Filomat 38:12 (2024), 4365–4380
https://doi.org/10.2298/FIL2412365L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation and Moduli of continuity of functions in Hölder class $H^{\alpha}[0,\mu),\ \mu>0$ and wavelet solutions of singular differential equations

Shyam Lal[a], **Abhilasha**[a,b,*]

[a]*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi*
[b]*Department of Mathematics, Institute of Integrated and Honors Studies, Kurukshetra University, Kurukshetra*

Abstract. This paper introduces a precise upper limit for solving the numerical solutions of singular differential equations by employing error analysis. The approach utilizes extended second-kind Chebyshev wavelets to solve the singular differential equation. By applying the exact upper bound based on the moduli of continuity in the Hölder space interval $[0,\mu)$, it becomes evident that the method exhibits convergence with a reduced number of wavelet coefficients. Additionally, the paper includes several numerical examples that effectively showcase the method's validity and applicability.

1. Introduction

Wavelet analysis is a contemporary field within mathematical research, finding extensive application in signal analysis, time-frequency analysis, and numerical analysis. This analytical technique revolves around function representation [12],[11],[5],[10] wherein functions are decomposed into a sum of wavelet basis functions. Each of these wavelet basis functions results from compressing and translating a mother wavelet function with desirable attributes of smoothness and locality. Therefore, wavelets are employed as foundational elements for the representation of signals and functions, characterized by distinct attributes that render them well-suited for a broad spectrum of applications [3],[7],[13] in science and engineering.

Wavelets are increasingly being used to solve differential equations [6],[20]. This includes both linear and nonlinear differential equations. The sparse representation property of wavelets makes solving these equations more efficient and accurate compared to traditional methods. The compact support of wavelets allows them to naturally incorporate initial and boundary conditions in solving differential equations. This property makes them suitable for capturing localized features in signals or functions and are adept at identifying singularities or discontinuities in signals. In the literature, a variety of wavelets can be found, including the Haar wavelet [16], [18], [17], Meyer wavelet [9], Legendre wavelet [15], Laguerre wavelet [21], and Chebyshev wavelet [22], Hermite wavelet [14] etc. These wavelets are utilized to the approximate functions in a specific space.

In the standard formulation of wavelets, they are indeed often defined on the interval $[0,1)$. This

2020 *Mathematics Subject Classification*. 41A50, 42C40, 65T60, 65L05, 65L99.
Keywords. Extended second kind Chebyshev wavelet, moduli of continuity, operational matrix of integration, wavelet approximations, singular differential equations.
Received: 22 December 2022; Revised: 22 August 2023; Accepted: 01 November 2023
Communicated by Miodrag Spalević
* Corresponding author: Abhilasha
Email addresses: shyam_lal@rediffmail.com (Shyam Lal), yadavabhilasha1942@gmail.com (Abhilasha)

interval is convenient for many theoretical and practical reasons. However, in this paper wavelets have been extended to different intervals, such as $[0, \mu)$, where μ is some positive number. Extending wavelets to $[0, \mu)$ involves scaling and translating the original wavelet function to fit within the new interval while preserving essential properties like orthogonality, compact support, and wavelet transform properties. This process can be useful when dealing with data that doesn't naturally conform to the $[0, 1)$ interval, and it allows for more flexibility in wavelet-based analysis. In this paper an operational matrix of integration based on the extended second kind Chebyshev wavelets is introduced. This matrix is designed for addressing non-linear initial value problems that exhibit singularities within the interval $[0, \mu)$. An essential objective of this research paper is to analyse the convergence of the wavelet series of functions having second derivatives in Hölder class $H^{\alpha}[0, \mu)$ using the moduli of continuity $W(f, \delta)$. The modulus of continuity of f, denoted by $W(f, \delta)$ [1], is established through the definition:

$$W(f, \delta) = \sup_{0<h\leq\delta} \|f(t + h) - f(t)\|_2,$$

for every t belonging to a finite interval, has the property $\lim_{\delta\to 0^+} W(f, \delta) = 0$. Due to this property, the better estimations of the rate of approximation of functions in different classes are obtained.

This paper is arranged as: Section 2. contains some definitions and preliminaries used in this paper. Section 3. gives the convergence and error analysis using moduli of continuity of functions whose second-order derivatives belong to Hölder class $H^{\alpha}[0, \mu)$, $\mu > 0$. Section 4. provides the detailed description of the method to solve singular differential equations using wavelet approximations. Section 5. encloses some numerical examples and their comparisons. Section 6 contains the final remarks, and lastly, the references used to support the content of this paper have been included.

2. Definitions and Preliminaries

2.1. Chebyshev wavelets of second kind on the interval $[0, \mu)$

The second kind of Chebyshev polynomial [8], denoted as $U_m(t)$, is a polynomial of degree m in the variable t. This polynomial is defined by the equation:

$$U_m(t) = \frac{\sin(m + 1)\theta}{\sin\theta}$$

when $t = \cos\theta$. These polynomials are defined on the interval [-1,1] and are orthogonal with respect to the weight function $w(t) = \sqrt{1 - t^2}$ as

$$\int_{-1}^{1} U_m(t)U_n(t)w(t)dt = \begin{cases} \frac{\pi}{2}, & m = n \\ 0, & \text{otherwise}, \end{cases}$$

for $m, n = 0, 1, 2, 3\ldots$.

The recurrence formula for these polynomials are $U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t)$, $m = 1, 2, 3, \ldots$.

Extended Chebyshev wavelets $\psi^{\mu}_{n,m}(t)=\psi(k, n, m, \mu, t)$ have five arguments, where $n = 1, 2, 3, \ldots, 2^{k-1}, k \in \mathbb{Z}^+, m$ being the degree of Chebyshev polynomials and t the normalized time. These are defined on the interval $[0, \mu)$ by

$$\psi^{\mu}_{n,m}(t) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\mu\pi}} U_m(\frac{2^k}{\mu}t - 2n + 1), & \frac{(n-1)\mu}{2^{k-1}} \leq t < \frac{n\mu}{2^{k-1}} \\ 0, & \text{otherwise}, \end{cases}$$

where $m = 0, 1, 2, \ldots, M$ and $n = 1, 2, \ldots, 2^{k-1}$ [22].

Here, the coefficient $\sqrt{\frac{2}{\mu\pi}}$ is for orthonormality, $(2n - 1)\mu 2^{-k}$ is the translation parameter and 2^{-k} is the dilation parameter. Also, the weight function $w(t) = \sqrt{1 - t^2}$ has to be dilated and translated as $w_n(t) = w(\frac{2^k}{\mu}t - 2n + 1)$.

2.2. Modulus of Continuity

The modulus of continuity [1] of a function $f \in L^2[0,\mu)$ is defined as

$$\begin{aligned} W(f,\delta) &= \sup_{0<h\leq\delta} \|f(t+h)-f(t)\|_2, \quad \forall t \in [0,\mu) \\ &= \sup_{0<h\leq\delta} \left(\int_0^\mu |f(t+h)-f(t)|^2 dt \right)^{1/2}. \end{aligned}$$

Remarkably, a noteworthy observation is that $W(f,\delta)$ exhibits a monotonically increasing pattern with the growth of δ and $W(f,\delta) \to 0$ as $\delta \to 0^+$ for $f \in L^2[0,\mu)$.

