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Approximation of fixed points for enriched nonexpansive mappings in Banach spaces

Kifayat Ullah[a], Junaid Ahmad[b], Muhammad Ijaz Ullah Khan[a], Muhammad Arshad[b]

[a]*Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan*
[b]*Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad - 44000, Pakistan*

Abstract. The three–step iteration scheme due to Thakur et al. [*J. Inequal. Appl.* **2014**, 328.] is analyzed with the new setting of mappings called enriched nonexpansive mappings. We establish weak convergence using the well-known Opial's condition and also prove strong convergence under various assumptions on the domain or on the mapping. Finally using an example of enriched nonexpansive mappings that is not nonexpansive, we show that the rate of convergence of the three-step Thakur iteration scheme is still more effective than the some other iterative schemes of the literature. The presented outcome is essentially novel and extend the corresponding announced results of the literature.

1. Introduction

As many know, if T is a given selfmap of a subset, namely, E of a Banach space B, then T is called nonexpansive if

$$\|Te - Te'\| \leq \|e - e'\| \text{ for every } e, e' \in E.$$

If there is a point $e_0 \in E$ such that $Te_0 = e_0$, then e_0 is called a fixed point for T and throughout the research, we shall write $F_T := \{e_0 \in E : Te_0 = e_0\}$. Browder [19] and Gohde [21] independently however differently proved the existence of a fixed point for nonexpansive operators on a certain subset of uniformly convex Banach space [20] (UCBS, for short). Fixed point theory of nonexpansive mappings is very important and its applications can be found in many areas of applied sciences. Thus it is always desirable to suggest some extension of these mappings in order to extend their area of applications. Thus, several authors suggested some extension of nonexpansive mappings along with related fixed point results (see e.g., [2–11] and others). Among the other things, Berinde [12] introduced the following generalization of nonexpansive mappings.

Definition 1.1. *Consider a subset E of a Banach space and $T : E \to E$. The map T is called enriched nonexpansive if and only if there is a constant $b \in [0, \infty)$ such that*

$$\|b(e - e') + Te - Te'\| \leq (b + 1)\|e - e'\| \textit{ for every } e, e' \in E.$$

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Email addresses: kifayatmath@yahoo.com (Kifayat Ullah), ahmadjunaid436@gmail.com (Junaid Ahmad), ijaz.03369090298@gmail.com (Muhammad Ijaz Ullah Khan), marshadzia@iiu.edu.pk (Muhammad Arshad)

Remark 1.2. *The class of all enriched nonexpansive mappings properly includes the class of all nonexpansive mappings. Because, if a mapping T is nonexpansive then it is 0–enriched nonexpansive. However, an example in the end of this paper shows that there are some examples of enriched nonexpansive mappings which are not nonexpansive.*

In [12], Berinde investigated some existence and approximation results for the class of enriched nonexpansive mappings using the Krasnoselskii iterative scheme [13] in the Hilbert space framework. Soon, Berinde [14] improved the results in [12] to the setting of Banach space framework. Once an existence of a fixed point for a given mapping is known then to compute its fixed point under a suitable faster iterative scheme is always requested. In the view of the proof of the remarkable Banach Contraction Principle (BCP) [18], the Picard iterative [1] converges in the strong sense to a contraction mapping of complete metric space. But it is known that for a nonexpansive mapping, the Picard iteration may not converge to its fixed point. Thus, to study fixed point construction for nonexpansive mappings and to obtain a better speed of convergence, we need some other iterative schemes, that is, Mann iteration [23], Ishikawa [22], three-step Noor [24], Agarwal iteration (S-iteration) [17], three-step Abbas iteration [15] etc.

