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On solvability of dynamic equation in Banach space of continuous functions over time scales

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Abstract. We investigate existence of solution of local dynamic initial value problem in the Banach space of continuous function from an interval in time scale to an arbitrary Banach space. We use the concept of measure of noncompactness and Meir-Keeler condensing operator involving L-function to obtain the existence of solutions of the problem. We also give an example to illustrate our result.

1. Introduction

In 1988, German Mathematician Stefan Hilger introduced the concept of time scale for the first time in the field of Mathematics through his Ph.D. thesis. After that, Hilger published two interesting articles on this topic [16, 17]. By a time scale, we mean any nonempty closed subset of $\mathbb{R}$. Time scale basically unifies the discrete and continuous calculi, to study them simultaneously rather separately. In recent times researchers are quiet actively working on dynamic equations to merge results from both differential and difference calculi. Dynamic equations are associated with several real-world phenomenon involving discrete as well as continuous variables, for example, we refer for population dynamics [19, 35], optimization theory [34], economics [3, 4], economic modelling [32], etc. Various aspects like qualitative and quantitative properties, stability of solution, applicability of a dynamic equation can be studied. Santos [26, 27], discussed about discontinuous dynamic equations and dynamic equations of first order, respectively, on time scales (also one can see [18, 31]). Bohner et al. [9] were studied on nonlinear dynamic initial value problems of first-order. The existence of solutions of the dynamic Cauchy problem in Banach spaces was studied by Cichoń et al. [10], Kubiaczyk and Sikorska-Nowak [20]. Shen [29] investigated the stability of dynamic equations on time scales. For details on time scale calculus one can see the monographs [7, 8].

For solvability of differential equations and integral equations in Banach spaces the concept of measure of noncompactness have play a significant role, because it has relaxed the domain of the operator to be compact. Kuratowski [21] was the first person to introduce this concept in the year 1930. In the year 1955, Darbo [12] introduced a fixed point theorem with the help of MNC, which is a generalization of both Banach contraction principle and Schauder's fixed point theorem. Recently many researchers apply this

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concept to solve different types of integral equations, differential equations, integro-differential equations, fractions differential equations, fractional integral equations, functional integral equations, etc (one can see the articles [1, 2, 5, 6, 11, 13–15, 24, 25, 28] and references therein).

Recently, Sanket et al. [30] were discussed the existence of solution of nonlocal initial value problem

$$y^{\Delta} + p(\tau)y^{\sigma}(\tau) = f(\tau, y)\ ,\ \tau \in \mathcal{L}^{k}$$
$$y(\tau_0) = y_0$$

using the fixed point theorem by Sadovskii.

The characterization of Meir-Keeler condensing operator using L-function discussed by Lim [22] (also see [23]), we are motivated to search for a solution of the following form of dynamic first order nonlocal initial value problem (IVP)

$$\begin{aligned} & y^{\Delta}(\tau) = l(\tau)y(\tau) + f(\tau, y)\ ,\ \tau \in \mathcal{L}^{k} \\ & y(\tau_0) = y_0, \end{aligned} \tag{1}$$

where y_0 is some fixed element in Y.

Here, $\mathbb{T}$ is a time scale and $\tau_0, S \in \mathbb{T}$ with $\tau_0 > 0$.

$$\mathcal{L}^{k} = \begin{cases} \mathcal{L} \setminus (\rho\,(\sup \mathcal{L})\,, \sup \mathcal{L}) \ \text{if} \ \sup \mathcal{L} < \infty \\ \mathcal{L}, \ \text{otherwise} \end{cases} \tag{2}$$

where $\mathcal{L} = [\tau_0, T]_{\mathbb{T}} = [\tau_0, T] \cup \mathbb{T} = \{\tau \in \mathbb{T} : \tau_0 \le \tau \le S\}$.

Throughout the article, we consider

Y is an arbitrary Banach space.

$C(\mathcal{L}, Y)$ is the family of continuous function from $\mathcal{L}$ into Y.

y is the unknown function to be determined.

$\mathcal{M}_Y$ denotes the collection of all nonempty and bounded subsets of Y.

y^{Δ} represents the delta derivative of y.