2.3. Hölder class $H^\alpha[0,\mu)$

A function f is said to be in Hölder class [2] $H^\alpha[0,\mu)$ of order $\alpha \in (0,1]$ if f is continuous on $[0,\mu)$ and satisfies the inequality,

$$f(x+t) - f(x) = O(|t|^\alpha), \qquad \forall\ x+t,\ x \in [0,\mu).$$

3. Moduli of Continuity of functions having second derivative in Hölder class $H^\alpha[0,\mu)$

3.1. Convergence Analysis

The following convergence theorem has been proved in this paper:

Theorem 3.1. *Suppose there exists a square integrable function denoted by $f(t)$ satisfying the condition $f'' \in H^\alpha[0,\mu)$, where $0<\alpha\leq 1$ and has the extended second kind Chebyshev wavelet expansion of the form*

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^\mu(t), \text{ where } c_{n,m} = \langle f, \psi_{n,m}^\mu \rangle_{w_n}, \tag{1}$$

then the series converges uniformly to the function $f(t)$ in the Hilbert space $L^2[0,\mu)$.

Proof. Consider the wavelet coefficient

$$\begin{aligned} c_{n,m} &= \int_0^\mu f(t)\psi_{n,m}^\mu(t) w_n(t) dt \\ &= \sqrt{\frac{2}{\mu\pi}}.2^{\frac{k}{2}} \int_{\frac{(n-1)\mu}{2^{k-1}}}^{\frac{n\mu}{2^{k-1}}} f(t) U_m(\frac{2^k t}{\mu} - 2n + 1) w(\frac{2^k t}{\mu} - 2n+1) dt \\ &= 2^{\frac{k}{2}} \sqrt{\frac{2\mu}{\pi}} \int_0^\pi f\left(\frac{(\cos\theta + 2n-1)\mu}{2^k}\right) U_m(\cos\theta) w(\cos\theta)\sin\theta \frac{d\theta}{2^k}, \qquad \frac{2^k t}{\mu} - 2n + 1 = \cos\theta \\ &= \frac{1}{2^{\frac{k}{2}}} \sqrt{\frac{2\mu}{\pi}} \int_0^\pi f\left(\frac{(\cos\theta + 2n-1)\mu}{2^k}\right) U_m(\cos\theta) w(\cos\theta)\sin\theta d\theta \\ &= \frac{1}{2^{\frac{(k-1)}{2}}} \sqrt{\frac{\mu}{\pi}} \int_0^\pi f\left(\frac{(\cos\theta + 2n-1)\mu}{2^k}\right) \sin(m+1)\theta \sin\theta d\theta \\ &= \frac{1}{2^{\frac{(k+1)}{2}}} \sqrt{\frac{\mu}{\pi}} \int_0^\pi f\left(\frac{(\cos\theta + 2n-1)\mu}{2^k}\right) (\cos m\theta - \cos(m+2)\theta)\, d\theta \\ &= \frac{1}{2^{\frac{(k+1)}{2}}} \sqrt{\frac{\mu}{\pi}} \left[f\left(\frac{(\cos\theta + 2n-1)\mu}{2^k}\right) \left(\frac{\sin m\theta}{m} - \frac{\sin(m+2)\theta}{m+2}\right)_0^\pi \right] \\ &- \frac{1}{2^{\frac{(k+1)}{2}}} \sqrt{\frac{\mu}{\pi}} \left[\int_0^\pi f'\left(\frac{(\cos\theta + 2n-1)\mu}{2^k}\right) \left(\frac{-\mu\sin\theta}{2^k}\right) \left(\frac{\sin m\theta}{m} - \frac{\sin(m+2)\theta}{m+2}\right) d\theta \right], \end{aligned}$$

$$= \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{\mu^{5/2}}{\sqrt{\pi}}\int_0^{\pi} f''\left(\frac{(\cos\theta+2n-1)\mu}{2^k}\right)\gamma_m(\theta)d\theta,$$

$$\text{where}\quad \gamma_m(\theta) = \frac{\sin\theta}{m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)-\frac{\sin\theta}{m+2}\left(\frac{\sin(m+1)\theta}{m+1}-\frac{\sin(m+3)\theta}{m+3}\right).$$

$$\begin{aligned}
c_{n,m} &= \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{\mu^{5/2}}{\sqrt{\pi}}\left(\int_0^{\pi}\left[f''\left(\frac{(\cos\theta+2n-1)\mu}{2^k}\right)-f''\left(\frac{(2n-1)\mu}{2^k}\right)\right]\gamma_m(\theta)d\theta\right.\\
&+ \left. f''\left(\frac{(2n-1)\mu}{2^k}\right)\int_0^{\pi}\gamma_m(\theta)d\theta\right)\\
&= I_1+I_2.\\
I_2 &= \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{\mu^{5/2}}{\sqrt{\pi}} f''\left(\frac{(2n-1)\mu}{2^k}\right)\int_0^{\pi}\gamma_m(\theta)d\theta = 0.\\
c_{n,m} &= \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{\mu^{5/2}}{\sqrt{\pi}}\int_0^{\pi}\left[f''\left(\frac{(\cos\theta+2n-1)\mu}{2^k}\right)-f''\left(\frac{(2n-1)\mu}{2^k}\right)\right]\gamma_m(\theta)d\theta\\
|c_{n,m}| &= \left|\frac{1}{2^{\frac{(5k+1)}{2}}}\frac{\mu^{5/2}}{\sqrt{\pi}}\int_0^{\pi}\left[f''\left(\frac{(\cos\theta+2n-1)\mu}{2^k}\right)-f''\left(\frac{(2n-1)\mu}{2^k}\right)\right]\gamma_m(\theta)d\theta\right|\\
&\le \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{\mu^{5/2}}{\sqrt{\pi}}\int_0^{\pi}\left|\frac{\mu\cos\theta}{2^k}\right|^{\alpha}|\gamma_m(\theta)|d\theta, \quad (\because f''\in H^{\alpha}[0,\mu))\\
&\le \frac{1}{2^{\frac{(5k+1)}{2}}}\frac{\mu^{\frac{5}{2}+\alpha}}{\sqrt{\pi}}.\frac{1}{2^{k\alpha}}\int_0^{\pi}|\gamma_m(\theta)|d\theta.\\
\int_0^{\pi}|\gamma_m(\theta)|d\theta &= \int_0^{\pi}\left|\frac{\sin\theta}{m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)-\frac{\sin\theta}{m+2}\left(\frac{\sin(m+1)\theta}{m+1}\right.\right.\\
&- \left.\left.\frac{\sin(m+3)\theta}{m+3}\right)\right|d\theta\\
&\le \left[\frac{1}{m}\left(\frac{1}{m-1}+\frac{1}{m+1}\right)+\frac{1}{m+2}\left(\frac{1}{m+1}+\frac{1}{m+3}\right)\right]\int_0^{\pi}d\theta\\
&= \pi\left[\left(\frac{1}{m-1}-\frac{1}{m}\right)+\left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\left(\frac{1}{m+2}\right.\right.\\
&- \left.\left.\frac{1}{m+3}\right)\right] = \frac{4\pi}{(m-1)(m+3)}.\\
|c_{n,m}| &\le \frac{4\sqrt{\pi}}{2^{(5k+1)/2}(m+3)(m-1)}.\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k\alpha}}, \text{ for } m>1. \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\text{For } m=0,\ c_{n,0} &= \int_0^{\mu} f(t)\psi^{\mu}_{n,0}(t)w_n(t)dt\\
&= \int_{\frac{(n-1)\mu}{2^{k-1}}}^{\frac{n\mu}{2^{k-1}}} f(t)2^{\frac{k}{2}}\sqrt{\frac{2}{\mu\pi}}U_0\left(\frac{2^k}{\mu}t-2n+1\right)w\left(\frac{2^k}{\mu}t-2n+1\right)dt\\
&= \int_{-1}^{1} f\left(\frac{(2n-1+v)\mu}{2^k}\right)2^{\frac{k}{2}}\sqrt{\frac{2\mu}{\pi}}U_0(v)w(v)\frac{dv}{2^k},\quad \frac{2^k}{\mu}t-2n+1=v\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\int_{-1}^{1}\left[f\left(\frac{(2n-1)\mu}{2^k}\right)+\frac{v\mu}{2^k}f'\left(\frac{(2n-1)\mu}{2^k}\right)+\left(\frac{v\mu}{2^k}\right)^2\frac{1}{2!}f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v\mu}{2^k}\right)\right]w(v)dv\\
&= I_1+I_2+I_3.
\end{aligned}$$