In 2013, Thakur et al. [29] suggested a new three-step iteration for finding fixed points of nonexpansive mapping in the setting of Banach spaces. They proved numerically that for nonexpansive mappings, this new iteration converges faster to a fixed point compared to some other three-step iterations. We now extend their results to the setting of enriched nonexpansive operators as: Consider a self enriched nonexpansive mapping T on a closed subset E of a Banch space with the constant $b \in [0, \infty)$. In this case, we can define the averaged mapping T_λ on E by the formula $T_\lambda e = (1-\lambda)e + \lambda Te$ where $\lambda = \frac{1}{b+1}$. It is easy to see that fixed point set of T_λ and the fixed point set of T are same. Now we define modified three-step Noor [24], three-step Abbas [15] and Thakur et al. [29] iteration as follows:

$$\begin{cases} e_1 \in E, \\ c_k = (1-\gamma_k)e_k + \gamma_k T_\lambda e_k, \\ d_k = (1-\beta_k)e_k + \beta_k T_\lambda c_k, \\ e_{k+1} = (1-\alpha_k)e_k + \alpha_k T_\lambda d_k, k \in \mathbb{N}, \end{cases} \tag{1}$$

where $\alpha_k, \beta_k, \gamma_k \in (0,1)$ and $\lambda = \frac{1}{b+1}$.

$$\begin{cases} e_1 \in E, \\ c_k = (1-\gamma_k)e_k + \gamma_k T_\lambda e_k, \\ d_k = (1-\beta_k)T_\lambda e_k + \beta_k T_\lambda c_k, \\ e_{k+1} = (1-\alpha_k)T_\lambda d_k + \alpha_k T_\lambda c_k, k \in \mathbb{N}, \end{cases} \tag{2}$$

where $\alpha_k, \beta_k, \gamma_k \in (0,1)$ and $\lambda = \frac{1}{b+1}$.

$$\begin{cases} e_1 \in E, \\ c_k = (1-\gamma_k)e_k + \gamma_k T_\lambda e_k, \\ d_k = (1-\beta_k)c_k + \beta_k T_\lambda c_k, \\ e_{k+1} = (1-\alpha_k)Te_k + \alpha_k T_\lambda d_k, k \in \mathbb{N}, \end{cases} \tag{3}$$

where $\alpha_k, \beta_k, \gamma_k \in (0,1)$ and $\lambda = \frac{1}{b+1}$.

Although, Thakur et al. [29] proved and studied some weak and strong convergence theorem under various assumptions on the domain E or on the mapping T and observed that this scheme essentially gives high accuracy when it is compared with three-step [24] and three-step Abbas [15] iterations. The main aim of this paper is construct some fixed point convergence results for enriched nonexpansive mappings using the iterative scheme (3). We support numerically our main results by considering an example in which the mapping is enriched nonexpansive but fails to form a nonexpansive mappings. We provide the numerical results in the form a table and graph that confirms the convergence numerically.

2. Preliminaries

First, we take a convex as well as a closed subset E of a UCBS B and also fix a point $p \in B$. In this case, we define the functional $r(p, \{e_k\})$ by the formula $r(p, \{e_k\}) := \limsup_{k\to\infty} \|p - e_k\|$, where $\{e_k\}$ denotes any bounded sequence in the space B.

Accordingly, we define the asymptotic radius $A(E, \{e_k\})$ of the sequence $\{e_k\}$ in the set E by the formula the $r(E, \{e_k\}) := \inf\{r(p, \{e_k\}) : p \in E\}$. While the asymptotic center $A(E, \{e_k\})$ of the sequence $\{e_k\}$ in the set E by the formula $A(E, \{e_k\}) := \{p \in E : r(p, \{w_k\}) = r(E, \{e_k\})\}$. In this case, the set $A(E, \{e_k\})$ admits one and only one element (see, e.g., [16, 28] and others).

Definition 2.1. *[25] A Banach space B is said to satisfy the Opial's property if we take any $\{e_k\}$ in B such that it converges weakly to $e \in B$, then one has*

$$\limsup_{k\to\infty} \|e_k - e\| < \limsup_{k\to\infty} \|e_k - e'\| \ \ \forall e' \in B - \{e\}.$$

The following facts are also necessary for establishing the main outcome.

Lemma 2.2. *Let us consider a subset E of a Banach space B such that $T : E \to E$.*

(i) If T is nonexpansive then so it is enriched nonexpansive.

