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On (Thick, Weak Thick)Spanier topology on *nth* homotopy group

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Abstract. In this paper, we introduce *nth* thick and *nth* weak thick Spanier groups as subgroups of the *nth* homotopy groups of a topological space. Then, we use these subgroups to define topologies on the nth homotopy groups that make them into topological groups. Finally, we present some properties of these topologies and examine the way they are related to the underlying space.

1. Introduction

One of the most important problems in homotopy theory is to understand the local behavior of a topological space. For example, to find the homotopy groups of a space, one needs to know its local properties. To compute the fundamental group of a space, two classical tools are commonly used, namely, Van-Kampen's theorem and the theory of covering spaces. But, these are appropriate only for locally well-behaved spaces, and papular instruments used to understand the higher homotopy groups of a space depend on its fundamental group and local behavior. Usually, in the topology literature, locally n-connected or n-semilocally simply connected spaces are considered as well-behaved spaces. On the other hand, many spaces that appear in topology, analysis and other branches of mathematics, including fractals and Menger spaces, are not locally well-behaved spaces. Such spaces are known as *locally complicated* or *locally wild* spaces.

It is beneficial to the study of a homotopy group, as well as any other group, to know about some of its subgroups. In his book [18], Spanier proved the equivalence of the semi-locally simple connectivity of a topological space and the triviality of a special subgroup of its fundamental group. He also used this subgroup to classify and study covering spaces of topological spaces. In [12], Fisher et al. referred to this group as the *Spanier group*. In [19], Wilkins equip the fundamental group of a space with a topology which he called the *Spanier subgroup topology*, what we refer to as the *Spanier topology*. Also, he explored the relation between the topology of a space X and the topology of its fundamental group. We know that when equipped with other topologies, this group and higher homotopy groups may not be topological groups. See [4] for more details.

In this paper we use the *nth* Spanier groups, defined in [3], to equip the nth homotopy group with topologies that make it into topological groups. Then, motivated by the work of Bogley and Sierdaski

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in [10], we define and study other topologies on $\pi_n(X, x_0)$ that make it into topological groups. Also, we provide conditions that ensure the metrizability of these topological groups. We denote the nth homotopy group with these topologies by τ_{Sp}, τ_{tSp} and τ_{wtSp}, to which we refer as the *Spanier topology*, the *thick Spanier topology* and the *weak thick Spanier topology*, respectively. Moreover, we examine the relationship between some local properties of the space X and the topologies τ_{Sp}, τ_{tSp} and τ_{wtSp} on the nth homotopy group.

In Section 2, we first explain some basic notions concerning topological spaces and their homotopy groups that will be used in other sections. Then, we recall various approaches to homotopy groups and relative homotopy groups. Next, we discuss the behavior of the homotopy group under changing the base point. In Definition 2.2 and Definition 2.3, we recall the definitions of Spanier group and thick Spanier group from [5, 12], respectively. At the end of this section, we describe the construction of the nth shape homotopy group of a space.

In Section 3, we state the definition of n-Spanier group from [3], and introduce a general version of the thick Spanier group. Also, we compare these subgroups and consider their equality under certain conditions. See Theorem 3.10. Next, we introduce the nth weak thick Spanier group, and we explore the relation between this group and the nth thick Spanier group. Moreover, in Example 3.7 we compare these three important subgroups of the nth homotopy group. Then, we show that these subgroups are equal for all paracompact spaces. Also, in Theorem 3.12, we prove the equality of the first Spanier group and the first thick (weak thick) Spanier group under certain conditions. Furthermore, in Theorem 3.8 we show that these three subgroups of the homotopy groups are equal for each topological group. One of the well-known subgroups of the nth homotopy group is the shape kernel of the canonical homomorphism from the nth homotopy group to the nth shape homotopy group (or $ker\Psi$.) In Proposition 3.15, we prove that the *nth* thick (weak thick) Spanier group lies in the kernel of this map.

In Section 4, we define the topologies τ_{Sp}, τ_{tsp} and $\tau wtSp$ (or the Spanier, thick Spanier and weak thick Spanier topologies) on the *nth* homotopy group, which induce topological group structures on $\pi_n(X, x_0)$. See Proposition 4.2. In Theorem 4.16, we provide conditions under which these topological groups are metrizable. Moreover, we examine the ways the topological properties of these topological groups are related to the local properties of the space (X, x_0). For example, let σ be a path from x_0 to x_1. Notice that the isomorphism $\sigma_\sharp$ is not necessarily continuous ([9]). But, we prove that the isomorphism $\sigma_\sharp(\pi_n(X, x_0), \tau) \longrightarrow (\pi_n(X, x_1), \tau)$ is a homeomorphism, where $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$. See Proposition 4.4. In Theorem 4.7, we show that if $(\pi_n(X, x_1), \tau)$ is Hausdorff, where $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$, then X is n-homotopically Hausdorff. But, in Example 4.9 we observe that the converse of Theorem 4.7 is not always true. Notice that converse of Theorem 4.7, in the special case $n = 1$, is Problem 8.11 of [8]. In Proposition 4.14, we state conditions under which the converse of Theorem 4.7 is true. Finally, we answer the following problem, posed by Brodskiy et al. in [8].

Let X be a path-connected space and $x_0 \in X$. Are the following conditions equivalent?

(i) $\pi_1^{Sp}(X, x_0) = \bigcap_{\mathcal{U}} \pi_1^{Sp}(\mathcal{U}, x_0)$ is trivial.

(ii) $p : (\hat{X}, \hat{x_0}) \longrightarrow (X, x_0)$ has the unique path lifting property.

(iii) X is 1-homotopically Hausdorff (or homotopically Hausdorff).

2. Preliminaries

This section summarizes some notions that will be used in the next sections. In this paper, we let

$$I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ 0 \le x_i \le 1, \ \ 1 \le i \le n\}, \quad \partial I^n = \{(x_1, \ldots, x_n) \in I^n : x_i \in \{0, 1\} \text{ for some i}\},$$
$$D^n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \textstyle\sum_{i=1}^{n+1} x_i^2 \le 1\right\}, \quad S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \textstyle\sum_{i=1}^{n+1} x_i^2 = 1\right\},$$
$$S^n_+ = \{(x_1, \ldots, x_{n+1}) \in S^n : x_{n+1} \ge 0\} \text{ and } \quad S^n_- = \{(x_1, \ldots, x_{n+1}) \in S^n : x_{n+1} \le 0\}.$$

Let X be a topological space and $x_0 \in X$.

(1) We say that X is *locally path-connected* at $x \in X$ if for every open set V containing x, there exists a path-connected open set U with $x \in U \subset V$. The space X is said to be *locally path-connected* if it is locally path-connected at each of its elements.

(2) An *n-loop* at x_0 is a continuous map $f : I^n \to X$ such that $f(\partial I^n) = \{x_0\}$. The n-loop f is *null-homotopic* if it is homotopic to the constant map $C_{x_0} : I^n \longrightarrow X$ defined by $C_{x_0}(s) = x_0$.

(3) The space X is called *n-semilocally simply connected* if for each $x \in X$, there exists an open subset U of X containing x such that every n-loop in U is null-homotopic in X.

(4) We call X a *locally n-connected* space if for every point x of X and any neighborhood U of x, there exists an open set V of x such that $V \subset U$ and for every $1 \leq k \leq n$, any k-loop in V is homotopically trivial in U.

(5) We say that X is *n-connected* if it is path-connected and any k-loop map in X is null-homotopic, for every $1 \leq k \leq n$.

(6) If $\pi_n(X, x_0)$ is the set of homotopy classes of n-loops $f : (I^n, \partial I^n) \to (X, x_0)$ at x_0, then $\pi_n(X, x_0)$ has a group structure with the operation $[f] + [g] = [f + g]$, where

$$(f+g)(x_1, x_2, \ldots, x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n), & x_1 \in [0, \frac{1}{2}] \\ g(2x_1 - 1, x_2, \ldots, x_n), & x_1 \in [\frac{1}{2}, 1]. \end{cases}$$

This group is called the *nth homotopy group* of X with base point x_0. See [16] for more details.
In (7), we present another definition of the *nth* homotopy group of X.

(7) If the boundary ∂I^n of I^n is equal to a point, the quotient space is homeomorphic to the n-sphere S^n with a base point $1 = (1, 0, ..., 0) \in S^n$, each element of $\pi_n(X, x_0)$ can be represented by a homotopy class relative to $1 \in S^n$ of the maps $f : (S^n, 1) \longrightarrow (X, x_0)$, and vice versa. So, this definition of the *nth* homotopy group is conceptually equivalent to the one above.
We use this equivalent definition of the nth homotopy group throughout this paper. See [2] for more details.

(8) Let A be a subspace of X. A useful generalization of the homotopy group $\pi_n(X, x_0)$ is provided by the relative homotopy group $\pi_n(X, A, x_0)$ with a base point $x_0 \in A$. Here, we are going to recall the definition of these groups from [14].
First, note that we can regard I^{n-1} as the face of I^n with the last coordinate $x_n = 0$. Let J^{n-1} be the union of the other faces of I^n. Suppose that $\pi_n(X, A, x_0)$ is the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, with homotopies through maps of the same form. The action taken in the relative homotopy groups is exactly the same as that in the homotopy groups. It is easy to see that $\pi_n(X, A, x_0)$ is a group for $n \geq 2$, and that it is a commutative group if $n \geq 3$.