$$
\begin{aligned}
I_1 &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\int_{-1}^{1} f\left(\frac{(2n-1)\mu}{2^k}\right)\sqrt{1-v^2}dv = \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\mu\pi}{2}} f\left(\frac{(2n-1)\mu}{2^k}\right)\\
I_2 &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}} f'\left(\frac{(2n-1)\mu}{2^k}\right)\int_{-1}^{1}\frac{v\mu}{2^k}\sqrt{1-v^2}dv = 0\\
I_3 &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\frac{1}{2!}\frac{\mu^2}{(2^k)^2}\int_{-1}^{1} v^2 f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v\mu}{2^k}\right)\sqrt{1-v^2}dv \ \ ,0<\theta<1\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\frac{1}{2!}\frac{\mu^2}{(2^k)^2} f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v_1\mu}{2^k}\right)2.\int_{0}^{1} v^2\sqrt{1-v^2}dv \ \ ,v_1\in(-1,1)\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\mu\pi}{2}}.\frac{1}{4}.\frac{1}{2!}\frac{\mu^2}{(2^{2k})} f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v_1\mu}{2^k}\right).\\
\text{Therefore, } c_{n,0} &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\mu\pi}{2}}\left[f\left(\frac{(2n-1)\mu}{2^k}\right)+\frac{1}{4.2!}.\frac{\mu^2}{2^{2k}} f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v_1\mu}{2^k}\right)\right] \qquad (3)\\
&\le \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\mu\pi}{2}}\left[A+\frac{1}{4.2!}.\frac{\mu^2}{2^{2k}}A''\right].
\end{aligned}
$$

$$
\begin{aligned}
\text{For } m=1,\ c_{n,1} &= \int_0^{\mu} f(t)\psi_{n,1}^{\mu}(t)w_n(t)dt\\
&= \int_{\frac{(n-1)\mu}{2^{k-1}}}^{\frac{n\mu}{2^{k-1}}} f(t)2^{\frac{k}{2}}\sqrt{\frac{2}{\mu\pi}}U_1\left(\frac{2^k}{\mu}t-2n+1\right)w\left(\frac{2^k}{\mu}t-2n+1\right)dt\\
&= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\int_{-1}^{1} f\left(\frac{(2n-1)\mu}{2^k}+\frac{v\mu}{2^k}\right)2v\sqrt{1-v^2}dv\\
&= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\int_{-1}^{1}\left[f\left(\frac{(2n-1)\mu}{2^k}\right)+\frac{v\mu}{2^k}f'\left(\frac{(2n-1)\mu}{2^k}\right)+\left(\frac{v\mu}{2^k}\right)^2\frac{1}{2!}f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v\mu}{2^k}\right)\right]\\
&\quad v\sqrt{1-v^2}dv, \quad 0<\theta<1\\
&= J_1+J_2+J_3.\\
J_1 &= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\int_{-1}^{1} f\left(\frac{(2n-1)\mu}{2^k}\right)v\sqrt{1-v^2}dv = 0\\
J_2 &= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}} f'\left(\frac{(2n-1)\mu}{2^k}\right)\frac{\mu}{2^k}\int_{-1}^{1} v^2\sqrt{1-v^2}dv\\
&= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}} f'\left(\frac{(2n-1)\mu}{2^k}\right)\frac{\mu}{2^k}.\frac{\pi}{8}\\
J_3 &= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\frac{1}{2!}\frac{\mu^2}{(2^k)^2}\int_{-1}^{1} v^3 f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v\mu}{2^k}\right)\sqrt{1-v^2}dv \ \ ,0<\theta<1\\
&= \frac{2}{2^{\frac{k}{2}}}\sqrt{\frac{2\mu}{\pi}}\frac{1}{2!}\frac{\mu^2}{(2^k)^2} f''\left(\frac{(2n-1)\mu}{2^k}+\frac{\theta v_2\mu}{2^k}\right)\int_{-1}^{1} v^3\sqrt{1-v^2}dv = 0 \ \ ,v_2\in(-1,1).\\
\text{Therefore, } c_{n,1} &= \frac{1}{2}.\sqrt{\frac{\pi}{2}}.\frac{\mu^{3/2}}{2^{\frac{3k}{2}}} f'\left(\frac{(2n-1)\mu}{2^k}\right) \qquad (4)\\
&\le \frac{1}{2}.\sqrt{\frac{\pi}{2}}.\frac{\mu^{3/2}}{2^{\frac{3k}{2}}}A'.
\end{aligned}
$$

Hence, from eqns. (2), (3), (4), the series (1) converges absolutely. Therefore, $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{\mu}(t)$ converges to $f(t)$ uniformly in $L^2[0,\mu)$. $\square$

3.2. Error Analysis

In this paper, the following theorem for error estimation has been proved:

Theorem 3.2. *Suppose there exists a square integrable function denoted by $f(t)$ satisfying the condition $f''(t) \in H^{\alpha}[0,\mu)$ i.e. $f''(x+t) - f''(x) = O(|t|^{\alpha}), 0 < \alpha \leq 1$, where $0 < \alpha \leq 1$ and has the extended second kind Chebyshev wavelet expansion as*

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{\mu}(t),$$

where $c_{n,m} = \langle f, \psi_{n,m}^{\mu}\rangle_{w_n}$ then the error bound using modulus of continuity $W(f - S_{2^{k-1},M}(f), \frac{1}{2^k})$ of $(f - S_{2^{k-1},M}(f))$ gives:
(i) for $f(t) = \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}^{\mu}(t)$,

$$W(f - S_{2^{k-1},0}(f), \frac{1}{2^k}) = \sup_{0<h\leq \frac{1}{2^k}} \|(f - S_{2^{k-1},0}(f))(.+h) - (f - S_{2^{k-1},0}(f))(.)\|_2$$

$$= O\left(\frac{\mu^{\alpha+2}}{2^{k\alpha}}\right), k \geq 1,$$

(ii) for $f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{1} c_{n,m}\psi_{n,m}^{\mu}(t)$,

$$W(f - S_{2^{k-1},1}(f), \frac{1}{2^k}) = \sup_{0<h\leq \frac{1}{2^k}} \|(f - S_{2^{k-1},1}(f))(.+h) - (f - S_{2^{k-1},1}(f))(.)\|_2$$

$$= O\left(\frac{\mu^{\alpha+2}}{2^{k(\alpha+2)}}\right), k \geq 1, \text{ and}$$

(iii) for $f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{\mu}(t)$,

$$W(f - S_{2^{k-1},M}(f), \frac{1}{2^k}) = \sup_{0<h\leq \frac{1}{2^k}} \|(f - S_{2^{k-1},M}(f))(.+h) - (f - S_{2^{k-1},M}(f))(.)\|_2$$

$$= O\left(\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k(\alpha+2)}.M^{\frac{3}{2}}}\right), k \geq 1, M \geq 2.$$

Proof. (i)For $m = 0$,
The error between $f(t)$ and its Chebyshev wavelet expansion in the interval $\left[\frac{(n-1)\mu}{2^{k-1}}, \frac{n\mu}{2^{k-1}}\right)$ is given by

$$e_n(f) = c_{n,0}\psi_{n,0}^{\mu} - f\chi_{\left[\frac{(n-1)\mu}{2^{k-1}}, \frac{n\mu}{2^{k-1}}\right)}, \quad \frac{(n-1)\mu}{2^{k-1}} \leq t < \frac{n\mu}{2^{k-1}}. \tag{5}$$

$$c_{n,0} = \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\mu\pi}{2}}\left[f\left(\frac{(2n-1)\mu}{2^k}\right) + \frac{1}{4.2!}.\frac{\mu^2}{2^{2k}}f''\left(\frac{(2n-1)\mu}{2^k} + \frac{\theta v_1\mu}{2^k}\right)\right] \text{ Using (3).}$$

