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Applications of Berezin symbols in summability theory and related problems

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Abstract. In this article, using the reproducing kernel and Berezin symbol techniques we prove that a sequence is Cesàro convergent, then it is Abel convergent. We prove criteria for Borel convergence of some sequences and series of complex numbers. Also, we answer if the Berezin transform of a bounded operator on the Dirichlet space $\mathcal{D}$ and the Fock space $\mathcal{F}(\mathbb{C})$ have radial limits almost everywhere. Moreover, we give an approximation problem for inner function by Berezin number of operators.

1. Introduction and background

By a reproducing kernel Hilbert space (RKHS) we mean a Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ of complex-valued functions on a (nonempty) set Θ, which has the property that point evaluations $f \to f(\mu)$ are continuous in $\mathcal{H}$ for all $\mu \in \Theta$. Then the Riesz representation theorem guarantees that there is a unique element $k_{\mathcal{H},\mu} \in \mathcal{H}$ for every $\mu \in \Theta$ such that $f(\mu) = \left\langle f, k_{\mathcal{H},\mu} \right\rangle$ for all $f \in \mathcal{H}$. The function $k_{\mathcal{H},\mu}$ is called the reproducing kernel of $\mathcal{H}$.

Let the sequence $(e_n)_{n\geq 1}$ be an orthonormal basis $(e_n)_{n\geq 1}$ of $\mathcal{H}$. Then, the reproducing kernel $k_{\mathcal{H},\mu}$ of $\mathcal{H}$ is given by (see Aronzajn [1])

$$k_{\mathcal{H},\mu} = \sum_n \overline{e_n(\mu)} e_n(z). \tag{1}$$

The Berezin symbol $\widetilde{A}$ of a bounded linear operator on $\mathcal{H}$ (shortly, $A \in \mathcal{B}(\mathcal{H})$) is defined by (see Zhu [22])

$$\widetilde{A}(\mu) := \left\langle A\widehat{k}_{\mathcal{H},\mu}, \widehat{k}_{\mathcal{H},\mu} \right\rangle \ (\mu \in \Theta),$$

where $\widehat{k}_{\mathcal{H},\mu} = \frac{k_{\mathcal{H},\mu}}{\|k_{\mathcal{H},\mu}\|}$ be the normalized reproducing kernel of $\mathcal{H}$. Important applications related with Berezin symbols can be found in [13, 14].

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The RKHS $\mathcal{H}$ is standard if the underlying set Θ is a subset of a topological space and the boundary of Θ is non-empty and has the property that $\left\{\widehat{k}_{\mu_n}\right\}$ is weakly convergent to 0 whenever $\{\mu_n\}$ is a sequence in Θ that is convergent to a point in boundary of Θ (see Nordgren and Rosenthal [16]). For example, Hardy, Bergman and Fock spaces are standard RKHS.

It is obvious that $\lim\limits_{n\to\infty}\widetilde{K}(\mu_n) = 0$ for a compact operator K on the standard RKHS $\mathcal{H}$, whenever $\{\mu_n\}$ is convergent to a point of $\partial\Theta$ (since compact operators send weakly convergent sequences into strongly convergent ones). So, the Berezin symbol of a compact operator on a standard RKHS vanishes on the boundary.

The Berezin set and the Berezin number of operator A are defined by (see Karaev [10])

$$Ber(A) := \left\{\widetilde{A}(\mu) : \mu \in \Theta\right\}$$

and

$$ber(A) := \sup\left\{\left|\widetilde{A}(\mu)\right| : \mu \in \Theta\right\},$$

respectively.

Recently, remarkable results for Berezin number of operators have been obtained by authors [2, 3, 6, 8, 21].

