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Multi-valued variational inclusion problem and convergence analysis

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Abstract. In this paper, under some new appropriate conditions imposed on the parameter and mappings involved in the resolvent operator associated with a $(\widehat{H},\eta)$-monotone operator, its Lipschitz continuity is proved and an estimate of its Lipschitz constant is computed. The resolvent operator method is used and a new equivalence relationship between a class of multi-valued variational inclusion problems involving $(\widehat{H},\eta)$-monotone operators and a class of fixed points problems is established. The obtained equivalence relationship is employed and a new iterative algorithm for solving the multi-valued variational inclusion problem is constructed. Under some suitable assumptions, the convergence analysis of the sequences generated by our proposed iterative algorithm is studied.

1. Introduction

In the last fifty years, variational inequality problems have been the focus of attention of researchers coming from mathematics, economics, and many other disciplines. This is mainly because many nonlinear problems arising in optimization, operations research, structural analysis, and engineering sciences can be transformed into variational inequality problems. For more details, we refer the interested reader, for example, to [6, 12, 16] and the references therein. Due to numerous theoretical and practical applications to problems in different areas of science, during the past few decades, there has been a major activity in the study of various extensions of variational inequality problems. In the last two decades, variational inclusions as the generalization of variational inequalities have attracted increasing attention. There is a rich literature on analysing and solving different classes of variational inclusion problems, see, for example, [1–3, 7, 11, 13–15, 18, 22, 23, 25–27, 29, 31, 34, 35] and the references therein.

Because of the importance and extremely wide applications of variational inequalities and variational inclusions in a huge variety of scientific fields, considerable attention has been paid to the construction and development of general methods, such as the projection method and its variant forms, linear approximation, descent method, Newton's method and the method based on the auxiliary principle technique, for the

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sensitivity analysis of solution sets of various kinds of them in recent years. The resolvent operator method, as a generalization of the projection method, is an important and useful tool for studying the approximation solvability of nonlinear variational inequalities and variational inclusions, see, for example, [1, 3–5, 7–10, 13–15, 18, 19, 22, 23, 25, 26, 28, 29, 31, 35] and the references therein.

It is worth pointing out that the concept of maximal monotonicity plays an important role in the study of various types of problems arising in the domain of optimization, nonlinear analysis, differential equations, and related fields. For this reason, in recent years, many efforts have been made to generalize the notion of maximal monotonicity for studying and analyzing variant classes of variational inequality (inclusion) problems based on different generalizations of maximal monotonicity. By the same taken, Huang and Fang [18] were the first to introduce the notion of maximal η-monotone operators and gave the definition of the resolvent operator for the maximal η-monotone operator in the setting of Hilbert spaces. In 2003, Fang and Huang [14] succeeded to introduce another extension of maximal monotone operator the so-called H-monotone operator and define the corresponding resolvent operator for solving a class of variational inclusion problems involving H-monotone operators. Subsequently, Fang et al. [15] introduced the concept of (H, η)-monotone operator as a unifying framework for the classes of maximal monotone operators, maximal η-monotone operators and H-monotone operators. At the same time, they defined the resolvent operator associated with an (H, η)-monotone operator and used it for finding the approximate solutions of a system of variational inclusions involving (H, η)-monotone operators. In 2008, Sun et al. [28] introduced and studied the notion of M-monotone operator in a Hilbert space setting as an extension of maximal monotone and H-monotone operators. In the same year, Zou and Huang [35] extended such an operator from the Hilbert space setting to a Banach space setting and called it $H(.,.)$-accretive. By defining the resolvent operator associated with an $H(.,.)$-accretive operator and employing it, the authors studied a class of variational inclusion problems involving $H(.,.)$-accretive operators in the setting of Banach spaces.

Recently, Ahmad et al. [1] introduced and studied the concept of $H(.,.)$-Co-monotone mapping in the context of Hilbert space as a unifying framework for the classes of maximal monotone operators, H-monotone operators and M-monotone operators. They defined the resolvent operator associated with an $H(.,.)$-Co-monotone mapping and used it to construct an iterative algorithm for solving a class of variational inclusion problems involving $H(.,.)$-Co-monotone mappings. Under some appropriate hypotheses, they established an existence and convergence result for the class of variational inclusion problems considered in [1]. Further generalizations of maximal monotone operators and the generalized operators mentioned above in different contexts along with discussions can be found in [1, 13, 22, 23, 25, 26, 29, 31] and the references therein.

Motivated and inspired by the excellent work mentioned above, in this paper, we pursue two goals. Our first objective is to study a new class of multi-valued variational inclusion problems (for short, MVIP) involving $(\widehat{H}, \eta)$-monotone operators. For this end, the Lipschitz continuity of the resolvent operator associated with a $(\widehat{H}, \eta)$-monotone operator is proved under some new assumptions imposed on the parameter and mappings involved in it and an estimate of its Lipschitz constant is computed. With the help of the resolvent operator technique, the equivalence between the MVIP and a fixed point problem is established. We apply the obtained equivalence relationship and suggest a new iterative algorithm for approximating a solution of the MVIP. Under some appropriate conditions, the strong convergence of the sequences generated by our proposed iterative algorithm to a solution of the MVIP is proved. The second purpose of this paper is to investigate and analyze the notion of $H(.,.)$-Co-monotone mapping introduced in [1] and to point out some comments regarding it. We show that under the hypotheses considered in [1], every $H(.,.)$-Co-monotone mapping is actually a $\widehat{H}$-monotone operator and is not a new one. Moreover, reviewing all results given in [1], some remarks concerning them are pointed out. Our results are new, and improve and generalize many known corresponding results.

2. Basic Definitions and Properties

Throughout this paper, we assume that X is a real Hilbert space endowed with a norm $\|.\|$ and an inner product $\langle .,. \rangle$, and $CB(X)$ is the family of all nonempty closed and bounded subsets of X. Further, let $D(.,.)$

be the Hausdorff metric on $CB(X)$ defined by

$$D(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\}, \quad \forall A,B\in CB(X).$$

For a given multi-valued mapping $\widehat{M}: X \rightrightarrows X$,

(i) the set Range($\widehat{M}$) defined by

$$\text{Range}(\widehat{M}) = \{y\in X : \exists x\in X : (x,y)\in \widehat{M}\} = \bigcup_{x\in X}\widehat{M}(x)$$

is called the range of $\widehat{M}$;

(ii) the set Graph($\widehat{M}$) defined by

$$\text{Graph}(\widehat{M}) = \{(x,u)\in X\times X : u\in \widehat{M}(x)\},$$

is called the graph of $\widehat{M}$.

In what follows, we recall some concepts and known results which will be used in the sequel.

Definition 2.1. *Let $T: X\to X$ and $\eta: X\times X\to X$ be the operators. T is said to be*

(i) *monotone, if*

$$\langle T(x)-T(y), x-y\rangle \geq 0, \quad \forall x,y\in X;$$

(ii) *η-monotone, if*

$$\langle T(x)-T(y), \eta(x,y)\rangle \geq 0, \quad \forall x,y\in X;$$

(iii) *strictly monotone, if T is monotone and equality holds if and only if $x=y$;*

(iv) *strictly η-monotone, if T is η-monotone and equality holds if and only if $x=y$;*

(v) *r-strongly monotone, if there exists a constant $r>0$ such that*

$$\langle T(x)-T(y), x-y\rangle \geq r\|x-y\|^2, \quad \forall x,y\in X;$$

(vi) *k-strongly η-monotone, if there exists a constant $k>0$ such that*

$$\langle T(x)-T(y), \eta(x,y)\rangle \geq k\|x-y\|^2, \quad \forall x,y\in X;$$

(vii) *ς-relaxed monotone, if there exists a constant $\varsigma>0$ such that*

$$\langle T(x)-T(y), x-y\rangle \geq -\varsigma\|x-y\|^2, \quad \forall x,y\in X;$$

(viii) *α-expansive, if there exists a constant $\alpha>0$ such that*

$$\|T(x)-T(y)\| \geq \alpha\|x-y\|, \quad \forall x,y\in X;$$

(ix) *γ-Lipschitz continuous, if there exists a constant $\gamma>0$ such that*

$$\|T(x)-T(y)\| \leq \gamma\|x-y\|, \quad \forall x,y\in X.$$

Definition 2.2. *[15, 18] Let $\eta: X\times X\to X$ be a vector-valued operator. A multi-valued operator $\widehat{M}: X\rightrightarrows X$ is said to be*

(i) *monotone, if*

$$\langle u-v, x-y\rangle \geq 0, \quad \forall (x,u),(y,v)\in \text{Graph}(\widehat{M});$$

(ii) *η-monotone, if*

$$\langle u-v, \eta(x,y)\rangle \geq 0, \quad \forall (x,u),(y,v)\in \text{Graph}(\widehat{M});$$

(iii) *ϱ-strongly monotone, if there exists a constant $\varrho > 0$ such that*

$$\langle u - v, x - y\rangle \geq \varrho\|x - y\|^2, \quad \forall (x, u), (y, v) \in \mathrm{Graph}(\widehat{M});$$

(iv) *ξ-strongly η-monotone, if there exists a constant $\xi > 0$ such that*

$$\langle u - v, \eta(x, y)\rangle \geq \xi\|x - y\|^2, \quad \forall (x, u), (y, v) \in \mathrm{Graph}(\widehat{M});$$

(v) *m-relaxed monotone, if there exists a constant $m > 0$ such that*

$$\langle u - v, x - y\rangle \geq -m\|x - y\|^2, \quad \forall (x, u), (y, v) \in \mathrm{Graph}(\widehat{M});$$

(vi) *maximal monotone, if $\widehat{M}$ is monotone and $(I + \lambda\widehat{M})(X) = X$, for every real constant $\lambda > 0$;*

(vii) *maximal η-monotone, if $\widehat{M}$ is η-monotone and $(I + \lambda\widehat{M})(X) = X$, for every real constant $\lambda > 0$.*

Fang and Huang [14] and subsequently Verma [29, 30] introduced and studied, respectively, the classes of $\widehat{H}$-monotone operators and $\widehat{H}$-maximal m-relaxed monotone operators as the generalizations of maximal monotone operators as follows.

Definition 2.3. *Let $\widehat{H} : X \to X$ be a single-valued operator and $\widehat{M} : X \rightrightarrows X$ be a multi-valued operator. $\widehat{M}$ is said to be*

(i) *$\widehat{H}$-monotone [14] if $\widehat{M}$ is monotone and $(\widehat{H} + \lambda\widehat{M})(X) = X$ holds for every real constant $\lambda > 0$;*

(ii) *$\widehat{H}$-maximal m-relaxed monotone (also referred to as $\widehat{H}$-maximal monotone in the literature) [29, 30] if $\widehat{M}$ is m-relaxed monotone and $(\widehat{H} + \lambda\widehat{M})(X) = X$ holds for every real constant $\lambda > 0$.*

Fang et al. [15] introduced and studied the class of $(\widehat{H}, \eta)$-monotone operators as a unifying framework for the classes of maximal monotone operators, maximal η-monotone operators and $\widehat{H}$-monotone operators as follows.

Definition 2.4. *[15] Let $\eta : X \times X \to X$ and $\widehat{H} : X \to X$ be two vector-valued operators. A multi-valued operator $\widehat{M} : X \rightrightarrows X$ is said to be $(\widehat{H}, \eta)$-monotone, if $\widehat{M}$ is η-monotone and $(\widehat{H} + \lambda\widehat{M})(X) = X$ holds, for every real constant $\lambda > 0$.*

The following new example illustrates that for given operators $\eta : X \times X \to X$ and $\widehat{H} : X \to X$, a $(\widehat{H}, \eta)$-monotone operator may be neither $\widehat{H}$-monotone nor maximal η-monotone.

Example 2.5. Let m and n be two arbitrary but fixed natural numbers such that n is even and $M_{m\times n}(\mathbb{C})$ be the vector space of all $m \times n$ matrices with complex entries over $\mathbb{C}$. Then

$$M_{m\times n}(\mathbb{C}) = \{A = \begin{pmatrix} a_{lj} \end{pmatrix} | a_{lj} \in \mathbb{C}, l = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$$

is a Hilbert space with the inner product $\langle A, B\rangle := tr(AB^*)$, for all $A, B \in M_{m\times n}(\mathbb{C})$, where tr denotes the trace, that is, the sum of diagonal entries, and B^* denotes the Hermitian conjugate (or adjoint) of the matrix B, that is, $B^* = \overline{B^t}$, the complex conjugate of the transpose B. The inner product defined above induces a norm on $M_{m\times n}(\mathbb{C})$ as follows:

$$\|A\| = \Big(\sum_{l=1}^{m}\sum_{j=1}^{n} |a_{lj}|^2\Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{C}).$$

For any $A = \begin{pmatrix} a_{lj} \end{pmatrix} \in M_{m\times n}(\mathbb{C})$, we have

$$A = \begin{pmatrix} a_{lj} \end{pmatrix} = \sum_{l=1}^{m} \sum_{j\in\{1,3,\ldots,\frac{n-2}{2}\}} (A_{l(2j-1)(2j+1)} + A_{l(2j)(2j+2)}),$$

that is, every $m \times n$ matrix $A \in M_{m\times n}(\mathbb{C})$ can be written as a linear combination of $\frac{mn}{2}$ matrices $A_{l(2j-1)(2j+1)}$ and $A_{l(2j)(2j+2)}$, where for each $l \in \{1, 2, \dots, m\}$ and $j \in \{1, 3, \dots, \frac{n-2}{2}\}$, $A_{l(2j-1)(2j+1)}$ is an $m \times n$ matrix in which the $(l, 2j-1)$ and $(l, 2j+1)$-entries equal to $a_{l(2j-1)} = x_{l(2j-1)} + iy_{l(2j-1)}$ and $a_{l(2j+1)} = x_{l(2j+1)} + iy_{l(2j+1)}$, respectively, and all other entries equal to zero, and $A_{l(2j)(2j+2)}$ is an $m \times n$ matrix in which the $(l, 2j)$ and $(l, 2j+2)$-entries equal to $a_{l(2j)} = x_{l(2j)} + iy_{l(2j)}$ and $a_{l(2j+2)} = x_{l(2j+2)} + iy_{l(2j+2)}$, respectively, and all other entries equal to zero. For each $l \in \{1, 2, \dots, m\}$ and $j \in \{1, 3, \dots, \frac{n-2}{2}\}$, yields

$$
\begin{aligned}
A_{l(2j-1)(2j+1)} + A_{l(2j)(2j+2)} &= \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & a_{l(2j-1)} & 0 & a_{l(2j+1)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & a_{l(2j)} & 0 & a_{l(2j+2)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \\
&= \frac{y_{l(2j-1)} + y_{l(2j+1)} - i(x_{l(2j-1)} + x_{l(2j+1)})}{2} \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & i_{l(2j-1)} & 0 & i_{l(2j+1)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \\
&\quad + \frac{y_{l(2j-1)} - y_{l(2j+1)} - i(x_{l(2j-1)} - x_{l(2j+1)})}{2} \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & i_{l(2j-1)} & 0 & -i_{l(2j+1)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \\
&\quad + \frac{y_{l(2j)} + y_{l(2j+2)} - i(x_{l(2j)} + x_{l(2j+2)})}{2} \begin{pmatrix} 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & i_{l(2j)} & 0 & i_{l(2j+2)}\cdots & 0 & 0 \\ \vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \\ 0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{y_{l(2j)}-y_{l(2j+2)}-i(x_{l(2j)}-x_{l(2j+2)})}{2}\begin{pmatrix}
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & i_{l(2j)} & 0 & -i_{l(2j+2)}\cdots & 0 & 0\\
\vdots & \vdots\cdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0\\
0 & 0\cdots & 0 & 0 & 0\cdots & 0 & 0
\end{pmatrix}\\
&=\frac{y_{l(2j-1)}+y_{l(2j+1)}-i(x_{l(2j-1)}+x_{l(2j+1)})}{2}N_{l(2j-1)(2j+1)}\\
&+\frac{y_{l(2j-1)}-y_{l(2j+1)}-i(x_{l(2j-1)}-x_{l(2j+1)})}{2}N'_{l(2j-1)(2j+1)}\\
&+\frac{y_{l(2j)}+y_{l(2j+2)}-i(x_{l(2j)}+x_{l(2j+2)})}{2}N_{l(2j)(2j+2)}\\
&+\frac{y_{l(2j)}-y_{l(2j+2)}-i(x_{l(2j)}-x_{l(2j+2)})}{2}N'_{l(2j)(2j+2)},
\end{aligned}
$$