(9) If J^{n-1} is equal to a point s_0, then $(I^n, \partial I^n, J^{n-1})$ admits a configuration equivalent to $(D^n, \partial D^n, s_0)$. This shows that an element of $\pi_n(X, A, x_0)$ can be as equally well-defined as a homotopy class of the maps $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$. From this point of view, addition is done via the map $c : D^n \vee D^n \to D^n$, collapsing $D^{n-1} \subset D^n$ to a point. In general, changing the base point in the homotopy group of a topological space X may change the homotopy group. We know that when a path σ exists between two points of a path-connected topological space X, the homotopy groups corresponding to these points are isomorphic. In what follows, we define the isomorphism induced by σ.

(10) Let $n \geq 2$, and define $\sigma_\sharp : \pi_n(X, x_1) \longrightarrow \pi_n(X, x_0)$ by $\sigma_\sharp[\alpha] = [F_1]$, where $F_1(s) = F(s, 1)$ for any $s \in I^n$, and $F : I^n \times [0, 1] \longrightarrow X$ is a homotopy with the following properties.

$$\begin{array}{rcll} F(s,0) & = & \alpha(s) & s \in I^n \\ F(s,t) & = & \overleftarrow{\sigma}(t) & s \in \partial I^n, t \in [0,1]. \end{array}$$

Then by [16, Theorem 2.5.6], the map $\sigma_\sharp$ is a well-defined isomorphism that only depends on the homotopy class of σ.

Theorem 2.1. *[7] If $\sigma, \tau : I \to X$ are paths, then given any $n \in \mathbb{N}$ the following statements are true for the isomorphisms $\sigma_\sharp, \tau_\sharp : \pi_n(X, x_1) \to \pi_n(X, x_0)$.*

1. *If $\sigma \simeq \tau \;\; rel(\partial I)$, then $\sigma_\sharp = \tau_\sharp$.*
2. *If $\sigma(1) = \tau(0)$, then $(\sigma * \tau)_\sharp = \sigma_\sharp \circ \tau_\sharp$.*
3. *If σ is the constant map, then $\sigma_\sharp$ is the identity map.*

4. *(Naturality): Let Y be a topological space and $\phi : X \to Y$ be a continuous map such that $\tau = \phi \circ \sigma$. Then, the following diagram commutes.*

$$\begin{array}{ccc} \pi_n(X, \sigma(1)) & \xrightarrow{\sigma_\sharp} & \pi_n(X, \sigma(0)) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \pi_n(Y, \tau(1)) & \xrightarrow[\tau_\sharp]{} & \pi_n(Y, \tau(0)) \end{array}$$

Notation. Throughout this paper we denote the collection of all open covers of X by $O(X)$, and we let

$$\widetilde{X} = \{[\alpha] : \alpha \text{ is a path in X with } \alpha(0) = x_0\}.$$

In Definition 2.2, we recall the definition of the Spanier group of a pointed space (X, x_0), as introduced by Fischer et al. in [12].

Definition 2.2. *Let (X, x_0) be a pointed topological space.*

1. *Suppose that $\mathcal{U}$ is an element of $O(X)$. The Spanier group of X at x_0 with respect to $\mathcal{U}$, denoted by $\pi^{Sp}(\mathcal{U}, x_0)$, is a subgroup of $\pi_1(X, x_0)$ which is generated by $[\sigma][\alpha][\overleftarrow{\sigma}]$, where $[\sigma] \in \widetilde{X}$ and $\alpha : I \to \mathcal{U}$ is a loop based at $\sigma(1)$, for some $U \in \mathcal{U}$.*
2. *The Spanier group of X at x_0, denoted by $\pi^{Sp}(X, x_0)$, is $\bigcap_{\mathcal{U} \in O(X)} \pi^{Sp}(\mathcal{U}, x_0)$. Equivalently, $\pi^{Sp}(X, x_0)$ is the inverse limit (or $\varprojlim \pi^{Sp}(\mathcal{U}, x_0)$) of the inverse system $\{\pi^{Sp}(\mathcal{U}, x_0), \pi^{Sp}(\mathcal{V}, x_0) \hookrightarrow \pi^{Sp}(\mathcal{U}, x_0), O(X)\}$, where $O(X)$ is directed by refinement.*

In [5], Brazas and Fabel defined the thick Spanier group as a subgroup of the fundamental group. A result established in this paper says that the Spanier group of a topological space is contained in the thick Spanier group. In the following definition, we define the thick Spanier group.

Definition 2.3. *Let (X, x_0) be a pointed topological space.*

1. *Suppose that $\mathcal{U}$ is an arbitrary open cover of X. The thick Spanier group of X with respect to $\mathcal{U}$, denoted by $\Pi^{Sp}(\mathcal{U}, x_0)$, is the subgroup generated by the elements*

   $$[\sigma][\alpha_1][\alpha_2][\overleftarrow{\sigma}],$$

   where $[\sigma] \in \widetilde{X}$ and for $i = 1, 2$, the maps $\alpha_i : I \to U_i$ are paths for some $U_i \in \mathcal{U}$.
2. *The subgroup $\Pi^{Sp}(X, x_0) = \bigcap_{\mathcal{U} \in O(X)} \pi^{Sp}(\mathcal{U}, x_0)$ of $\pi_1(X, x_0)$ is called thick Spanier group. Equivalently, $\Pi^{Sp}(X, x_0)$ is the inverse limit (or $\varprojlim \Pi^{Sp}(\mathcal{U}, x_0)$) of the inverse system $\{\Pi^{Sp}(\mathcal{U}, x_0), \Pi^{Sp}(\mathcal{V}, x_0) \hookrightarrow \Pi^{Sp}(\mathcal{U}, x_0), O(X)\}$, where $O(X)$ is directed by refinement.*

2.1. Čech expansion and the nth shape homotopy group

In this section, we recall the construction of the nth shape homotopy group via Čech expansion. See [15] for more details.

Let $O(X)$ be the set of all open covers of X, and

$$O(X, x_0) = \{(\mathcal{U}, U_0) : \mathcal{U} \in O(X), U_0 \in \mathcal{U}\}.$$

It is easy to see that $O(X)$ is a directed set by refinement. We say that $(\mathcal{V}, V_0)$ *refines* $(\mathcal{U}, U_0)$ if $\mathcal{V}$ refines $\mathcal{U}$ as a covering and $V_0 \subset U_0$.

The *nerve of a covering* $(\mathcal{U}, U_0) \in O(X, x_0)$ is an abstract simplicial complex $N(\mathcal{U})$ whose vertex set is the set of elements of $\mathcal{U}$ and the vertices $U_0, U_1, \ldots, U_n \in \mathcal{U}$ span an n-simplex in $N(\mathcal{U})$ if $\bigcap_{i=1}^{n} U_i \neq \emptyset$. The vertex U_0 is taken to be the base point of the geometric realization $|N(\mathcal{U})|$. If $(\mathcal{V}, V_0)$ refines $(\mathcal{U}, U_0)$, we can construct a simplicial map $P_{\mathcal{U}\mathcal{V}} : N(\mathcal{V}) \to N(\mathcal{U})$, called *projection map*, as follows.

Given a vertex $V \in N(\mathcal{V})$, there exists $U \in \mathcal{U}$ such that $V \subset U$, $(\mathcal{V}, V_0)$ refines $(\mathcal{U}, U_0)$), the map $P_{\mathcal{U}\mathcal{V}}$ sends the vertex V of $N(\mathcal{V})$ to the vertex U of $N(\mathcal{U})$, and V_0 must be sent to U_0. We know that such an assignment of vertices induces a simplicial map. Now, $P_{\mathcal{U}\mathcal{V}}$ induces a map $|P_{\mathcal{U}\mathcal{V}}| : |N(\mathcal{V})| \to |N(\mathcal{U})|$, which is unique up to the based homotopy. Thus, the induced homomorphism $p_{\mathcal{U}\mathcal{V}*} : \pi_n(|N(\mathcal{V}|, V_0) \longrightarrow \pi_n(|N(\mathcal{U}|, U_0)$ is independent of the choice of the simplicial map.

An open covering $\mathcal{U}$ of X is said to be *normal* if it admits a partition of unity subordinated to $\mathcal{U}$. Let Λ be the subset of $O(X, x_0)$ consisting of all pairs $(\mathcal{U}, U_0)$, where $\mathcal{U}$ is a normal open covering of X, such that there exists a partition of unity $\{\varphi_u\}_{u\in\mathcal{U}}$ subordinated to $\mathcal{U}$ with $\varphi_{u_0}(x_0) = 1$. We know that for a paracompact Hausdorff space X, Λ is cofinal in $O(X, x_0)$.