Let $(a_n)_{n\geq 0}$ be any sequence of complex numbers. The diagonal operator $D_{(a_n)}$ on $\mathcal{H}$ for any bounded sequence $(a_n)_{n\geq 0}$ is defined by the formula $D_{(a_n)}e_n(z) := a_n e_n(z)$, $n \geq 0$, with respect to the orthonormal basis $(e_n(z))_{n\geq 0}$ of $\mathcal{H}$. A simple calculus shows [12] that the Berezin symbol of diagonal operator is

$$\widetilde{D}_{(a_n)}(\mu) = \frac{1}{\sum_{n=0}^{\infty}\left|e_n(\mu)\right|^2}\sum_{n=0}^{\infty}a_n\left|e_n(\mu)\right|^2, \ \ \mu \in \Theta. \tag{2}$$

Recall that [15] the Dirichlet space $\mathcal{D}$ is the Hilbert space of analytic functions $f = \sum\limits_{n=0}^{\infty}a_n z^n$ on the unit disk $\mathbb{D}$ with $\int\limits_{\mathbb{D}}\left|f'(z)\right|^2 dA/\pi = \sum\limits_{n=0}^{\infty}(n+1)\left|a_n\right|^2 < \infty$, where dA denotes the usual Lebesgue measure on $\mathbb{D}$. Since the sequence $\left\{z^n/\sqrt{n+1} : n \geq 0\right\}$ is an orthonormal basis of Dirichlet space, the reproducing kernel of $\mathcal{D}$ is given by formula (1)

$$k_\mu(z) = \sum_{n=0}^{\infty}\frac{(\overline{\mu}z)^n}{n+1} = \frac{1}{\overline{\mu}z}\log\frac{1}{1-\overline{\mu}z}.$$

Following by [19], note that the Fock space (or Segal-Bargmann space) is the space of entire functions that are square-integrable with respect to Gaussian measure on the complex plane, that is, the space of all analytic functions f on $\mathbb{C}$ for which $\int\limits_{\mathbb{C}}\left|f(z)\right|^2 d\mu(z) < \infty$, where $d\mu(z) = e^{-\frac{|z|^2}{2}}d\frac{A(z)}{2}$. Since $\left\{z^n/(n!2^n)^{1/2} : n \geq 0\right\}$ is an orthonormal basis in $\mathcal{F}$, the reproducing kernel in $\mathcal{F}$ by formula (1) is given by $k_{\mathcal{F},\mu}(z) = e^{\overline{\mu}z/2}$.

Let $L_a^2 = L_a^2(\mathbb{D})$ represent the Bergman space of all analytic functions on $\mathbb{D}$ satisfying $\left\|f\right\|_{L_a^2}^2 := \int\limits_{\mathbb{D}}\left|f(z)\right|^2 dA(z) < \infty$, where $dA(z)$ represent Lebesgue area measure on the unit disc $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ is equal to 1. It is well known (see Zhu [22]) that since $\left\{\sqrt{n+1}z^n\right\}_{n\geq 0}$ is an orthonormal basis of the Bergman space L_a^2, the reproducing kernel of Bergman space is $k_{L_a^2,\mu}(z) = 1/(1-\overline{\mu}z)^2$, $\mu, z \in \mathbb{D}$.

In this article, using the reproducing kernel and Berezin symbol techniques we prove that a sequence $(a_n)_{n\geq 0}$ is Cesàro convergent to L, then $(a_n)_{n\geq 0}$ is Abel convergent to L. We give criteria for Borel convergence of some sequences and series of complex numbers. Also, we answer if the Berezin transform of a bounded operator on the Dirichlet space $\mathcal{D}$ and the Fock space $\mathcal{F}(\mathbb{C})$ have radial limits almost everywhere. Moreover, we give an approximation problem for inner function by Berezin number of operators.

2. A new proof for a relationship between Cesaro and Abel convergences

Recall [17] that a sequence $(a_n)_{n\geq 0}$ of complex numbers a_n is said to be Cesàro convergent (written $(C, 1)$ convergent) to a finite number L if the limit

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} a_k = L.$$

It is Abel convergent (written (A) convergent) to L if the limit

$$\lim_{x\to 1^-} (1-x)\sum_{n=0}^{\infty} a_n x^n = L.$$

The following theorem was proved in [17]. But, we here give a different proof using the Berezin symbol technique.