where for each $l \in \{1,2,\dots,m\}$ and $j \in \{1,3,\dots,\frac{n-2}{2}\}$, $N_{l(2j-1)(2j+1)}$ is an $m \times n$ matrix in which the $(l,2j-1)$ and $(l,2j+1)$-entries equal to i and all other entries equal to zero, $N'_{l(2j-1)(2j+1)}$ is an $m \times n$ matrix with the $(l,2j-1)$-entry i, the $(l,2j+1)$-entry $-i$, and all other entries equal to zero, $N_{l(2j)(2j+2)}$ is an $m \times n$ matrix with the $(l,2j)$ and $(l,2j+2)$-entries i, and all other entries equal to zero, and $N'_{l(2j)(2j+2)}$ is an $m \times n$ matrix in which the $(l,2j)$ and $(l,2j+2)$-entries equal to i and $-i$, respectively, and all other entries equal to zero. Accordingly, for any $A \in M_{m\times n}(\mathbb{C})$, we have

$$
\begin{aligned}
A &= \sum_{l=1}^{m}\sum_{j\in\{1,3,\dots,\frac{n-2}{2}\}}(A_{l(2j-1)(2j+1)}+A_{l(2j)(2j+2)})\\
&=\sum_{l=1}^{m}\sum_{j\in\{1,3,\dots,\frac{n-2}{2}\}}\Big[\frac{y_{l(2j-1)}+y_{l(2j+1)}-i(x_{l(2j-1)}+x_{l(2j+1)})}{2}N_{l(2j-1)(2j+1)}\\
&+\frac{y_{l(2j-1)}-y_{l(2j+1)}-i(x_{l(2j-1)}-x_{l(2j+1)})}{2}N'_{l(2j-1)(2j+1)}\\
&+\frac{y_{l(2j)}+y_{l(2j+2)}-i(x_{l(2j)}+x_{l(2j+2)})}{2}N_{l(2j)(2j+2)}\\
&+\frac{y_{l(2j)}-y_{l(2j+2)}-i(x_{l(2j)}-x_{l(2j+2)})}{2}N'_{l(2j)(2j+2)}\Big].
\end{aligned}
$$

Therefore, the set

$$\Big\{N_{l(2j-1)(2j+1)}, N'_{l(2j-1)(2j+1)}, N_{l(2j)(2j+2)}, N'_{l(2j)(2j+2)} : l=1,2,\dots,m;\, j=1,3,\dots,\frac{n-2}{2}\Big\}$$

spans the Hilbert space $M_{m\times n}(\mathbb{C})$. Taking $E_{l(2j-1)(2j+1)} := \frac{1}{\sqrt{2}}N_{l(2j-1)(2j+1)}$, $E'_{l(2j-1)(2j+1)} := \frac{1}{\sqrt{2}}N'_{l(2j-1)(2j+1)}$, $E_{l(2j)(2j+2)} := \frac{1}{\sqrt{2}}N_{l(2j)(2j+2)}$ and $E'_{l(2j)(2j+2)} := \frac{1}{\sqrt{2}}N'_{l(2j)(2j+2)}$, for each $l \in \{1,2,\dots,m\}$ and $j \in \{1,3,\dots,\frac{n-2}{2}\}$, it follows that the set

$$\mathfrak{B} = \Big\{E_{l(2j-1)(2j+1)}, E'_{l(2j-1)(2j+1)}, E_{l(2j)(2j+2)}, E'_{l(2j)(2j+2)} : l=1,2,\dots,m;\, j=1,3,\dots,\frac{n-2}{2}\Big\}$$

also spans the Hilbert space $M_{m\times n}(\mathbb{C})$. It can be easily seen that the set $\mathfrak{B}$ is linearly independent and orthonormal and so $\mathfrak{B}$ is an orthonormal basis for the Hilbert space $M_{m\times n}(\mathbb{C})$.

Let the mappings $\widehat{M} : M_{m\times n}(\mathbb{C}) \rightrightarrows M_{m\times n}(\mathbb{C})$, $\eta : M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ and $\widehat{H} : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ be defined, respectively, by

$$\widehat{M}(A) = \begin{cases} \Phi, & A = E'_{s(2k)(2k+2)}, \\ -A + E'_{s(2k)(2k+2)}, & A \neq E'_{s(2k)(2k+2)}, \end{cases}$$

$$\eta(A, B) = \begin{cases} \alpha(B - A), & A, B \neq E'_{s(2k)(2k+2)}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

and $\widehat{H}(A) = \beta A + \gamma E'_{s(2k)(2k+2)}$, for all $A, B \in M_{m\times n}(\mathbb{C})$, where

$$\Phi = \Big\{E_{l(2j-1)(2j+1)} - E'_{s(2k)(2k+2)}, E'_{l(2j-1)(2j+1)} - E'_{s(2k)(2k+2)}, E_{l(2j)(2j+2)} - E'_{s(2k)(2k+2)},$$
$$E'_{l(2j)(2j+2)} - E'_{s(2k)(2k+2)} : l = 1, 2, \ldots, m; j = 1, 3, \ldots, \frac{n-2}{2}\Big\},$$

α, β and γ are arbitrary real constants such that $\beta < 0 < \alpha$, $s \in \{1, 2, \ldots, m\}$ and $k \in \{1, 3, \ldots, \frac{n-2}{2}\}$ are arbitrary but fixed natural numbers, and $\mathbf{0}$ is the zero vector of the space $M_{m\times n}(\mathbb{C})$, that is, the zero $m \times n$ matrix. Then, for all $A, B \in M_{m\times n}(\mathbb{C})$, $A \neq B \neq E'_{s(2k)(2k+2)}$, we get

$$\begin{aligned} \langle \widehat{M}(A) - \widehat{M}(B), A - B\rangle &= \langle -A + E'_{s(2k)(2k+2)} + B - E'_{s(2k)(2k+2)}, A - B\rangle \\ &= \langle B - A, A - B\rangle = -\|A - B\|^2 = -\sqrt{\sum_{l=1}^{m}\sum_{j=1}^{n} |a_{lj} - b_{lj}|^2} < 0, \end{aligned}$$

i.e., $\widehat{M}$ is not monotone and so $\widehat{M}$ is not $\widehat{H}$-monotone. For any given $A, B \in M_{m\times n}(\mathbb{C})$, $A \neq B \neq E'_{s(2k)(2k+2)}$, we yield

$$\begin{aligned} \langle \widehat{M}(A) - \widehat{M}(B), \eta(A, B)\rangle &= \langle -A + E'_{s(2k)(2k+2)} + B - E'_{s(2k)(2k+2)}, \alpha(B - A)\rangle \\ &= \alpha\langle B - A, B - A\rangle = \alpha\|B - A\|^2 = \alpha\sqrt{\sum_{l=1}^{m}\sum_{j=1}^{n} |a_{lj} - b_{lj}|^2} > 0. \end{aligned}$$

Taking into account that for each of the cases when $A \neq B = E'_{s(2k)(2k+2)}$, $B \neq A = E'_{s(2k)(2k+2)}$ and $A = B = E'_{s(2k)(2k+2)}$, we have $\eta(A, B) = \mathbf{0}$, it follows that

$$\langle u - v, \eta(A, B)\rangle = 0, \quad \forall (A, u), (B, v) \in \text{Graph}(\widehat{M}).$$

Therefore, $\widehat{M}$ is an η-accretive mapping.

In virtue of the facts that for any $A \in M_{m\times n}(\mathbb{C})$, $A \neq E'_{s(2k)(2k+2)}$,

$$\|(I + \widehat{M})(A)\| = \|E'_{s(2k)(2k+2)}\| = 1 > 0$$

and

$$\begin{aligned} (I + \widehat{M})(E'_{s(2k)(2k+2)}) = \Big\{&E_{l(2j-1)(2j+1)}, E'_{l(2j-1)(2j+1)}, E_{l(2j)(2j+2)}, E'_{l(2j)(2j+2)} : \\ &l = 1, 2, \ldots, m; j = 1, 3, \ldots, \frac{n-2}{2}\Big\} = \mathfrak{B}, \end{aligned}$$

where I is the identity mapping on $X = M_{m\times n}(\mathbb{C})$, we deduce that $\mathbf{0} \notin (I + \widehat{M})(M_{m\times n}(\mathbb{C}))$. Accordingly, $I + \widehat{M}$ is not surjective and so $\widehat{M}$ is not a maximal η-monotone operator.

For any $\lambda > 0$ and $A \in M_{m\times n}(\mathbb{C})$, taking $B = \frac{1}{\beta-\lambda}A + \frac{\gamma+\lambda}{\lambda-\beta}E'_{s(2k)(2k+2)}$ ($\lambda \neq \beta$, because $\beta < 0$), we obtain

$$(\widehat{H} + \lambda\widehat{M})(B) \quad = (\widehat{H} + \lambda\widehat{M})\left(\tfrac{1}{\beta-\lambda}A + \tfrac{\gamma+\lambda}{\lambda-\beta}E'_{s(2k)(2k+2)}\right) = A.$$

Consequently, for every $\lambda > 0$, $\widehat{H} + \lambda\widehat{M}$ is surjective and so $\widehat{M}$ is a $(\widehat{H}, \eta)$-monotone operator.

We now present a new example which shows that for given single-valued operators $\widehat{H} : X \to X$ and $\eta : X \times X \to X$, a maximal η-monotone operator need not be $(\widehat{H}, \eta)$-monotone.

Example 2.6. Let $m, n \in \mathbb{N}$ and $M_{m\times n}(\mathbb{F})$ be the space of all $m \times n$ matrices with real or complex entries. Then

$$M_{m\times n}(\mathbb{F}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{F}, i = 1, 2, \dots, m; j = 1, 2, \dots, n; \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

is a Hilbert space with respect to the Hilbert-Schmidt norm

$$\|A\| = \Big(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{F})$$

induced by the Hilbert-Schmidt inner product

$$\langle A, B \rangle = tr(A^* B) = \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{a}_{ij} b_{ij}, \quad \forall A, B \in M_{m\times n}(\mathbb{F}),$$

where tr denotes the trace, that is, the sum of the diagonal entries, and A^* denotes the Hermitian conjugate (or adjoint) of the matrix A, that is, $A^* = \overline{A^t}$, the complex conjugate of the transpose A, and the bar denotes complex conjugation and superscript denotes the transpose of the entries. Let us denote by $D_n(\mathbb{R})$ the space of all diagonal $n \times n$ matrices with real entries, that is, the (i, j)-entry is an arbitrary real number if $i = j$, and is zero if $i \neq j$. Then

$$D_n(\mathbb{R}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{R}, a_{ij} = 0 \text{ if } i \neq j; i, j = 1, 2, \dots, n\}$$

is a subspace of $M_{n\times n}(\mathbb{R}) = M_n(\mathbb{R})$ with respect to the operations of addition and scalar multiplication defined on $M_n(\mathbb{R})$, and the Hilbert-Schmidt inner product on $D_n(\mathbb{R})$, and the Hilbert-Schmidt norm induced by it become

$$\langle A, B \rangle = tr(A^* B) = tr(AB)$$

and

$$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{tr(AA)} = \Big(\sum_{i=1}^{n} a_{ii}^2 \Big)^{\frac{1}{2}},$$

respectively. Let us now define the operators $\widehat{M}, \widehat{H} : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $\eta : D_n(\mathbb{R}) \times D_n(\mathbb{R}) \to D_n(\mathbb{R})$, respectively, as

$$\widehat{M}(A) = \widehat{M}(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} \frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} n_i a_{ij}} \end{pmatrix},$$

$$\widehat{H}(A) = \widehat{H}(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} \xi_i + \rho_i \sin^{k_i} t_i a_{ij} \end{pmatrix}$$

and

$$\eta(A, B) = \eta(\begin{pmatrix} a_{ij} \end{pmatrix}, \begin{pmatrix} b_{ij} \end{pmatrix}) = \begin{pmatrix} \theta_i \cos^{p_i} m_i a_{ij} \cos^{P_i} m_i b_{ij} (\sin^{q_i} n_i b_{ij} - \sin^{q_i} n_i a_{ij}), \end{pmatrix}$$

for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, where $\xi_i \in \mathbb{R}$, $\alpha_i, \beta_i, \gamma_i, \theta_i, p_i, m_i, n_i, t_i > 0$ and p_i, q_i and k_i are arbitrary but fixed even natural numbers for $i = 1, 2, \dots, n$.

For any $A, B \in D_n(\mathbb{R})$, we yield

$$\begin{aligned}
&\langle \widehat{M}(A) - \widehat{M}(B), \eta(A, B)\rangle \\
&= tr\Big(\Big(\ \tfrac{\alpha_i}{\beta_i+\gamma_i \sin^{q_i} n_i a_{ij}} - \tfrac{\alpha_i}{\beta_i+\gamma_i \sin^{q_i} n_i b_{ij}} \ \Big) \\
&\qquad \Big(\ \theta_i \cos^{p_i} m_i a_{ij} \cos^{P_i} m_i b_{ij} (\sin^{q_i} n_i b_{ij} - \sin^{q_i} n_i a_{ij}) \ \Big)\Big) \\
&= tr\Big(\Big(\ \tfrac{\alpha_i \gamma_i \theta_i \cos^{p_i} m_i a_{ij} \cos^{P_i} m_i b_{ij} (\sin^{q_i} n_i a_{ij} - \sin^{q_i} n_i b_{ij})^2}{(\beta_i+\gamma_i \sin^{q_i} n_i a_{ij})(\beta_i+\gamma_i \sin^{q_i} n_i b_{ij})} \ \Big)\Big) \\
&= \sum_{i=1}^{n} \frac{\alpha_i \gamma_i \theta_i \cos^{p_i} m_i a_{ij} \cos^{P_i} m_i b_{ij} (\sin^{q_i} n_i a_{ij} - \sin^{q_i} n_i b_{ij})^2}{(\beta_i + \gamma_i \sin^{q_i} n_i a_{ij})(\beta_i + \gamma_i \sin^{q_i} n_i b_{ij})} \geq 0,
\end{aligned}$$

which means that $\widehat{M}$ is an η-monotone operator.