Substituting the values of $c_{n,0}$ in eq. (5),

$$|e_n(f)| = \left| f\left(\frac{(2n-1)\mu}{2^k}\right) + \frac{1}{4.2!}.\frac{\mu^2}{2^{2k}}f''\left(\frac{(2n-1+\theta v_1)\mu}{2^k}\right) - f\left(\frac{(2n-1)\mu}{2^k}\right)\right.$$

$$
\begin{aligned}
&- \quad \frac{v\mu}{2^k} f'\left(\frac{(2n-1)\mu}{2^k}\right) - \frac{v^2}{2!}\left(\frac{\mu^2}{2^{2k}}\right) f''\left(\frac{(2n-1+\theta v)\mu}{2^k}\right)\Bigg| \\
&\leq \quad \left|\frac{v^2\mu^2}{2!.2^{2k}.4} f''\left(\frac{(2n-1+\theta v_1)\mu}{2^k}\right) - \frac{v^2\mu^2}{2!.2^{2k}} f''\left(\frac{(2n-1+\theta v)\mu}{2^k}\right)\right. \\
&- \quad \left.\frac{v\mu}{2^k} f'\left(\frac{(2n-1)\mu}{2^k}\right)\right| \\
&\leq \quad \left(\frac{v^2\mu^2}{2!.2^{2k}}\right)\left|\frac{\theta(v_1-v)\mu}{2^k}\right|^{\alpha} + \left(\frac{v^2\mu^2}{2^{k\alpha}}\right) \quad (\because f'' \in H^{\alpha}[0,\mu), 0<\alpha\leq 1) \\
&\leq \quad \frac{\mu^{\alpha+2}}{(2!).2^{k(\alpha+2)}} + \left(\frac{\mu^2}{2^{k\alpha}}\right) \leq \left(\frac{2\mu^{\alpha+2}}{2^{k\alpha}}\right). \quad (\because v\in[-1,1) \text{ and } 0<\theta<1)
\end{aligned}
$$

$$
\begin{aligned}
\|e_n\|_2^2 &= \int_{\frac{(n-1)\mu}{2^{k-1}}}^{\frac{n\mu}{2^{k-1}}} |e_n(f)|^2 |w_n(t)| dt \leq \int_{-1}^{1} \frac{4\mu^{2\alpha+4}}{2^{2k\alpha}} \frac{\sqrt{1-v^2}}{2^k} dv = \frac{2\pi\mu^{2\alpha+4}}{2^{k(2\alpha+1)}}. \\
\|f - S_{2^{k-1},0}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}} \|e_n\|_2^2 \leq \sum_{n=1}^{2^{k-1}} \frac{2\pi\mu^{2\alpha+4}}{2^{k(2\alpha+1)}} = \frac{\pi\mu^{2\alpha+4}}{2^{2k\alpha}}. \\
\|f - S_{2^{k-1},0}(f)\|_2 &\leq \frac{\mu^{\alpha+2}\sqrt{\pi}}{2^{k\alpha}}. \\
W\left(f - S_{2^{k-1},0}(f), \frac{1}{2^k}\right) &= \sup_{0<h\leq\frac{1}{2^k}} \|(f - S_{2^{k-1},0}(f))(t+h) - (f - S_{2^{k-1},0}(f))(t)\|_2 \\
&\leq \sup_{0<h\leq\frac{1}{2^k}} \left[\|(f - S_{2^{k-1},0}(f))(t+h)\|_2 + \|(f - S_{2^{k-1},0}(f))(t)\|_2\right] \\
&\leq 2\|f - S_{2^{k-1},0}(f)\|_2 \leq 2\frac{\mu^{\alpha+2}\sqrt{\pi}}{2^{k\alpha}} = O\left(\frac{\mu^{\alpha+2}}{2^{k\alpha}}\right), \ k\geq 1.
\end{aligned}
$$

(ii)For $m = 0, 1,$

The error between $f(t)$ and its Chebyshev wavelet expansion in the interval $\left[\frac{(n-1)\mu}{2^{k-1}}, \frac{n\mu}{2^{k-1}}\right)$ is given by

$$
e_n(f) = c_{n,0}\psi^{\mu}_{n,0} + c_{n,1}\psi^{\mu}_{n,1} - f\chi_{[\frac{(n-1)\mu}{2^{k-1}}, \frac{n\mu}{2^{k-1}})}, \quad \frac{(n-1)\mu}{2^{k-1}} \leq t < \frac{n\mu}{2^{k-1}} \tag{6}
$$

$$
\begin{aligned}
c_{n,0} &= \frac{1}{2^{\frac{k}{2}}}\sqrt{\frac{\mu\pi}{2}}\left[f\left(\frac{(2n-1)\mu}{2^k}\right) + \frac{1}{4.2!}.\frac{\mu^2}{2^{2k}} f''\left(\frac{(2n-1)\mu}{2^k} + \frac{\theta v_1\mu}{2^k}\right)\right], \ \text{Using (3)} \\
c_{n,1} &= \frac{1}{2}.\sqrt{\frac{\pi}{2}}.\frac{\mu^{3/2}}{2^{\frac{3k}{2}}} f'\left(\frac{(2n-1)\mu}{2^k}\right), \ \text{Using (4).}
\end{aligned}
$$

Substituting the values of $c_{n,0}$ *and* $c_{n,1}$ in eq. (6)

$$
\begin{aligned}
|e_n(f)| &= \left| f\left(\frac{(2n-1)\mu}{2^k}\right) + \frac{1}{4.2!}.\frac{\mu^2}{2^{2k}} f''\left(\frac{(2n-1+\theta v_1)\mu}{2^k}\right) + \frac{v\mu}{2^k} f'\left(\frac{(2n-1)\mu}{2^k}\right)\right. \\
&- \left. f\left(\frac{(2n-1)\mu}{2^k}\right) - \frac{v\mu}{2^k} f'\left(\frac{(2n-1)\mu}{2^k}\right) - \frac{v^2}{2!}\left(\frac{\mu^2}{2^k}\right) f''\left(\frac{(2n-1+\theta v)\mu}{2^k}\right)\right| \\
&\leq \left|\frac{v^2\mu^2}{2!.2^{2k}.4} f''\left(\frac{(2n-1+\theta v_1)\mu}{2^k}\right) - \frac{v^2\mu}{2!.2^{2k}} f''\left(\frac{(2n-1+\theta v)\mu}{2^k}\right)\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \left(\frac{v^2\mu^2}{2!.2^{2k}}\right)\left|f''\left(\frac{(2n-1+\theta v_1)\mu^2}{2^k}\right)-f''\left(\frac{(2n-1+\theta v)\mu}{2^k}\right)\right| \\
&\leq \left(\frac{v^2\mu^2}{2!.2^{2k}}\right)\left|\frac{\theta(v_1-v)\mu}{2^k}\right|^{\alpha} \qquad (\because f''\in H^{\alpha}[0,\mu)) \\
&\leq \frac{\mu^{\alpha+2}}{(2!).2^{k(\alpha+2)}}. \qquad (\because v\in[-1,1)\ \ and\ \ 0<\theta<1)
\end{aligned}$$