(ii) Suppose B has Opial's property and T is nonexpansive. If there is a weakly convergent sequence $\{e_k\}$ with limit e_0 and also satisfying $\lim_{k\to\infty} \|e_k - Te_k\| = 0$, then $Te_0 = e_0$, that is, $e_0 \in F_T$.

We also need they following key characterization of a UCBS.

Lemma 2.3. *[26] Let B is a UCBS, $k \in \mathbb{N}$ and $0 < u \leq \gamma_k \leq v < 1$. If there is a some $z \geq 0$ such that $\{d_k\}$ and $\{e_k\}$ in B are any sequences with $\limsup_{k\to\infty} \|d_k\| \leq z$, $\limsup_{k\to\infty} \|e_k\| \leq z$ and $\lim_{k\to\infty} \|\gamma_k d_k + (1 - \gamma_k)e_k\| = z$, then $\lim_{k\to\infty} \|d_k - e_k\| = 0$.*

3. Main results

Now we are ready to achieve our objective. First we establish a very elementary lemma as follows.

Lemma 3.1. *Take any convex closed subset E of a UCBS B and set an enriched nonexpansive mapping $T : E \to E$ such that $F_T \neq \emptyset$. Suppose $\{e_k\}$ is a sequence of iterates generated by (3). Then eventually, we have $\lim_{k\to\infty} \|e_k - e_0\| = 0$ exists for all $e_0 \in F_T$.*

Proof. Let $e_0 \in F_T$. It follows that $e_0 \in F_{T_\lambda}$. Since T is enriched nonexpansive so there is a b such that

$$\|b(e - e') + Te - Te'\| \leq (b + 1)\|e - e'\| \text{ for every } e, e' \in E.$$

Put $\lambda = \frac{1}{1+b}$, then we get

$$\|T_\lambda e - T_\lambda e'\| \leq \|e - e'\| \text{ for every } e, e' \in E. \tag{4}$$

Now using (4), we get

$$\begin{aligned}
\|c_k - e_0\| &= \|(1 - \gamma_k)e_k + \gamma_k T_\lambda e_k - e_0\| \\
&\leq (1 - \gamma_k)\|e_k - e_0\| + \gamma_k\|T_\lambda e_k - e_0\| \\
&\leq (1 - \gamma_k)\|e_k - e_0\| + \gamma_k\|e_k - e_0\| \\
&= \|e_k - e_0\|,
\end{aligned}$$

and

$$\begin{aligned} \|d_k - e_0\| &= \|(1-\beta_k)c_k + \beta_k T_\lambda c_k - e_0\| \\ &\leq (1-\beta_k)\|c_k - e_0\| + \beta_k \|T_\lambda c_k - e_0\| \\ &\leq (1-\beta_k)\|c_k - e_0\| + \beta_k \|c_k - e_0\| \\ &= \|c_k - e_0\|. \end{aligned}$$

While using the above inequilities, we have

$$\begin{aligned} \|e_{k+1} - e_0\| &= \|(1-\alpha_k)T_\lambda e_k + \alpha_k F_{T_\lambda} d_k - e_0\| \\ &\leq (1-\alpha_k)\|T_\lambda e_k - e_0\| + \alpha_k \|T_\lambda d_k - e_0\| \\ &\leq (1-\alpha_k)\|e_k - e_0\| + \alpha_k \|d_k - e_0\| \\ &\leq (1-\alpha_k)\|e_k - e_0\| + \alpha_k \|c_k - e_0\| \\ &\leq (1-\alpha_k)\|e_k - e_0\| + \alpha_k \|e_k - e_0\| \\ &= \|e_k - e_0\|. \end{aligned}$$

Lastly, we got $\|e_{k+1} - e_0\| \leq \|e_k - e_0\|$. This means that $\{\|e_k - e_0\|\}$ is essentially nonincreaing and bounded so that $\lim_{k\to\infty} \|e_k - e_0\|$ exists, where $e_0 \in F_{T_\lambda} = F_T$ is any point. □

Using Lemmas 2.3 and 3.1, we have the following existence result.