Let us define for each $i \in \{1, 2, \dots, n\}$, the function $f_i : \mathbb{R} \to \mathbb{R}$ as $f_i(x); = \xi_i + \rho_i \sin^{k_i} t_i x + \frac{\alpha_i}{\beta_i+\gamma_i \sin^{q_i} n_i x}$, for all $x \in \mathbb{R}$. Then, for any $A \in D_n(\mathbb{R})$, we obtain

$$\begin{aligned}
(\widehat{H} + \widehat{M})(A) &= (\widehat{H} + \widehat{M})(\big(\ a_{ij} \ \big)) \\
&= \Big(\ \xi_i + \rho_i \sin^{k_i} t_i a_{ij} + \tfrac{\alpha_i}{\beta_i+\gamma_i \sin^{q_i} n_i a_{ij}} \ \Big) \\
&= \Big(\ f_i(a_{ij}) \ \Big).
\end{aligned}$$

Taking into account that for each $i \in \{1, 2, \dots, n\}$,

$$f_i(x) = \xi_i + \rho_i \sin^{k_i} t_i x + \frac{\alpha_i}{\beta_i + \gamma_i \sin^{q_i} n_i x} \geq \xi_i + \frac{\alpha_i}{\beta_i + \gamma_i},$$

it follows that $x \notin f_i(\mathbb{R})$ for all $x \in (-\infty, \frac{\xi_i(\beta_i+\gamma_i)+\alpha_i}{\beta_i+\gamma_i})$, that is, for each $i \in \{1, 2, \dots, n\}$, $f_i(\mathbb{R}) \neq \mathbb{R}$. Consequently, $(\widehat{H} + \widehat{M})(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, i.e., $\widehat{H} + \widehat{M}$ is not surjective, and so $\widehat{M}$ is not $(\widehat{H}, \eta)$-monotone. Now, let $\lambda > 0$ be an arbitrary real constant and assume that for each $i \in \{1, 2, \dots, n\}$, the function $g_i : \mathbb{R} \to \mathbb{R}$ is defined by $g_i(x) := x + \frac{\lambda\alpha_i}{\beta_i+\gamma_i \sin^{q_i} n_i x}$, for all $x \in \mathbb{R}$. Then, for any $A = \big(\ a_{ij} \ \big) \in D_n(\mathbb{R})$, we have

$$(I + \lambda\widehat{M})(A) = (I + \lambda\widehat{M})(\big(\ a_{ij} \ \big)) = \Big(\ a_{ij} + \tfrac{\lambda\alpha_i}{\beta_i+\gamma_i \sin^{q_i} n_i a_{ij}} \ \Big) = \Big(\ g_i(a_{ij}) \ \Big),$$

where I is the identity operator on $D_n(\mathbb{R})$. Since $g_i(\mathbb{R}) = \mathbb{R}$ for each $i \in \{1, 2, \dots, n\}$, we conclude that $(I + \lambda\widehat{M})(D_n(\mathbb{R})) = D_n(\mathbb{R})$, that is, $I + \lambda\widehat{M}$ is surjective. Taking into account the arbitrariness in the choice of $\lambda > 0$, it follows that $\widehat{M}$ is a maximal η-monotone operator.

According to Example 2.5, for given vector-valued operators $\widehat{H} : X \to X$ and $\eta : X \times X \to X$, a $(\widehat{H}, \eta)$-monotone operator need not be maximal η-monotone. As a consequent of Theorem 2.1 in [21], we obtain the following conclusion in which the sufficient conditions for a $(\widehat{H}, \eta)$-monotone operator $\widehat{M}$ to be maximal η-monotone are stated.

Lemma 2.7. *Let $\eta : X \times X \to X$ be a vector-valued operator, $\widehat{H} : X \to X$ be a strictly η-monotone operator, $\widehat{M} : X \rightrightarrows X$ be a $(\widehat{H}, \eta)$-monotone operator, and let $x, u \in X$ be two given points. If $\langle u - v, \eta(x, y)\rangle \geq 0$ holds, for all $(v, y) \in \mathrm{Graph}(\widehat{M})$, then $(u, x) \in \mathrm{Graph}(\widehat{M})$, that is, $\widehat{M}$ is a maximal η-monotone operator.*

Invoking Example 2.6, for given vector-valued operators $\widehat{H} : X \to X$ and $\eta : X \times X \to X$, a maximal η-monotone operator may not be $(\widehat{H}, \eta)$-monotone. The next result provides the sufficient conditions for a maximal η-monotone operator to be $(\widehat{H}, \eta)$-monotone. Before dealing with it, we need to recall the following definitions.

Definition 2.8. *[14, Definition 2.2] A single-valued operator $\widehat{H}: X \to X$ is said to be coercive if*

$$\lim_{\|x\| \to +\infty} \frac{\langle \widehat{H}(x), x \rangle}{\|x\|} = +\infty.$$

Definition 2.9. [14, Definition 2.3] *An operator $A: X \to X$ is said to be bounded if $A(B)$ is bounded for every bounded subset B of X. A is said to be hemi-continuous if for any fixed $x, y, z \in X$, the function $t \mapsto \langle A(x + ty), z \rangle$ is continuous at 0^+.*

Lemma 2.10. *Let $\eta: X \times X \to X$ be a vector-valued operator and $\widehat{H}: X \to X$ be a bounded, coercive, hemi-continuous and η-monotone operator. If $\widehat{M}: X \rightrightarrows X$ is a maximal η-monotone operator, then $\widehat{M}$ is $(\widehat{H}, \eta)$-monotone.*

Proof. Since $\widehat{H}$ is bounded, coercive, hemi-continuous and η-monotone, in the light of Corollary 32.26 of [32], it follows that $\widehat{H} + \lambda\widehat{M}$ is surjective for every $\lambda > 0$, i.e., the range of $\widehat{H} + \lambda\widehat{M}$ is precisely X for all $\lambda > 0$. Thus, $\widehat{M}$ is a $(\widehat{H}, \eta)$-monotone operator. This is the desired result. □

Theorem 2.11. *Let $\eta: X \times X \to X$ be a vector-valued operator, $\widehat{H}: X \to X$ be a strictly η-monotone operator and $\widehat{M}: X \rightrightarrows X$ be an η-monotone operator. Then, for every real constant $\lambda > 0$, the operator $(\widehat{H} + \lambda\widehat{M})^{-1}$ from* Range$(\widehat{H} + \lambda\widehat{M})$ *to X is single-valued.*

Proof. Suppose, by contradiction, that there exists $z \in \text{Range}(\widehat{H} + \lambda\widehat{M})$ such that $x, y \in (\widehat{H} + \lambda\widehat{M})^{-1}(z)$ and $x \neq y$. Then, we have $z \in (\widehat{H} + \lambda\widehat{M})(x)$ and $z \in (\widehat{H} + \lambda\widehat{M})(y)$, and so there exist $u \in \widehat{M}(x)$ and $v \in \widehat{M}(y)$ such that

$$\widehat{H}(x) + \lambda u = \widehat{H}(y) + \lambda v. \tag{1}$$

Taking into account that $\widehat{M}$ and $\widehat{H}$ are η-monotone, by (1), yields

$$0 \leq \langle \lambda(u - v), \eta(x, y) \rangle = -\langle \widehat{H}(x) - \widehat{H}(y), \eta(x, y) \rangle \leq 0,$$

which implies that $\langle \widehat{H}(x) - \widehat{H}(y), \eta(x, y) \rangle = 0$. Since $\widehat{H}$ is strictly η-monotone, it follows that $x = y$ which is in contradiction to our assumption. □

With the goal of defining the resolvent operator associated with a $(\widehat{H}, \eta)$-monotone operator, Fang et al. [15] presented the following assertion which is an immediate consequence of the previous theorem.

Lemma 2.12. [15, Lemma 2.1] *Let $\eta: X \times X \to X$ be a vector-valued operator, $\widehat{H}: X \to X$ be a strictly η-monotone and $\widehat{M}: X \rightrightarrows X$ be a $(\widehat{H}, \eta)$-monotone operator. Then, for every real constant $\lambda > 0$, the operator $(\widehat{H} + \lambda\widehat{M})^{-1}$ is single-valued.*

Letting $\eta(x, y) = x - y$ for all $x, y \in X$, we obtain the following result due to Fang and Huang [14] as a direct consequence of the previous conclusion.

Lemma 2.13. [14, Theorem 2.1] *Let $\widehat{H}: X \to X$ be a strictly monotone operator and $\widehat{M}: X \rightrightarrows X$ be an $\widehat{H}$-monotone operator. Then, for every real constant $\lambda > 0$, the operator $(\widehat{H} + \lambda\widehat{M})^{-1}$ is single-valued.*

Based on Lemma 2.12, for an arbitrary real constant $\lambda > 0$, Fang et al. [15] defined the resolvent operator $R^{\widehat{H},\eta}_{\widehat{M},\lambda}$ associated with a $(\widehat{H}, \eta)$-monotone operator $\widehat{M}$ as follows.

Definition 2.14. [15, Definition 2.4] *Let $\eta : X \times X \to X$ be a vector-valued operator, $\widehat{H} : X \to X$ be a strictly η-monotone operator and $\widehat{M} : X \rightrightarrows X$ be a $(\widehat{H}, \eta)$-monotone operator. The resolvent operator $R^{\widehat{H},\eta}_{\widehat{M},\lambda} : X \to X$ is defined by*

$$R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) = (\widehat{H} + \lambda\widehat{M})^{-1}(u), \quad \forall u \in X,$$

where $\lambda > 0$ is an arbitrary real constant.

Taking $\eta(x, y) = x - y$, for all $x, y \in X$ in Definition 2.14, we obtain the definition of the resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda}$ associated with a $\widehat{H}$-monotone operator and an arbitrary real constant $\lambda > 0$ as follows.

Definition 2.15. [14, Definition 2.4] *Let $\widehat{H} : X \to X$ be a strictly monotone operator and $\widehat{M} : X \rightrightarrows X$ be a $\widehat{H}$-monotone operator. The resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda} : X \to X$ is defined by*

$$R^{\widehat{H}}_{\widehat{M},\lambda}(u) = (\widehat{H} + \lambda\widehat{M})^{-1}(u), \quad \forall u \in X,$$

where $\lambda > 0$ is an arbitrary real constant.

It should be remarked that in the rest of the paper, we say that $\widehat{M}$ is a $(\widehat{H}, \eta)$-ς-strongly (resp. $\widehat{H}$-ς-strongly) monotone operator, means that $\widehat{M}$ is a ς-strongly η-monotone (resp. ς-strongly monotone) operator and $(\widehat{H} + \lambda\widehat{M})(X) = X$, for every real constant $\lambda > 0$.

In the next theorem, the Lipschitz continuity of the resolvent operator $R^{\widehat{H},\eta}_{\widehat{M},\lambda}$ associated with a $(\widehat{H}, \eta)$-monotone operator $\widehat{M}$ and an arbitrary real constant $\lambda > 0$ under some appropriate conditions is proved and an estimate of its Lipschitz constant is computed. Before turning to the main result of this section, let us recall the following definition.

Definition 2.16. *A vector-valued operator $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau\|x - y\|$ for all $x, y \in X$.*

Theorem 2.17. *Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous operator, $\widehat{H} : X \to X$ be a π-strongly η-monotone operator and let $\widehat{M} : X \rightrightarrows X$ be a $(\widehat{H}, \eta)$-ς-strongly monotone operator. Then, the resolvent operator $R^{\widehat{H},\eta}_{\widehat{M},\lambda} : X \to X$ is $\frac{\tau}{\lambda\varsigma+\pi}$-Lipschitz continuous, i.e.,*

$$\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\| \leq \frac{\tau}{\lambda\varsigma + \pi}\|u - v\|, \quad \forall u, v \in X.$$

Proof. Owing to the fact that $\widehat{M}$ is a $(\widehat{H}, \eta)$-monotone operator, for any given points $u, v \in X$ with $\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\| \neq 0$, we have

$$R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) = (\widehat{H} + \lambda\widehat{M})^{-1}(u) \text{ and } R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v) = (\widehat{H} + \lambda\widehat{M})^{-1}(v),$$

whence we deduce that

$$\lambda^{-1}(u - \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u))) \in \widehat{M}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u)) \text{ and } \lambda^{-1}(v - \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v))) \in \widehat{M}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)).$$

In the light of the fact that $\widehat{M}$ is ς-strongly η-monotone, yields

$$\lambda^{-1}\langle u - \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u)) - (v - \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v))), \eta(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u), R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v))\rangle \geq \varsigma\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\|^2.$$

Taking into account that $\lambda^{-1} > 0$, the preceding inequality implies that

$$\begin{aligned}\langle u - v, \eta(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u), R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v))\rangle \geq{}& \lambda\varsigma\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\|^2 \\ &+ \langle \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u)) - \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)), \eta(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u), R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v))\rangle.\end{aligned}$$

Using τ-Lipschitz continuity of the operator η, π-strong η-monotonicity of $\widehat{H}$ and the last inequality, we obtain

$$\begin{aligned}\|u - v\|\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\| &\geq \|u - v\|\|\eta(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u), R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v))\| \\ &\geq \lambda\varsigma\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\|^2 \\ &\quad + \langle \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u)) - \widehat{H}(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)), \eta(R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u), R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v))\rangle \\ &\geq \lambda\varsigma\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\|^2 + \pi\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\|^2 \\ &= (\lambda\varsigma + \pi)\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\|^2.\end{aligned}$$

Since $\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\| \neq 0$, from the preceding inequality, it follows that

$$\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}(u) - R^{\widehat{H},\eta}_{\widehat{M},\lambda}(v)\| \leq \frac{\tau}{\lambda\varsigma + \pi}\|u - v\|.$$

This gives the desired result. □

Defining the operator η as $\eta(x, y) = x - y$ for all $x, y \in X$, we obtain the following corollary as an immediate consequence of the previous theorem.

Corollary 2.18. *Let $\widehat{H} : X \to X$ be a π-strongly monotone operator and let $\widehat{M} : X \rightrightarrows X$ be a $\widehat{H}$-ς-strongly monotone operator. Then, the resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda} : X \to X$ is $\frac{1}{\lambda\varsigma+\pi}$-Lipschitz continuous, i.e.,*

$$\|R^{\widehat{H}}_{\widehat{M},\lambda}(u) - R^{\widehat{H}}_{\widehat{M},\lambda}(v)\| \leq \frac{1}{\lambda\varsigma + \pi}\|u - v\|, \quad \forall u, v \in X.$$

3. Formulation, Iterative Algorithms and Convergence Results

For given single-valued operators $P, F, G : X \to X$ and multi-valued operators $R, S, T : X \rightrightarrows CB(X)$ and $\widehat{M} : X \rightrightarrows X$, we consider the problem of finding $x \in X$, $u \in R(x)$, $v \in S(x)$ and $w \in T(x)$ such that

$$0 \in P(u) - (F(v) - G(w)) + \widehat{M}(x), \tag{2}$$

which is called a *multi-valued variational inclusion problem* (MVIP).

It should be remarked that the MVIP (2) includes many variational inequalities/inclusions and complementarity problems as special cases, see, for example, [14, 17, 20, 33] and the references therein.

The next conclusion gives us a characterization of a solution of the MVIP (2) and plays a key role in establishing our main results in the sequel.

Lemma 3.1. *Let P, F, G, R, S and T be the same as in the* MVIP (2)*, and let $\eta : X \times X \to X$ be a vector-valued operator. Suppose further that $\widehat{H} : X \to X$ is a strictly η-monotone operator and $\widehat{M} : X \rightrightarrows X$ is a $(\widehat{H}, \eta)$-monotone operator. Then, $(x, u, v, w) \in X \times R(x) \times S(x) \times T(x)$ is a solution of the* MVIP (2) *if (x, u, v, w) satisfies*

$$x = R^{\widehat{H},\eta}_{\widehat{M},\lambda}[\widehat{H}(x) - \lambda(P(u) - (F(v) - G(w)))], \tag{3}$$

where $\lambda > 0$ is a real constant.

Proof. It follows directly from Definition 2.14 and some simple arguments. □

Taking $\eta(x,y) = x - y$ for all $x, y \in X$, we obtain the following result as an immediate consequence of Lemma 3.1 which provides a characterization of the solution of the problem (2) involving $\widehat{H}$-monotone operator.

Lemma 3.2. *Assume that P, F, G, R, S and T are the same as in the* MVIP (2). *Furthermore, let $\widehat{H} : X \to X$ be a strictly monotone operator and $\widehat{M} : X \rightrightarrows X$ be a $\widehat{H}$-monotone operator. Then, $(x, u, v, w) \in X \times R(x) \times S(x) \times T(x)$ is a solution of the* MVIP (2) *if*

$$x = R^{\widehat{H}}_{\widehat{M},\lambda}[\widehat{H}(x) - \lambda(P(u) - (F(v) - G(w)))],$$

where $\lambda > 0$ is a real constant.