$f : \mathcal{L} \times Y \longrightarrow Y$ is a non-linear function.

$l : \mathcal{L} \longrightarrow \mathbb{R}$ is a function which is both regressive and rd-continuous.

2. Preliminaries and definitions

We recall few existing definitions along with a few associated results for development of our primary result.

Definition 2.1. *[8, Definition 1.58] Let there be function $v : \mathbb{T} \longrightarrow Y$ which is continuous at every right dense point of $\mathbb{T}$, and also ensures the existence of its limits at all left dense points in $\mathbb{T}$, then the function v is called a rd-continuous function. We denote by $C_{rd}(\mathbb{T}, Y)$, the collection of all rd-continuous functions $f : \mathbb{T} \longrightarrow Y$.*

Definition 2.2. *[19, Definition 5] Similarly as above, a function $f : \mathbb{T} \times Y \longrightarrow Y$ which is continuous on Y for each $t \in \mathbb{T}$ is called a rd-continuous function on $\mathbb{T} \times Y$ if f is rd-continuous on $\mathbb{T}$ for every $y \in Y$. We denote by $C_{rd}(\mathbb{T} \times Y, Y)$, the collection of all rd-continuous functions $f : \mathbb{T} \times Y \longrightarrow Y$.*

Definition 2.3. *[8, Definition 2.25] A function $l : \mathbb{T} \longrightarrow \mathbb{R}$ is called a regressive function if for all $\tau \in \mathbb{T}^{k}$, the quantity $1 + \mu(\tau)l(\tau)$ is always a nonzero quantity, where $\mu : \mathbb{T} \longrightarrow \mathbb{R}$ is the graininess function on $\mathbb{T}$, defined as $\mu(\tau) = \sigma(\tau) - \tau$. We denote by $\mathcal{R}(\mathbb{T}, \mathbb{R})$, the collection of all regressive functions $l : \mathbb{T} \longrightarrow \mathbb{R}$.*

Definition 2.4. *[8, Definition 2.30] On a time scale $\mathbb{T}$, exponential function denoted by $e_l(\cdot, \tau_0)$ is the unique solution of the dynamic initial value problem*

$y^{\Delta}(\tau) = l(\tau)y,\ \ y(\tau_0) = 1,\ \tau_0 \in \mathbb{T}\ ,\ \tau \in \mathbb{T}^{k}$ and $l : \mathbb{T} \longrightarrow \mathbb{R}$ is a regressive function.

For $l, m \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ we define

$l \oplus m := l + m + \mu l m\ ,\ \ominus l := \frac{-l}{1+\mu l}\ ,\ l \ominus m := l \oplus \ominus(m)\ ,\ \ominus(\ominus l) := l.$

Theorem 2.5. [8, Theorem 2.36] If $l, m : \mathbb{T} \longrightarrow \mathbb{R}$ are functions both of which is regressive and rd-continuous simultaneously, for $t, u, s \in \mathbb{T}$, the following properties

i. $e_0(\tau, u) \equiv 1$ and $e_l(\tau, \tau) \equiv 1$;
ii. $e_l(\sigma(\tau), u) = (1 + \mu(\tau) l(\tau)) e_l(\tau, u)$;
iii. $e_l(\tau, u) = \frac{1}{e_l(\tau,u)} = e_{\ominus l}(\tau, u)$;
iv. $e_l(\tau, s) e_l(s, u) = e_l(\tau, u)$;
v. $e_l(\tau, u), e_m(\tau, u) = e_{l \oplus m}(\tau, u)$;
vi. $e_l(\tau, u)/e_m(\tau, u) = e_{l \ominus m}(\tau, u)$

hold.

Throughout the articl, we denote $E := \max\left\{\sup_{\tau,u\in\mathcal{L}} |e_l(\tau, u)|, \sup_{\tau,u\in\mathcal{L}} |e_l(\tau, \sigma(u))|\right\}$.

By $C(\mathcal{L}, Y)$, we denote the family of all continuous functions from $\mathcal{L}$ to Y, where $(Y, \|\cdot\|_Y)$ is a Banach space. Also to note here is that $C(\mathcal{L}, Y)$ is a Banach space with the norm $\|\cdot\|$ defined as $\|y\| := \sup_{\tau\in\mathcal{L}} \|y(\tau)\|_Y$.