Lemma 3.2. *Take any convex closed subset E of a UCBS B and set an enriched nonexpansive mapping $T : E \to E$. Suppose $\{e_k\}$ is a sequence of iterates generated by (3). Then eventually, we have $F_T \neq \emptyset$ if and only if the sequence $\{e_k\}$ is bounded in E and satisfying the property $\lim_{k\to\infty} \|T_\lambda e_k - e_k\| = 0$ where $\lambda = \frac{1}{b+1}$.*

Proof. Suppose the fixed point set F_T is nonempty and fix any element $e_0 \in F_T$. Applying Lemma 3.1 on e_0, we have $\lim_{k\to\infty} \|e_k - e_0\|$ exists as well as the sequence of iterates $\{e_k\}$ is bounded. We want to prove $\lim_{k\to\infty} \|T_\lambda e_k - e_k\| = 0$. For this, since $\lim_{k\to\infty} \|e_k - e_0\|$, we put

$$\lim_{k\to\infty} \|e_k - e_0\| = z. \tag{5}$$

But we have proved in the Lemma 3.1 that

$$\|c_k - e_0\| \leq \|e_k - e_0\|.$$

Accordingly, we get

$$\Rightarrow \limsup_{k\to\infty} \|c_k - e_0\| \leq \limsup_{k\to\infty} \|e_k - e_0\| = z. \tag{6}$$

Also form (4), we get

$$\limsup_{k\to\infty} \|T_\lambda c_k - e_0\| \leq \limsup_{k\to\infty} \|c_k - e_0\| = z. \tag{7}$$

Again, from the proof of Lemma 3.1,

$$\|e_{k+1} - e_0\| \leq (1-\alpha_k)\|e_k - e_0\| + \alpha_k \|c_k - e_0\|.$$

It follows that

$$\|e_{k+1} - e_0\| - \|e_k - e_0\| \leq \frac{\|e_{k+1} - e_0\| - \|e_k - e_0\|}{\alpha_k} \leq \|c_k - e_0\| - \|c_k - e_0\|.$$

Accordingly, we can get $\|e_{k+1} - e_0\| \leq \|c_k - e_0\|$.

$$\Rightarrow z \leq \liminf_{k\to\infty} \|c_k - e_0\|. \tag{8}$$

From (6) and (8), we get

$$z = \lim_{k\to\infty} \|c_k - e_0\|. \tag{9}$$

From (9), we have

$$z = \lim_{k\to\infty} \|e_k - e_0\| = \lim_{k\to\infty} \|(1-\gamma_k)(e_k - e_0) + \gamma_k(T_\lambda e_k - e_0)\|.$$

Now all the requirements of Lemma 2.3 are proved and so

$$\lim_{k\to\infty} \|T_\lambda e_k - e_k\| = 0.$$

In the conversely part, we consider the sequence of iterates $\{e_k\} \subseteq E$ as a bounded sequence and assume that $\lim_{k\to\infty} \|e_k - T_\lambda e_k\| = 0$. Suppose $e_0 \in A(E, \{e_k\})$ denotes any element. The aim is to show that $T_\lambda e_0 \in A(E, \{e_k\})$. To do this, we use (4) as follows:

$$\begin{aligned}
r(T_\lambda e_0, \{e_k\}) &= \limsup_{k\to\infty} \|e_k - Te_0\| = \limsup_{k\to\infty} \|T_\lambda e_k - T_\lambda e_0\| \\
&= \limsup_{k\to\infty} (\|e_k - T_\lambda e_k + T_\lambda e_k - Te_0\|) \\
&\leq \limsup_{k\to\infty} (\|e_k - T_\lambda e_k\| + \|T_\lambda e_k - T_\lambda e_0\|) \\
&\leq \limsup_{k\to\infty} (\|e_k - T_\lambda e_k\| + \|e_k - e_0\|) \\
&= \limsup_{k\to\infty} \|e_k - T_\lambda e_k\| + \limsup_{k\to\infty} \|e_k - e_0\| \\
&= \limsup_{k\to\infty} \|e_k - e_0\| = r(e_0, \{e_k\}).
\end{aligned}$$

Subsequently, we see that $T_\lambda e_0 \in A(E, \{e_k\})$. But the set $A(E, \{e_k\})$ admits one point, it follows that $e_0 = T_\lambda e_0$. Since $F_{T_\lambda} = F_T$, we conclude that $F_T \neq \emptyset$. □

The first conergence result of this section is the following.