Theorem 2.1. *If the sequence $(a_n)_{n\geq 0}$ is $(C, 1)$ convergent to L, then $(a_n)_{n\geq 0}$ is (A) convergent to L.*

Proof. Let $(a_n)_{n\geq 0}$ be $(C, 1)$ convergent to L and $s_n = \frac{1}{n+1}\sum_{k=0}^{n} a_k$. Then we have

$$(n+1)\, s_n = \sum_{k=0}^{n} a_k \text{ and } n s_{n-1} = \sum_{k=0}^{n-1} a_k.$$

From above equations, we obtain

$$a_k = (k+1)\, s_k - k s_{k-1}. \tag{3}$$

Now, we calculate the Berezin symbol of diagonal operator with $(s_n)_{n\geq 0}$ on the Bergman space using formula (2) for all $\mu \in \mathbb{D}$ that

$$\begin{aligned}\widetilde{D}_{(s_n)}(\mu) &= \frac{1}{\sum_{n=0}^{\infty}(n+1)\left(\left|\mu\right|^2\right)^n}\sum_{n=0}^{\infty} s_n\,(n+1)\left(\left|\mu\right|^2\right)^n \\ &= \left(1-\left|\mu\right|^2\right)^2\sum_{n=0}^{\infty} s_n\,(n+1)\left(\left|\mu\right|^2\right)^n\end{aligned}$$

and therefore, $\widetilde{D}_{(s_n)}$ is a radial function on $\mathbb{D}$, that is $\widetilde{D}_{(s_n)}(\mu) = \widetilde{D}_{(s_n)}\left(\left|\mu\right|\right)$.

Let $\left|\mu\right|^2 = x$. Then

$$\widetilde{D}_{(s_n)}\left(\sqrt{x}\right) = (1-x)^2\sum_{n=0}^{\infty} s_n\,(n+1)\, x^n. \tag{4}$$

Expanding the formula (4), we have that

$$\begin{aligned}\widetilde{D}_{(s_n)}\left(\sqrt{x}\right) &= (1-x)\sum_{n=0}^{\infty}\left[(n+1)\, s_n - n s_{n-1}\right] x^n \\ &= (1-x)\sum_{n=0}^{\infty} a_n x^n. \text{ (by (3))}\end{aligned}$$

Then, we get from above formula

$$\begin{aligned}(1-x)\sum_{n=0}^{\infty}a_nx^n &= (1-x)\sum_{n=0}^{\infty}(a_n-L)\,x^n+L\,(1-x)\sum_{n=0}^{\infty}x^n\\ &= \widetilde{D}_{s_n-L}\left(\sqrt{x}\right)+L\end{aligned}$$

Since $s_n - L \to 0$ as $n \to \infty$, we have that D_{s_k-L} is a compact operator. Therefore, its Berezin symbol vanishes on the boundary, that is,

$$\lim_{x\to1^-}\widetilde{D}_{s_n-L}\left(\sqrt{x}\right)=0.$$

Then we conclude from the last equality

$$\lim_{x\to1^-}(1-x)\sum_{n=0}^{\infty}a_nx^n=L,$$

which shows that $(a_n)_{n\geq0}$ is (A) convergent to L. □

3. Boundary behavior of Berezin symbols of operators

Zorboska in [23] formulated the following natural and fundamental problem: it is not known if the Berezin transform of a bounded operator on L_a^2 must have radial limits almost everywhere on the unit circle. Some authors tried to answer this question. Motivated by this question, one can ask this question: does the Berezin transform of a bounded operator on Dirichlet space $\mathcal{D}$ have radial limits almost everywhere on the unit circle? Is similar question also valid in case of the Fock space $\mathcal{F}(\mathbb{C})$ over the complex plane? In this section, we try to answer these questions.

Definition 3.1. *The sequence $(a_n)_{n\geq0}$ of complex numbers a_n is Borel summable (written (B)-summable) to l if the limit*

$$\lim_{t\to\infty}e^{-t}\sum_{n=0}^{\infty}a_n\frac{t^n}{n!}=l$$

exists.

Definition 3.2. *The sequence $(a_n)_{n\geq0}$ of complex numbers $(a_n)_{n\geq0}$ is said to be summable to a finite number ζ by the logarithmic method (L) (or (L)-summable to ζ) if the limit*

$$\lim_{x\to1^-}-\frac{1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}=\zeta.$$

exists.

Let K^{∞} and K_{-}^{∞} denote the set of all bounded non-Borel summable sequences and the set of all bounded non-(L) summable sequences, respectively (see [9, 18] for Borel and logarithmic type summability).

The following result shows that there exist linear bounded operators on the Dirichlet space $\mathcal{D}$ such that their Berezin symbols have no radial boundary values on a unit circle.