$$\begin{aligned}
\|e_n\|_2^2 &= \int_{\frac{(n-1)\mu}{2^{k-1}}}^{\frac{n\mu}{2^{k-1}}} |e_n(f)|^2|w_n(t)|dt \\
&\leq \int_{-1}^{1}\frac{\mu^{2\alpha+4}}{(2!)^2.2^{2k(\alpha+2)}}\frac{\sqrt{1-v^2}}{2^k}dv = \frac{\pi\mu^{2\alpha+4}}{2(2!)^2 2^{k(2\alpha+5)}}. \\
\|f-S_{2^{k-1},1}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}}\|e_n\|_2^2 \leq \sum_{n=1}^{2^{k-1}}\frac{\pi\mu^{2\alpha+4}}{2(2!)^2 2^{k(2\alpha+5)}} = \frac{\pi\mu^{2\alpha+4}}{16.2^{2k(\alpha+2)}} \\
\therefore \|f-S_{2^{k-1},1}(f)\|_2 &\leq \frac{\mu^{\alpha+2}\sqrt{\pi}}{4.2^{k(\alpha+2)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
W\left(f-S_{2^{k-1},1}(f),\frac{1}{2^k}\right) &= \sup_{0<h\leq\frac{1}{2^k}}\|(f-S_{2^{k-1},1}(f))(t+h)-(f-S_{2^{k-1},1}(f))(t)\|_2 \\
&= O\left(\frac{\mu^{\alpha+2}}{2^{k(\alpha+2)}}\right),\ k\geq 1.
\end{aligned}$$

(ii) For $m\geq 2$,

$$\begin{aligned}
\|f-S_{2^{k-1},M}\|_2^2 &= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty}|c_{n,m}|^2. \\
|c_{n,m}| &\leq \frac{4\sqrt{\pi}}{2^{(5k+1)/2}(m+3)(m-1)}.\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k\alpha}},\ \text{for}\ \ m>1,\ \ \text{Using (2)}. \\
\therefore \|f-S_{2^{k-1},M}(f)\|_2^2 &= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty}\frac{16\pi}{2^{(5k+1)}(m+3)^2(m-1)^2}.\frac{\mu^{5+2\alpha}}{2^{2k\alpha}} \\
&= \frac{16\pi}{2^{(5k+1)}}\frac{\mu^{5+2\alpha}}{2^{2k\alpha}}\sum_{m=M+1}^{\infty}\frac{2^{k-1}}{(m+3)^2(m-1)^2} \\
&\leq \frac{16\pi}{2^{(4k+2)}}\frac{\mu^{5+2\alpha}}{2^{2k\alpha}}\left[\frac{4}{3M^3}\right]. \\
\|f-S_{2^{k-1},M}(f)\|_2 &\leq \frac{8\sqrt{\pi}}{2^{(2k+1)}}\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k\alpha}}\frac{4}{\sqrt{3}.M^{3/2}} \\
&= O\left(\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k(\alpha+2)}.M^{3/2}}\right),\ k\geq 1,\ \ M\geq 2. \qquad (7)
\end{aligned}$$

Hence,

$$W\left(f-S_{2^{k-1},M}(f),\frac{1}{2^k}\right) = \sup_{0<h\leq\frac{1}{2^k}}\|(f-S_{2^{k-1},M}(f))(t+h)-(f-S_{2^{k-1},M}(f))(t)\|_2$$

$$= O\left(\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k(\alpha+2)}.M^{3/2}}\right),\ k \geq 1,\ \ M \geq 2.$$

This completes the proof of theorem (3.2). □

Following corollary is derived from theorem (3.2):

Corollary 3.3. *Suppose there exists a square integrable function denoted by $f(t)$ satisfying the condition $f'' \in H^{\alpha}[0,\mu)$, where $0 < \alpha \leq 1$ and has the extended second kind Chebyshev wavelet expansion is*

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{\mu}_{n,m}(t),\ \text{where } c_{n,m} = \langle f, \psi^{\mu}_{n,m}\rangle_{w_n}$$

then the Chebyshev wavelet approximation $E_{2^{k-1},M}$ of f by $S_{2^{k-1},M}$ satisfies
(i) for $f(t) = \sum_{n=1}^{\infty} c_{n,0}\psi^{\mu}_{n,0}(t)$,

$$\begin{aligned} E_{2^{k-1},0} &= \min \|f - S_{2^{k-1},0}(f)\|_2 \\ &= O\left(\frac{\mu^{\alpha+2}}{2^{k\alpha}}\right), k \geq 1, \end{aligned}$$

(ii) for $f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{1} c_{n,m}\psi^{\mu}_{n,m}(t)$,

$$\begin{aligned} E_{2^{k-1},1} &= \min \|f - S_{2^{k-1},1}(f)\|_2 \\ &= O\left(\frac{\mu^{\alpha+2}}{2^{k(\alpha+2)}}\right), k \geq 1, \ \ \textit{and} \end{aligned}$$

(iii) for $f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{\mu}_{n,m}(t)$

$$\begin{aligned} E_{2^{k-1},M} &= \min \|f - S_{2^{k-1},M}(f)\|_2 \\ &= O\left(\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k(\alpha+2)}.M^{3/2}}\right),\ k \geq 1,\ \ M \geq 2. \end{aligned}$$

Proof. : Following the proof of theorem (3.2), $\|f - S_{2^{k-1},0}(f)\|_2 = O\left(\frac{\mu^{\alpha+2}}{2^{k\alpha}}\right), k \geq 1$,

$\|f - S_{2^{k-1},1}(f)\|_2 = O\left(\frac{\mu^{\alpha+2}}{2^{k(\alpha+2)}}\right), k \geq 1$,

$\|f - S_{2^{k-1},M}(f)\|_2 = O\left(\frac{\mu^{\frac{5}{2}+\alpha}}{2^{k(\alpha+2)}.M^{3/2}}\right),\ k \geq 1,\ \ M \geq 2.$ □

Remark: The proof of corollary (3.3) can be developed independently, parallel to the proof of theorem (3.2).

4. Solving singular differential equations via extended second kind Chebyshev wavelets

4.1. Expanding and Approximating Functions with Extended Chebyshev Wavelet of the Second Kind

A function $f \in L^2[0,\mu)$ can be expanded in terms of extended second kind Chebyshev wavelet as

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{\mu}_{n,m}(t), \\ &\quad \text{where} \quad c_{n,m} = \int_0^{\mu} f(t)\psi^{\mu}_{n,m}(t)w_n(t)dt \\ &= S_{2^{k-1},M} + \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\psi^{\mu}_{n,m}(t), \quad \text{where } S_{2^{k-1},M} = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\psi^{\mu}_{n,m}(t). \end{aligned}$$

Upon truncating the function $f(t)$ using $S_{2^{k-1},M}$, the approximation becomes

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \psi^{\mu}_{n,m}(t) = C^T \psi^{\mu}(t),$$

where C and $\psi^{\mu}(t)$ are $2^{k-1}(M+1)$ vectors of the form
$C = [c_{1,0}\ c_{1,1}\ ...\ c_{1,M}\ c_{2,0}\ c_{2,1}\ ...\ c_{2,M}\ ...\ c_{2^{k-1},0}\ ...\ c_{2^{k-1},M}]^T$ and
$\psi^{\mu}(t) = [\psi^{\mu}_{1,0}\ \psi^{\mu}_{1,1}\ ...\psi^{\mu}_{1,M}\ \psi^{\mu}_{2,0}\ \psi^{\mu}_{2,1}\ ...\ \psi^{\mu}_{2,M}\ ...\ \psi^{\mu}_{2^{k-1},0}\ ...\ \psi^{\mu}_{2^{k-1},M}]^T$.