Theorem 3.3. *Take any convex compact subset E of a UCBS B and set an enriched nonexpansive mapping $T : E \to E$. Suppose $\{e_k\}$ is a sequence of iterates generated by (3). Then eventually, we have $\{e_k\}$ converges strongly to a fixed point of T.*

Proof. To establish the proof, we use the assumption of compactness of E and (4) as: since $\{e_k\}$ is a sequence in E, we may choose its subsequence $\{e_{k_r}\}$ with the property that $\lim_{r\to\infty} \|e_{k_r} - u_0\| = 0$, for some $u_0 \in E$. Hence

$$\begin{aligned}
\|e_{k_r} - T_\lambda u_0\| &= \|e_{k_r} - T_\lambda e_{k_r} + T_\lambda e_{k_r} - T_\lambda u_0\| \\
&\leq \|e_{k_r} - T_\lambda e_{k_r}\| + \|T_\lambda e_{k_r} - T_\lambda u_0\| \\
&\leq \|e_{k_r} - T_\lambda e_{k_r}\| + \|e_{k_r} - u_0\|.
\end{aligned}$$

Consequently, we get

$$\|e_{k_r} - T_\lambda u_0\| \leq \|e_{k_r} - T_\lambda e_{k_r}\| + \|e_{k_r} - u_0\|. \tag{10}$$

Now using the facts that $\lim_{r\to\infty} \|e_{k_r} - T_\lambda e_{k_r}\| = 0$ (see, Theorem 3.2) as well as $\lim_{r\to\infty} \|e_{k_r} - u_0\| = 0$ in the connection with (10), we get $\lim_{r\to\infty} \|e_{k_r} - T_\lambda u_0\| = 0$. Accoringly, it follows that $T_\lambda u_0 = u_0$, this means that u_0 is a fixed point for T_λ. It follows that u_0 is also a fixed point of T. According to Lemma 3.1, $\lim_{k\to\infty} \|e_k - u_0\|$ essentially exists. Eventually, u_0 must be the strong limit for $\{e_k\}$. □

Theorem 3.4. *Take any convex convex subset E of a UCBS B and set an enriched nonexpansive mapping $T : E \to E$. Suppose $\{e_k\}$ is a sequence of iterates generated by (3). Then eventually, we have $\{e_k\}$ converges strongly to a fixed point of T if the condition $\liminf_{k\to\infty} dist(e_k, F_{T_\lambda}) = 0$ holds.*

Proof. The proof is elementary and, hence, omitted. □

Senter and Dotson [27] essentially provided the following condition of selfmaps.

Definition 3.5. *[27] Take any convex convex subset E of a UCBS B a selfmap $T : E \to E$ is said to satisfy the condition (I) provided that there is a function f such that $f(u) = 0$ for $u = 0$, $f(v) > 0$ for $u > 0$ and $\|e - Te\| \geq f(dist(e_k, F_T))$ for each choice of $e \in E$.*

Theorem 3.6. *Take any convex convex subset E of a UCBS B and set an enriched nonexpansive mapping $T : E \to E$. Suppose $\{e_k\}$ is a sequence of iterates generated by (3). Then eventually, we have $\{e_k\}$ converges strongly to a fixed point of T if the operator T_λ satisfies condition (I).*

Proof. First we can write in the view of Theorem 3.2 that

$$\liminf_{k\to\infty} \|e_k - T_\lambda e_k\| = 0. \tag{11}$$

But T_λ satisfies condition (I), that is,

$$\|e_k - T_\lambda e_k\| \geq f(\mathrm{dist}(e_k, F_{T_\lambda})).$$

It now follows form (11), that

$$\liminf_{k\to\infty} f(\mathrm{dist}(e_k, F_{T_\lambda})) = 0.$$

From the properties of f, we have

$$\liminf_{k\to\infty} \mathrm{dist}(e_k, F_{T_\lambda}) = 0.$$

It follows from the Theorem 3.4 that, $\{e_k\}$ strongly converges to a fixed point of T. □

We finish the section with the following weak convergence result.