Lemma 3.3. [24] *Let X be a complete metric space and $T : X \rightrightarrows CB(X)$ be a multi-valued mapping. Then, for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \leq (1 + \varepsilon)D(T(x), T(y)),$$

where $D(.,.)$ is the Hausdorff metric on $CB(X)$.

Based on the fixed point formulation (3) and using Nadler's technique [24], we can construct the following iterative algorithm with mixed errors for finding an approximate solution of the MVIP (2).

Algorithm 3.4. *Let P, F, G, R, S and T be the same as in the* MVIP (2). *Suppose that $\eta : X \times X \to X$ is a vector-valued operator, $\widehat{H} : X \to X$ is a strictly η-monotone operator, and $\widehat{M} : X \rightrightarrows X$ is a $(\widehat{H}, \eta)$-monotone operator. For any given $x_0 \in X$, $u_0 \in R(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, define the sequences $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ in X by the iterative schemes*

$$x_{n+1} = (1 - \alpha)x_n + \alpha R^{\widehat{H},\eta}_{\widehat{M},\lambda}[\widehat{H}(x_n) - \lambda(P(u_n) - (F(v_n) - G(w_n)))], \tag{4}$$

$$u_n \in R(x_n); \|u_n - u_{n+1}\| \leq (1 + (1 + n)^{-1})D(R(x_n), R(x_{n+1})), \tag{5}$$

$$v_n \in S(x_n); \|v_n - v_{n+1}\| \leq (1 + (1 + n)^{-1})D(S(x_n), S(x_{n+1})), \tag{6}$$

$$w_n \in T(x_n); \|w_n - w_{n+1}\| \leq (1 + (1 + n)^{-1})D(T(x_n), T(x_{n+1})), \tag{7}$$

where $n = 0, 1, 2 \ldots$; $\lambda > 0$ is a real constant; $\alpha \in (0, 1]$ is a relaxation parameter, $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ are two sequences in X to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

$$\begin{cases} \lim\limits_{n\to\infty} \|p_n\| = \lim\limits_{n\to\infty} \|q_n\| = 0; \\ \sum\limits_{n=0}^{\infty} \|p_n - p_{n-1}\| < \infty, \sum\limits_{n=0}^{\infty} \|q_n - q_{n-1}\| < \infty. \end{cases} \tag{8}$$

If $\alpha = 1$ and $p_n = q_n = 0$ for all $n \geq 0$, then Algorithm 3.4 collapses to the following iterative algorithm.

Algorithm 3.5. *Suppose that $P, F, G, R, S, T, \widehat{H}$ and $\widehat{M}$ are the same as in Lemma 3.2. For any given $x_0 \in X$, $u_0 \in R(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, define the sequences $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ in X by the iterative processes*

$$x_{n+1} = R^{\widehat{H}}_{\widehat{M},\lambda}[\widehat{H}(x_n) - \lambda(P(u_n) - (F(v_n) - G(w_n)))],$$

$$u_n \in R(x_n); \|u_n - u_{n+1}\| \leq (1 + (1 + n)^{-1})D(R(x_n), R(x_{n+1})),$$

$$v_n \in S(x_n); \|v_n - v_{n+1}\| \leq (1 + (1 + n)^{-1})D(S(x_n), S(x_{n+1})),$$

$$w_n \in T(x_n); \|w_n - w_{n+1}\| \leq (1 + (1 + n)^{-1})D(T(x_n), T(x_{n+1})),$$

where $n = 0, 1, 2, \ldots$; and $\lambda > 0$ is a real constant.

Before proceeding to the study of convergence analysis of the sequences generated by our proposed iterative algorithms, we need to recall the following definitions.

Definition 3.6. *A multi-valued operator $T : X \rightrightarrows CB(X)$ is said to be D-Lipschitz continuous with constant (or δ-D-Lipschitz continuous) if there exits a constant $\delta > 0$ such that*

$$D(T(x), T(y)) \leq \delta \|x - y\|, \quad \forall x, y \in X.$$

Definition 3.7. *Let $\widehat{H}, P : X \to X$ be single-valued operators and $T : X \rightrightarrows CB(X)$ be a multi-valued operator. Then, $\widehat{H}$ is said to be*

(i) *r-strongly monotone with respect to P, if there exists a constant $r > 0$ such that*

$$\langle \widehat{H}(x) - \widehat{H}(y), P(x) - P(y) \rangle \geq r\|x - y\|^2, \quad \forall x, y \in X;$$

(ii) *θ-strongly monotone with respect to P and T, if there exists a constant $\theta > 0$ such that*

$$\langle \widehat{H}(x) - \widehat{H}(y), P(x) - P(y) \rangle \geq \theta\|x - y\|^2, \quad \forall x, y \in X, u \in T(x), v \in T(y).$$

Definition 3.8. [1, Definition 2.6] *For a given mapping $G : X \to X$, a multi-valued mapping $T : X \rightrightarrows X$ is said to be*

(i) *relaxed Lipschitz continuous with respect to G, if there exists a constant $k > 0$ such that*

$$\langle G(u) - G(v), x - y \rangle \leq -k\|x - y\|^2, \quad \forall x, y \in X, u \in T(x), v \in T(y);$$

(ii) *relaxed monotone with respect to G, if there exists a constant $c > 0$ such that*

$$\langle G(u) - G(v), x - y \rangle \geq -c\|x - y\|^2, \quad \forall x, y \in X, u \in T(x), v \in T(y).$$

Comparing Definition 2.6 in [1] and Definition 3.8, one can notice that a small change has been made in Definition 3.8. In fact, by a careful reading the proof of Theorem 4.1 of [1], we found that in parts (i) and (ii) of [1, Definition 2.6], "for some $u \in T(x), v \in T(y)$" must be replaced by "for all $u \in T(x), v \in T(y)$", as we have done in parts (i) and (ii) of Definition 3.8.

Theorem 3.9. *Let the operators $P, F, G : X \to X$ be Lipschitz continuous with constants ρ, ξ and r, respectively, the operators $R, S, T : X \rightrightarrows CB(X)$ be D-Lipschitz continuous with constants e, h and d, respectively, S is k-relaxed Lipschitz continuous with respect to F, and T is c-relaxed monotone with respect to G. Suppose that $\eta : X \times X \to X$ is a τ-Lipschitz continuous operator, $\widehat{H} : X \to X$ is a π-strongly η-monotone, δ-strongly monotone with respect to P and R, and t-Lipschitz continuous operator, and $\widehat{M} : X \rightrightarrows X$ is a $(\widehat{H}, \eta)$-ς-strongly monotone operator. If there exits a real constant $\lambda > 0$ such that*

$$\sqrt{t^2 - 2\lambda\delta + \lambda^2\rho^2e^2} + \sqrt{1 - 2\lambda(k - c) + \lambda^2(\xi h + rd)^2} < \frac{\lambda\varsigma + \pi}{\tau} - 1, \tag{9}$$

$$2\lambda\delta \leq t^2 + \lambda^2\rho^2e^2, \; 2\lambda(k - c) \leq 1 + \lambda^2(\xi h + rd)^2, \tag{10}$$

then, the iterative sequences $\{x_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ generated by Algorithm 3.4 converge strongly to x, u, v and w, respectively, and (x, u, v, w) is a solution of the MVIP (2).

Proof. Making use of (4) and Theorem 2.17, for each $n \in \mathbb{N}$, yields

$$\begin{aligned}
\|x_{n+1} - x_n\| \leq\ & (1 - \alpha)\|x_n - x_{n-1}\| + \alpha\|R^{\widehat{H},\eta}_{\widehat{M},\lambda}[\widehat{H}(x_n) - \lambda(P(u_n) - (F(v_n) - G(w_n)))] \\
& - R^{\widehat{H},\eta}_{\widehat{M},\lambda}[\widehat{H}(x_{n-1}) - \lambda(P(u_{n-1}) - (F(v_{n-1}) - G(w_{n-1})))]\| \\
& + \alpha\|p_n - p_{n-1}\| + \|q_n - q_{n-1}\| \\
\leq\ & (1 - \alpha)\|x_n - x_{n-1}\| + \frac{\alpha\tau}{\lambda\varsigma + \pi}\big(\|\widehat{H}(x_n) - \widehat{H}(x_{n-1}) - \lambda(P(u_n) - P(u_{n-1}))\| \\
& + \|x_n - x_{n-1} + \lambda(F(v_n) - F(v_{n-1}) - (G(w_n) - G(w_{n-1})))\| \\
& + \|x_n - x_{n-1}\| + \alpha\|p_n - p_{n-1}\| + \|q_n - q_{n-1}\|.
\end{aligned} \tag{11}$$

Taking into account that $\widehat{H}$ is δ-strongly monotone with respect to P and R and t-Lipschitz continuous, and the operator P is ρ-Lipschitz continuous, using (5), e-D-Lipschitz continuity of R and utilizing well known property of the norm arising from inner product in Hilbert space X, it follows that

$$\begin{aligned}
&\|\widehat{H}(x_n) - \widehat{H}(x_{n-1}) - \lambda(P(u_n) - P(u_{n-1}))\|^2 \\
&= \|\widehat{H}(x_n) - \widehat{H}(x_{n-1})\|^2 - 2\lambda\langle \widehat{H}(x_n) - \widehat{H}(x_{n-1}), P(u_n) - P(u_{n-1})\rangle \\
&\quad + \lambda^2\|P(u_n) - P(u_{n-1})\|^2 \\
&\leq t^2\|x_n - x_{n-1}\|^2 - 2\lambda\delta\|x_n - x_{n-1}\|^2 + \lambda\rho^2\|u_n - u_{n-1}\|^2 \\
&\leq (t^2 - 2\lambda\delta + \lambda^2\rho^2e^2(1 + n^{-1})^2)\|x_n - x_{n-1}\|^2,
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
&\|\widehat{H}(x_n) - \widehat{H}(x_{n-1}) - \lambda(P(u_n) - P(u_{n-1}))\| \\
&\leq \sqrt{t^2 - 2\lambda\delta + \lambda^2\rho^2e^2(1 + n^{-1})^2}\|x_n - x_{n-1}\|.
\end{aligned} \tag{12}$$

Since S is k-relaxed Lipschitz continuous with respect to F, the operator T is c-relaxed monotone with respect to G, the operators F and G are Lipschitz continuous with constants ξ and r, respectively, S and T are D-Lipschitz continuous multi-valued operators with constants h and d, respectively, making use of (6) and (7) and well known property of the norm arising from inner product in Hilbert space X, yields

$$\begin{aligned}
&\|x_n - x_{n-1} + \lambda(F(v_n) - F(v_{n-1}) - (G(w_n) - G(w_{n-1})))\|^2 \\
&= \|x_n - x_{n-1}\|^2 + 2\lambda\langle F(v_n) - F(v_{n-1}) - (G(w_n) - G(w_{n-1})), x_n - x_{n-1}\rangle \\
&\quad + \lambda^2\|F(v_n) - F(v_{n-1}) - (G(w_n) - G(w_{n-1}))\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\lambda\langle F(v_n) - F(v_{n-1}), x_n - x_{n-1}\rangle \\
&\quad - 2\lambda\langle G(w_n) - G(w_{n-1}), x_n - x_{n-1}\rangle \\
&\quad + \lambda^2(\|F(v_n) - F(v_{n-1})\|^2 + \|G(w_n) - G(w_{n-1})\|)^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\lambda k\|x_n - x_{n-1}\|^2 + 2\lambda c\|x_n - x_{n-1}\|^2 \\
&\quad + \lambda^2(\xi\|v_n - v_{n-1}\| + r\|w_n - w_{n-1}\|)^2 \\
&\leq (1 - 2\lambda(k - c) + \lambda^2(\xi h + rd)^2(1 + n^{-1})^2)\|x_n - x_{n-1}\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_n - x_{n-1} + \lambda(F(v_n) - F(v_{n-1}) - (G(w_n) - G(w_{n-1})))\| \\
&\leq \sqrt{1 - 2\lambda(k - c) + \lambda^2(\xi h + rd)^2(1 + n^{-1})^2}\|x_n - x_{n-1}\|.
\end{aligned} \tag{13}$$

Substituting (12) and (13) into (11), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 - \alpha)\|x_n - x_{n-1}\| + \frac{\alpha\tau}{\lambda\varsigma + \pi}\Big(\sqrt{t^2 - 2\lambda\delta + \lambda^2\rho^2e^2(1 + n^{-1})^2} \\
&\quad + \sqrt{1 - 2\lambda(k - c) + \lambda^2(\xi h + rd)^2(1 + n^{-1})^2} + 1\Big)\|x_n - x_{n-1}\| \\
&\quad + \alpha\|p_n - p_{n-1}\| + \|q_n - q_{n-1}\| \\
&= (1 - \alpha)\|x_n - x_{n-1}\| + \alpha\varphi(n)\|x_n - x_{n-1}\| + \alpha\|p_n - p_{n-1}\| + \|q_n - q_{n-1}\|,
\end{aligned} \tag{14}$$

where for each $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(n) = \frac{\tau}{\lambda\varsigma + \pi}\Big(&\sqrt{t^2 - 2\lambda\delta + \lambda^2\rho^2e^2(1 + n^{-1})^2} \\
&+ \sqrt{1 - 2\lambda(k - c) + \lambda^2(\xi h + rd)^2(1 + n^{-1})^2} + 1\Big).
\end{aligned}$$

Clearly, $\varphi(n) \to \varphi$, as $n \to \infty$, where

$$\varphi = \frac{\tau}{\lambda\varsigma + \pi}\left(\sqrt{t^2 - 2\lambda\delta + \lambda^2\rho^2 e^2} + \sqrt{1 - 2\lambda(k-c) + \lambda^2(\xi h + rd)^2} + 1\right).$$

Let $\vartheta(n) = 1 - \alpha + \alpha\varphi(n)$ for each $n \in \mathbb{N}$. Then, we know that $\vartheta(n) \to \vartheta$, as $n \to \infty$, where $\vartheta = 1 - \alpha + \alpha\varphi$. Evidently, (10) and (11) imply that $\varphi \in (0,1)$, and so $\vartheta \in (0,1)$. Hence, there exist $\hat{\vartheta} \in (0,1)$ (take $\hat{\vartheta} = \frac{\vartheta+1}{2} \in (\vartheta, 1)$) and $n_0 \in \mathbb{N}$ such that $\vartheta(n) \leq \hat{\vartheta}$, for all $n \geq n_0$. Then, for all $n > n_0$, by (14), we deduce that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \hat{\vartheta}\|x_n - x_{n-1}\| + \alpha\|p_n - p_{n-1}\| + \|q_n - q_{n-1}\| \\
&\leq \hat{\vartheta}[\hat{\vartheta}\|x_{n-1} - x_{n-2}\| + \alpha\|p_{n-1} - p_{n-2}\| + \|q_{n-1} - q_{n-2}\|] \\
&\quad + \alpha\|p_n - p_{n-1}\| + \|q_n - q_{n-1}\| \\
&= \hat{\vartheta}^2\|x_{n-1} - x_{n-2}\| + \alpha(\hat{\vartheta}\|p_{n-1} - p_{n-2}\| + \|p_n - p_{n-1}\|) \\
&\quad + \hat{\vartheta}\|q_{n-1} - q_{n-2}\| + \|q_n - q_{n-1}\| \\
&\leq \dots \\
&\leq \hat{\vartheta}^{n-n_0}\|x_{n_0+1} - x_{n_0}\| + \alpha\sum_{i=1}^{n-n_0}\hat{\vartheta}^{i-1}\|p_{n-(i-1)} - p_{n-i}\| \\
&\quad + \sum_{i=1}^{n-n_0}\hat{\vartheta}^{i-1}\|q_{n-(i-1)} - q_{n-i}\|.
\end{aligned} \tag{15}$$

Making use of (15) it follows that for any $m \geq n > n_0$,

$$\begin{aligned}
\|x_m - x_n\| &\leq \sum_{i=n}^{m-1}\|x_{i+1} - x_i\| \leq \sum_{i=n}^{m-1}\hat{\vartheta}^{i-n_0}\|x_{n_0+1} - x_{n_0}\| \\
&\quad + \alpha\sum_{i=n}^{m-1}\sum_{j=1}^{i-n_0}\hat{\vartheta}^{j-1}\|p_{i-(j-1)} - p_{i-j}\| + \sum_{i=n}^{m-1}\sum_{j=1}^{i-n_0}\hat{\vartheta}^{j-1}\|q_{i-(j-1)} - q_{i-j}\|.
\end{aligned} \tag{16}$$

Now, in the light of the fact that $\hat{\vartheta} < 1$, (8) and (16) imply that $\|x_m - x_n\| \to 0$, as $n \to \infty$ and so $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. The completeness of X ensures the existence of a $x \in X$ such that $x_n \to x$, as $n \to \infty$. Employing (5) and e-D-Lipschitz continuity of the operator R, we obtain

$$\|u_n - u_{n+1}\| \leq (1 + (1+n)^{-1})D(R(x_n), R(x_{n+1})) \leq (1 + (1+n)^{-1})e\|x_{n+1} - x_n\|,$$

from which we deduce that $\{u_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Thereby, there exists $u \in X$ such that $u_n \to u$, as $n \to \infty$. We now show that $u \in R(x)$. Since $u_n \in R(x_n)$ for each $n \geq 0$, applying (5) and e-D-Lipschitz continuity of R, it follows that

$$\begin{aligned}
d(u, R(x)) &= \inf\{\|u - z\| : z \in R(x)\} \\
&\leq \|u - u_n\| + d(u_n, R(x)) \\
&\leq \|u - u_n\| + D(R(x_n), R(x)) \\
&\leq \|u - u_n\| + e\|x_n - x\|.
\end{aligned}$$

The right-hand side of the above inequality tends to zero, as $n \to \infty$. Taking into account that $R(x)$ is closed, it follows that $u \in R(x)$. By the same arguments as above, one can prove that $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are also Cauchy sequences in X and $v_n \to v$ and $w_n \to w$, as $n \to \infty$, for some $v \in S(x)$ and $w \in T(x)$, respectively.