Theorem 3.3. *Let $(a_n)_{n\geq0}\in K_{-}^{\infty}$ be any sequence. Then the Berezin symbol of the diagonal operator $D_{\{a_n\}}\dfrac{z^n}{\sqrt{n+1}}=a_n\dfrac{z^n}{\sqrt{n+1}}$ has no radial limits anywhere on the unit circle $\mathbb{T}$, where $\left\{z^n/\sqrt{n+1}:n\geq0\right\}$ is an orthonormal basis of Dirichlet space $\mathcal{D}$.*

Proof. The Berezin symbol of the diaoganal operator on the Dirichlet space $\mathcal{D}$ is as following for all $\mu \in \mathbb{D}$ (see [20]):

$$\widetilde{D}_{\{a_n\}}(\mu) = \left\langle D_{\{a_n\}}\widehat{k}_\mu, \widehat{k}_\mu \right\rangle$$
$$= -\frac{1}{\log\left(1 - |\mu|^2\right)} \sum_{n=0}^{\infty} a_n \frac{\left(|\mu|^2\right)^{n+1}}{n+1}$$

and so, $\widetilde{D}_{\{a_n\}}$ is a radial function on $\mathbb{D}$, that is $\widetilde{D}_{\{a_n\}}(\mu) = \widetilde{D}_{\{a_n\}}\left(|\mu|\right)$.

In view of the fact that $(a_n)_{n\geq 0}$ is not (L) convergent sequence, we get from above equation that $\widetilde{D}_{\{a_n\}}$ has radial limits nowhere on a unit circle, which gives the desired result. □

Next result proves that there exist linear bounded operators on the Fock space $\mathcal{F}(\mathbb{C})$ such that their Berezin symbols have no radial boundary values on the complex plane $\mathbb{C}$.

Theorem 3.4. *Let $(a_n)_{n\geq 0} \in K^\infty$ be any sequence, and let $D_{\{a_n\}}$ be a diagonal operator with respect to the standard orthonormal basis $e_n(z) = \left\{z^n / (n!2^n)^{1/2}\right\}_{n\geq 0}$ of the Fock space $\mathcal{F}(\mathbb{C})$. Then the Berezin symbol $\widetilde{D}_{\{a_n\}}$ has no radial limits anywhere on the complex plane $\mathbb{C}$.*

Proof. By calculating the Berezin symbol of the diaoganal operator on the Fock space $\mathcal{F}(\mathbb{C})$, we have (see [5])

$$\widetilde{D}_{\{a_n\}}(\mu) = \left\langle D_{\{a_n\}}\widehat{k}_\mu, \widehat{k}_\mu \right\rangle = \frac{1}{e^{|\mu|^2/2}} \sum_{n=0}^{\infty} a_n \frac{\left(|\mu|^2/2\right)^n}{n!}$$
$$= e^{-|\mu|^2/2} \sum_{n=0}^{\infty} a_n \frac{\left(|\mu|^2/2\right)^n}{n!}$$

for all $\mu \in \mathbb{C}$ and hence, $\widetilde{D}_{\{a_n\}}$ is a radial function on $\mathbb{C}$, that is $\widetilde{D}_{\{a_n\}}(\mu) = \widetilde{D}_{\{a_n\}}\left(|\mu|\right)$.

Since by condition of theorem, $(a_n)_{n\geq 0}$ is not (B)-summable, it follows from above equation that the Berezin symbol $\widetilde{D}_{\{a_n\}}$ of the operator $D_{\{a_n\}}$ has radial limits nowhere on the complex plane $\mathbb{C}$. □

4. Berezin symbols and Borel convergence of sequences and series

In this section, we give in terms of Berezin symbols of weighted shift operator W_Λ,

$$W_\Lambda \frac{z^n}{(n!2^n)^{1/2}} = \mu_n \frac{z^{n+1}}{((n+1)!2^{n+1})^{1/2}},\ n \geq 0, \tag{5}$$

on the Fock space $\mathcal{F}$ some criterion for Borel convergence of sequences and series of complex numbers. Recall that $\left\{\frac{z^n}{(n!2^n)^{1/2}} : n \geq 0\right\}$ is an orthonormal basis in $\mathcal{F}$.