For each $(\mathcal{U}, U_0) \in \Lambda$, choose a pointed map $p_{\mathcal{U}} : (X, x_0) \to (|N(\mathcal{U})|, U_0)$ such that $p^{-1}_{\mathcal{U}}(St(U, N(\mathcal{U}))) \subseteq U$ for all $U \in \mathcal{U}$, where $St(U, N(\mathcal{U}))$ denotes the open star of the vertex of $N(\mathcal{U})$ which corresponds to U. (For example, define $p_{\mathcal{U}}$ based on a locally finite partition of unity subordinated to $\mathcal{U}$.) Again, such a map $p_{\mathcal{U}}$ is unique up to the pointed homotopy, and we denote its pointed homotopy class by $[p_{\mathcal{U}}]$. Then, $[p_{\mathcal{U}\mathcal{V}} \circ p_{\mathcal{V}}] = [p_{\mathcal{U}}]$. The so-called (pointed) *Čech expansion*

$$(X, x_0)\ \underset{\longrightarrow}{[p_{\mathcal{U}}]}\ ((N(\mathcal{U}, *), [p_{\mathcal{U}\mathcal{V}}], C)$$

is an $\mathbf{HPol}_*$-expansion. The nth *shape homotopy group of a space X based at x_0*, denoted by $\check{\pi}_n(X, x_0)$, is defined as follows.

$$\pi_n(X, x_0) = \varprojlim(\pi_n(|N(\mathcal{U})|, U_0), p_{\mathcal{U}\mathcal{V}*}, \Lambda).$$

Since the maps $p_{\mathcal{U}}$ induce homomorphisms $p_{\mathcal{U}*} : \pi_n(X, x_0) \to \pi_n(|N(\mathcal{U})|, U_0)$ such that $p_{\mathcal{U}*} = p_{\mathcal{U}\mathcal{V}*} \circ p_{\mathcal{V}*}$, whenever $(\mathcal{V}, V_0)$ refines $(\mathcal{U}, U_0)$, we obtain an induced homomorphism

$$\Psi_X : \pi_n(X, x_0) \longrightarrow \check{\pi}_n(X, x_0)$$

given by $\Psi_X([\alpha]) = [\alpha_{\mathcal{U}}]$, where $\alpha_{\mathcal{U}} = p_{\mathcal{U}} \circ \alpha$. We say that X is *n-shape injective (bijective)* if the homomorphism φ is injective (bijective).

3. The *n*th thick and weak thick Spanier groups and their properties

Throughout the paper, $\sigma_\sharp[\alpha]$ is an n-loop such that σ is a path from $\sigma(0)$ to $\sigma(1)$, and α is an n-loop at $\sigma(1)$. To begin with, we recall the construction of the nth Spanier groups from [3].

Definition 3.1. *Let (X, x_0) be a pointed space, and $\mathcal{U} = \{U_i | i \in I\} \in O(X)$. Let $\pi_n^{Sp}(\mathcal{U}, x_0)$ be the subgroup of $\pi_n(X, x_0)$ which is spanned by all homotopy classes of the form $\Pi_{j=1}^{n} \sigma_{j\sharp}[\nu_j]$, where for every $1 \le j \le n$, $\sigma_j(0) = x_0$ and the n-loop ν_j lies in one of the neighborhoods $U_j \in \mathcal{U}$. This group is called the n-Spanier group with respect to $\mathcal{U}$. Moreover, the subgroup $\pi_n^{Sp}(X, x_0) = \bigcap_{\mathcal{U}\in O(X)} \pi_n^{Sp}(\mathcal{U}, x_0)$ is said to be the n-Spanier group.*

Let (X, x_0) be a pointed space, and $s_0 = (1, 0, 0, ..., 0) \in \partial D^n$. Suppose that $f_1, f_2 : (D^n, s_0) \longrightarrow (X, x_0)$ are pointed continuous maps such that $f_1(z) = f_2(z)$, for every $z \in \partial D^n$. Then the map $f_1 \circledast f_2 : S^n \longrightarrow X$, defined by the following formula, is an n-loop at x_0.

$$f_1 \circledast f_2(z) = \begin{cases} f_1(z) & z \in S^n_+ \\ f_2(z) & z \in S^n_-. \end{cases}$$

In the following proposition, we show that the definition above is independent of the choice of maps in their relative homotopy classes under homotopy.

Proposition 3.2. *Let (X, x_0) be a pointed space, $s_0 = (1, 0, 0, ..., 0) \in \partial D^n$ and $f_i, g_i : (D^n, s_0) \longrightarrow (X, x_0)$ be pointed continuous functions, for $i = 1, 2$, such that for each $z \in \partial D^n$, $f_1(z) = f_2(z)$ and $g_1(z) = g_2(z)$. If $f_i \simeq g_i$ rel$\{\partial D^n\}$, then $[f_1 \circledast f_2] = [g_1 \circledast g_2]$.*

Proof. Let F be a relative homotopy from f_1 to g_1, and G be a relative homotopy from f_2 to g_2. The map $H : S^n \times [0, 1] \longrightarrow X$, defined by $H(z, t) = F_t \circledast G_t(z)$, is a relative homotopy from $f_1 \circledast f_2$ to $g_1 \circledast g_2$ because for every z in S^n, $H(z, 0) = f_1 \circledast f_2(z), H(z, 1) = g_1 \circledast g_2(z)$ and for every $z \in \partial D^n, t \in [0, 1]$, $H(z, t) = f_1 \circledast f_2(z) = g_1 \circledast g_2(z)$. H. Therefore, $[f_1 \circledast f_2] = [g_1 \circledast g_2]$. □

Definition 3.3. *Let $\mathcal{U}$ be an open cover of a pointed space (X, x_0).*

(1) The n-thick Spanier group with respect to $\mathcal{U}$ is the subgroup of $\pi_n(X, x_0)$ which is spanned by elements of the form $\sigma_\sharp[f_1 \circledast f_2]$, where $\sigma \in \tilde{X}$, $f_1, f_2 : (D^n, s_0) \longrightarrow (X, x_0)$ are pointed continuous maps such that for any $z \in \partial D^n$, $f_1(z) = f_2(z)$, and $Im f_i \subseteq U_i$ for some $U_i \in \mathcal{U}$ $(i = 1, 2)$. We denote this by $\pi_n^{tSp}(\mathcal{U}, x_0)$.

(2) The subgroup

$$\pi_n^{tSp}(X, x_0) = \bigcap_{\mathcal{U} \in O(X)} \pi_n^{tSp}(\mathcal{U}, x_0)$$

is said to be the n-thick Spanier group of X. Equivalently, $\pi_n^{tSp}(X, x_0)$ is the inverse limit (or $\varprojlim \pi_n^{tSp}(\mathcal{U}, x_0)$) of the inverse system $\{\pi_n^{tSp}(\mathcal{U}, x_0), \pi_n^{tSp}(\mathcal{V}, x_0) \hookrightarrow \pi_n^{tSp}(\mathcal{U}, x_0), O(X)\}$, where $O(X)$ is directed by refinement.

If $n = 1$, then Definition 3.3 and Definition 2.3 are equivalent. It is easy to see that if an open cover $\mathcal{V}$ of X refines $\mathcal{U} \in O(X)$, then $\pi_n^{Sp}(\mathcal{U}, x_0) \subseteq \pi_n^{tSp}(\mathcal{U}, x_0)$ and $\pi_n^{tSp}(\mathcal{V}, x_0) \subseteq \pi_n^{tSp}(\mathcal{U}, x_0)$. It is also clear that $\pi_n^{Sp}(X, x_0)$ is a subgroup of $\pi_n^{tSp}(X, x_0)$.

Example 3.4. *Let $\mathcal{U}$ be an open cover of a locally $(n$-$1)$-connected metric space X. Then, $\mathcal{U}$ is called $(2, n)$-set simple if each element of this cover is $(n-1)$-connected and each n-loop in X that lies in the union of two elements of $\mathcal{U}$ is contractible in X (See [13]). Clearly, for each $(2, n)$-set simple open cover of X, the n-thick Spanier group and the n-Spanier group of X with respect to $\mathcal{U}$ are equal to the trivial group.*

In our next example, we show that the n-Spanier group and the n-thick Spanier group of a cover are not equal in general.

Example 3.5. *Let $U = S^n - \{N\}$ and $V = S^n - \{S\}$, where N and S are the north pole and the south pole of S^n, respectively. It is obvious that $\mathcal{U} = \{U, V\}$ is an open cover of S^n, and that U and V are contractible open subsets of S^n. So, $\pi_n^{Sp}(\mathcal{U}, x_0)$ is the trivial group. On the other hand, the generator of $\pi_n(S^n, x_0)$ is an element of $\pi_n^{tSp}(\mathcal{U}, x_0)$, and so $\pi_n^{tSp}(\mathcal{U}, x_0) = \pi_n(S^n, x_0) = \mathbb{Z}$.*

Definition 3.6. *Let $\mathcal{U}$ be an open cover of a pointed space (X, x_0). The n-weak thick Spanier group with respect to $\mathcal{U}$ is the subgroup of $\pi_n(X, x_0)$ which is generated by elements of the form $\sigma_\sharp[\alpha]$, where $\alpha(I^n) \subseteq U_1 \cup U_2$ for some $U_1, U_2 \in \mathcal{U}$. We denote this by $\pi_n^{wtSp}(\mathcal{U}, x_0)$. The n-weak thick Spanier group of X is defined as follows.*

$$\pi_n^{wtSp}(X, x_0) = \bigcap_{\mathcal{U} \in O(X)} \pi_n^{wtSp}(\mathcal{U}, x_0).$$

Equivalently, $\pi_n^{wtSp}(X, x_0)$ is the inverse limit (or $\varprojlim \pi_n^{wtSp}(\mathcal{U}, x_0)$) of the inverse system

$$\{\pi_n^{wtSp}(\mathcal{U}, x_0), \pi_n^{wtSp}(\mathcal{V}, x_0) \hookrightarrow \pi_n^{wtSp}(\mathcal{U}, x_0), O(X)\},$$

where $O(X)$ is directed by refinement.

