (1-\alpha) \sum_{v_k \in N_s^+} w_{sk} \frac{\beta_k}{\beta_s} x_k,$$

i.e.,

$$\begin{aligned}
x_s^T(\mu_\alpha(G)I_p - \alpha w_s^+)x_s &= (1-\alpha)\sum_{v_k\in N_s^+}\frac{\beta_k}{\beta_s}x_s^T w_{sk}x_k \\
&\le (1-\alpha)\sum_{v_k\in N_s^+}|\frac{\beta_k}{\beta_s}x_s^T w_{sk}x_k| \\
&\le (1-\alpha)\sum_{v_k\in N_s^+}\frac{\beta_k}{\beta_s}\rho(w_{sk})\sqrt{x_s^T x_s}\sqrt{x_k^T x_k} \text{ (using (1))} \\
&\le (1-\alpha)x_s^T x_s \sum_{v_k\in N_s^+}\frac{\beta_k}{\beta_s}\rho(w_{sk}). \qquad (17)
\end{aligned}$$

From the inequality (17) and (2), we have

$$\mu_\alpha(G) - \alpha\rho(w_s^+) \le \frac{x_s^T(\mu_\alpha(G)I_p - \alpha w_s^+)x_s}{x_s^T x_s} \le (1-\alpha)\sum_{v_k\in N_s^+}\frac{\beta_k}{\beta_s}\rho(w_{sk}). \text{ (as } x_s^T x_s > 0)$$

Thus

$$\begin{aligned}
\mu_\alpha(G) &\le \alpha\rho(w_s^+) + (1-\alpha)\sum_{v_k\in N_s^+}\frac{\beta_k}{\beta_s}\rho(w_{sk}) \\
&\le \alpha\rho(w_s^+) + (1-\alpha)\frac{\max\limits_{v_k\in N_s^+}\{\beta_k\}}{\beta_s}\sum_{v_k\in N_s^+}\rho(w_{sk}) \\
&= \alpha\rho(w_s^+) + (1-\alpha)\max_{v_k\in N_s^+}\{\beta_k\} \\
&= \alpha\rho(w_s^+) + (1-\alpha)\max_{v_k\in N_s^+}\{\sum_{v_t\in N_k^+}\rho(w_{kt})\} \\
&\le \max_{(v_i,v_j)\in E(G)}\{\alpha\rho(\sum_{v_k\in N_i^+} w_{ik}) + (1-\alpha)\sum_{v_k\in N_j^+}\rho(w_{jk})\}.
\end{aligned}$$

Then we complete the first part of the theorem.

Next, suppose that G is an weight-regular digraph and w_{ij} has a common eigenvector corresponding to the largest eigenvalue $\rho(w_{ij})$ for all v_i, v_j. Let $\rho(w_i^+) = \gamma$ for any $v_i \in V(G)$, and x be a common eigenvector of w_{ij} corresponding to the largest eigenvalue $\rho(w_{ij})$ for all v_i, v_j. Then by Lemma 2.3, x is also an eigenvector of w_i^+ corresponding to the largest eigenvalue $\rho(w_i^+)$ for all i, and $\rho(w_i^+) = \sum_{v_j\in N_i^+}\rho(w_{ij})$. Hence, we can easily verified $A_\alpha(G)\mathbf{X} = \gamma\mathbf{X}$, where $\mathbf{X} = \{x^T, x^T, \ldots, x^T\}^T$. So $\gamma \le \mu_\alpha(G)$. On the other hand, $\alpha\rho(\sum_{v_k\in N_i^+} w_{ik}) + (1-\alpha)\sum_{v_k\in N_j^+}\rho(w_{jk}) = \gamma$. Since

$$\mu_\alpha(G) \le \max_{(v_i,v_j)\in E(G)}\{\alpha\rho(\sum_{v_k\in N_i^+} w_{ik}) + (1-\alpha)\sum_{v_k\in N_j^+}\rho(w_{jk})\} = \gamma.$$

Then

$$\mu_\alpha(G) = \gamma = \max_{(v_i,v_j)\in E(G)}\{\alpha\rho(\sum_{v_k\in N_i^+} w_{ik}) + (1-\alpha)\sum_{v_k\in N_j^+}\rho(w_{jk})\}.$$

Hence the theorem is proved. □

From Theorem 2.7, we have the following two corollaries.

Corollary 2.8. *Let $G = (V(G), E(G))$ be a strongly connected weighted digraph, where the weight w_{ij} of each arc $(v_i, v_j) \in E(G)$ is a positive number. Then*

$$\mu_\alpha(G) \leq \max_{(v_i,v_j)\in E(G)} \left\{\alpha w_i^+ + (1-\alpha)w_j^+\right\},$$

where $w_i^+ = \sum\limits_{v_k\in N_i^+} w_{ik}$.

Corollary 2.9. *Let $G = (V(G), E(G))$ be a strongly connected unweighted digraph. Then*

$$\mu_\alpha(G) \leq \max_{(v_i,v_j)\in E(G)} \left\{\alpha d_i^+ + (1-\alpha)d_j^+\right\},$$

where d_i^+ is the outdegree of vertex v_i in the digraph G.

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