Theorem 4.1. *Let $\{a_n\}_{n\geq 0}$ be a bounded sequence of complex numbers, and W_a be an associated weighted shift operator acting on the Fock space $\mathcal{F}$ by the formula (5). Then*

(a) the series $\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}}$ is Borel convergent if and only if

$$\frac{\left|\widetilde{W}_a\right|\left(\sqrt{2t}\right)}{\sqrt{2t}} = O\left(e^{-t}\right) \text{ as } t \to \infty;$$

(b) the sequence $\left(\frac{a_n}{\sqrt{2n+2}}\right)_{n\geq 0}$ *is Borel convergent if and only if*

$\left|\widetilde{W}_a\right|\left(\sqrt{2t}\right) = O\left(\sqrt{2t}\right)$ *as* $t \to \infty$.

First let us give the following.

Lemma 4.2. *For any operator* $A \in \mathcal{B}(\mathcal{F})$ *we obtain*

$$\widetilde{A}(\mu) = e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{(n!2^n)(m!2^m)}} \langle Ae_n, e_m \rangle \overline{\mu}^n \mu^m \tag{6}$$

for $\mu \in \mathbb{C}$, *where* $e_n(z) = \left\{\frac{z^n}{(n!2^n)^{1/2}}\right\}_{n\geq 0}$ *is the orthonormal basis for the space* $\mathcal{F}$.

Proof. Actually,

$$\begin{aligned}
\widetilde{A}(\mu) &= \left\langle A\widehat{k}_\mu, \widehat{k}_\mu \right\rangle = \left\langle A\frac{k_\mu}{\left\|k_{L_a^2,\mu}\right\|}, \frac{k_\mu}{\left\|k_{L_a^2,\mu}\right\|} \right\rangle \\
&= e^{-|\mu|^2/2} \left\langle \sum_{n\geq 0} \overline{e_n(\mu)} Ae_n(z), \sum_{n\geq 0} \overline{e_n(\mu)} e_n(z) \right\rangle \\
&= e^{-|\mu|^2/2} \sum_{n,m=0}^{\infty} \overline{e_n(\mu)} e_m(\mu) \langle Ae_n, e_m \rangle \\
&= e^{-|\mu|^2/2} \sum_{n,m=0}^{\infty} \frac{\overline{\mu}^n}{\sqrt{n!2^n}} \frac{\mu^m}{\sqrt{m!2^m}} \langle Ae_n, e_m \rangle \\
&= e^{-|\mu|^2/2} \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{(n!2^n)(m!2^m)}} \langle Ae_n, e_m \rangle \overline{\mu}^n \mu^m,
\end{aligned}$$

which proves (6). □

Proof. [Proof of Theorem 4.1]Let W_a be a weighted shift operator on $\mathcal{F}$. Then it follows from formula (5) that

$$\begin{aligned}
\widetilde{W}_a(\mu) &= e^{-|\mu|^2/2} \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{(n!2^n)(m!2^m)}} \langle a_n e_{n+1}, e_m \rangle \overline{\mu}^n \mu^m \\
&= \mu e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{|\mu|^{2n}}{n!2^n} \\
&= \mu e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{\left(|\mu|^2/2\right)^n}{n!}
\end{aligned}$$

or

$$\widetilde{W}_a(\mu) = \mu e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{\left(|\mu|^2/2\right)^n}{n!} \quad (\mu \in \mathbb{C}). \tag{7}$$

As a result,

$$\left|\frac{\widetilde{W}_a(\mu)}{\mu e^{-|\mu|^2/2}}\right| = \left|\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{\left(|\mu|^2/2\right)^n}{n!}\right| \tag{8}$$

and

$$\left|\frac{\widetilde{W}_a(\mu)}{\mu}\right| = \left|e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{\left(|\mu|^2/2\right)^n}{n!}\right| \tag{9}$$

for all $\mu \in \mathbb{C}$. Let $t = |\mu|^2/2$. Then (8) and (9) has the form

$$\frac{\left|\widetilde{W}_a\right|\left(\sqrt{2t}\right)}{\sqrt{2t}e^{-t}} = \left|\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{(t)^n}{n!}\right| \tag{10}$$

and

$$\frac{\left|\widetilde{W}_a\right|\left(\sqrt{2t}\right)}{\sqrt{2t}} = \left|e^{-t} \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{(t)^n}{n!}\right| \tag{11}$$

for all $t \in R^+$. It follows now immediately from (10) and (11) the assertions (a) and (b) of the theorem. □