Let $\mathcal{U}$ be an open cover of the pointed space (X, x_0). According to the definitions of the nth Spanier group, the nth weak thick Spanier group and the nth thick Spanier group,

$$\pi_n^{Sp}(\mathcal{U}, x_0) \subseteq \pi_n^{tSp}(\mathcal{U}, x_0) \subseteq \pi_n^{wtSp}(\mathcal{U}, x_0).$$

Moreover, if $\mathcal{V}$ is an open cover of X which refines $\mathcal{U}$, then $\pi_n^{wtSp}(\mathcal{V}, x_0) \subseteq \pi_n^{wtSp}(\mathcal{U}, x_0)$. In parts (i) and (ii) of the next example, we compute the nth Spanier group, the nth thick Spanier group and the nth weak thick Spanier group. In (iii), we show that in general, these subgroups are not equal to each other.

Example 3.7. *(i) Let K be a geometric simplicial complex, and $\mathcal{W} = \{St(v, K) : v \text{ is a vertex of } K\}$ be an open cover of K. Since for any vertex v of K, the open star $St(v, K)$ is contractible and path-connected,*

$$\pi_n^{Sp}(\mathcal{W}, w_0) = \pi_n^{tSp}(\mathcal{W}, w_0) = \pi_n^{wtSp}(\mathcal{W}, w_0) = 1.$$

(ii) *As shown in Example 3.5, $\pi_n^{Sp}(\mathcal{U}, x_0)$ is the trivial group and $\pi_n^{tSp}(\mathcal{U}, x_0) = \pi_n(S^n, x_0) = \mathbb{Z}$. Hence, $\pi_n^{wtSp}(\mathcal{U}, x_0) = \mathbb{Z}$.*

(iii) *Let $U_1 = \{e^{2\pi it} : 0 < t < \frac{1}{4}\}, U_2 = \{e^{2\pi it} : \frac{1}{2} < t < \frac{3}{4}\}$, $U_3 = \{e^{2\pi it} : \frac{1}{8} < t < \frac{5}{8}\}$ and $U_4 = \{e^{2\pi it} : \frac{5}{8} < t < \frac{9}{8}\}$. If $V_1 = U_1 \cup U_2$ and $V_2 = U_3 \cup U_4$, then $\mathcal{V} = \{V_1, V_2\}$ is an open cover of S^1. The subgroup $\pi_1^{tSp}(\mathcal{V}, x_0)$ is trivial, because no pair of paths $\gamma_1 : I \longrightarrow V_1$ and $\gamma_2 : I \longrightarrow V_2$ satisfying $\gamma_1 \circledast \gamma_2(I) = S^1$ can be found. Since $\pi_1^{wtSp}(\mathcal{V}, x_0)$ contains a generator of $\pi_1(S^1, x_0)$, $\pi_1^{wtSp}(\mathcal{V}, x_0) = \pi_1(S^1, x_0) \simeq \mathbb{Z}$. If $X = S^n$, using a similar method one finds an open cover $\mathcal{V}$ such that*

$$\pi_n^{wtSp}(\mathcal{V}, x_0) = \pi_n(S^n, x_0) \simeq \mathbb{Z}, \text{ and } \pi_n^{tSp}(\mathcal{V}, x_0) = 1.$$

As mentioned in Example 3.5 and Example 3.7, in general, the *nth* Spanier group, the *nth* thick Spanier group and the *nth* weak thick Spanier group of an open cover may be different subgroups of the *nth* homotopy group. In the following theorems, we present sufficient conditions that ensure the equality of these subgroups for an open cover.

Theorem 3.8. *If $\mathcal{U}$ is an open cover of (X, x_0) such that $U_1 \cap U_2$ is non-empty and path-connected for every pair $U_1, U_2 \in \mathcal{U}$, then $\pi_1^{Sp}(\mathcal{U}, x_0) = \pi_1^{tSp}(\mathcal{U}, x_0) = \pi_1^{wtSp}(\mathcal{U}, x_0)$. Moreover, if every open cover $\mathcal{U}$ of X admits a refinement $\mathcal{V}$ with this property, then*

$$\pi_1^{Sp}(X, x_0) = \pi_1^{tSp}(X, x_0) = \pi_1^{wtSp}(X, x_0).$$

Proof. Let $\mathcal{U}$ be an open cover of X, and $\sigma_\sharp[\alpha] = [\sigma\alpha\overleftarrow{\sigma}]$ be a generator of $\pi_1^{wtSp}(\mathcal{U}, x_0)$, where $\alpha(I) \subseteq U_1 \cup U_2$ for some $U_1, U_2 \in \mathcal{U}$. First, assume that $\sigma(1) = x_1 \in U_1 \cap U_2$. By [2, Lemma 3.2.5], there exist loops $\mu_1, ..., \mu_k$ such that $[\alpha] = [\mu_1][\mu_2]\cdots[\mu_k]$, and for each $1 \leq i \leq k$, $\mu_i(I)$ is contained in U_1 or U_2. Now, for any $1 \leq i \leq k$, $\sigma_\sharp[\mu_i] \in \pi_1^{Sp}(\mathcal{U}, x_0)$ and hence $\sigma_\sharp[\alpha] = \sigma_\sharp[\mu_1]\sigma_\sharp[\mu_2]\cdots\sigma_\sharp[\mu_k] \in \pi_1^{Sp}(\mathcal{U}, x_0)$. Let $x_1 \notin U_1 \cap U_2$ and $y = \alpha(t_0) \in U_1 \cap U_2$. Suppose that $\tau : I \longrightarrow X$ is a part of the loop α such that $\tau(0) = y$ and $\tau(1) = x_1$. Clearly, $\beta = \tau\alpha\overleftarrow{\tau}$ is a loop at y which satisfies $\beta(I) \subseteq U_1 \cup U_2$. Since $\sigma_\sharp[\alpha] = (\sigma\overleftarrow{\tau})_\sharp[\beta]$ and $y \in U_1 \cap U_2$, by the paragraph above, $\sigma_\sharp[\alpha] = (\sigma\overleftarrow{\tau})_\sharp[\beta] \in \pi_1^{Sp}(\mathcal{U}, x_0)$. □

As already mentioned, there are two equivalent definitions for relative homotopic groups as well as for homotopic groups. So, we can redefine $f_1 \circledast f_2$ using the cubical approach. If f and g are pointed continuous maps from $(I^n, \partial I^n, J^{n-1})$ to (X, A, x_0), where $A = f(\partial I^n) = g(\partial I^n)$, then the map $f \circledast g$ is defined as follows.

$$f \circledast g(x_1, x_2, x_3, \ldots, x_n) = \begin{cases} f(x_1, x_2, x_3, \ldots, 1 - 2x_n) & 0 \leq x_n \leq \frac{1}{2} \\ g(x_1, x_2, x_3, \ldots, 2x_n - 1) & \frac{1}{2} \leq x_n \leq 1. \end{cases}$$

Lemma 3.9. *Let f_1, f_2, g_1 and g_2 be maps from I^n to X such that $f_1|\partial I^n = f_2|\partial I^n = g_1|\partial I^n = g_2|\partial I^n = A$. Then*

$$[(f_1 \circledast g_1) + (f_2 \circledast g_2)] = [(f_1 + f_2) \circledast (g_1 + g_2)], \tag{1}$$

where + in the left side of (1) is the sum in $\pi_n(X, x_0)$, and + in the right side of (1) is the sum in $\pi_n(X, A, x_0)$.

Proof. If $(x_1, x_2, \ldots, x_n) \in I^n$, then

$$
\begin{aligned}
(f_1 \circledast g_1) + (f_2 \circledast g_2)(x_1, x_2, \ldots, x_n) &= \begin{cases} f_1 \circledast g_1(2x_1, x_2, \ldots, x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ f_2 \circledast g_2(2x_1 - 1, x_2, \ldots, x_n)) & \frac{1}{2} \leq x_1 \leq 1 \end{cases} \\
&= \begin{cases} f_1(2x_1, x_2, \ldots, 1 - 2x_n) & 0 \leq x_1 \leq \frac{1}{2}, 0 \leq x_n \leq \frac{1}{2} \\ g_1(2x_1, x_2, \ldots, 2x_n - 1) & 0 \leq x_1 \leq \frac{1}{2}, \frac{1}{2} \leq x_n \leq 1 \\ f_2(2x_1 - 1, x_2, \ldots, 1 - 2x_n) & \frac{1}{2} \leq x_1 \leq 1, 0 \leq x_n \leq \frac{1}{2} \\ g_2(2x_1 - 1, x_2, \ldots, 2x_n - 1) & \frac{1}{2} \leq x_1 \leq 1, \frac{1}{2} \leq x_n \leq 1 \end{cases} \\
&= \begin{cases} f_1(2x_1, x_2, \ldots, 1 - 2x_n)) & 0 \leq x_1 \leq \frac{1}{2}, 0 \leq x_n \leq \frac{1}{2} \\ f_2(2x_1 - 1, x_2, \ldots, 1 - 2x_n)) & \frac{1}{2} \leq x_1 \leq 1, 0 \leq x_n \leq \frac{1}{2} \\ g_1(2x_1, x_2, \ldots, 2x_n - 1)) & 0 \leq x_1 \leq \frac{1}{2}, \frac{1}{2} \leq x_n \leq 1 \\ g_2(2x_1 - 1, x_2, \ldots, 2x_n - 1)) & \frac{1}{2} \leq x_1 \leq 1, \frac{1}{2} \leq x_n \leq 1 \end{cases} \\
&= \begin{cases} f_1 + f_2(x_1, x_2, \ldots, 1 - 2x_n) & 0 \leq x_n \leq \frac{1}{2} \\ g_1 + g_2(x_1, x_2, \ldots, 2x_n - 1)) & \frac{1}{2} \leq x_n \leq 1 \end{cases} \\
&= (f_1 + f_2) \circledast (g_1 + g_2)(x_1, x_2, \ldots, x_n).
\end{aligned}
$$