Theorem 3.7. *Take any convex convex subset E of a UCBS B and set an enriched nonexpansive mapping $T : E \to E$. Suppose $\{e_k\}$ is a sequence of iterates generated by (3). Then eventually, we have $\{e_k\}$ converges weakly to a fixed point of T if the space B satisfies Opial's condition.*

Proof. This proof can be completed as follows. Since the Banch space B is uniformly convex it follows directly that it is also reflexive. Moreover, Theorem 3.2 hypothesis suggests that $\{e_k\}$ is essentially bounded in E. It follows that there must be a subsequence which we may denote by $\{e_{k_s}\}$ of the iterative sequence $\{e_k\}$ such that $e_1 \in E$ is the weak limit of it. Applying Theorem 3.2 on $\{e_{k_s}\}$, we have $\lim_{k\to\infty} \|T_\lambda e_{k_s} - e_{k_s}\| = 0$. In the view of Lemma 2.2 and (4), we have $e_1 \in F_{T_\lambda}$.
Claim. We show that e_1 is the weak limit of $\{e_k\}$.
If $e_2 \in E$ (such that $e_2 \neq e_1$) is also weak limit of $\{e_k\}$ then it means that one can find another subsequence, namely, $\{e_{k_t}\}$ such that it converges weakly to e_1. But using the same techniques as previous, we have $e_2 \in F_{T_\lambda}$. Hence bearing Lemma 3.1 in mind along with the Opial's condition, it follows that

$$\begin{aligned}
\lim_{k\to\infty} \|e_k - e_1\| &= \lim_{s\to\infty} \|e_{k_s} - e_1\| < \lim_{s\to\infty} \|e_{k_s} - e_2\| \\
&= \lim_{k\to\infty} \|e_k - e_2\| = \lim_{t\to\infty} \|e_{k_t} - e_2\| \\
&< \lim_{t\to\infty} \|e_{k_t} - e_1\| = \lim_{k\to\infty} \|e_k - e_1\|.
\end{aligned}$$

Subsequently, a contradiction is found. This contradiction suggests that $e_1 \in E$ is the weak limit of $\{e_k\}$ and fixed point of T. □

4. Example

We now check the convergence of our iterative scheme numerically. To do this, use an example of enriched nonexpansive mappings which is not nonexpansive. It has been shown that our iterative scheme converges to a fixed point and the convergence is faster corresponding to some other iterative schemes of the literature.

Example 4.1. *Consider $E = [0.5, 2]$ with the absolute valued norm. In this case, we set $T : E \to E$ such that $Te = \frac{1}{e}$ $\forall e \in E$. It follows that T is enriched nonexpansive with $b = 1.5$. Notice that T is not nonexpansive because $|T1 - T0.5| > |1 - 0.5|$. Also $F_T = \{1\}$, that is, F_T is nonempty. By our main results, the sequence of iterates (3) converges to 1. This fact is displayed in the Table 1 and Figure 1.*

Table 1: Comparison of three-step Thakur and other iterative schemes.