4.2. Operational Integration Matrix for extended Second Kind Chebyshev Wavelets

In this segment, we derive the operational matrix of integration for the extended second-kind Chebyshev wavelet with parameters k=2, M=3, and $\mu = 3$. In this scenario, the wavelet basis functions are delineated as follows:

$$\left.\begin{array}{l}
\psi^{\mu}_{1,0}(t) = 2\sqrt{\frac{2}{3\pi}} \\
\psi^{\mu}_{1,1}(t) = \frac{4}{3}\sqrt{\frac{2}{3\pi}}(4t-3) \\
\psi^{\mu}_{1,2}(t) = \frac{2}{9}\sqrt{\frac{2}{3\pi}}(64t^2-96t+27) \\
\psi^{\mu}_{1,3}(t) = \frac{2}{27}\sqrt{\frac{2}{3\pi}}(512t^3-1152t^2+750t-108)
\end{array}\right\} 0 \le t < \frac{3}{2}$$

$$\left.\begin{array}{l}
\psi^{\mu}_{2,0}(t) = 2\sqrt{\frac{2}{3\pi}} \\
\psi^{\mu}_{2,1}(t) = \frac{2}{3}\sqrt{\frac{2}{3\pi}}(8t-18) \\
\psi^{\mu}_{2,2}(t) = \frac{2}{9}\sqrt{\frac{2}{3\pi}}(64t^2-96t+315) \\
\psi^{\mu}_{2,3}(t) = \frac{2}{27}\sqrt{\frac{2}{3\pi}}(512t^3-3456t^2+7632t-5508)
\end{array}\right\} \frac{3}{2} \le t < 3$$

Integrating above functions from 0 to t and expressing in terms of basis wavelet functions,

$$\int_0^t \psi^{\mu}_{1,0}(t')dt' = \begin{cases} 2\sqrt{\dfrac{2}{3\pi}}t, & 0 \le t < \frac{3}{2} \\ \sqrt{\dfrac{6}{\pi}}, & \frac{3}{2} \le t < 3 \end{cases}$$
$$= \frac{3}{4}\psi^{\mu}_{1,0}(t) + \frac{3}{8}\psi^{\mu}_{1,1}(t) + \frac{3}{2}\psi^{\mu}_{2,0}(t)$$

$$\int_0^t \psi^{\mu}_{1,1}(t')dt' = \begin{cases} \frac{4}{3}\sqrt{\dfrac{2}{3\pi}}(2t^2-3t), & 0 \le t < \frac{3}{2} \\ 0, & \frac{3}{2} \le t < 3 \end{cases}$$
$$= \frac{-9}{16}\psi^{\mu}_{1,0}(t) + \frac{3}{16}\psi^{\mu}_{1,2}(t).$$

Similarly, $\int_0^t \psi^{\mu}_{1,2}(t')dt' = \frac{1}{4}\psi^{\mu}_{1,0}(t) + \frac{-1}{8}\psi^{\mu}_{1,1}(t) + \frac{1}{8}\psi^{\mu}_{1,3}(t) + \frac{1}{2}\psi^{\mu}_{2,0}(t)$

$$\int_0^t \psi^{\mu}_{1,3}(t')dt' = \frac{-3}{16}\psi^{\mu}_{1,0}(t) + \frac{3}{32}\psi^{\mu}_{1,2}(t)$$

$$\int_0^t \psi^{\mu}_{2,0}(t')dt' = \frac{3}{4}\psi^{\mu}_{2,0}(t) + \frac{3}{8}\psi^{\mu}_{2,1}(t)$$

$$\int_0^t \psi^{\mu}_{2,1}(t')dt' = \frac{-9}{16}\psi^{\mu}_{2,0}(t) + \frac{3}{16}\psi^{\mu}_{2,2}(t)$$

$\int_0^t \psi^{\mu}_{2,2}(t')dt' = \frac{1}{4}\psi^{\mu}_{2,0}(t) + \frac{-1}{8}\psi^{\mu}_{2,1}(t) + \frac{1}{8}\psi^{\mu}_{2,3}(t)$

$\int_0^t \psi^{\mu}_{2,3}(t')dt' = \frac{-3}{16}\psi^{\mu}_{2,0}(t) + \frac{-3}{32}\psi^{\mu}_{2,2}(t).$

Thus, $\int_0^t \psi^{\mu}_{8\times1}(t')dt' = P_{8\times8}\psi^{\mu}_{8\times1}(t)$, where $\psi^{\mu}(t) = [\psi^{\mu}_{1,0}\ \psi^{\mu}_{1,1}\ \psi^{\mu}_{1,2}\ \psi^{\mu}_{1,3}\ \psi^{\mu}_{2,0}\ \psi^{\mu}_{2,1}\ \psi^{\mu}_{2,2}\ \psi^{\mu}_{2,3}\ ]^T$.

Hence, operational matrix of integration using extended second kind Chebyshev wavelet is given by

$$P_{8\times8} = \begin{pmatrix} \frac{3}{4} & \frac{3}{8} & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 \\ \frac{-9}{16} & 0 & \frac{3}{16} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{-1}{8} & 0 & \frac{1}{8} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{-3}{16} & 0 & \frac{-3}{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-9}{16} & 0 & \frac{3}{16} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{-1}{8} & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & \frac{-3}{16} & 0 & \frac{-3}{32} & 0 \end{pmatrix} \tag{8}$$

4.3. Description of the method

Consider the non-linear singular differential equation given by

$$u''(t) + \frac{mu'(t)}{t} + h(t,u) = d(t), \quad 0 \le t < \mu,\ m \ge 1 \tag{9}$$

with initial conditions

$$u(0) = \beta,\ u'(0) = \gamma. \tag{10}$$

To solve the differential equation (9), we begin by multiplying it by t:

$$tu''(t) + mu'(t) + th(t,u) = td(t), \quad 0 \le t < \mu,\ m \ge 1 \tag{11}$$

We also approximate $u''(t)$ as a linear combination:

$$u''(t) \approx \sum_{n=1}^{2}\sum_{m=0}^{3} c_{n,m}\psi^{\mu}_{n,m}(t) = C^T\psi^{\mu}(t). \tag{12}$$

Integrating equation (11) twice with respect t twice from 0 to t, we get the following approximations for

$$\begin{aligned} u'(t) &\approx C^TP\psi^{\mu}(t) + \gamma \\ &= C^TP\psi^{\mu}(t) + A^T\psi^{\mu}(t), \quad (13) \\ u(t) &\approx C^TP^2\psi^{\mu}(t) + t\gamma + \beta \\ &= C^TP^2\psi^{\mu}(t) + B^T\psi^{\mu}(t), \quad (14) \end{aligned}$$

where A and B can be determined from the initial conditions (10). We also make approximations for t, $h(t,u)$ and $d(t)$:

$$\begin{aligned} t &\approx E^T\psi^{\mu}(t), \quad (15) \\ h(t,u) &\approx \sum_{j=0}^{n}\frac{1}{j!}\left((t-t_0)\frac{\partial}{\partial t} + (u-u_0)\frac{\partial}{\partial y}\right)^j h(t_0,u_0) \\ &= F^T\psi^{\mu}(t), \quad (16) \\ d(t) &\approx \sum_{j=0}^{n}\frac{g^{(j)}(t_0)}{j!}(t-t_0)^j \end{aligned}$$

$$= \quad G^T\psi^\mu(t). \tag{17}$$

Substituting the values from equations (12)-(17) into equation (11), we obtain the following equation:

$$E^T\psi^\mu(t)C^T\psi^\mu(t) + m(C^TP + A^T)\psi^\mu(t) + E^T\psi^\mu(t)F^T\psi^\mu(t) = E^T\psi^\mu(t)G^T\psi^\mu(t). \tag{18}$$

By solving the system of algebraic equations (18), the differential equation (9) can be solved.