Therefore,

$$[(f_1 \circledast g_1) + (f_2 \circledast g_2)] = [(f_1 + f_2) \circledast (g_1 + g_2)].$$

□

Theorem 3.10. *If $\mathcal{U}$ is an open cover of (X, x_0) such that for every $U_1, U_2 \in \mathcal{U}$, $U_1 \cap U_2$ is $(n-1)$-connected (or empty), then $\pi_n^{Sp}(\mathcal{U}, x_0) = \pi_n^{tSp}(\mathcal{U}, x_0)$. Also, if every open cover $\mathcal{U}$ of X admits a refinement $\mathcal{V}$ with this property, then $\pi_n^{Sp}(X, x_0) = \pi_n^{tSp}(X, x_0)$.*

Proof. Let $\mathcal{U}$ be an open cover of X as stated, and $k = \sigma_\sharp[f_1 \circledast f_2]$ be a generator of $\pi_n^{tSp}(X, x_0)$, where $f_1, f_2 : D^n \to X$, $f_1|_{\partial D^n} = f_2|_{\partial D^n}$ and there exist $U_1, U_2 \in \mathcal{U}$ such that the image of f_i is a subset of U_i for each $i \in \{1, 2\}$. Hence, the image of the map $g = f_1|_{\partial D^n} = f_2|_{\partial D^n}$ is a subset of $U_1 \cap U_2$. Since $U_1 \cap U_2$ is $(n-1)$-connected, there exists a map $\hat{g} : D^n \to U_1 \cap U_2$ such that $\hat{g}|_{\partial D^n} = g$. Thus, $\hat{g}, f_1$ and f_2 are maps from D^n to X such that $\hat{g}|_{\partial D^n} = f_1|_{\partial D^n} = f_2|_{\partial D^n}$. If $A = f_1(\partial D^n) = f_2(\partial D^n) = \hat{g}(\partial D^n)$, then f_1, f_2 and $\hat{g}$ are elements of $\pi_n(X, A, x_0)$. If $h_1 = \sigma_\sharp[f_1 \circledast \hat{g}]$ and $h_2 = \sigma_\sharp[\hat{g} \circledast f_2]$, it is obvious that $Im(f_1 \circledast \hat{g}) \subset U_1$ and $Im(\hat{g} \circledast f_2) \subset U_2$. So, the maps h_1 and h_2 are in $\pi_n^{Sp}(\mathcal{U}, x_0)$. To complete the proof, we need to show that $k = h_1 + h_2$. Since $\hat{g}(\partial D^n) = A$, $\hat{g}$ is the zero element of $\pi_n(X, A, x_0)$. So, by Lemma 3.9 and Theorem 2.1,

$$
\begin{aligned}
k &= \sigma_\sharp[f_1 \circledast f_2] = \sigma_\sharp[(f_1 + \hat{g}) \circledast (\hat{g} + f_2)] \\
&= \sigma_\sharp[(f_1 \circledast \hat{g}) + (\hat{g} \circledast f_2)] = \sigma_\sharp[f_1 \circledast \hat{g}] + \sigma_\sharp[\hat{g} \circledast f_2] \\
&= h_1 + h_2 \in \pi_n^{Sp}(\mathcal{U}, x_0).
\end{aligned}
$$

□

Let $\mathcal{U}$ be an open cover of a topological space X. We recall that the *star of x with respect to $\mathcal{U}$*, denoted by $\mathrm{St}(x, \mathcal{U})$, is the union of all $U \in \mathcal{U}$ which contain x. An open refinement $\mathcal{V}$ of $\mathcal{U}$ is called a *barycentric refinement* of $\mathcal{U}$ if for each $x \in X$, there exists $U \in \mathcal{U}$ such that $\mathrm{St}(x, \mathcal{V}) \subset U$. [11]

Theorem 3.11. *If (X, x_0) is a T_1 and paracompact space, then $\pi_n^{Sp}(X, x_0) = \pi_n^{tSp}(X, x_0) = \pi_n^{wtSp}(X, x_0)$. Moreover, this identity holds if X is metrizable.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is T_1 and paracompact, by [11, Theorem 5.1.12] there exists an open barycentric refinement $\mathcal{V}$ of $\mathcal{U}$. Suppose that $\sigma_\sharp[\alpha]$ is a generator of $\pi_n^{wtSp}(\mathcal{V}, x_0)$, where $\alpha(I^n) \subseteq V_1 \cup V_2$ for some $V_1, V_2 \in \mathcal{V}$. Since $\mathcal{V}$ is a barycentric refinement of $\mathcal{U}$, there exists $U \in \mathcal{U}$ such that $V_1 \cup V_2 \subseteq St(\sigma(1), \mathcal{V}) \subset U$. This implies that $\sigma_\sharp[\alpha]$ is an element of $\pi_n^{Sp}(\mathcal{U}, x_0)$. Now, it is easy to see that $\pi_n^{Sp}(X, x_0) = \pi_n^{tSp}(X, x_0) = \pi_n^{wtSp}(X, x_0)$. □

Theorem 3.12. *If G is a path-connected topological group, then for every $x \in G$,*

$$\pi_n^{Sp}(G,x) = \pi_n^{tSp}(G,x) = \pi_n^{wtSp}(G,x).$$

Proof. We only show that $\pi_n^{wtSp}(G,x) \subseteq \pi_n^{Sp}(G,x)$, for every $x \in G$. Let $x \in G$, $[\beta]$ be a generator of $\pi_n^{wtSp}(G,x)$, and $\mathcal{W}$ be an open cover of G. Then, there exist a path σ from x to $\sigma(1) = y$ and an n-loop $\alpha : I^n \longrightarrow G$ at y such that $\sigma_\sharp[\alpha] = [\beta]$ and $\alpha(I^n) \subseteq W_1 \cup W_2$ for some W_1 and W_2 in $\mathcal{W}$. If $W_1 \cap W_2$ is empty, since the set $\alpha(I^n)$ is connected, $[\beta] \in \pi_n^{Sp}(\mathcal{W},x)$. Now assume that $z \in W_1 \cap W_2 = W$. Since G is a topological group, there exist open neighborhoods U and V of e such that $zU \subseteq W$, $V^2 \subseteq U$ and V is symmetric. If $\mathcal{V} = \{gV : g \in G\}$, then $\mathcal{V}$ is an open cover of G such that for some $g_1, g_2 \in G$, $y \in \alpha(I^n) \subseteq g_1V \cup g_2V$. Since V is symmetric, $y^{-1}g_1 \in V$ or $y^{-1}g_2 \in V$. Let $\tau : I \longrightarrow G$ be a path from z to y. Define an n-loop $\gamma : I^n \longrightarrow G$ by $\gamma(s) = zy^{-1}\alpha(s)$ at z, and a map $H : I^n \times I \longrightarrow G$ by $H(s,t) = \overleftarrow{\tau}(t)z^{-1}\gamma(s)$. Then, for any $s \in I^n$, $H(s,0) = \overleftarrow{\tau}(0)z^{-1}\gamma(s) = yz^{-1}zy^{-1}\alpha(s) = \alpha(s)$, and for every $s \in \partial I^n$,

$$H(s,t) = \overleftarrow{\tau}(t)z^{-1}\gamma(s) = \overleftarrow{\tau}(t)z^{-1}zy^{-1}\alpha(s) = \overleftarrow{\tau}(t)y^{-1}y = \overleftarrow{\tau}(t).$$

Since for each $s \in I^n$, $H(s,1) = \overleftarrow{\tau}(1)z^{-1}\gamma(s) = zz^{-1}\gamma(s) = \gamma(s)$, $\tau_\sharp[\alpha] = [\gamma]$ and

$$\gamma(I^n) = zy^{-1}\alpha(I^n) \subseteq zy^{-1}(g_1V \cup g_2V) \subseteq zy^{-1}g_1V \cup zy^{-1}g_2V \subseteq zV^2 \cup zV^2 = zV^2 \subseteq zU \subseteq W.$$

Then, $[\beta] = (\sigma\overleftarrow{\tau})_\sharp[\gamma] \in \pi_n^{Sp}(\mathcal{W},x)$. Since $\mathcal{W}$ is an arbitrary open cover, $[\beta] \in \pi_n^{Sp}(G,x)$. □

Problem 3.13. *Is there a topological group whose nth weak thick (or thick) Spanier group is a non-trivial group?*

The authors do not know the answer to this question.

Lemma 3.14. *Let $h : (X,x_0) \longrightarrow (Y,y_0)$ be a pointed map, and $\sigma : [0,1] \longrightarrow X$ be a path with $\sigma(0) = x_0$. Then, the following statements are true.*

(i) For all $[\beta] \in \pi_n(X,\sigma(1))$, $(h \circ \sigma)_\sharp[h \circ \beta] = h_* \circ \sigma_\sharp[\beta]$.