k	**Thakur** (3)	**Abbas** (2)	**Noor** (1)	**Krasnoselskii** ([14])
1	1.85	1.85	1.85	1.85
2	1.14111313785184	1.19465860862694	1.24736675474255	1.40478378378378
3	1.01397464201861	1.02851054525053	1.060576797608	1.16918742600807
4	1.00117331554965	1.00337778114075	1.01386365906343	1.06246395477466
5	1.00009664654083	1.0003850717474	1.00311704493221	1.02123706649332
6	1.00000794783156	1.00004369219038	1.0006979430334	1.00694601683207
7	1.00000065351039	1.00000495486203	1.00015613261145	1.00223901626052
8	1.00000005373429	1.00000056186593	1.00003492021548	1.00071818588141
9	1.00000000441824	1.00000006371340	1.00000780980140	1.00022999472512
10	1.00000000036328	1.00000000722484	1.00000174662046	1.00007361629307
11	1.00000000002987	1.00000000081926	1.00000039062144	1.00002355905623
12	1.00000000000245	1.00000000009290	1.00000008736015	1.0000075390860
13	1.00000000000020	1.00000000001053	1.00000001953757	1.00000241252706
14	1.00000000000001	1.00000000000119	1.00000000436946	1.00000077201064
15	1	1.00000000000013	1.00000000097720	1.00000024704360
16	1	1.00000000000001	1.00000000021854	1.00000007905397
17	1	1	1.00000000004887	1.00000002529727
18	1	1	1.00000000001093	1.00000000809512
19	1	1	1.00000000000244	1.00000000259044
20	1	1	1.00000000000054	1.00000000082894
21	1	1	1.00000000000012	1.00000000026526
22	1	1	1.00000000000002	1.00000000008488
23	1	1	1	1.00000000002716
24	1	1	1	1.00000000000869
25	1	1	1	1.00000000000278
26	1	1	1	1.00000000000089
27	1	1	1	1.00000000000028
28	1	1	1	1.00000000000009
29	1	1	1	1.00000000000002
30	1	1	1	1

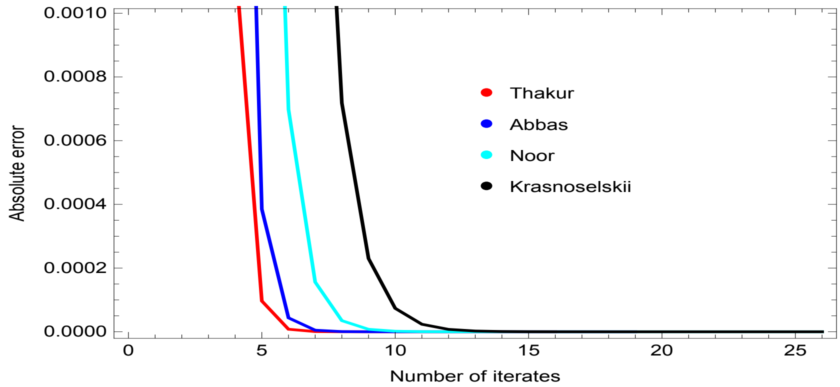

Figure 1: Three-step Thakur vs three-step Abbas and three-step Noor iterations.

5. Conclusions and Future work

This paper suggested the Thakur et al. [29] type three-step scheme to find the fixed points of enriched nonexpansive selfmaps on a Banach space setting. Convergence results are established under appropriate mild assumptions. Interestingly, these results extend the results of Thakur et al. [29] from nonexpansive selfmaps to the broader class of mappings so-called enriched nonexpansive selmaps. We have also performed some numerical computations to validate the claims and results of the paper. The next plan of the authors is to extend these results to the setting of multi-valued enriched nonexpansive mappings.