Definition 2.6. *Let G be a subset of $\mathcal{M}_c(\mathcal{L}, Y)$. For $\varepsilon > 0$ and $g \in G$, denote $\omega(g, \varepsilon)$ as the modulus of continuity of g, defined as*

$$\omega_g = \omega(g, \varepsilon) = \sup\{|g(\tau_1) - g(\tau_2)| : \ \tau_1, \tau_2 \in \mathcal{L}, |\tau_1 - \tau_2| \le \varepsilon\}. \tag{3}$$

Furthermore let $\omega(G, \varepsilon)$ and $\omega_0(G)$ be defined by

$$\omega(G, \varepsilon) = \sup\{\omega(g, \varepsilon) : g \in G\} \tag{4}$$

$$\omega_0(G) = \lim_{\varepsilon\longrightarrow 0} \omega(G, \varepsilon). \tag{5}$$

Using Arzela-Ascoli theorem we can show that $\lim_{\varepsilon\longrightarrow 0} \omega_g = 0$, whenever g is uniformly continuous and $\omega_0(G) = 0$, when G is an equicontinuous family.

Definition 2.7. *[21] For a nonempty bounded subset B of Y, the Kuratowski MNC of B is denoted as $\chi(B)$ which is defined as the smallest possible $\varepsilon > 0$ so that B is covered by a finitely many sets of measures $\le \varepsilon$. i.e.,*

$$\chi(B) := \inf\{\varepsilon > 0 : \text{finite number of sets of diameter} \le \varepsilon \text{ completely covers } B\}.$$

Theorem 2.8. *[33, Proposition 11.3] Let I and J be bounded subsets of a Banach space Y and χ be an arbitrary MNC in $\mathcal{M}_Y$, then the properties*

(i) *If $I \subset J$ then $\chi(I) \le \chi(J)$;*
(ii) *$\chi(I) = \chi(\bar{I})$, where $\bar{I}$ denotes the closure of I;*
(iii) *$\chi(I) = 0$ iff I is relatively compact;*
(iv) *$\chi(I \cap J) = \max\{\chi(I), \chi(J)\}$;*
(v) *$\chi(\alpha I) = |\alpha|\chi(I)$ $(\alpha \in \mathbb{R})$;*
(vi) *$\chi(I + J) \le \chi(I) + \chi(J)$;*
(vii) *$\chi(conv\ I) = \chi(I)$, where $conv(I)$ denotes the convex extension of I;*
(viii) *$\chi(I) \le diam(I)$, where $diam(I)$ is the diameter of I*

hold.

The following lemma stated in reference to time scale in [20, Lemma 2.7] is an essential tool for development of MNC in our proposed Banach space $C(\mathcal{L}, Y)$ with the help of any arbitrary MNC in the space Y.

Lemma 2.9. *[20, Lemma 2.7] Let $G \subset C(\mathcal{L}, Y)$ consisting equicontinuous functions in $C(\mathcal{L}, Y)$ and let*

$$G(\tau) := \{g(\tau) \in Y : g \in G\}, \quad \tau \in \mathcal{L}$$
$$G(\mathcal{L}) := \bigcup_{\tau \in \mathcal{L}} G(t).$$

Then $\chi_C(G) = \sup_{\tau \in \mathcal{L}} \chi(G(t))$, where $\chi_c(G)$ is the MNC in $C(\mathcal{L}, Y)$ and the function $t \longrightarrow \chi(G(\tau))$ is continuous.

Theorem 2.10. *[10, MVT] For an rd-continuous function $f : \mathcal{L} \longrightarrow Y$, then*

$$\int_{\mathcal{I}} f(\tau)\,\Delta(\tau) \in \mu_\Delta(\mathcal{I})\,\overline{conv} f(\mathcal{I}),$$

for any $\mathcal{I} \subset \mathcal{L}$ with its Lebesgue delta measure $\mu_\Delta(\mathcal{I})$.

3. Auxiliary results

This section will be comprised of few results for Meir-Keeler condensing operators associated with a L-function. We will consider χ as the MNC.