(ii) If $\mathcal{W}$ is an open cover of Y such that $h^{-1}(\mathcal{W})$ is an open cover of X, then

$$h_*(\pi_n^{wtSp}(h^{-1}(\mathcal{W}),x_0)) \subseteq \pi_n^{wtSp}(\mathcal{W},y_0).$$

Proof. The proof is straightforward. □

As Brazas et al. proved in [5], the thick Spanier group lies in the kernel of the canonical map from the fundamental group to the first shape homotopy groups. In the following proposition, we show that this holds for the nth thick (weak thick) Spanier group.

Proposition 3.15. *For any space X, if Ψ_X is the canonical map from the nth homotopy group to the nth shape homotopy groups, then $\pi_n^{tSp}(X,x_0) \subseteq \pi_n^{wtSp}(X,x_0) \subseteq ker\Psi_X$.*

Proof. Let $(\mathcal{U},U_0)$ be a pointed open cover of X. The set $\mathcal{V} = \{P_{\mathcal{U}}^{-1}(St(U,N(\mathcal{U}))) : U \in \mathcal{U}\}$ is an open refinement of $\mathcal{U}$, because for each $U \in \mathcal{U}$, the set $P_{\mathcal{U}}^{-1}(St(U,N(\mathcal{U})))$ is a subset of U. Let g be a generator of $\pi_n^{wtSp}(X,x_0)$. Since g lies in $\pi_n^{wtSp}(\mathcal{V},x_0)$, there exist $\sigma \in \tilde{X}$ and $\alpha : I^n \rightarrow P_{\mathcal{U}}^{-1}(St(U,N(\mathcal{U}))) \cup P_{\mathcal{U}}^{-1}(St(V,N(\mathcal{U})))$, for some $U, V \in \mathcal{U}$, such that $g = \sigma_\sharp[\alpha]$. So, $P_{\mathcal{U}} \circ \sigma$ is a path in $|N(\mathcal{U})|$ with initial point U_0 and end point $P_{\mathcal{U}}(\sigma(1))$, and $(P_{\mathcal{U}} \circ \alpha)(I^n) \subset St(U,N(\mathcal{U})) \cup St(V,N(\mathcal{U}))$. Since $\mathcal{W} = \{St(U,N(\mathcal{U})) : U \in \mathcal{U}\}$ is an open cover of $|N(\mathcal{U})|$ by open stars, $P_{\mathcal{U}\#}(\sigma_\sharp[\alpha]) = (P_{\mathcal{U}} \circ \sigma)_\sharp[(P_{\mathcal{U}} \circ \alpha)]$ by Lemma 3.14. But the right side of the latter equation is a generator of $\pi_n^{wtSp}(\mathcal{W},U_0)$, and by (i) of Example 3.7, this group is the trivial group. So, g lies in $\ker P_{\mathcal{U}\#}$. □

4. Some topologies on the nth homotopy group

Homotopy groups can be equipped with various topologies. Relative homotopy defines an equivalence relation on the space of continuous maps from the n-sphere to a topological space with the compact-open topology. So, this relation induces a quotient topology on the homotopy groups. See [4, 6] for more details.

Perhaps, this was the first topology defined on the homotopy groups. Many authors have defined other topologies on the homotopy groups. They have explored the relationship between the topological properties of a topological space and its homotopy groups, for example, the relation between the Hausdorffness of a topological space X and the Hausdorffness of its homotopy groups.

In this section we use the notion of subgroup topology, introduced by Brodskiy in [10], and our subgroups of homotopy groups to define some topologies on the homotopy groups. To begin with, let us recall the definition of subgroup topology from [10].

Definition 4.1. *Let G be a group, and Σ be a non-empty family of subgroups of G such that for any $S, S' \in \Sigma$ there exists $S'' \in \Sigma$ such that $S'' \subseteq S \cap S'$. Then, the following statements are true.*

(S1) The set $\{gS : g \in G, S \in \Sigma\}$ is a base for a topology τ_Σ on G which is called the subgroup topology induced by Σ.
(S2) The topology τ_Σ is discrete if and only if Σ contains the trivial group.
(S3) The space (G, τ_Σ) is topological group if all members of Σ are normal subgroups of G ([19]).
(S4) The space (G, τ_Σ) is Hausdorff if and only if $\cap_{S\in\Sigma} S$ is the trivial group.

Proposition 4.2. *Let (X, x_0) be a pointed space and $n \in \mathbb{N}$. Then, the following statements are true.*

1. *The sets $\Sigma_{Sp} = \{\pi_n^{Sp}(\mathcal{U}, x_0) : \mathcal{U} \in O(X)\}$, $\Sigma_{tSp} = \{\pi_n^{tSp}(\mathcal{U}, x_0) : \mathcal{U} \in O(X)\}$ and $\Sigma_{wtsp} = \{\pi_n^{wtSp}(\mathcal{U}, x_0) : \mathcal{U} \in O(X)\}$ induce subgroup topologies τ_{Sp}, τ_{tSp} and τ_{wtSp} on $\pi_n(X, x_0)$, respectively.*
2. *The space $(\pi_n(X, x_0), \tau)$ is a topological group if $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$.*

Proof. (i) It suffices to show that Σ_{Sp} induces a topology τ_{Sp} on $\pi_n(X, x_0)$. The other cases can be proved similarly. Let $\pi_n^{Sp}(\mathcal{U}, x_0)$ and $\pi_n^{Sp}(\mathcal{V}, x_0)$ be elements of Σ_{Sp}. The set $\mathcal{W} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ is a refinement of $\mathcal{U}$ and $\mathcal{V}$ such that $\pi_n^{Sp}(\mathcal{W}, x_0) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0) \cap \pi_n^{Sp}(\mathcal{V}, x_0)$. By $(S1)$, the set Σ_{Sp} induces a subgroup topology τ_{Sp} on $\pi_n(X, x_0)$. (ii) Since the elements of $\Sigma_{Sp}, \Sigma_{tSp}$ and Σ_{wtSp} are normal in $\pi_n(X, x_0)$, by $(S3)$, the proof is straightforward. □

Definition 4.3. *Let (X, x_0) be a pointed space. We call the topologies τ_{Sp}, τ_{tSp} and τ_{wtSp} the Spanier topology, the thick Spanier topology, and the weak thick Spanier topology, respectively.*

Proposition 4.4. *Let $\sigma : I \longrightarrow X$ be a path from x_0 to x_1 in a topological space X. Then, $\sigma_\sharp : (\pi_n(X, x_1), \tau) \longrightarrow (\pi_n(X, x_0), \tau)$ is a homeomorphism, where $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$.*

Proof. Let $1_1 = [c_{x_1}]$ be the identity element of $\pi_n(X, x_1)$, and $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$. Since for each open cover $\mathcal{U}$ of X, $\sigma_\sharp(\pi_n^{Sp}(\mathcal{U}, x_1)) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0)$, $\sigma_\sharp(\pi_n^{tSp}(\mathcal{U}, x_1)) \subseteq \pi_n^{tSp}(\mathcal{U}, x_0)$, and $\sigma_\sharp(\pi_n^{wtSp}(\mathcal{U}, x_1)) \subseteq \pi_n^{wtSp}(\mathcal{U}, x_0)$, the map $\sigma_\sharp$ is continuous at 1_1. By Proposition 4.2, $(\pi_n(X, x_0), \tau)$ and $(\pi_n(X, x_1), \tau)$ are topological groups. So, $\sigma_\sharp$ is continuous at every point of $(\pi_n(X, x_1), \tau)$. Similarly, we can prove that $\overleftarrow{\sigma}_\sharp$ is also continuous. □

Proposition 4.5. *If $f : (X, x_0) \longrightarrow (Y, y_0)$ is a pointed continuous map, then the induced homomorphism $f_* : (\pi_n(X, x_0), \tau) \longrightarrow (\pi_n(Y, y_0), \tau)$ is continuous, where $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$.*

Proof. Let $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$, and $\mathcal{U}$ be an open cover of Y. Then, $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X. It is easy to prove that $f_*(\pi_n^{Sp}(f^{-1}(\mathcal{U}), x_0)) \subseteq \pi_n^{Sp}(\mathcal{U}, y_0)$, $f_*(\pi_n^{tSp}(f^{-1}(\mathcal{U}), x_0)) \subseteq \pi_n^{tSp}(\mathcal{U}, y_0)$ and $f_*(\pi_n^{wtSp}(f^{-1}(\mathcal{U}), x_0)) \subseteq \pi_n^{wtSp}(\mathcal{U}, y_0)$. Hence, f_* is continuous at the identity element of $\pi_n(X, x_0)$. Since $(\pi_n(X, x_0), \tau)$ and $(\pi_n(Y, y_0), \tau)$ are topological groups, the homomorphism f_* is continuous. □

Proposition 4.6. *Let $\{(X_i, x_i) : i \in I\}$ be a family of pointed spaces, $x = \{x_i : i \in I\}$ and $X = \prod_{i\in I} X_i$. If $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$ and τ' is the product topology on $\prod_{i\in I} \pi_n(X_i, x_i)$, then the isomorphism $\psi : \pi_n(X, x) \longrightarrow \prod_{i\in I} \pi_n(X_i, x_i)$ defined by $\psi([\alpha]) = \{P_{i*}([\alpha])\}_{i\in I}$ is continuous, where $P_i : X \longrightarrow X_i$ is the canonical projection into the ith component of X.*