Corollary 4.3. *We have:*

(a) $Ber(W_a) = \left\{\mu e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{\left(|\mu|^2/2\right)^n}{n!} : \mu \in \mathbb{C}\right\}$

(b) $ber(W_a) = \sup\left\{|\mu| e^{-|\mu|^2/2} \left|\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{2n+2}} \frac{\left(|\mu|^2/2\right)^n}{n!}\right| : \mu \in \mathbb{C}\right\}$

(c) If $\sup_{n\geq 0} \frac{|a_n|}{\sqrt{2n+2}} < +\infty$*, then* $ber(W_a)$ *does not have an upper bound.*

Proof. The proofs of (a) and (b) are clear from the formula (7). Now we prove (c). Actually, it follows from the condition $\sup_{n\geq 0} \frac{|a_n|}{\sqrt{2n+2}} < +\infty$, $n \geq 0$, that

$$\begin{aligned} ber(W_a) &\leq \sup_{\mu\in\mathbb{C}} \left(|\mu| e^{-|\mu|^2/2} \sum_{n=0}^{\infty} \frac{\left(|\mu|^2/2\right)^n}{n!}\right) \sup_{n\geq 0} \frac{|a_n|}{\sqrt{2n+2}} \\ &\leq \sup_{n\geq 0} \frac{|a_n|}{\sqrt{2n+2}} \sup_{\mu\in\mathbb{C}} \left(|\mu| e^{-|\mu|^2/2} e^{|\mu|^2/2}\right) \\ &= \sup_{n\geq 0} \frac{|a_n|}{\sqrt{2n+2}} \sup_{\mu\in\mathbb{C}} \left(|\mu|\right), \end{aligned}$$

for all $\mu \in \mathbb{C}$. This shows that $ber(W_a)$ does not have an upper bound. □

5. Approximation of inner functions and Berezin number

In this section, we give an approximation problem for inner function by Berezin number of operators, which improves the results in [11, Theorem 5.1].

Theorem 5.1. *Let Φ be an inner function. For any $A \in \{T_\Phi\}' = \left\{L \in \mathcal{B}\left(H^2\right) : LT_\Phi = T_\Phi L\right\}$, we set $N_{\Phi,A} = T_\Phi\left(I - AT_\Phi T_\Phi^*\right)$. Let $K_{\epsilon,\Phi} = \{z \in \mathbb{D} : |\Phi(z)| \leq \epsilon\}$, where $0 < \epsilon < 1$, be a level set of Φ for every ϵ. Then*

$$\left\|\Phi - \widetilde{N}_{\Phi,A}\right\|_{L^\infty\left(K_{\epsilon,\Phi}\right)} \leq \frac{\left(ber\left(A^2\right) + \|A\|^2\right)^{1/2}}{\sqrt{2}}\epsilon^3.$$

Proof. Calculating in the same manner as in Theorem 5.1 of [11], we have

$$\widetilde{N}_{\Phi,A}(\lambda) = \left\langle N_{\Phi,A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle = \Phi(\lambda)\left(1 - \widetilde{A}(\lambda)|\Phi(\lambda)|^2\right)$$

from which we have that

$$\left|\Phi(\lambda) - \widetilde{N}_{\Phi,A}\right| = \left|\widetilde{A}(\lambda)\right||\Phi(\lambda)|^3$$

for $\lambda \in \mathbb{D}$. It is known that following inequality was obtained in a different way earlier by Buzano in [4]:

$$|\langle a, e\rangle\langle e, b\rangle| \leq \frac{1}{2}\left(|\langle a, b\rangle| + \|a\|\|b\|\right) \tag{12}$$

Putting the $e = \widehat{k}_\lambda$, $a = A\widehat{k}_\lambda$, $b = A^*\widehat{k}_\lambda$ in (12), we have that

$$\begin{aligned}\left|\Phi(\lambda) - \widetilde{N}_{\Phi,A}\right| &= \left|\widetilde{A}(\lambda)\right||\Phi(\lambda)|^3 \\ &\leq \frac{\left(\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left\|A\widehat{k}_\lambda\right\|\left\|A^*\widehat{k}_\lambda\right\|\right)^{1/2}}{\sqrt{2}}\epsilon^3 \\ &\leq \frac{\left(ber\left(A^2\right) + \|A\|^2\right)^{1/2}}{\sqrt{2}}\epsilon^3\end{aligned}$$