Definition 3.1. *[1, Definition 2.1] An operator T on a nonempty bounded subset M of Y is said to be a Meir-Keeler condensing operator if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\chi(T(Q)) < \varepsilon$, whenever $\varepsilon \leq \chi(Q) < \varepsilon + \delta$.*

Theorem 3.2. *[1, Theorem 2.2] Let M be a nonempty, bounded, closed and convex subset of Y and $F : M \longrightarrow M$ is a continuous and Meir-Keeler operator, then F has at least one fixed point and the set of all fixed points of F in M is compact.*

Definition 3.3. *[22, Definition 2] A function $\psi : [0, \infty) \longrightarrow [0, \infty)$ satisfying*

i. $\psi(0) = 0,\ \psi(w) > 0, \forall\, w \in (0, \infty)$

ii. for every $z \in (0, \infty)$, $\exists\, \delta > 0$ such that $\psi(s) \leq z, \forall\, s \in [z, z + \delta]$

and is called a L-function.

Theorem 3.4. *[1, Theorem 2.6] A continuous operator T on a nonempty bounded subset M of Y is a Meir-Keeler condensing operator if and only if there exists an L-function ψ for which $\chi(T(Q)) < \psi(\chi(Q))$ for every subset Q of M with $\chi(Q) \neq 0$.*

As a consequence of above two theorems, we can establish the following theorem, which will be our base theorem to achieve our desired result.

Theorem 3.5. *Let G be a continuous operator on a nonempty bounded, closed and convex subset M of Y. Then F has fixed point if and only if there exists a L-function ψ such that $\chi(G(Q)) < \psi(\chi(Q))$ for all $Q \in \mathcal{M}_M$ with $\chi(Q) \neq 0$.*

4. Main results

The following Lemma [9, Lemma 3.1] is important to state here which converts the given differential equation into an integral equation.

Lemma 4.1. *Let $y_0 \in Y$ and $l \in \mathcal{R}(\mathcal{L}, Y)$. Assume that $f \in C_{rd}(\mathcal{L} \times Y, Y)$. Then y is a solution of (1) if and only if y satisfies the integral equation*

$$y(\tau) = e_l(\tau, \tau_0) + \int_{\tau_0}^{t} e_l(\tau, \sigma(u))\, f(u, y(u))\,\Delta u. \tag{6}$$

Proof. Let y be a solution of (1). We have from time scale calculus,

$$\mu(\tau)\, y^{\Delta}(\tau) = y(\sigma(\tau)) - y(\tau).$$

So,

$$y(\tau) = y(\sigma(\tau)) - \mu(\tau)\, y^{\Delta}(\tau).$$

Now from (1), we have

$$\begin{aligned} y^{\Delta}(\tau) &= l(\tau)y(\tau) + f(\tau, y) \\ &= l(\tau)[y(\sigma(\tau)) - \mu(\tau)y^{\Delta}(\tau)] + f(\tau, y) \\ &= l(\tau)y(\sigma(\tau)) - l(\tau)\,\mu(\tau)\, y^{\Delta}(\tau) + f(\tau, y) \end{aligned} \tag{7}$$

$$\begin{aligned} i.e.,\ [1 + l(\tau)\,\mu(\tau)]y^{\Delta}(\tau) &= l(\tau)\, y(\sigma(\tau)) + f(\tau, y) \\ \Rightarrow y^{\Delta}(\tau) &= \frac{l(\tau)}{1+\mu(\tau)\, l(\tau)} y(\sigma(\tau)) + \frac{f(\tau, y)}{1+\mu(\tau)\, l(\tau)} \\ \Rightarrow y^{\Delta}(\tau) &= -\frac{-l(\tau)}{1+\mu(\tau)\, l(\tau)} y(\sigma(\tau)) + \frac{f(\tau, y)}{1+\mu(\tau)\, l(\tau)} \\ \Rightarrow y^{\Delta}(\tau) &= -\ominus l\; y(\sigma(\tau)) + \frac{f(\tau, y)}{1+\mu(\tau)\, l(\tau)}. \end{aligned} \tag{8}$$