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References

[1] Picard, E.M. Memorie sur la theorie des equations aux derivees partielles et la methode des approximation ssuccessives. *J. Math. Pure Appl.* **1890**, *6*, 145–210.
[2] Goebel, K.; Kirk, W.A. A fixed point theorem for asymptotically nonexpansive mappings. *Proc. Amer. Math. Soc.* **1972**, *35*, 171–174.
[3] Aoyama, K.; Kohsaka, F. Fixed point theorem for α-nonexpansive mappings in Banach spaces. *Nonlinear Anal. Ser. Theory, Methods Appl.* **2011**, *74*, 4387–4391.
[4] Bae, J.S. Fixed point theorems of generalized nonexpansive mappings. *J. Korean Math. Soc.* **1984**, *21*, 233–248.
[5] Bogin, J. A generalization of a fixed point theorem of Goebel, Kirk and Shimi. *Canad. Math. Bull.* **1976**, *19*, 7–12.
[6] Goebel, K.; Kirk, W.A.; Shimi, T.N. A fixed point theorem in uniformly convex spaces. *Boll. Della Unione Mat. Ital.* **1973**, *7*, 67–75.
[7] Llorens-Fuster, E.; Moreno-Galvez, E. The Fixed Point Theory for some generalized nonexpansive mappings. *Abstr. Appl. Anal.* **2011**, *2011*, 1–15.
[8] Suzuki, T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2008**, *340*, 1088–1095.
[9] Pant, R.; Shukla, R. Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces. *Numer. Funct. Anal. Optim.* **2017**, *38*, 248–266.
[10] Garcia-Falset, J.; Llorens-Fuster, E.; Suzuki, T. Fixed point theory for a class of generalized nonexpasive mappings. *J. Math. Anal. Appl.* **2011**, *375*, 185–195.
[11] Patir, B.; Goswami, N.; Mishra, V.N. Some results on fixed point theory for a class of generalized nonexpansive mappings. *Fixed Point Theory Appl.* **2018**.
[12] Berinde, V. Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces. *Carpathian J. Math.* **2019**, *35*, 293–304.
[13] Krasnoselskii, M.A.: Two observations about method of successive approximations. Usp. Mat. Nauk 10, 123-127 (1955)
[14] Berinde, V. Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition. Carpathian J. Math. 36 (2020), no. 1, 27–34.
[15] Abbas, M.; Nazir, T. A new faster iteration process applied to constrained minimization and feasibility problems. *Math. Vesnik* **2014**, *66*, 223–234.

[16] Agarwal, R.P.; O'Regan, D.; Sahu, D.R. *Fixed Point Theory for Lipschitzian-Type Mappings with Applications Series*; Topological Fixed Point Theory and Its Applications; Springer: New York, NY, USA, 2009; Volume 6.
[17] Agarwal, R.P.; O'Regon, D.; Sahu, D.R. Iterative construction of fixed points of nearly asymtotically non-expansive mappings. *J. Nonlinear Convex Anal.* **2007**, *8*, 61–79.
[18] Banach, S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181.
[19] Browder, F.E. Nonexpansive nonlinear operators in a Banach space. *Proc. Natl. Acad. Sci. USA* **1965**, *54*, 1041–1044.
[20] Clarkson, J.A. Uniformly convex spaces. *Trans. Am. Math. Soc.* **1936**, 40, 396–414.
[21] Gohde, D. Zum Prinzip der Kontraktiven Abbildung. *Math. Nachr.* **1965**, *30*, 251–258.
[22] Ishikawa, S. Fixed points by a new iteration method. *Proc. Am. Math. Soc.* **1974**, *44*, 147–150.
[23] Mann, W.R. Mean value methods in iteration. *Proc. Am. Math. Soc.* **1953**, *4*, 506–510.
[24] Noor, M.A. New approximation schemes for general variational inequalities. *J. Math. Anal. Appl.* **2000**, *251*, 217–229.
[25] Opial, Z. Weak and strong convergence of the sequence of successive approximations for non-expansive mappings. *Bull. Am. Math. Soc.* **1967**, *73*, 591–597.
[26] Schu, J. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bull. Austral. Math. Soc.* **1991**, *43*, 153–159.
[27] Senter, H.F.; Dotson, W.G. Approximating fixed points of non-expansive mappings. *Proc. Am. Math. Soc.* **1974**, *44*, 375–380.
[28] Takahashi, W. *Nonlinear Functional Analysis*; Yokohoma Publishers: Yokohoma, Japan, 2000.
[29] Thakur, B.S.; Thakur, D.; Postolache, M. New iteration scheme for numerical reckoning fixed points of nonexpansive mappings. *J. Inequal. Appl.* **2014**, 328.