for all $\lambda \in \mathbb{D}$ and $0 < \epsilon < 1$. Particularly,

$$\left|\Phi(\lambda) - \widetilde{N}_{\Phi,A}\right| \leq \frac{\left(ber\left(A^2\right) + \|A\|^2\right)^{1/2}}{\sqrt{2}}\epsilon^3$$

for all $\lambda \in K_{\epsilon,\Phi}$, and therefore

$$\left\|\Phi - \widetilde{N}_{\Phi,A}\right\|_{L^\infty\left(K_{\epsilon,\Phi}\right)} \leq \frac{\left(ber\left(A^2\right) + \|A\|^2\right)^{1/2}}{\sqrt{2}}\epsilon^3, \ 0 < \epsilon < 1$$

which gives the desired result. □

References

[1] N. Aronzajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68(1950), 337-404.

[2] M. Bakherad, Some Berezin number inequalities for operator matrices, Czechoslovak Math. J., 68(2018), 997-1009.

[3] P. Bhunia, K. Paul, A. Sen, Inequalities involving Berezin norm and Berezin number, Complex Anal. Oper. Theory, 17(7)(2023), 1-15.

[4] M.L. Buzano, Generalizzazione della disiguaglianza di Cauchy-Schwartz (Italian). Rend. Sem. Mat. Univ. e Politech. Torino, 31(1974), 405-409.
[5] M.T. Garayev, M. Gürdal M., U. Yamancı, Berezin symbols and Borel summability, Quaestiones Mathematicae, 40(3)(2017), 403-411.
[6] M.T. Garayev, U.Yamancı, Čebyšev's type inequalities and power inequalities for the Berezin number of operators, Filomat, 33(8)(2019), 2307-2316.
[7] I.Ts. Gohberg, M.G. Krein, Introduction to the theory of linear nonselfadjoint operator, Translations of Mathematical Monographs 18, Amer. Math. Soc., Provindence, RI, 1969.
[8] M. Hajmohamadi, R. Lashkaripour, M. Bakherad, Improvements of Berezin number inequalities, Linear Multilinear Algebra, 68(6)(2020), 1218-1229.
[9] K. Ishiguro, On the summability methods of logarithmic type, Proc. Jpn. Acad., 38(1962), 703-705.
[10] M.T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, J. Funct. Anal., 238(2006),181-192.
[11] M.T. Karaev, S. Saltan, Some results on Berezin symbols, Complex Variables: Theory and Applications, 50(2005), 185-193.
[12] M.T. Karaev, (e)-convergence and related problem, C. R. Math. Acad. Sci. Paris, 348 (2010), 1059-1062.
[13] M.T. Karaev, M. Gürdal, U. Yamancı, Special operator classes and their properties, Banach J. Math. Anal., 7(2)(2013), 75-86.
[14] M.T. Karaev, M. Gürdal, U. Yamancı, Some results related with Berezin symbols and Toeplitz operators, Math. Inequal. Appl., 17(3)(2014), 1031-1045.
[15] D. Marshall, C. Sundberg, Interpolating sequences for the multipliers of the Dirichlet space, Preprint; see http://www.math.washington.edu/~marshall/preprints/preprints.html, 1994.
[16] E. Nordgren, P. Rosenthal, Boundary values of Berezin symbols, Oper. Theory Adv. Appl., 73 (1994), 362-368.
[17] R.E. Powell, S.M. Shah, Summability theory and applications, Van Nostrand Reinhold, London, 1972.
[18] B. Sawyer, B. Watson, Borel's methods of summability: theory and applications, Oxford University Press Inc., New York, 1994.
[19] K. Stroethoff, The Berezin transform and operators on spaces of analytic functions, Banach Center Publ., 38(1997), 361-380.
[20] U. Yamancı, On the summability methods of logarithmic type and the Berezin symbol, Turk. J. Math., 42(2018), 2417-2422.
[21] U. Yamancı, İ.M. Karlı, Further refinements of the Berezin number inequalities on operators, Linear Multilinear Algebra, 70(20)(2022), 5237-5246.
[22] K. Zhu, Operator theory in function spaces, Marcel Dekker, Ins. 1990.
[23] N. Zorboska, Berezin transforms and radial operators, Proc. Amer. Math. Soc., 131(2003), 793-800.