Comparing (8) with

$$\begin{aligned} & y^{\Delta}(\tau) = -l(\tau)y(\sigma(\tau)) + f(\tau, y), \quad \tau \in \mathcal{L}^k \\ & y(\tau_0) = y_0 \end{aligned} \tag{9}$$

the only solution is

$$y(\tau) = e_{\ominus l}(\tau, \tau_0) + \int_{\tau_0}^{t} e_{\ominus l}(\tau, s)\, f(s, y(s))\, \Delta s. \tag{10}$$

So (8) also has a unique solution y, given by

$$\begin{aligned} y(\tau) &= e_{\ominus\ominus l}(\tau, \tau_0)\, y_0 + \int_{\tau_0}^{\tau} e_{\ominus\ominus l}(\tau, s)\, \frac{f(u, y(u))}{1+\mu(u)l(s)} \Delta u \\ &= e_l(\tau, \tau_0)\, y_0 + \int_{\tau_0}^{t} e_l(\tau, u)\, \frac{f(u, y(u))}{1+\mu(u)l(u)} \Delta u \\ &= e_l(\tau, \tau_0)\, y_0 + \int_{\tau_0}^{\tau} \frac{f(u, y(u))}{e_l(u, \tau)(1+\mu(u)l(u))} \Delta u \\ &= e_u(\tau, \tau_0)\, y_0 + \int_{\tau_0}^{\tau} \frac{f(u, y(u))}{e_l(\sigma(u), \tau)} \Delta u \\ &= e_l(\tau, \tau_0)\, y_0 + \int_{\tau_0}^{t} e_u(\tau, \sigma(u))\, f(u, y(u)) \Delta u. \end{aligned}$$

This proofs the lemma. □

Now we establish our main theorem.

Theorem 4.2. *Let $f : \mathcal{L} \times Y \longrightarrow Y$ be a rd-continuous function. Assume that*

(i) there exist $N, M \in [0, \infty)$ for which $|f(u, y(u))| \leq M + N\|y\|_Y$,

(ii) there is a continuous L-function ψ, for which $\chi\Big(e_q(\mathcal{I}, \sigma(\mathcal{I}))\, u(\mathcal{I}, Q)\Big) \leq \frac{1}{T}\psi(\chi(Q(\tau)))$, for each and every nonempty, bounded subset Q of Y, where $\mathcal{I} = [\tau_0, \tau]_{\mathcal{L}}$.

Then the IVP (1) has a solution, whenever $EN(T - \tau_0) < 1$.

Proof. We take $r > 0$ and the ball B_r such that

$$\frac{E\|y_0\|_Y + EMT}{(1 - EN(T - \tau_0))} \leq r \text{ and } B_r := \{y \in C(\mathcal{L}, Y) : \|y\| \leq r\}.$$

Clearly, the ball $B_r \subset C(\mathcal{L}, Y)$ is closed and bounded. Now we examine B_r to be an equicontinuous family of functions.
Consider $\tau_1, \tau_2 \in \mathcal{L}$ with $\tau_2 \geq \tau_1$. Then from (6)