Proof. Let $\tau = \tau_{Sp}$, and $\prod_{i\in I} B_i$ be a base element of τ'. Then, there exists a finite subset J of I such that for any $i \in I \setminus J$, $B_i = \pi_n(X_i, x_i)$. If $B_i = \pi_n^{Sp}(\mathcal{U}_i, x_i)$ for each $i \in J$, then the set

$$W = \{\textstyle\prod_{i\in I} U_i : U_i \in \mathcal{U}_i \text{ for } i \in J \text{ and } U_i = X_i \text{ for } i \in I \setminus J\}$$

is an open cover of X. We show that $\psi(\pi_n^{Sp}(\mathcal{W}, x)) \subseteq \prod_{i\in I} B_i$. Let $\sigma_\sharp[\gamma]$ be a generator of $\pi_n^{Sp}(\mathcal{W}, x)$ such that $\gamma(I^n) \subseteq \prod_{i\in I} U_i$ for some $\prod_{i\in I} U_i \in \mathcal{W}$. Then,

$$\psi(\sigma_\sharp[\gamma]) = \{P_{i*}\sigma_\sharp[\gamma]\}_{i\in I} = \{(P_i \circ \sigma)_\sharp[P_i \circ \gamma]\}_{i\in I} \in \textstyle\prod_{i\in I} B_i.$$

Therefore, ψ is continuous at the identity element of $(\pi_n(X, x), \tau)$. Since $(\pi_n(X, x), \tau)$ and $(\prod_{i\in I} \pi_n(X_i, x_i), \tau')$ are topological groups, ψ is continuous at any element of $\pi_n(X, x)$. The other cases can be proved similarly. □

A topological space X is said to be *n-homotopically Hausdorff* at $x \in X$ if for any essential n-loop $\alpha : (I^n, \partial I^n) \longrightarrow (X, x)$, there exists an open neighborhood U of x such that no n-loops at x are homotopic (in X) to α rel ∂I^n. The space X is called *n-homotopically Hausdorff* if it is n-homotopically Hausdorff at x, for every $x \in X$. See [13] for more details.

Theorem 4.7. *Let X be a path-connected space, $x_0 \in X$ and $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$. If $(\pi_n(X, x_0), \tau)$ is Hausdorff for every $n \in \mathbb{N}$, then X is n-homotopically Hausdorff.*

Proof. Let $\tau = \tau_{Sp}$, and suppose that$(\pi_n(X, x_0), \tau)$ is a Hausdorff space. Then by (S4), $\bigcap_{\mathcal{U}\in O(X)} \pi_n^{Sp}(\mathcal{U}, x_0) = \pi_n^{Sp}(X, x_0)$ is the trivial group. Suppose that X is not n-homotopically Hausdorff. Then there exists an essential n-loop α with base point x such that for each $\mathcal{U} \in O(X)$, there exist an open set $V \in \mathcal{U}$ and an n-loop $\beta_V : I^n \longrightarrow V$ with base point x such that $[\alpha] = [\beta_V]$. Since X is path-connected, there exists a path σ from x_0 to x such that $\sigma_\sharp[\alpha] = \sigma_\sharp[\beta_V] \in \pi_n^{Sp}(\mathcal{U}, x_0)$. Since $\mathcal{U}$ is arbitrary, $\sigma_\sharp[\alpha]$ is the identity element of $\pi_n^{Sp}(X, x_0)$, and so $[\alpha] = 1$, which is a contradiction. Therefore, X is n-homotopically Hausdorff. If $\tau \in \{\tau_{tSp}, \tau_{wtSp}\}$ and $(\pi_n(X, x_0), \tau)$ are Hausdorff spaces, then by (S4), $\pi_n^{Sp}(X, x_0)$ or $\pi_n^{wtSp}(X, x_0)$ is the trivial group. This implies that $\pi_n^{Sp}(X, x_0)$ is the trivial group, and so X is n-homotopically Hausdorff. □

Corollary 4.8. *If X is a π_n-injective space, then X is n-homotopically Hausdorff.*

Proof. Since X is a π_n-injective space (that is, $\ker\Psi = 1$), by Proposition 3.15 and (S3), $(\pi_n(X, x_0), \tau)$ is Hausdorff, where $\tau \in \{\tau_{Sp}, \tau_{tSp}, \tau_{wtSp}\}$. Therefore, by Theorem 4.7, X is n-homotopically Hausdorff. □

The following example shows that the converse of Theorem 4.7 is not true in general. In Proposition 4.14, we will show that there are some conditions under which the converse of Theorem 4.7 is true.

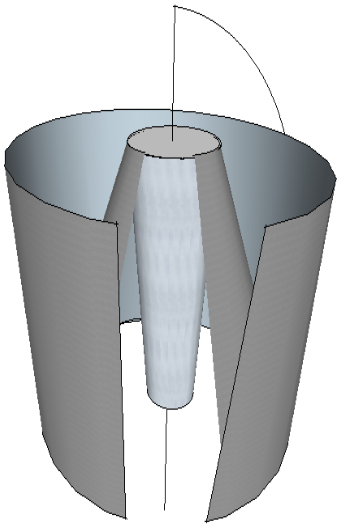
Example 4.9. *Let* $A^n = \left(\bigcup_{i\in\mathbb{N}} \left(S_i^n \times [0, 1]\right)\right) \cup \left(\bigcup_{i\in\mathbb{N}} \left(B_i^n \times [0, 1]\right)\right)$, *where*

$$S_i^n = \left\{(x_1, ..., x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = \tfrac{1}{i^2}\right\} \text{ and } B_i^n = \left\{(x_1, ..., x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = (\tfrac{2i+1}{2i(i+1)})^2\right\}.$$

It is easy to show that if $x \in B_i^n$, then $\frac{2i+2}{2i+1}x$ and $\frac{2i}{2i+1}x$ are in S_i^n and S_{i+1}^n, respectively. Hence, the following relation is an equivalence relation on A^n. For any $i \in \mathbb{N}$ and $x \in B_i^n$,

$$(x, 0) \sim \left(\tfrac{2i+2}{2i+1}x, 0\right) \text{ and } (x, 1) \sim \left(\tfrac{2i}{2i+1}x, 1\right),$$

and the other points of A^n are only related to themselves. Let $W^n = \frac{A^n}{\sim}$ be the subspace of $\mathbb{R}^{n+2}$. The space W^n is n-semilocally simply connected, and its n-Spanier groups are trivial. By Theorem 4.7, the topological space W^n is n-homotopically Hausdorff. Now, let $\mathcal{W}^1 = W^1 \cup \{(0, 0, b) \in \mathbb{R}^3 : 0 \leq b \leq 1\} \cup C$, where C is a single arc that intersects the central axis of W^1 only at its endpoint (Figure 4.9).



The space $\mathcal{W}^1$

In general, let $\mathcal{W}^n = W^n \cup \{(0, ..., 0, b) \in \mathbb{R}^{n+1} : 0 \leq b \leq 1\} \cup C$, where C is a single arc that connects the central axis to W^n. This arc C cannot intersect W^n or the central axis at any points other than its endpoints. The n-Spanier group of the space $\mathcal{W}^n$ is non-trivial ([3]). Since

$$\pi_n^{Sp}(\mathcal{W}^n, x_0) \subseteq \pi_n^{tSp}(\mathcal{W}^n, x_0) \subseteq \pi_n^{wtSp}(\mathcal{W}^n, x_0),$$

the subgroups $\pi_n^{tSp}(\mathcal{W}^n, x_0)$ and $\pi_n^{wtSp}(\mathcal{W}^n, x_0)$ are also non-trivial. By (S3), the topological spaces $(\pi_n(\mathcal{W}^n, x_0), \tau_{Sp})$, $(\pi_n(\mathcal{W}^n, x_0), \tau_{tSp})$ and $(\pi_n(\mathcal{W}^n, x_0), \tau_{wtSp})$ are not Hausdorff. But, by [17, Theorem 2.9], the space $\mathcal{W}^n$ is n-homotopically Hausdorff.

The concept of small loop transfer was defined by Brodskiy et al. in [9]. Now, we are going to introduce the concept of small n-loop transfer as an extension of this.