5. Numerical Examples

1. Consider the singular differential equation [19]:

$$u''(t) + \frac{2u'(t)}{t} + u^s(t) = 0, \quad 0 \le t < 3, \ s \ge 1 \tag{19}$$

with initial conditions $u(0) = 1, \ u'(0) = 0$.
Substituting $s = 1$ and 5, the eq. (19) reduces to

$$\text{linear equation} \quad u''(t) + \frac{2u'(t)}{t} + u(t) = 0, \quad u(0) = 1, \ u'(0) = 0. \tag{20}$$

$$\text{non-linear equation} \quad u''(t) + \frac{2u'(t)}{t} + u^5(t) = 0, \quad u(0) = 1, \ u'(0) = 0. \tag{21}$$

The exact solutions of eqs. (20) and (21) are

$$u(t) = \frac{\sin t}{t} \quad \text{and} \quad u(t) = \frac{1}{\sqrt{1 + \frac{t^2}{3}}}.$$

.
On the other hand, when considering a numerical approach to solving equations (20) and (21), a specific method outlined in section (4.3) is employed. The outcome of this numerical procedure is documented comprehensively within Table (1). This table encapsulate the results of computations carried out through the prescribed methodology. Table (1) presents comparison between the numerical solutions of the linear and non-linear singular differential equations with an index of $s = 1$ and $s = 5$ and their corresponding exact solutions. The evaluation of absolute errors underscores a remarkable alignment between the numerical results attained through the application of the extended second-kind Chebyshev wavelet and the true values at these specific points. Moreover, Figure (1) and (2) visually illustrate the plotted graphs depicting both the precise and approximate solutions achieved by utilizing the extended second-kind Chebyshev wavelet. It is noteworthy that these graphical representations showcase an almost ubiquitous concurrence between the exact and approximate solutions profiles.

Table 1: Extended Chebyshev solution of singular differential equations for $s = 1$ and $s = 5$

t	Sol.of eqn.(20)	Exact sol.	Abs. error	Sol.of eqn.(21)	Exact sol.	Abs. error
0.0001	0.99999008	0.99999998	9.9×10^{-6}	0.9999961	0.99999999	3.89×10^{-6}
0.01	0.99997393	0.99998333	9.4×10^{-6}	0.99997737	0.99998333	5.96×10^{-6}
0.1	0.99832697	0.99833416	7.19×10^{-6}	0.99833045	0.99833749	7.04×10^{-6}
0.3	0.98505748	0.98506735	9.87×10^{-6}	0.98532655	0.98532928	2.73×10^{-6}
0.6	0.94106118	0.94107078	9.6×10^{-6}	0.94490722	0.94491118	3.96×10^{-6}
1.2	0.77669224	0.77669923	6.99×10^{-6}	0.82199073	0.82199493	4.2×10^{-6}
1.5	0.66499416	0.66499665	2.49×10^{-6}	0.75592675	0.75592894	2.19×10^{-6}
1.8	0.54018146	0.54102646	8.25×10^{-4}	0.69352324	0.69337524	1.48×10^{-4}
2.1	0.41105103	0.41105208	1.05×10^{-6}	0.63629466	0.63628476	9.9×10^{-6}
2.3	0.32421944	0.32421965	2.1×10^{-7}	0.601566387	0.60156611	2.77×10^{-7}
2.5	0.23938069	0.23938885	8.16×10^{-6}	0.569501445	0.56949479	6.655×10^{-6}
2.9	0.08249762	0.08249976	2.14×10^{-6}	0.51275802	0.51276432	6.3×10^{-6}

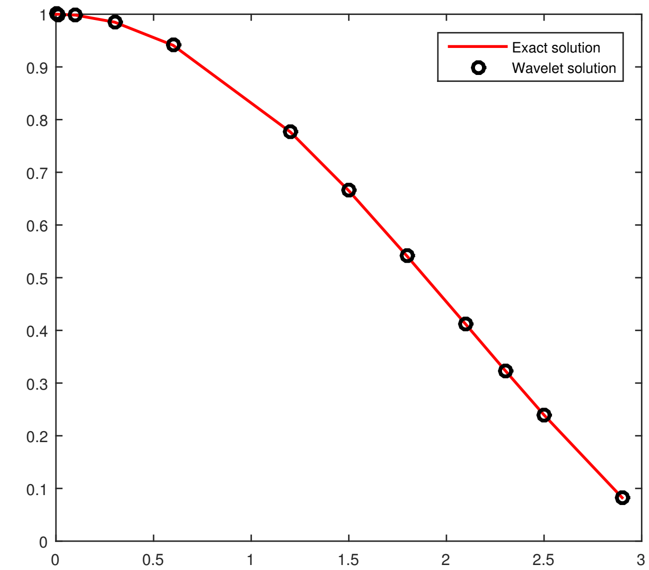

Figure 1: Comparison of extended second kind Chebyshev and exact solution of eqn. 20

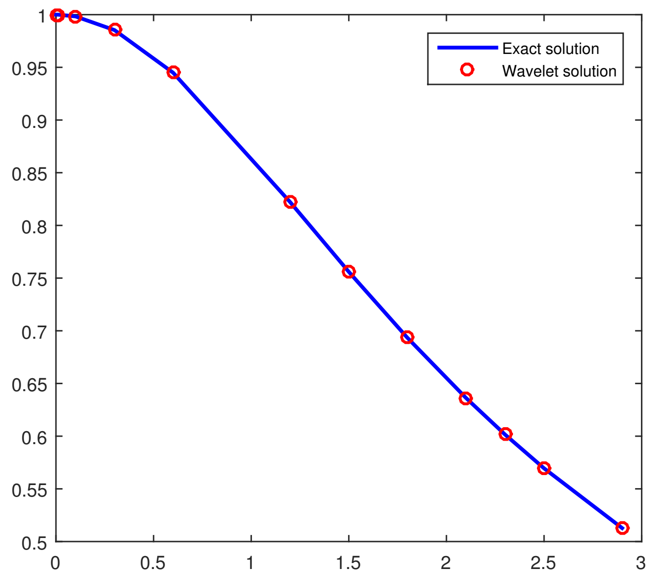

Figure 2: Comparison of extended second kind Chebyshev and exact solution of eqn. 21

2. Consider the non-linear singular differential equation whose analytic solution is not known in the literature.

$$u''(t) + \frac{2u'(t)}{t} + t^2 log^2(t) log^3(u) = 0, \quad t \geq 0, \tag{22}$$

with initial conditions $u(0) = e,\ u'(0) = 0$.
The numerical solutions of equation (22) have been obtained using the computational approach detailed in section (4.3). These results are presented in Table (2). Furthermore, Figure (3) displays the graphical representations of the numerical solutions for eqn. 22.
Examining the information in Table (2), it becomes evident that the proposed methodology yields highly accurate solutions while demanding a reduced computational work.

Table 2: Extented Chebyshev solution of non-linear singular eqn. 22

t	Sol. of eqn. 22	Comparison with [4]
0.0001	2.71828676	2×10^{-8}
0.01	2.71828676	2×10^{-8}
0.1	2.71828676	2×10^{-8}
0.3	2.717793942	1.472×10^{-5}
0.6	2.710893672	2.092×10^{-5}
1.2	2.68954882	2.79×10^{-5}
1.5	2.667693675	1.717×10^{-5}
1.8	2.643273457	5.2×10^{-5}
2.1	2.59983386	2.32×10^{-5}
2.3	2.571128569	4.02×10^{-5}
2.5	2.552859927	3.56×10^{-4}
2.9	2.512833638	2.9×10^{-4}