Taking into consideration the facts that the operators $R^{\widehat{H},\eta}_{\widehat{M},\lambda}$, $\widehat{H}$, P, F and G are continuous, $x_n \to x$, $u_n \to u$, $v_n \to v$ and $w_n \to w$, as $n \to \infty$, utilizing (4) and (8), it follows that

$$x = R^{\widehat{H},\eta}_{\widehat{M},\lambda}[\widehat{H}(x) - \lambda(P(u) - (F(v) - G(w)))].$$

Now, based on Lemma 3.1, (x, u, v, w) is a solution of the MVIP (2). This completes the proof. □

As a direct consequence of the above theorem, we obtain the following assertion concerning the convergence analysis of the sequences generated by Algorithm 3.5.

Corollary 3.10. *Let*

(i) *the operator $\widehat{H} : X \to X$ be π-strongly monotone, δ-strongly monotone with respect to P and R, and t-Lipschitz continuous;*
(ii) *$\widehat{M} : X \rightrightarrows X$ be a $\widehat{H}$-ς-strongly monotone operator;*
(iii) *the operators $R, S, T : X \rightrightarrows CB(X)$ be D-Lipschitz continuous with constants e, h and r, respectively;*
(iv) *the operators $P, F, G : X \to X$ be Lipschitz continuous with constants ρ, ξ and r, respectively;*
(v) *S be k-relaxed Lipschitz continuous with respect to F;*
(vi) *T be c-relaxed monotone with respect to G;*
(vii) *there exists a constant $\lambda > 0$ such that*

$$\sqrt{t^2 - 2\lambda\delta + \lambda^2\rho^2 e^2} + \sqrt{1 - 2\lambda(k - c) + \lambda^2(\xi h + rd)^2} < \lambda\varsigma + \pi - 1,$$

and (10) *holds.*

Then, the iterative sequences $\{x_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ generated by Algorithm 3.5 converge strongly to x, u, v and w, respectively, and (x, u, v, w) is a solution of the MVIP (2)*.*

4. $H(.,.)$-Co-Monotone Mappings and Some Comments

This section is devoted to the investigation and analysis of the $H(.,.)$-Co-monotone mapping introduced in [1] and pointing out some facts concerning it and the results appeared in [1]. At the same time, we show that our results improve and generalize the corresponding results of [1] and some known results in the literature.

Definition 4.1. [1, Definition 2.4] *Let $H : X \times X \to X$ and $A, B : X \to X$ be the mappings. Then*

(i) *$H(A, .)$ is said to be cocoercive with respect to A with constant μ (or μ-cocoercive with respect to A), if there exists a constant $\mu > 0$ such that*

$$\langle H(A(x), u) - H(A(y), u), x - y\rangle \geq \mu\|A(x) - A(y)\|^2, \quad \forall x, y, u \in X;$$

(ii) *$H(., B)$ is said to be relaxed cocoercive with respect to B with constant γ (or γ-cocoercive with respect to B), if there exists a constant $\gamma > 0$ such that*

$$\langle H(u, B(x)) - H(u, B(y)), x - y\rangle \geq -\gamma\|B(x) - B(y)\|^2, \quad \forall x, y, u \in X;$$

(iii) *$H(A, B)$ is said to be mixed Lipschitz continuous with respect to A and B with constant t, if there exists a constant $t > 0$ such that*

$$\|H(A(x), B(x)) - H(A(y), B(y))\| \leq t\|x - y\|, \quad \forall x, y \in X;$$

(iv) *$H(A, B)$ is said to be mixed strongly monotone with respect to A and B with constant ρ, if there exists a constant $\rho > 0$ such that*

$$\langle H(A(x), B(x)) - H(A(y), B(y)), x - y\rangle \leq \rho\|x - y\|^2, \quad \forall x, y \in X;$$

(v) *$H(A, B)$ is said to be symmetric cocoercive (or $\mu\gamma$-symmetric cocoercive) with respect to A and B, if $H(A, .)$ is cocoercive with respect to A with constant μ (or μ-cocoercive with respect to A) and $H(., B)$ is relaxed cocoercive with respect to B with constant γ (or γ-relaxed cocoercive with respect to B).*

Proposition 4.2. *Suppose that $A, B : X \to X$ and $H : X \times X \to X$ are the mappings such that A is η-expansive, B is γ-Lipschitz continuous, and $H(A, B)$ is $\mu\gamma$-symmetric cocoercive with respect to A and B. Furthermore, let $\widehat{H} : X \to X$ be a mapping defined by $\widehat{H}(x) := H(A(x), B(x))$ for all $x \in X$. If $\mu\eta^2 > \gamma\sigma^2$ (resp. $\mu\eta^2 = \gamma\sigma^2$ and $\mu\eta^2 < \gamma\sigma^2$) then $\widehat{H}$ is $(\mu\eta^2 - \gamma\sigma^2)$-strongly monotone and hence it is strictly monotone (resp. monotone and $(\gamma\sigma^2 - \mu\eta^2)$-relaxed monotone).*

Proof. Taking into account the assumptions, for all $x, y \in X$, yields

$$\begin{aligned}\langle \widehat{H}(x) - \widehat{H}(y), x - y\rangle &= \langle H(A(x), B(x)) - H(A(y), B(y)), x - y\rangle \\ &= \langle H(A(x), B(x)) - H(A(y), B(x)), x - y\rangle \\ &\quad + \langle H(A(y), B(x)) - H(A(y), B(y)), x - y\rangle \\ &\geq \mu\|A(x) - A(y)\|^2 - \gamma\|B(x) - B(y)\|^2 \\ &= (\mu\eta^2 - \gamma\sigma^2)\|x - y\|^2.\end{aligned}$$

The preceding inequality implies that the mapping $\widehat{H}$ is $(\mu\eta^2 - \gamma\sigma^2)$-strongly monotone and so it is strictly monotone (resp. monotone and $(\gamma\sigma^2 - \mu\eta^2)$-relaxed monotone) provided that $\mu\eta^2 > \gamma\sigma^2$ (resp. $\mu\eta^2 = \gamma\sigma^2$ and $\mu\eta^2 < \gamma\sigma^2$). This gives the desired result. □

In the light of Proposition 4.2, every bifunction $H : X \times X \to X$ which is $\mu\gamma$-symmetric cocoercive with respect to the mappings A and B, where A is η-expansive and B is σ-Lipschitz continuous, is actually a univariate $(\mu\eta^2 - \gamma\sigma^2)$-strongly monotone (resp. monotone and $(\gamma\sigma^2 - \mu\eta^2)$-relaxed monotone) mapping provided that $\mu\eta^2 > \gamma\sigma^2$ (resp. $\mu\eta^2 = \gamma\sigma^2$ and $\mu\eta^2 < \gamma\sigma^2$) and is not a new one.

Definition 4.3. [1, Definition 2.5] *Let $f, g : X \to X$ and $M : X \times X \rightrightarrows X$ be the mappings. Then*

(i) *$M(f, .)$ is said to be α-strongly monotone with respect to f, if there exits a constant $\alpha > 0$ such that*

$$\langle u - v, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y, w \in X, u \in M(f(x), w), v \in M(f(y), w);$$

(ii) *$M(., g)$ is said to be β-relaxed monotone with respect to g, if there exits a constant $\beta > 0$ such that*

$$\langle u - v, x - y\rangle \geq -\beta\|x - y\|^2, \quad \forall x, y, w \in X, u \in M(w, g(x)), v \in M(w, g(y));$$

(iii) *$M(f, g)$ is said to be symmetric with respect to f and g with constants α and β (or $\alpha\beta$-symmetric monotone with respect to f and g), if $M(f, .)$ is strongly monotone with respect to f with constant α (or α-strongly monotone with respect to f) and $M(., g)$ is relaxed monotone with respect to g with constant β (or β-relaxed monotone with respect to g).*

It is worthwhile to stress that Definition 4.3 is presented making a small change in the context of Definition 2.5 of [1]. In fact, by a careful reading the results in [1], we found that in parts (i) and (ii) of Definition 2.5 of [1], "for some $u \in M(f(x), w), v \in M(f(y), w)$" and "for some $u \in M(w, g(x)), v \in M(w, g(y))$" must be replaced by "for all $u \in M(f(x), w), v \in M(f(y), w)$" and "for all $u \in M(w, g(x))$, $v \in M(w, g(y))$", respectively, as we have done in parts (i) and (ii) of Definition 4.3.

Proposition 4.4. *Let $f, g : X \to X$ and $M : X \times X \rightrightarrows X$ be the mappings such that $M(f, g)$ is $\alpha\beta$-symmetric monotone with respect to f and g. Assume further that $\widehat{M} : X \rightrightarrows X$ is a multi-valued mapping defined by $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$. Then $\widehat{M}$ is $(\alpha - \beta)$-strongly monotone (resp. monotone and $(\beta - \alpha)$-relaxed monotone) provided that $\alpha > \beta$ (resp. $\alpha = \beta$ and $\alpha < \beta$).*

Proof. Taking into consideration the facts that $M(f, .)$ is α-strongly monotone with respect to f and $M(., g)$ is β-relaxed monotone with respect to g, for all $x, y \in X$, $u \in \widehat{M}(x)$ and $v \in \widehat{M}(y)$, we obtain

$$\begin{aligned}\langle u - v, x - y\rangle &= \langle u - w + w - v, x - y\rangle \\ &= \langle u - w, x - y\rangle + \langle w - v, x - y\rangle \\ &\geq \alpha\|x - y\|^2 - \beta\|x - y\|^2 \\ &= (\alpha - \beta)\|x - y\|^2,\end{aligned}$$

for all $w \in M(f(y), g(x))$. Now, if $\alpha > \beta$ (resp. $\alpha = \beta$ and $\alpha < \beta$), then the preceding inequality implies that $\widehat{M}$ is $(\alpha - \beta)$-strongly monotone (resp. monotone and $(\beta - \alpha)$-relaxed monotone). This completes the proof. $\square$

Note, in particular, that in view of Proposition 4.4, every $\alpha\beta$-symmetric monotone bifunction $M : X \times X \rightrightarrows X$ with respect to the mappings f and g is actually a univariate $(\alpha - \beta)$-strongly monotone (resp. monotone and $(\beta - \alpha)$-relaxed monotone) mapping and is not a new one.

Recently, Ahmad et al. [1] introduced a class of monotone operators, the so-called $H(.,.)$-Co-monotone mappings as a generalization of $\widehat{H}$-monotone operators as follows.

Definition 4.5. [1, Definition 3.1] *Let $A, B, f, g : X \to X$ and $H : X \times X \to X$ be the single-valued mappings. Let $M : X \times X \rightrightarrows X$ be a multi-valued mapping. The mapping M is said to be $H(.,.)$-Co-monotone with respect to A, B, f and g, if $H(A, B)$ is symmetric cocoercive (or $\mu\gamma$-symmetric cocoercive) with respect to A and B, $M(f, g)$ is symmetric monotone (or $\alpha\beta$-symmetric monotone) with respect to f and g, and $(H(A, B) + \lambda M(f, g))(X) = X$ for all $\lambda > 0$.*

Remark 4.6. *It should be pointed out that by defining the mapping $\widehat{M} : X \rightrightarrows X$ as $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$, Proposition 4.4 implies that $\widehat{M}$ is $(\alpha - \beta)$-strongly monotone (resp. monotone and $(\beta - \alpha)$-relaxed monotone) provided that $\alpha > \beta$ (resp. $\alpha = \beta$ and $\alpha < \beta$). Then, by defining the mapping $\widehat{H} : X \to X$ as $\widehat{H}(x) := H(A(x), B(x))$ for all $x \in X$, in view of Definition 4.5 we have $(\widehat{H} + \lambda\widehat{M})(X) = (H(A, B) + \lambda M(f, g))(X) = X$ for all $\lambda > 0$. Consequently, every $H(.,.)$-Co-monotone mapping is actually a $\widehat{H}$-$(\alpha - \beta)$-strongly monotone (resp. $\widehat{H}$-monotone and $\widehat{H}$-maximal $(\beta - \alpha)$-relaxed monotone) operator provided that $\alpha > \beta$ (resp. $\alpha = \beta$ and $\alpha < \beta$) and is not a new one.*

In support of the definition of $H(.,.)$-Co-monotone mapping, the authors [1] presented the following example.

Example 4.7. Let $S_2(\mathbb{R})$ be the space of all symmetric 2×2 matrices with real entries, that is, consisting of all 2×2 matrices with real entries such that $(1, 2)$-entry equals to $(2, 1)$-entry. Then,

$$\begin{aligned} S_2(\mathbb{R}) &= \left\{ A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{R}, a_{12} = a_{21}, i, j = 1, 2 \right\} \\ &= \left\{ \begin{pmatrix} x & z \\ z & y \end{pmatrix} | x, y, z \in \mathbb{R} \right\} \end{aligned}$$

is a subspace of $M_{2\times2}(\mathbb{R}) = M_2(\mathbb{R})$ with respect to the operations of addition and scalar multiplication defined on $M_2(\mathbb{R})$ in Example 2.6, and the Hilbert-Schmidt inner product on $S_2(\mathbb{R})$ and the Hilbert-Schmidt norm induced by it become the same as in Example 2.6.