$$\begin{aligned}
&\|y(\tau_2) - y(\tau_1)\|_Y \\
&= \Bigg\| e_l(\tau_2, \tau_0)\, y_0 + \int_{\tau_0}^{\tau_2} e_l(\tau_2, \sigma(u))\, f(u, y(u))\Delta u - [e_l(\tau_1, \tau_0)\, y_0 + \int_{\tau_0}^{\tau_1} e_l(\tau_1, \sigma(u))\, f(u, y(u))\Delta u] \Bigg\|_Y \\
&\leq \|e_l(\tau_2, \tau_0)\, y_0 - e_l(\tau_1, \tau_0)\, y_0\|_Y + \Bigg\| \int_{\tau_0}^{\tau_2} e_l(\tau_2, \sigma(u))\, f(u, y(u))\Delta u - \int_{\tau_0}^{\tau_1} e_l(\tau_1, \sigma(u))\, f(u, y(u))\Delta u \Bigg\|_Y \\
&\leq |e_l(\tau_2, \tau_0) - e_l(\tau_1, \tau_0)|\, \|y_0\|_Y + \Bigg\| e_l(\tau_2, \tau_0) \int_{\tau_0}^{\tau_1} e_l(\tau_0, \sigma(u))\, f(u, y(u))\Delta u \\
&+ e_l(\tau_2, \tau_0) \int_{\tau_1}^{\tau_2} e_l(\tau_0, \sigma(u))\, f(u, y(u))\Delta u - e_l(\tau_1, \tau_0) \int_{\tau_0}^{\tau_1} e_l(\tau_0, \sigma(u))\, f(u, y(u))\Delta u \Bigg\|_Y \\
&\leq |e_l(\tau_2, \tau_0) - e_l(\tau_1, \tau_0)|\, \|y_0\|_Y + |e_l(\tau_2, \tau_0) - e_l(\tau_1, \tau_0)| \int_{\tau_0}^{\tau_1} |e_l(\tau_0, \sigma(u))|\, \|f(u, y(u))\|_Y \Delta u \\
&+ |e_l(\tau_2, \tau_0)| \int_{\tau_1}^{\tau_2} |e_l(\tau_0, \sigma(u))|\, \|f(u, y(u))\|_Y \Delta u.
\end{aligned} \tag{11}$$

A similar result will occur when we take $\tau_2 \leq \tau_1$. As $e_q(\cdot, \tau_0)$ is continuous, we can clearly see that right hand side of (11) tends to zero as t_2 approaches τ_1. Thus equicontinuity of the set B_r is established. Now consider the mapping $G : B_r \longrightarrow C(\mathcal{L}, Y)$ defined by

$$G(y)(\tau) := e_l(\tau, \tau_0) + \int_{\tau_0}^{t} e_l(\tau, \sigma(u))\, f(u, y(u))\, \Delta u. \tag{12}$$

We claim that G is invariant of B_r and is also continuous.

Now for $y \in B_r$, we have

$$\begin{aligned}
\|G(y)(\tau)\|_Y &= \|e_l(\tau,\tau_0)\, y_0 + \int_{\tau_0}^{\tau} e_l(\tau,\sigma(u))\, f(u,y(u))\Delta u\|_Y \\
&\le \|e_l(\tau,\tau_0)\, y_0\|_Y + \left\| \int_{\tau_0}^{t} e_l(\tau,\sigma(u))\, f(u,y(u))\Delta u \right\|_Y \\
&\le |e_l(\tau,\tau_0)|\, \|y_0\|_Y + \int_{\tau_0}^{\tau} |e_l(\tau,\sigma(u))|\, \|f(u,y(u))\|_Y \Delta u \\
&\le E\|Y_0\|_Y + E\int_{\tau_0}^{\tau} (M + N\|y\|_Y)\Delta u \\
&\le E\|y_0\|_Y + E(M+Nr)(T-\tau_0) \\
&\le r
\end{aligned}$$

which shows that G is bounded by r, i.e., $G : B_r \longrightarrow B_r$.

Next to show G is continuous in B_r. For this let $\varepsilon > 0$ be arbitrary such that $\|x - y\| \le \varepsilon$ for $y, z \in B_{r_0}$. We have

$$\begin{aligned}
\|G(y)(\tau) - G(z)(\tau)\| &= \|e_l(\tau,\tau_0)\, y_0 + \int_{\tau_0}^{\tau} e_l(\tau,\sigma(u))\, f(u,y(u))\Delta u \\
&\quad - [e_l(\tau,\tau_0)\, z_0 + \int_{\tau_0}^{\tau} e_l(\tau,\sigma(u))\, f(u,z(u))\Delta u]\| \\
&\le \int_{\tau_0}^{t} |e_l(\tau,\sigma(u))|\, \|f(u,y(u)) - f(u,z(u))\|_Y \Delta u \\
&\le E\int_{\tau_0}^{\tau} \|f(u,y(u)) - f(u,z(u))\|_Y \Delta u \\
&\le E(T-\tau_0)\, \omega_{f_2}(\mathcal{L},\varepsilon), \qquad (13)
\end{aligned}$$

where $\omega_{f_2}(\mathcal{L},\varepsilon) := \{\|f(u,y(u)) - f(u,z(u))\| : u \in \mathcal{L}, y, z \in B_r \text{with } \|y - z\| \le \varepsilon\}$.