Definition 4.10. *A topological space X is said to be a* small n-loop transfer space *at $x \in X$ if for every path σ in X with $\sigma(0) = x$ and every open neighborhood U of x, there is an open set V containing $\sigma(1)$ such that for any n-loop β in V at $\sigma(1)$, there exists an n-loop α in U at x such that $\sigma_\sharp[\beta] = [\alpha]$. A topological space is called a* small n-loop transfer space *if it is small n-loop transfer at each of its points.*

Proposition 4.11. *Any path-connected topological group is a small n-loop transfer space.*

Proof. Let G be a topological group and $x \in G$. Let σ be a path in G from x to y, and U be an open neighborhood of x. Then, $V = yx^{-1}U$ is an open neighborhood of y. If β is an n-loop at y in V, then the map $\alpha : I^n \longrightarrow U$ defined by $\alpha(t) = xy^{-1}\beta(t)$ is an n-loop at x. The homotopy map $H : I^n \times I \longrightarrow G$ given by $H(s, t) = \sigma(t)x^{-1}\alpha(s)$ satisfies the following conditions.

$$H(s, 0) = \alpha(s), H(s, 1) = \beta(s), \text{for all } s \in I^n$$

$$H(s, t) = \sigma(t), \text{for all } s \in \partial I^n, t \in I.$$

Hence by (10), $\sigma_\sharp[\beta] = [\alpha]$. Therefore, G is a small n-loop transfer space. □

Definition 4.12. *[13] A space X is said to be n-homotopically Hausdorff at $x \in X$ if for any essential n-loop $\alpha : (I^n, \partial I^n) \longrightarrow (X, x)$, there exists an open neighborhood U of x such that no n-loops at x with image in U are homotopic (in X) to α rel ∂I^n. The space X is called n-homotopically Hausdorff if it is n-homotopically Hausdorff at x, for every $x \in X$.*

Definition 4.13. *[14] A path-connected space X is said to be abelian if the action of the fundamental group on each homotopy group of X is trivial.*

Proposition 4.14. *Let (X, x_0) be an abelian, locally path-connected, n-homotopically Hausdorff and small n-loop transfer space. Then, $(\pi_n(X, x_0), \tau_{Sp})$ is a Hausdorff space. Moreover, if X is a T_1 and paracompact space, then $(\pi_n(X, x_0), \tau_{tSp})$ and $(\pi_n(X, x_0), \tau_{wtSp})$ are Hausdorff spaces.*

Proof. Let $\alpha : I^n \longrightarrow X$ be an essential n-loop at x_0. Since the space is locally path-connected and n-homotopically Hausdorff, there exists a path-connected open neighborhood U of x_0 such that α is not homotopic to any n-loop at x_0 in U. Let σ be a path from x_0 to $\sigma(1)$. Since X is a small n-loop transfer space, there exists a path-connected open neighborhood V_σ of $\sigma(1)$ such that $\sigma_\sharp(\pi_n(V_\sigma, \sigma(1)) \subseteq \pi_n(U, x_0)$. We claim that for an open cover $\mathcal{V} = \{V_\sigma : \sigma \text{ is a path from } x_0 \text{ to } \sigma(1)\}$ of X, $[\alpha] \notin \pi_n^{Sp}(\mathcal{V}, x_0)$. If $[\alpha] \in \pi_n^{Sp}(\mathcal{V}, x_0)$, then there exist paths $\sigma_1, ..., \sigma_m$ from x_0 and n-loops $\gamma_1, ..., \gamma_m$ at $\sigma_1(1), ..., \sigma_m(1)$, respectively, such that $[\alpha] = \sigma_{1\sharp}[\gamma_1]...\sigma_{m\sharp}[\gamma_m]$, where $\gamma_i(I^n) \subseteq V_{\lambda_i}$ for some $V_{\lambda_i} \in \mathcal{V}$ (for $i = 1, 2, ..., m$). Since X is path-connected, there exists a path δ_i from $\lambda_i(1)$ to $\sigma_i(1)$ in V_{λ_i} such that $\delta_{i\sharp}[\gamma_i]$ is an n-loop at $\lambda_i(1)$ in V_{λ_i}, for each $1 \leq i \leq m$. Since X is a small n-loop transfer space, there exists an n-loop f_i at x_0 in U such that $\lambda_{i\sharp}\delta_{i\sharp}[\gamma_i] = [f_i]$, for all $1 \leq i \leq m$. On the other hand, since the space X is abelian, $(\lambda_i\delta_i)_\sharp = \sigma_{i\sharp}$ for every $1 \leq i \leq m$. Thus, $[\alpha] = [f_1]...[f_m] = [f_1 f_2...f_m]$, which is a contradiction because $f_1...f_m(I^n) \subseteq U$. Therefore, the topological group $(\pi_n(X, x_0), \tau_{Sp})$ is a T_0 space, and so it is Hausdorff. If X is a T_1 and paracompact space, then the proof is straightforward by Theorem 3.11. □

Proposition 4.15. *Let (X, x_0) be a path-connected and small n-loop transfer space. If X is first countable at x_0, then $(\pi_n(X, x_0), \tau_{Sp})$ is also first countable.*

Proof. Let $\{U_i : i \in \mathbb{N}\}$ be a local base at x_0. Since (X, x_0) is a small n-loop transfer space, for any $i \in \mathbb{N}$ and every path σ from x_0 to $\sigma(1)$, there exists an open neighborhood V_i^σ of $\sigma(1)$ such that $\sigma_\sharp\pi_n(V_i^\sigma, \sigma(1)) \subseteq \pi_n(U_i, x_0)$. Since X is a path-connected space, the set $\mathcal{V}_i = \{V_i^\sigma : \sigma \text{ is a path with } \sigma(0) = x_0\}$ is an open cover of X. We show that the set $\{\pi_n^{Sp}(\mathcal{V}_i, x_0) : i \in \mathbb{N}\}$ is a local base of $(\pi_n(X, x_0), \tau_{Sp})$ at the identity element. Let us consider the open neighborhood $\pi_n^{Sp}(\mathcal{U}, x_0)$ of this, where $\mathcal{U}$ is an open cover of X. Let $x_0 \in U$ for some $U \in \mathcal{U}$. Then there exists $j \in \mathbb{N}$ such that $x_0 \in U_j \subseteq U$. It is easy to see that $\pi_n(U_j, x_0) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0)$. Since for every path σ with $\sigma(0) = x_0$, $\sigma_\sharp\pi_n(V_j^\sigma, \sigma(1)) \subseteq \pi_n(U_j, x_0)$, we obtain

$$\pi_n^{Sp}(\mathcal{V}_j, x_0) \subseteq \bigcup_\sigma \sigma_\sharp\pi_n(V_j^\sigma, \sigma(1)) \subseteq \pi_n(U_j, x_0).$$

Thus $\pi_n^{Sp}(\mathcal{V}_j, x_0) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0)$, which implies that the set $\{\pi_n^{Sp}(\mathcal{V}_i, x_0) : i \in \mathbb{N}\}$ is a local base at the identity element of the topological space $(\pi_n(X, x_0), \tau_{Sp})$. By Proposition 4.2, this space is first countable. □

Theorem 4.16. *Let (X, x_0) be path-connected, locally path-connected and n-homotopically Hausdorff, and assume that the space X is abelian. If X is first countable at x_0 and a small n-loop transfer space, then $(\pi_n(X, x_0), \tau_{Sp})$ is metrizable. Moreover, if X is a T_1 and paracompact space, then $(\pi_n(X, x_0), \tau_{tSp})$ and $(\pi_n(X, x_0), \tau_{wtSp})$ are metrizable.*

Proof. The proof can be completed using Propositions 4.2, 4.15 and 4.14 and Theorem 3.3.12 of [1]. □

Recall that a map $f : X \longrightarrow Y$ has the *unique path lifting property* if for any two paths $\alpha, \beta : [0, 1] \longrightarrow X$, we obtain $\alpha = \beta$ whenever $f \circ \alpha = f \circ \beta$ and $\alpha(0) = \beta(0)$.

Now, we recall the definition of Whisker topology on the set $\widetilde{X}$ from [8].

Definition 4.17. *The basis of the Whisker topology on $\widetilde{X}$ is the collection of sets of the form*

$$B([\sigma], U) = \left\{[\lambda] \in \widetilde{X} : [\lambda] = [\sigma\delta], \text{for some } \delta : I \longrightarrow U, \delta(0) = \sigma(1)\right\},$$

where $\sigma \in \widetilde{X}$ and U is an open neighborhood of $\sigma(1)$.

The set $\widetilde{X}$ with the Whisker topology is a topological space, denoted by $\widehat{X}$.

In what follows, we use Proposition 4.14 to give an answer to the problem posed by Brodskiy et al. in [8, Problem 8.11]. First, let us recall this problem.

Problem 4.18. *([8, Problem 8.11]) Let X be a path-connected space and $x_0 \in X$. Are the following conditions equivalent?*

(i) $\pi_1^{Sp}(X, x_0) = \bigcap_{\mathcal{U}} \pi_1^{Sp}(\mathcal{U}, x_0)$ *is trivial.*

(ii) $p : (\widehat{X}, \hat{x}_0) \longrightarrow (X, x_0)$ *has the unique path lifting property.*

(iii) X is 1-homotopically Hausdorff (or homotopically Hausdorff).

The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ follow from [8]. But, under what conditions is the implication $(iii) \Longrightarrow (i)$ established?

Notice that by $(S3)$, Problem 4.18 can be transformed into a strong problem as follows.

Problem 4.19. *Let X be a path-connected space and $x_0 \in X$. Given any $n \in \mathbb{N}$, are the following conditions equivalent?*

(i) The space $(\pi_n(X, x_0), \tau_{Sp})$ *is Hausdorff.*

(ii) The space X is n-homotopically Hausdorff.

Answer:

The implication $(i) \Longrightarrow (ii)$ follows from Theorem 4.7. Now, if we add to (ii) the assumption of being a locally path-connected, small n-loop transfer space, and assume that the space X is abelian, then (i) follows by Proposition 4.14. Thus, the answer to Problem 4.18 is given with $n = 1$. So, we obtain the following theorem.

Theorem 4.20. *Let X be a path-connected space and $x_0 \in X$. If X is a locally path-connected, small 1-loop transfer space and $\pi_1(X, x_0)$ is abelian, then the following conditions are equivalent.*

(i) $\pi_1^{Sp}(X, x_0) = \bigcap_{\mathcal{U}} \pi_1^{Sp}(\mathcal{U}, x_0)$ *is trivial.*

(ii) $p : (\widehat{X}, \hat{x}_0) \longrightarrow (X, x_0)$ *has the unique path lifting property.*

(iii) The space X is 1-homotopically Hausdorff.

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