Define the operators $A, B, f, g : X \to X$, respectively, by

$$\begin{aligned} &A\left(\begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}\right) := \begin{pmatrix} 2x_1 & \frac{a}{2} \\ \frac{a}{2} & 2x_2 \end{pmatrix}, B\left(\begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}\right) := \begin{pmatrix} 0 & 2a \\ 2a & 0 \end{pmatrix}, \\ &f\left(\begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}\right) := \begin{pmatrix} 3x_1 & 2a \\ 2a & 2x_2 \end{pmatrix}, g\left(\begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}\right) := \begin{pmatrix} -\frac{x_1}{2} & -\frac{a}{2} \\ -\frac{a}{2} & -\frac{x_2}{3} \end{pmatrix}, \end{aligned}$$

for all $\begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix} \in S_2(\mathbb{R})$. Suppose further that the bifunctions $H, M : X \times X \to X$ are defined, respectively, by $H(x, y) = x - y$ and $M(x, y) = x + y$, for all $x = \begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 & a \\ a & y_2 \end{pmatrix} \in X = S_2(\mathbb{R})$. Taking into account that for all $x, y \in X$,

$$\begin{aligned} &\langle H(A(x), .) - H(A(y), .), x - y \rangle = 2(x_1 - y_1)^2 + (a - b)^2 + 2(x_2 - y_2)^2, \\ &\langle H(., B(x)) - H(., B(y)), x - y \rangle = -4(a - b)^2, \\ &\|A(x) - A(y)\|^2 \leq 2[2(x_1 - y_1)^2 + (a - b)^2 + 2(x_2 - y_2)^2], \end{aligned}$$

and $\|B(x)-B(y)\|^2 = 8(a-b)^2$, the authors [1] deduced that for all $x, y \in X$,

$$\langle H(A(x),.) - H(A(y),.), x-y\rangle \geq \frac{1}{2}\|A(x)-A(y)\|^2$$

and

$$\langle H(.,B(x)) - H(.,B(y)), x-y\rangle \geq -\frac{1}{2}\|B(x)-B(y)\|^2,$$

i.e., $H(A,B)$ is $\frac{1}{2}$-cocoercive with respect to A and $\frac{1}{2}$-relaxed cocoercive with respect to B, and so $H(A,B)$ is $\mu\gamma$-symmetric cocoercive with respect to A and B with $\mu = \gamma = \frac{1}{2}$.

At the same time, in the light of the facts that for all $x, y \in X$,

$$\langle M(f(x),.) - M(f(y),.), x-y\rangle \geq 2(x_1-y_1)^2 + 4(a-b)^2 + 2(x_2-y_2)^2,$$
$$\langle M(.,g(x)) - M(.,g(y)), x-y\rangle \geq -\frac{1}{2}\|x-y\|^2,$$

and

$$\|x-y\|^2 = (x_1-y_1)^2 + 2(a-b)^2 + (x_2-y_2)^2,$$

they concluded that for all $x, y \in X$,

$$\langle M(f(x),.) - M(f(y),.), x-y\rangle \geq 2\|x-y\|^2$$

and

$$\langle M(.,g(x)) - M(.,g(y)), x-y\rangle \geq -\frac{1}{2}\|x-y\|^2,$$

which means that $M(f,g)$ is 2-strongly monotone with respect to f and $\frac{1}{2}$-relaxed monotone with respect to g. In other words, $M(f,g)$ is $\alpha\beta$-symmetric monotone with respect to f and g with $\alpha = 2$ and $\beta = \frac{1}{2}$. Finally, relying on the fact that for all $x \in X$,

$$\begin{aligned}[H(A,B) + M(f,g)](x) &= H(A(x),B(x)) + M(f(x),g(x))\\ &= A(x) - B(x) + f(x) + g(x)\\ &= \begin{pmatrix} \frac{9x_1}{2} & 0 \\ 0 & \frac{7x_2}{2} \end{pmatrix},\end{aligned}$$

they deduced that $[H(A,B) + M(f,g)](X) \neq X$, which ensures M is not an $H(.,.)$-Co-monotone mapping. Let us now define the operators $\widehat{H}, \widehat{M} : X \to X$, respectively, by

$$\widehat{H}(x) = \widehat{H}\Big(\begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}\Big) := \begin{pmatrix} 2x_1 & -\frac{3a}{2} \\ -\frac{3a}{2} & 2x_2 \end{pmatrix}$$

and

$$\widehat{M}(x) = \widehat{M}\Big(\begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}\Big) := \begin{pmatrix} \frac{5}{2}x_1 & \frac{3a}{2} \\ \frac{3a}{2} & \frac{5}{3}x_2 \end{pmatrix},$$

for all $x = \begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix} \in S_2(\mathbb{R})$.

Since $\alpha = 2 > \beta = \frac{1}{2}$, according to Proposition 4.4, it is expected that $\widehat{M}$ to be $(\alpha - \beta) = \frac{3}{2}$-strongly monotone. We now show that our observation is compatible with this fact. In fact, taking into account that for all $x = \begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 & b \\ b & y_2 \end{pmatrix} \in X = S_2(\mathbb{R})$,

$$\begin{aligned}
&\langle \widehat{M}(x) - \widehat{M}(y), x - y \rangle \\
&= \Big\langle \begin{pmatrix} \frac{5}{2}x_1 & \frac{3a}{2} \\ \frac{3a}{2} & \frac{5}{3}x_2 \end{pmatrix} - \begin{pmatrix} \frac{5}{2}y_1 & \frac{3b}{2} \\ \frac{3b}{2} & \frac{5}{3}y_2 \end{pmatrix}, \begin{pmatrix} x_1 - y_1 & a - b \\ a - b & x_2 - y_2 \end{pmatrix} \Big\rangle \\
&= \Big\langle \begin{pmatrix} \frac{5}{2}(x_1 - y_1) & \frac{3}{2}(a - b) \\ \frac{3}{2}(a - b) & \frac{5}{3}(x_2 - y_2) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 & a - b \\ a - b & x_2 - y_2 \end{pmatrix} \Big\rangle \\
&= tr\Big(\begin{pmatrix} \frac{5}{2}(x_1 - y_1)^2 + \frac{3}{2}(a - b)^2 & (a - b)\big(\frac{5}{2}(x_1 - y_1) + \frac{3}{2}(x_2 - y_2)\big) \\ (a - b)\big(\frac{3}{2}(x_1 - y_1) + \frac{5}{3}(x_2 - y_2)\big) & \frac{3}{2}(a - b)^2 + \frac{5}{3}(x_2 - y_2)^2 \end{pmatrix} \Big) \\
&= \frac{5}{2}(x_1 - y_1)^2 + 3(a - b)^2 + \frac{5}{3}(x_2 - y_2)^2 \\
&\geq \frac{3}{2}[(x_1 - y_1)^2 + 2(a - b)^2 + (x_2 - y_2)^2] \\
&= \frac{3}{2}\|x - y\|^2,
\end{aligned}$$

it follows that $\widehat{M}$ is $\frac{3}{2}$-strongly monotone which is compatible with the assertion derived in Proposition 4.4. But, thanks to the fact that for each $x = \begin{pmatrix} x_1 & a \\ a & x_2 \end{pmatrix} \in X = S_2(\mathbb{R})$,

$$(\widehat{H} + \widehat{M})(x) = \begin{pmatrix} 2x_1 & -\frac{3a}{2} \\ -\frac{3a}{2} & 2x_2 \end{pmatrix} + \begin{pmatrix} \frac{5}{2}x_1 & \frac{3a}{2} \\ \frac{3a}{2} & \frac{5}{3}x_2 \end{pmatrix} = \begin{pmatrix} \frac{9x_1}{2} & 0 \\ 0 & \frac{11x_2}{3} \end{pmatrix},$$

we deduce that $(\widehat{H} + \widehat{M})(X)$ is not precisely X for $\lambda = 1$, that is, $\widehat{H} + \widehat{M}$ is not surjective, and so $\widehat{M}$ is not $\widehat{H}$-$\frac{3}{2}$-strongly monotone. Since in the light of Proposition 4.4, every $H(.,.)$-Co-monotone mapping M with $\alpha > \beta$ is actually $\widehat{H}$-$(\alpha - \beta)$-strongly monotone, the fact that $\widehat{M}$ is not $\widehat{H}$-$\frac{3}{2}$-strongly monotone implies that M is not $H(.,.)$-Co-monotone. Therefore, this fact is also compatible with our above observation.

Note 1 In the rest of the paper, we say that M is an $H(.,.)$-Co-monotone mapping with respect to A, B, f and g, means that $H(A, B)$ is symmetric cocoercive with respect to A and B with constants μ and γ, respectively, and $M(f, g)$ is symmetric monotone with respect to f and g with constants α and β, respectively, where $\mu > \gamma$ and $\alpha > \beta$.

Remark 4.8. *It is important to emphasize that by a careful checking the results in [1], we found that in the context of Note 1 and results of [1], four constants μ, γ, α and β must be satisfied the conditions $\mu > \gamma$ and $\alpha > \beta$, as these conditions are added to the context of Note 1.*

In order to define the resolvent operator associated with an $H(.,.)$-Co-monotone mapping, Ahmad et al. [1] presented the following conclusion in which sufficient conditions for the mapping $(H(A, B) + \lambda M(f, g))^{-1}$ to be single-valued for all $\lambda > 0$ are stated.

Theorem 4.9. [1, Theorem 3.1] *Let X be a real Hilbert space. Let $A, B, f, g : X \to X$ and $H : X \times X \to X$ be the mappings. Let $M : X \times X \rightrightarrows X$ be an $H(.,.)$-Co-monotone mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous and $\eta > \sigma$. Then the mapping $(H(A, B) + \lambda M(f, g))^{-1}$ is single-valued for all $\lambda > 0$.*

Proof. Define $\widehat{H} : X \to X$ and $\widehat{M} : X \rightrightarrows X$ by $\widehat{H}(x) := H(A(x), B(x))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$. In the light of the assumptions and Remark 4.8, since $\mu > \gamma$, Proposition 4.2 implies that $\widehat{H}$ is strictly monotone. At the same time, from Proposition 4.4 and thanks to Remark 4.8, it follows that $\widehat{M}$ is a $\widehat{H}$-monotone operator. Therefore, all the conditions of Lemma 2.13 are satisfied. Consequently, according to Lemma 2.13, for every real constant $\lambda > 0$, the operator $(\widehat{H} + \lambda\widehat{M})^{-1} = (H(A, B) + \lambda M(f, g))^{-1}$ is single-valued. This gives the desired result. □

Here it is to be noted that by a careful reading the proof of Theorem 3.1 of [1], we found that the two constants η and σ must be satisfied the condition $\eta > \sigma$, as we have added it to the context of Theorem 4.9.

Based on Theorem 4.9 (that is, [1, Theorem 3.1]), Ahmad et al. [1] defined the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ associated with an $H(.,.)$-Co-monotone mapping M and an arbitrary real constant $\lambda > 0$ as follows.

Definition 4.10. [1, Definition 3.2] *Let $A, B, f, g : X \to X$ and $H : X\times X \to X$ be the mappings and $M : X\times X \rightrightarrows X$ be an $H(.,.)$-Co-monotone mapping with respect to A, B, f and g. Suppose further that A is η-expansive and B is σ-Lipschitz continuous and $\eta > \sigma$. The resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)} : X \to X$ is defined by*

$$R^{H(.,.)}_{\lambda,M(.,.)}(u) = (H(A, B) + \lambda M(f, g))^{-1}(u), \quad \forall u \in X \text{ and } \lambda > 0.$$

Comparing the contexts of Definition 3.2 in [1] and Definition 4.10, one can easily see that a correct version of Definition 3.2 of [1], that is, Definition 4.10, is provided by adding the assumptions of η-expansivity of the mapping A and σ-Lipschitz continuity of the mapping B with $\eta > \sigma$ to the context of [1, Definition 3.2]. In fact, in the light of Theorem 4.9, the above-mentioned conditions are essential for the mapping $(H(A, B) + \lambda M)^{-1}$ to be single-valued for all $\lambda > 0$, and can not be dropped.

Defining the mappings $\widehat{H} : X \to X$ and $\widehat{M} : X \rightrightarrows X$ as $\widehat{H}(x) := H(A(x), B(x))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$, by the argument similar to that of Theorem 4.9, it follows that $\widehat{H}$ is a strictly monotone operator and $\widehat{M}$ is a $\widehat{H}$-monotone operator. Then, accordance with Definition 2.15, for every real constant $\lambda > 0$, the resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda} = R^{H(.,.)}_{\lambda,M(.,.)} : X \to X$ associated with $\widehat{H}$-monotone operator $\widehat{M}$ and real constant $\lambda > 0$, is defined by

$$R^{\widehat{H}}_{\widehat{M},\lambda}(u) = R^{H(.,.)}_{\lambda,M(.,.)}(u) = (\widehat{H} + \lambda\widehat{M})^{-1}(u) = (H(A, B) + \lambda M(f, g))^{-1}(u), \quad \forall u \in X.$$

Indeed, the notion of the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ associated with an $H(.,.)$-Co-monotone mapping M and an arbitrary real constant $\lambda > 0$ is actually the same concept of the resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda}$ associated with a $\widehat{H}$-monotone operator $\widehat{M}$, and is not a new one. Therefore, Definition 4.10 is exactly the same Definition 2.15 and is not a new one.

Section 3 in [1] is closed with the following assertion in which, under some appropriate conditions, the Lipschitz continuity of the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ is proved and an estimate of its Lipschitz constant is calculated.

Theorem 4.11. [1, Theorem 3.2] *Let $A, B, f, g : X \to X$ and $H : X \times X \to X$ be the mappings. Suppose that $M : X \times X \rightrightarrows X$ is an $H(.,.)$-Co-monotone mapping with respect to A, B, f and g. Let A be η-expansive and B be σ-Lipschitz continuous such that $\alpha > \beta$, $\mu > \gamma$ and $\eta > \sigma$. Then the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)} : X \to X$ is Lipschitz continuous, i.e.,*

$$\|R^{H(.,.)}_{\lambda,M(.,.)}(u) - R^{H(.,.)}_{\lambda,M(.,.)}(v)\| \leq \theta\|x - y\|, \quad \forall u, v \in X,$$

where $\theta = \frac{1}{\lambda(\alpha-\beta)+\mu\eta^2-\gamma\sigma^2}$.

Proof. Let us define the mappings $\widehat{H} : X \to X$ and $\widehat{M} : X \rightrightarrows X$ as $\widehat{H}(x) := H(A(x), B(x))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$. Thanks to the assumptions, Propositions 4.2 and 4.4, and in view of Remark 4.8, it follows that the operator $\widehat{H}$ is $(\mu\eta^2 - \gamma\sigma^2)$-strongly monotone, $\widehat{M}$ is $\widehat{H}$-$(\alpha - \beta)$-strongly monotone, and the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ associated with the $H(.,.)$-Co-monotone mapping M and real constant $\lambda > 0$ becomes actually the same resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda}$ associated with the $\widehat{H}$-$(\alpha - \beta)$-strongly monotone operator $\widehat{M}$ and given real constant λ. Taking $\pi = \mu\eta^2 - \gamma\sigma^2$ and $\varsigma = \alpha - \beta$, all the conditions of Corollary

2.18 are satisfied and so according to Corollary 2.18, for every real constant $\lambda > 0$, the resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda} = R^{H(.,.)}_{\lambda,M(.,.)} : X \to X$ is $\frac{1}{\lambda\varsigma+\pi} = \frac{1}{\lambda(\alpha-\beta)+\mu\eta^2-\gamma\sigma^2}$-Lipschitz continuous, i.e.,

$$\begin{aligned}\|R^{H(.,.)}_{\lambda,M(.,.)}(u) - R^{H(.,.)}_{\lambda,M(.,.)}(v)\| &= \|R^{\widehat{H}}_{\widehat{M},\lambda}(u) - R^{\widehat{H}}_{\widehat{M},\lambda}(v)\| \leq \tfrac{1}{\lambda\varsigma+\pi}\|u-v\| \\ &= \tfrac{1}{\lambda(\alpha-\beta)+\mu\eta^2-\gamma\sigma^2}\|u-v\|, \quad \forall u,v \in X.\end{aligned}$$

This completes the proof. □

Let X be a real Hilbert space, $P, F, G, f, g : X \to X$ be single-valued operators, $R, S, T : X \rightrightarrows CB(X)$ and $M : X\times X \rightrightarrows X$ be multi-valued operators. Recently, Ahmad et al. [1] considered and studied the variational inclusion problem of finding $x \in X$, $u \in R(x)$, $v \in S(x)$ and $w \in T(x)$ such that

$$0 \in P(u) - (F(v) - G(w)) + M(f(x), g(x)). \tag{17}$$

Utilizing the notion of the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$, they gave a characterization of the solution of the problem (17) in order to construct an iterative algorithm for approximating its solutions as follows.