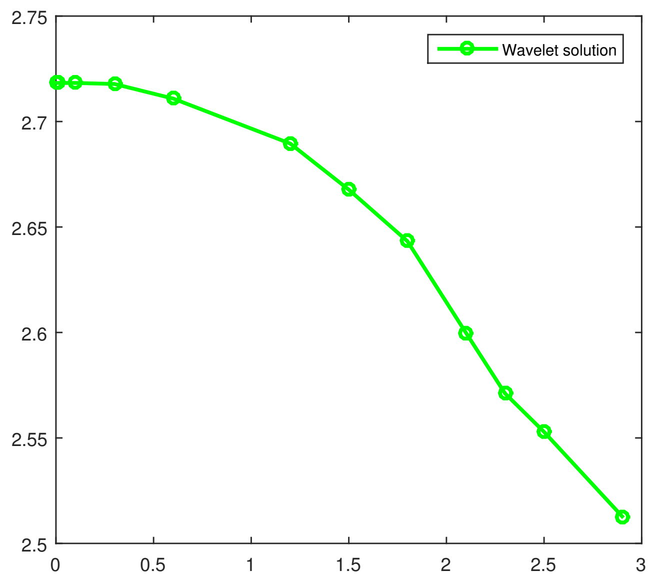

Figure 3: Approximate solution of eqn. 22 using extended Chebyshev wavelet

6. Conclusions

In conclusion, this study establishes the uniform convergence of wavelet series for functions in $H^\alpha[0,\mu)$, supported by theorems (3.1) and (3.2). The study reveals that the moduli of continuity $W(f - S_{2^{k-1},M}(f), \frac{1}{2^k})$ demonstrate remarkable convergence properties, rapidly vanishing as k and M tend to infinity. Through corollary (3.3), it is shown that these moduli of continuity surpass the approximations $E_{2^{k-1},0}, E_{2^{k-1},1}, E_{2^{k-1},M}$ in precision. The paper introduces an extended second-kind Chebyshev wavelet operational matrix of integration method, enabling efficient solution of non-linear singular differential equations. This extension allows wavelets to be adapted to signals or data that are defined on a different domain or have different characteristics. The method's accuracy and efficiency surpass previous approaches ([4],[19]), and its potential extends to a wider range of non-linear singular differential equations. Notably, this accuracy is achieved while utilizing a reduced number of basis functions and computational efforts in contrast to alternative techniques. This work marks a significant advancement in both theoretical understanding and practical application of wavelet-based methods for solving complex mathematical problems.

Declarations

Competing Interests

The authors have no competing interests to declare that are relevant to the content of this article.

Funding

Shyam Lal, one of the authors, is thankful to DST-CIMS for encouragement to this work.
Abhilasha, one of the authors, is grateful to University Grants Commission(India) for providing financial assistance in the form of Junior Research Fellowship vide NTA Ref. No. 201610030018 for her research work.

Acknowlegdement

The authors are grateful to the referees for their suggestions to improve the quality of this article.

Ethics Approval

This article does not contain any studies with human participants or animals, performed by any of the authors.

References

[1] C. K. Chui, *An Introduction to Wavelets (Wavelet Analysis and its Applications)*, (Volume I), Academic Press, Cambridge, 1992.
[2] G. Das, T. Ghosh, B. K. Ray, *Degree of approximation of functions by their Fourier series in the generalised Hölder metric*, Proc. Indian Acad. Sci.Math. Sci., **106** (1996), 139–153.
[3] S. P. Jakhar, A. Nandal, A. Dhaka, B. Jiang, L. Zhou, V. N. Mishra, *Fractal Feature based Image Resolution Enhancement using Wavelet-Fractal Transformation in Gradient Domain*, Journal of Circuits, Systems and Computers **32**(2) (2023), Manuscript ID: 2350035. DOI: https://doi.org/10.1142/S0218126623500354
[4] H. Kaur, R. C. Mittal, V. Mishra, *Haar wavelet approximate solutions for the generalized Lane–Emden equations arising in astrophysics*, Computer Physics Communications **184** (2013), 2169–2177
[5] K. Khatri, V. N. Mishra, *Degree of Approximation by the $(T.E^1)$ Means of Conjugate Series of Fourier Series in the Hölder Metric*, Iranian Journal of Science and Technology, Transactions A: Science **43**(4) (2019), 1591–1599. DOI: 10.1007/s40995-017-0272-3.
[6] S. Liao, *A new analytic algorithm of Lane–Emden type equations*, Applied Mathematics and Computation **142** (2003), 1–16.
[7] H. R. Marasi, V. N. Mishra, M. Daneshbastam, *A constructive approach for solving system of fractional differential equations*, Waves, Wavelets and Fractals Advanced Analysis **3**(1) (2017), 40–47. DOI: 10.1515/wwfaa-2017-0004
[8] J. C. Mason, D. C. Handscomb, *Chebyshev Polynomials*, CRC Press LLC, Boca Raton, 2003.
[9] Y. Meyer, *Wavelets: their past and their future, Progress in Wavelet Analysis and Applications* (Toulouse, 1992) (Y.Meyer and S.Roques, eds) Frontieres, Gif-sur-Yvette, 9-18, 1993.
[10] V. N. Mishra, K. Khatri, *Degree of approximation of functions $\widetilde{f} \in H_w$ class by the (Np.E1) means in the Hölder metric*, International Journal of Mathematics and Mathematical Sciences **2014** (2014), Article ID 837408, 9 pages.

[11] L. N. Mishra, M. Raiz, L. Rathour, V. N. Mishra, *Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces*, Yugoslav Journal of Operations Research **32**(3) (2022), 377–388. DOI: https://doi.org/10.2298/YJOR210915005M.

[12] V. N. Mishra, L. N. Mishra, *Trigonometric Approximation of Signals (Functions) in L_p- norm*, International Journal of Contemporary Mathematical Sciences **7**(19) (2012), 909–918.

[13] N. T. Negero, G. F. Duressa, L. Rathour, V. N. Mishra, *A novel fitted numerical scheme for singularly perturbed delay parabolic problems with two small parameters*, Partial Differential Equations in Applied Mathematics **8** (2023), 1–8. DOI: https://doi.org/10.1016/j.padiff.2023.100546

[14] K. Parand, M. Dehghan, A. R. Rezaeia, S. M. Ghaderia, *An approximation algorithm for the solution of the nonlinear Lane–Emden type equations arising in astrophysics using Hermite functions collocation method*, Comput. Phys. Comm. **181** (2010), 1096–1108.

[15] M. Razzaghi, S. Yousefi, *The Legendre wavelets operational matrix of integration*, International Journal of Systems Science **32** (1959), 495–502.

[16] D. K. Ruch, P. J. Van Fleet, *Wavelet Theory: An elementary approach with applications*, John Wiley, New York, 1959.

[17] K. P. Sumana, L. N. Achala, V. N. Mishra, *Numerical Simulation of Elliptic Partial Differential Equations using 3-scale Haar wavelets*, Adv. Stud. Contemp. Math. (Kyungshang) **31**(3) (2021), 355–375. DOI: http://dx.doi.org/10.17777/ascm2021.31.3.355.

[18] K. P. Sumana, L. N. Achala, V. N. Mishra, *Numerical solution of time-delayed Burgers' equations using Haar wavelets*, Adv. Stud. Contemp. Math. (Kyungshang) **29**(3) (2019), 411–437. DOI: 10.17777/ascm2019.29.3.411.

[19] A. M. Wazwaz, *A new algorithm for solving differential equations of Lane–Emden type*, Appl. Math. Comput. **118** (2001), 287–310.

[20] M. M. Woldaregay, T. W. Hunde, V. N. Mishra, *Fitted exact difference method for solutions of a singularly perturbed time delay parabolic PDE*, Partial Differential Equations in Applied Mathematics **8** (2023), 1–12. https://doi.org/10.1016/j.padiff.2023.100556

[21] F. Zhou, X. Xu, *Numerical solutions for the linear and nonlinear singular boundary value problems using Laguerre wavelets*, Adv. Differ. Equ. **17** (2016).

[22] L. Zhu, Q. Fan, *Solving fractional nonlinear Fredholm integro-differential equation by the second kind Chebyshev wavelet*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), 2333–2341.