Lemma 4.12. [1, Lemma 4.1] *Let X be a real Hilbert space. Let $A, B, f, g, P, F, G : X \to X$ and $R, S, T : X \rightrightarrows CB(X)$ be the mappings. Let $H : X \times X \to X$ and $M : X \times X \rightrightarrows X$ be the mappings such that M is $H(.,.)$-Co-monotone with respect to A, B, f and g. Suppose further that A is η-expansive and B is σ-Lipschitz continuous such that $\eta > \sigma$. Then (x, u, v, w), where $x \in X$, $u \in R(x)$, $v \in S(x)$ and $w \in T(x)$, is a solution of the problem* (17) *if (x, u, v, w) satisfies the following equation:*

$$x = R^{H(.,.)}_{\lambda,M(.,.)}[H(A(x), B(x)) - \lambda(P(u) - (F(v) - G(w)))].$$

Proof. By defining the mapping $\widehat{H} : X \to X$ and $\widehat{M} : X \rightrightarrows X$ as before, from the assumptions, Propositions 4.2 and 4.4, and Remark 4.8 it follows that $\widehat{H}$ is strictly monotone, $\widehat{M}$ is $\widehat{H}$-monotone, and the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ associated with the $H(.,.)$-Co-monotone mapping M and real constant $\lambda > 0$ is exactly the same resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda}$ associated with the $\widehat{H}$-monotone operator $\widehat{M}$ and λ. In the meanwhile, the problem (17) involving $H(.,.)$-Co-monotone mapping M becomes actually the same MVIP (2) involving $\widehat{H}$-monotone operator $\widehat{M}$. Then, all the conditions of Lemma 3.2 are satisfied and so based on Lemma 3.2, $(x, u, v, w) \in X \times R(x) \times S(x) \times T(x)$ is a solution of the problem (17) if

$$\begin{aligned}x &= R^{\widehat{H}}_{\widehat{M},\lambda}[\widehat{H}(x) - \lambda(P(u) - (F(v) - G(w)))] \\ &= R^{H(.,.)}_{\lambda,M(.,.)}[H(A(x), B(x)) - \lambda(P(u) - (F(v) - G(w)))],\end{aligned}$$

where $R^{H(.,.)}_{\lambda,M(.,.)} = R^{\widehat{H}}_{\widehat{M},\lambda} = (\widehat{H} + \lambda\widehat{M})^{-1} = (H(A, B) + \lambda M(f, g))^{-1}$ and $\lambda > 0$ is an arbitrary real constant. This is the same desired result. □

It should be remarked that Lemma 4.12 is a correct version of Lemma 4.1 in [1]. In fact, in view of the conclusion of [1, Lemma 4.1], the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ must be single-valued. According to Theorem 4.9 and Definition 4.10, for $R^{H(.,.)}_{\lambda,M(.,.)}$ to be single-valued, the mappings A and B must be η-expansive and σ-Lipschitz continuous, respectively, such that $\eta > \sigma$. By adding these assumptions to the context of [1, Lemma 4.1], its correct version, that is Lemma 4.12 is presented. In the meanwhile, it is notable that contrary to the claim in [1], the characterization of the solution for the problem (17) involving $H(.,.)$-Co-monotone mapping M with respect to A, B, f and g, presented in Lemma 4.12 is actually the same characterization of the solution for the problem (2) involving $\widehat{H}$-monotone operator $\widehat{M}$ given in Lemma 3.2, and is not a new one.

Based on Lemma 4.12, Ahmad et al. [1] constructed the following iterative algorithm for finding an approximate solution of the problem (17) involving an $H(.,.)$-Co-monotone mapping M with respect to A, B, f and g.

Algorithm 4.13. [1, Algorithm 4.1] *For any $x_0 \in X$, $u_0 \in R(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, compute the sequences $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ by the following iterative scheme:*

$$x_{n+1} = R_{\lambda,M(.,.)}^{H(.,.)}[H(A(x_n), B(x_n)) - \lambda(P(u_n) - (F(v_n) - G(w_n)))], \tag{18}$$

$$u_n \in R(x_n); \|u_n - u_{n+1}\| \leq D(R(x_n), R(x_{n+1})), \tag{19}$$

$$v_n \in S(x_n); \|v_n - v_{n+1}\| \leq D(S(x_n), S(x_{n+1})), \tag{20}$$

$$w_n \in T(x_n); \|w_n - w_{n+1}\| \leq D(T(x_n), T(x_{n+1})), \tag{21}$$

where $n = 0, 1, 2, \dots$; and $\lambda > 0$ is a constant.

By a careful reading Algorithm 4.13, we found that the sequences $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ generated by Algorithm 4.13 are not well defined necessarily. In fact, for any given $x_0 \in X$, $u_0 \in R(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, the authors computed $x_{n+1} \in X$ by induction of n using the iterative scheme (18), and then they claimed that one can choose $u_{n+1} \in R(x_{n+1})$, $v_{n+1} \in S(x_{n+1})$ and $w_{n+1} \in T(x_{n+1})$ satisfying (19)–(21). In the light of Lemma 3.3, if X is a complete metric space and $T : X \to CB(X)$ is a multi-valued mapping, then for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that $d(u, v) \leq (1 + \varepsilon)D(T(x), T(y))$. But, the following example illustrates that for given $x, y \in X$ and $u \in T(x)$, there may not be a point $v \in T(y)$ such that $d(u, v) \leq D(T(x), T(y))$.

Example 4.14. Consider $X = l^{\infty}(\mathbb{Z}) = \{z = \{z_n\}_{n=-\infty}^{\infty} : \sup\limits_{n\in\mathbb{Z}} |z_n| < \infty, z_n \in \mathbb{C}\}$ the Banach space consisting of all bounded complex sequences $z = \{z_n\}_{n=-\infty}^{\infty}$ with the supremum norm $\|z\|_{\infty} = \sup\limits_{n\in\mathbb{Z}} |z_n|$. Any element $z = \{z_n\}_{n=-\infty}^{\infty} = \{x_n + iy_n\}_{n=-\infty}^{\infty} \in l^{\infty}(\mathbb{Z})$ can be written as

$$\begin{aligned}
z &= \sum_{p=-\infty}^{\infty} (\dots, 0, 0, \dots, 0, x_{pq+1} + iy_{pq+1}, x_{pq+2} + iy_{pq+2}, \dots, x_{(p+1)k} + iy_{(p+1)k}, 0, 0, \dots) \\
&= \sum_{p=-\infty}^{\infty} \sum_{m=pq+1}^{\frac{(2p+1)q}{2}} (\dots, 0, 0, \dots, 0, x_m + iy_m, 0, 0, \dots, 0, x_{(2p+1)q-m+1} + iy_{(2p+1)q-m+1}, 0, 0, \dots) \\
&= \sum_{p=-\infty}^{\infty} \sum_{m=pq+1}^{\frac{(2p+1)q}{2}} \Big[\frac{y_m + y_{(2p+1)q-m+1} - i(x_m + x_{(2p+1)q-m+1})}{2} w_{m,(2p+1)q-m+1} \\
&\quad + \frac{y_m - y_{(2p+1)q-m+1} - i(x_m - x_{(2p+1)q-m+1})}{2} w'_{m,(2p+1)q-m+1} \Big],
\end{aligned}$$

where $q \geq 2$ is an arbitrary but fixed even natural number; for each $p \in \mathbb{Z}$ and $m \in \{pq + 1, pq + 2, \dots, \frac{(2p+1)q}{2}\}$,

$$w_{m,(2p+1)q-m+1} = (\dots, 0, 0, \dots, 0, i_m, 0, 0, \dots, 0, i_{(2p+1)q-m+1}, 0, 0, \dots),$$

i at the mth and $((2p + 1)q - m + 1)$th coordinates and all other coordinates are zero, and

$$w'_{m,(2p+1)q-m+1} = (\dots, 0, 0, \dots, 0, i_m, 0, 0, \dots, 0, -i_{(2p+1)q-m+1}, 0, 0, \dots),$$

i and $-i$ at the mth and $((2p + 1)q - m + 1)$th coordinates, respectively, and all other coordinates are zero. Thus, the set

$$\mathfrak{B} = \left\{ w_{m,(2p+1)q-m+1}, w'_{m,(2p+1)q-m+1} : p \in \mathbb{Z}, m = pq + 1, pq + 2, \dots, \frac{(2p + 1)q}{2} \right\}$$

spans the real Banach space $l^{\infty}(\mathbb{Z})$. It can be easily seen that $\mathfrak{B}$ is linearly independent and so it is a Schauder basis for the real Banach space $l^{\infty}(\mathbb{Z})$. Define the set-valued mapping $T : l^{\infty}(\mathbb{Z}) \to CB(l^{\infty}(\mathbb{Z}))$ by

$$T(x) = \begin{cases} \Phi_1, & x \neq w'_{s,(2\theta+1)q-s+1}, \\ \Phi_2, & x = w'_{s,(2\theta+1)q-s+1}, \end{cases}$$

where

$$\Phi_1 = \Big\{\Big\{\frac{\alpha}{\sqrt[k]{e^{n^\beta !}}\prod\limits_{j=1}^{r} n^{\gamma j}!}i\Big\}_{n=-\infty}^{\infty}, w'_{m,(2p+1)q-m+1} : p \in \mathbb{Z}; m = pq+1, pq+2, \ldots, \frac{(2p+1)q}{2}\Big\},$$

$$\Phi_2 = \Big\{w_{m,(2p+1)q-m+1} : p \in \mathbb{Z}; m = pq+1, pq+2, \ldots, \frac{(2p+1)q}{2}\Big\},$$

$\alpha \in [-1, 0)$ is an arbitrary but fixed real number, $k \in \mathbb{N}\backslash\{1\}$ and $r \in \mathbb{N}$ are arbitrary but fixed natural numbers, β, γ are two arbitrary but fixed even natural numbers, and $s = \theta q + t$ for an arbitrary but fixed $\theta \in \mathbb{Z}$ and $t \in \{1, 2, \ldots, \frac{q}{2}\}$. Take $w'_{s,(2\theta+1)q-s+1} \neq x \in X$ arbitrarily, $y = w'_{s,(2\theta+1)q-s+1}$ and $u = \Big\{\frac{\alpha}{\sqrt[k]{e^{n^\beta !}}\prod_{j=1}^{r} n^{\gamma j}!}i\Big\}_{n=-\infty}^{\infty}$. If $a = \Big\{\frac{\alpha}{\sqrt[k]{e^{n^\beta !}}\prod_{j=1}^{r} n^{\gamma j}!}i\Big\}_{n=-\infty}^{\infty}$, then in virtue of the fact that $\alpha < 0$, for any $p \in \mathbb{Z}$ and $m \in \{pq+1, pq+2, \ldots, \frac{(2p+1)q}{2}\}$, we get

$$\begin{aligned}
d(a, w_{m,(2p+1)q-m+1}) &= \|\Big\{\frac{\alpha}{\sqrt[k]{e^{n^\beta !}}\prod\limits_{j=1}^{r} n^{\gamma j}!}i\Big\}_{n=-\infty}^{\infty} - w_{m,(2p+1)q-m+1}\|_\infty \\
&= \sup\Big\{\Big|\frac{\alpha}{\sqrt[k]{e^{n^\beta !}}\prod\limits_{j=1}^{r} n^{\gamma j}!}\Big|, \Big|\frac{\alpha}{\sqrt[k]{e^{m^\beta !}}\prod\limits_{j=1}^{r} m^{\gamma j}!} - 1\Big|, \\
&\quad \Big|\frac{\alpha}{\sqrt[k]{e^{((2p+1)q-m+1)^\beta !}}\prod\limits_{j=1}^{r}((2p+1)q-m+1)^{\gamma j}!} - 1\Big| : \\
&\quad n \in \mathbb{Z}; n \neq m, (2p+1)q-m+1\Big\} \\
&= \Big|\frac{\alpha}{\sqrt[k]{e^{m^\beta !}}\prod\limits_{j=1}^{r} m^{\gamma j}!} - 1\Big| = 1 - \frac{\alpha}{\sqrt[k]{e^{m^\beta !}}\prod\limits_{j=1}^{r} m^{\gamma j}!}.
\end{aligned}$$

Since $\alpha \in [-1, 0)$, it follows that

$$d(a, T(y)) = \inf_{b \in T(y)} d(a, b) = \inf\Big\{1 - \frac{\alpha}{\sqrt[k]{e^{m^\beta !}}\prod\limits_{j=1}^{r} m^{\gamma j}!} : p \in \mathbb{Z}; m = pq+1, pq+2, \ldots, \frac{(2p+1)q}{2}\Big\} = 1.$$

For the case when $a = w'_{\sigma,(2\delta+1)q-\sigma+1}$ for some $\delta \in \mathbb{Z}$ and $\sigma \in \{\delta q+1, \delta q+2, \ldots, \frac{(2\delta+1)q}{2}\}$, then for each $p \in \mathbb{Z}$ and $m \in \{pq+1, pq+2, \ldots, \frac{(2p+1)q}{2}\}$, one has

$$d(a, w_{m,(2p+1)q-m+1}) = \begin{cases} \|w'_{\sigma,(2\delta+1)q-\sigma+1} - w_{\sigma,(2\delta+1)q-\sigma+1}\|_\infty, & m = \sigma, \\ \|w'_{\sigma,(2\delta+1)q-\sigma+1} - w_{m,(2p+1)q-m+1}\|_\infty, & m \neq \sigma, \end{cases}$$

$$= \begin{cases} 2 & m = \sigma, \\ 1 & m \neq \sigma \end{cases}$$

and so

$$d(a, T(y)) = \inf_{b \in T(y)} d(a, b) = 1.$$

Taking into account the above-mentioned fact, we conclude that

$$\sup_{a \in T(x)} d(a, T(y)) = 1.$$

If $b = w_{\hat{m},(2\hat{p}+1)q-\hat{m}+1}$ for some $\hat{p} \in \mathbb{Z}$ and $\hat{m} \in \{\hat{p}q+1, \hat{p}q+2, \dots, \frac{(2\hat{p}+1)q}{2}\}$, thanks to the fact $\alpha < 0$, we obtain

$$\begin{aligned}
d\Big(\Big\{\frac{\alpha}{\sqrt[k]{e^{n^{\beta}!}}\prod_{j=1}^{r} n^{\gamma j}!}i\Big\}_{n=-\infty}^{\infty}, w_{\hat{m},(2\hat{p}+1)q-\hat{m}+1}\Big) &= \Big\|\Big\{\frac{\alpha}{\sqrt[k]{e^{n^{\beta}!}}\prod_{j=1}^{r} n^{\gamma j}!}i\Big\}_{n=-\infty}^{\infty} - w_{\hat{m},(2\hat{p}+1)q-\hat{m}+1}\Big\|_{\infty} \\
&= \sup\Big\{\Big|\frac{\alpha}{\sqrt[k]{e^{n^{\beta}!}}\prod_{j=1}^{r} n^{\gamma j}!}\Big|, \Big|\frac{\alpha}{\sqrt[k]{e^{\hat{m}^{\beta}!}}\prod_{j=1}^{r} \hat{m}^{\gamma j}!} - 1\Big|, \\
&\quad \Big|\frac{\alpha}{\sqrt[k]{e^{((2\hat{p}+1)q-\hat{m}+1)^{\beta}!}}\prod_{j=1}^{r}((2\hat{p}+1)q-\hat{m}+1)^{\gamma j}!} - 1\Big| : \\
&\quad n \in \mathbb{Z}; n \neq \hat{m}, (2\hat{p}+1)q-\hat{m}+1\Big\} \\
&= \Big|\frac{\alpha}{\sqrt[k]{e^{\hat{m}^{\beta}!}}\prod_{j=1}^{r} \hat{m}^{\gamma j}!} - 1\Big| \\
&= 1 - \frac{\alpha}{\sqrt[k]{e^{\hat{m}^{\beta}!}}\prod_{j=1}^{r} \hat{m}^{\gamma j}!},
\end{aligned}$$

and for each $p \in \mathbb{Z}$ and $m \in \{pq+1, pq+2, \dots, \frac{(2p+1)q}{2}\}$,

$$d(w'_{m,(2p+1)q-m+1}, w_{\hat{m},(2\hat{p}+1)q-\hat{m}+1}) = \begin{cases} \|w'_{\hat{m},(2\hat{p}+1)q-\hat{m}+1} - w_{\hat{m},(2\hat{p}+1)q-\hat{m}+1}\|_{\infty}, & m = \hat{m}, \\ \|w'_{m,(2p+1)q-m+1} - w_{\hat{m},(2\hat{p}+1)q-\hat{m}+1}\|_{\infty}, & m \neq \hat{m}, \end{cases}$$

$$= \begin{cases} 2 & m = \hat{m}, \\ 1 & m \neq \hat{m}. \end{cases}$$

Since $\alpha < 0$, it follows that

$$d(T(x), b) = \inf_{a \in T(x)} d(a, b) = 1$$

and so

$$\sup_{b \in T(y)} d(T(x), b) = 1.$$

Accordingly,

$$D(T(x), T(y)) = \max\Big\{\sup_{a \in T(x)} d(a, T(y)), \sup_{b \in T(y)} d(T(x), b)\Big\} = 1.$$

Owing to the fact that for each $p \in \mathbb{Z}$ and $m \in \{pq+1, pq+2, \dots, \frac{(2p+1)q}{2}\}$,

$$\Big\|\Big\{\frac{\alpha}{\sqrt[k]{e^{n^{\beta}!}}\prod_{j=1}^{r} n^{\gamma j}!}i\Big\}_{n=-\infty}^{\infty} - w_{m,(2p+1)q-m+1}\Big\|_{\infty} = 1 - \frac{\alpha}{\sqrt[k]{e^{m^{\beta}!}}\prod_{j=1}^{r} m^{\gamma j}!} > 1,$$

because $\alpha \in [-1, 0)$, we deduce that for any $v \in T(y)$,

$$d(u, v) = \|u - v\|_{\infty} > D(T(x), T(y)).$$

It is significant to mention that if $T(y)$ is compact, then such a point v does exist. In fact, if $T : X \to C(X)$, where $C(X)$ is the family of all the nonempty compact subsets of X, then for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that $d(u, v) \leq D(T(x), T(y))$. In the light of the above-mentioned arguments, we now present the correct version of Algorithm 4.13 only by making some small changes in relations (19)–(21) as follows.

Algorithm 4.15. *Let $X, A, B, f, g, P, F, G, R, S, T, H$ and M be the same as in Lemma 4.12. For any given $x_0 \in X$, $u_0 \in R(x_0)$, $v_0 \in S(x_0)$ and $w_0 \in T(x_0)$, compute the sequences $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ by the following iterative schemes:*

$$\begin{aligned}
&x_{n+1} = R^{H(.,.)}_{\lambda,M(.,.)}[H(A(x_n), B(x_n)) - \lambda(P(u_n) - (F(v_n) - G(w_n)))],\\
&u_n \in R(x_n); \|u_n - u_{n+1}\| \leq (1 + (1 + n)^{-1})D(R(x_n), R(x_{n+1})),\\
&v_n \in S(x_n); \|v_n - v_{n+1}\| \leq (1 + (1 + n)^{-1})D(S(x_n), S(x_{n+1})),\\
&w_n \in T(x_n); \|w_n - w_{n+1}\| \leq (1 + (1 + n)^{-1})D(T(x_n), T(x_{n+1})),
\end{aligned}$$

where $n = 0, 1, 2, \ldots$; and $\lambda > 0$ is a constant.

It should be noticed that by defining the mappings $\widehat{H} : X \to X$ and $\widehat{M} : X \rightrightarrows X$ as $\widehat{H}(x) := H(A(x), B(x))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$, using the assumptions, Propositions 4.2 and 4.4 and Remark 4.8 we conclude that $\widehat{H}$ is strictly monotone, $\widehat{M}$ is $\widehat{H}$-monotone and the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ associated with the $H(.,.)$-Co-monotone mapping M and real constant $\lambda > 0$ is exactly the same resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda}$ associated with the $\widehat{H}$-monotone operator $\widehat{M}$ and λ. In virtue of these facts, for each $n \geq 0$, we yield

$$\begin{aligned}
x_{n+1} &= R^{H(.,.)}_{\lambda,M(.,.)}[H(A(x_n), B(x_n)) - \lambda(P(u_n) - (F(v_n) - G(w_n)))]\\
&= R^{\widehat{H}}_{\widehat{M},\lambda}[\widehat{H}(x_n) - \lambda(P(u_n) - (F(v_n) - G(w_n)))].
\end{aligned}$$

Then, Algorithm 4.15 becomes actually the same Algorithm 3.5 and is not a new one.

Definition 4.16. [1, Definition 2.7] *Let $H : X \times X \to X$, $A, B, P : X \to X$ and $T : X \rightrightarrows CB(X)$ be the mappings. The mapping H is said to be mixed strongly monotone with respect to P and T, if there exists a constant $\delta > 0$ such that*

$$\begin{aligned}
&\langle H(A(x_n), B(x_n)) - H(A(x_{n-1}), B(x_{n-1})), P(u_n) - P(u_{n-1})\rangle \geq \delta\|x_n - x_{n-1}\|^2,\\
&\qquad\qquad \forall x_n, y_n \in X, u_n \in T(x_n), u_{n-1} \in T(x_{n-1}).
\end{aligned}$$

As a matter of fact, a small change has been made in the context of Definition 4.16. In fact, Definition 4.16 plays a key role in the study of convergence analysis of the proposed iterative algorithm in [1]. But, by a careful reading the proof of Theorem 4.1 in [1], we found that in Definition 2.7 of [1], "for some $u_n \in T(x_n), u_{n-1} \in T(x_{n-1})$" must be replaced by "for all $u_n \in T(x_n), u_{n-1} \in T(x_{n-1})$" as we have done in Definition 4.16. Note, in particular, that by defining the mapping $\widehat{H} : X \to X$ as $\widehat{H}(x) := H(A(x), B(x))$ for all $x \in X$, we note that Definition 4.16 reduces to the definition of strong monotonicity of the mapping $\widehat{H}$ with respect to P and T presented in Definition 3.7(ii) and is not a new one.

The following conclusion regarding the strong convergence of the iterative sequences generated by the suggested algorithm in [1] is the most important result that appeared in [1].

Theorem 4.17. [1, Theorem 4.1] *Let X be a real Hilbert space and $A, B, f, g, P, F, G : X \to X$ and $H : X \times X \to X$ be the single-valued mappings. Let $R, S, T : X \rightrightarrows CB(X)$ be the multi-valued mappings and $M : X \times X \rightrightarrows X$ be a multi-valued mapping such that M is $H(.,.)$-Co-monotone with respect to A, B, f and g. Assume that*

(i) *$H(A, B)$ is μ-cocoercive with respect to A and μ-relaxed cocoercive with respect to B;*
(ii) *A is η-expansive;*
(iii) *B is σ-Lipschitz continuous;*
(iv) *$H(A, B)$ is mixed Lipschitz continuous with constant $t > 0$;*
(v) *$H(A, B)$ is mixed strongly monotone with respect to P and R with constant $\delta > 0$;*
(vi) *$M(f, g)$ is α-strongly monotone with respect to f and β-relaxed monotone with respect to g;*
(vii) *R, S and T are D-Lipschitz continuous mappings with constants e, h and d, respectively;*

(viii) *P, F and G are Lipschitz continuous mappings with constants ρ, ξ and r, respectively;*
(ix) *S is relaxed Lipschitz continuous with respect to F with constant $k > 0$;*
(x) *T is relaxed monotone with respect to G with constant $c > 0$;*
(xi) *there exists a constant $\lambda > 0$ such that*

$$\sqrt{t^2 - 2\lambda\delta + \lambda^2\rho^2e^2} + \sqrt{1 - 2\lambda(k-c) + \lambda^2(\xi h + rd)^2} < \lambda(\alpha - \beta) + \mu\eta^2 - \gamma\sigma^2 - 1, \tag{22}$$

$$2\lambda\delta \le t^2 + \lambda^2\rho^2e^2,\ 2\lambda(k-c) \le 1 + \lambda^2(\xi h + rd)^2, \tag{23}$$

$$\mu > \gamma, \eta > \sigma, \alpha > \beta. \tag{24}$$

Then, the iterative sequences $\{x_n\}_{n=0}^{\infty}, \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ generated by Algorithm 4.15 converge strongly to x, u, v and w, respectively, and (x, u, v, w) is a solution of the problem (17).

Proof. By defining the mappings $\widehat{H} : X \to X$ and $\widehat{M} : X \rightrightarrows X$ by $\widehat{H}(x) := H(A(x), B(x))$ and $\widehat{M}(x) := M(f(x), g(x))$ for all $x \in X$, thanks to the assumptions, Propositions 4.2 and 4.4 and Remark 4.8 it follows that $\widehat{H}$ is $(\mu\eta^2 - \gamma\sigma^2)$-strongly monotone, $\widehat{M}$ is $\widehat{H}$-$(\alpha - \beta)$-strongly monotone, the resolvent operator $R^{H(.,.)}_{\lambda, M(.,.)}$ becomes actually the resolvent operator $R^{\widehat{H}}_{\widehat{M},\lambda}$, and the problem (17) involving $H(.,.)$-Co-monotone mapping M is exactly the same MVI (2) involving $\widehat{H}$-$(\alpha - \beta)$-strongly monotone mapping $\widehat{M}$. At the same time, owing to the facts mentioned above, Algorithm 4.15 reduces to Algorithm 3.5. Now, taking $\varsigma = \alpha - \beta$ and $\pi = \mu\eta^2 - \gamma\sigma^2$, we observe that all the conditions of Corollary 3.10 are satisfied. Therefore, the statement follows from Corollary 3.10 immediately. □

By comparing the contexts of Theorem 4.1 of [1] and Theorem 4.17, one can observe that some changes have been made in the context of Theorem 4.17. In fact, in the context of [1, Theorem 4.1], the condition $k > c$ must be replaced by $\mu > \gamma$ and $\eta > \sigma$, as we have done in (24). Furthermore, the constant $\lambda > 0$, in addition to (22), must be satisfied (23), as we have added the required assumptions to assumption (xi) in the context of Theorem 4.17. At the same time, as it was pointed out Algorithm 4.13 (that is, [1, Algorithm 4.1]) is not well defined necessarily and Algorithm 4.15 is its correct version. For this reason, in the context of Theorem 4.17, Algorithm 4.13 is replaced by Algorithm 4.15.

In support of Theorem 4.1 in [1], the authors presented an example and asserted that assumptions (i) to (xi) of the aforesaid theorem are satisfied for the variational inclusion problem (17). We end this paper by investigating and analyzing [1, Example 4.1] and point out that contrary to the claim of the authors in [1], all the conditions of [1, Theorem 4.1] do not hold.

Example 4.18. Consider $X = \mathbb{R}^2$ with usual inner product. Let $A, B, f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined, respectively, by

$$A(x) := (\frac{1}{3}x_1 - x_2, x_1 + \frac{1}{3}x_2),$$
$$B(x) := (-\frac{1}{2}x_1 + \frac{1}{2}x_2, -\frac{1}{2}x_1 - \frac{1}{2}x_2),$$
$$f(x) := (4x_1 - \frac{4}{3}x_2, \frac{4}{3}x_1 + 4x_2),$$
$$g(x) := (\frac{13}{4}x_1 - \frac{3}{4}x_2, \frac{3}{4}x_1 + \frac{13}{4}x_2),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Suppose that $H, M : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are defined, respectively, by $H(x, y) := x + y$ and $M(x, y) := x - y$, for all $x, y \in \mathbb{R}^2$. In the final section of [1], the authors showed that $H(A, B)$ is $\frac{13}{10}$-cocoercive with respect to A, 1-relaxed cocoercive with respect to B, the operator A is $\frac{\sqrt{10}}{3}$-expansive, the operator B is $\frac{1}{\sqrt{2}}$-Lipschitz continuous, $M(f, g)$ is 4-strongly monotone with respect to f and $\frac{1}{4}$-relaxed monotone with respect to g. At the same time, they claimed that M is an $H(.,.)$-Co-monotone mapping with respect to

A, B, f and g. By a careful reading the proof of [1, Theorem 4.1], we found that the multi-valued mapping $M : X \times X \rightrightarrows X$ was assumed to be $H(.,.)$-Co-monotone, and the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)} : X \to X$ must be $\theta = \frac{1}{\lambda(\alpha-\beta)+\mu\eta^2-\gamma\sigma^2}$-Lipschitz continuous. Invoking Theorems 4.9 and 4.11, $\alpha > \beta$, $\mu > \gamma$ and $\eta > \sigma$ are the necessary conditions for the resolvent operator $R^{H(.,.)}_{\lambda,M(.,.)}$ associated with an $H(.,.)$-Co-monotone mapping M and a real constant $\lambda > 0$ to be well defined and $\theta = \frac{1}{\lambda(\alpha-\beta)+\mu\eta^2-\gamma\sigma^2}$-Lipschitz continuous. But, taking $\mu = \frac{3}{10}$, $\gamma = 1$, $\eta = \frac{\sqrt{10}}{3}$, $\sigma = \frac{1}{\sqrt{2}}$, $\alpha = 4$ and $\beta = \frac{1}{4}$, we have $\alpha > \beta$, $\eta > \sigma$ and $\mu < \gamma$ such that $\mu\eta^2 < \gamma\sigma^2$. Hence, contrary to the claim in [1], all the conditions of [1, Theorem 4.1] are not satisfied.

5. Conclusion

The introduction of the notion of M-monotone operator in a Hilbert space setting, as a generalization of maximal monotone operator and H-monotone operator, was first made by Sun et al. [28] in 2008. In later years, the efforts in this direction have been continued and further generalizations of them were introduced and the resolvent operators associated with these generalizations were defined. In one of these attempts and with goal of providing a unifying framework for the classes of maximal monotone operators, H-monotone operators and M-monotone operators, Ahmad et al. [1] introduced and studied the concept of $H(.,.)$-Co-monotone mapping in the context of Hilbert space and defined the resolvent operator associated with such a mapping. In this paper, we have proved the Lipschitz continuity of the resolvent operator associated with a $(\widehat{H}, \eta)$-monotone operator under some new appropriate conditions imposed on the parameter and mappings involved in it and computed an estimate of its Lipschitz constant. We have employed the resolvent operator method and established a new equivalence relationship between a class of multi-valued variational inclusion problems involving $(\widehat{H}, \eta)$-monotone operators (for short, MVIP) and a class of fixed points problems. We have used the obtained equivalence relationship and constructed a new iterative algorithm for solving the MVIP. The convergence analysis of the sequences generated by our suggested iterative algorithm has been studied under some suitable assumptions. In the final section, we have investigated and analyzed the concept of $H(.,.)$-Co-monotone mapping introduced in [1] and pointed out some comments about it. We have shown that under the hypotheses considered in [1], every $H(.,.)$-Co-monotone mapping is actually a $\widehat{H}$-monotone operator and is not a new one. Further remarks concerning the results given in [1] have been provided.

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