Since $f : \mathcal{L} \times B_r \longrightarrow Y$ is rd-continuous, so $f(u.\cdot)$ is continuous on B_r for each $u \in \mathcal{L}$. Also B_r being closed and bounded subset of $C(\mathcal{L}, Y)$, $f(u,\cdot)$ is equicontinuous on B_r. Thus by equicontinuity of $f(u,\cdot)$ and Arezela-Ascoli theorem ensures that $\omega_{f_2} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and hence the RHS of (13) tends to zero as $\varepsilon \longrightarrow 0$. This shows that G is continuous on B_r.

Now for any $\tau \in \mathcal{L}$ and any nonempty subset $Q \subset B_r$, we have

$$\begin{aligned}
\chi(G(Q)(\tau)) &= \chi\left(e_l(\tau,\tau_0)\, y_0 + \int_{\tau_0}^{\tau} e_l(\tau,\sigma(u))\, f(u,y(u))\Delta u\ ;\ y \in Q \right) \\
&\le \chi(e_l(\tau,\tau_0)\, y_0) + \chi\left(\int_{\tau_0}^{\tau} e_l(t,\sigma(u))\, f(u,Q(u))\Delta u \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \chi\left(\int_{\tau_0}^{\tau} e_l(\tau,\sigma(u)) f(u, Q(u))\Delta u\right)\\
&\leq \chi\left(\mu_\Delta(\mathcal{I})\overline{conv}\left(e_l(\tau,\sigma(\mathcal{I})) f(\mathcal{I}, Q(\mathcal{I}))\right)\right) \;;\; \mathcal{I} = [\tau_0, \tau]\\
&\leq (T-\tau_0)\,\chi(\overline{conv}\left(e_l(\tau,\sigma(\mathcal{I}))\, u(\mathcal{I}, Q(\mathcal{I}))\right))\\
&\leq (T-\tau_0)\,\chi(e_l(\tau,\sigma(\mathcal{I})) f(\mathcal{I}, Q(\mathcal{I})))\\
&\leq (T-\tau_0)\,\frac{1}{T}\psi(\chi(Q(\tau)))\,, \tau \in \mathcal{L}\\
&\leq \frac{(T-\tau_0)}{T}\psi\left(\sup_{\tau\in\mathcal{L}}\chi(Q(\tau))\right)\\
&\leq \frac{(T-\tau_0)}{T}\psi(\chi_c(Q(\tau))).
\end{aligned}$$

So for any $\tau \in \mathcal{L}$ we get that

$$\chi(G(Q)(\tau)) \leq \frac{(T-\tau_0)}{T}\psi(\chi_c(Q(\tau))).$$

So,

$$\begin{aligned}
\sup_{\tau\in\mathcal{L}}\{\chi(G(Q)(\tau))\} &\leq \frac{(T-\tau_0)}{T}\psi(\chi_c(Q(\tau)))\\
\Rightarrow \chi_c(G(Q)) &\leq \frac{(T-\tau_0)}{T}\psi(\chi_c(D(\tau)))\\
\Rightarrow \chi_c(G(Q)) &< \psi(\chi_c(Q(\tau))), \text{ as } \left(\frac{T-\tau_0}{T}\right) < 1.
\end{aligned}$$

Thus G has a fixed point, given by Theorem 3.5.

5. Examples

Here we validate our result with an example. We take $\mathbb{R}$ as our Banach space Y to simplify our results.

Example 5.1. *Let* $\mathbb{T} := [0.5, 1] \cup [1.5, 2]$ *be the time scale and consider the local IVP*

$$\begin{aligned}
&y^\Delta(\tau) = l(\tau)\,y(\tau) + f(\tau, y)\,,\ \tau \in \mathbb{T}^k := [0.5, 2]^k_{\mathbb{T}}\\
&y(\tau_0) = A_0,
\end{aligned} \tag{14}$$

where $f(\tau, y) = \cos\left(\frac{1}{3+\sin^2\tau}\right) + 2y,\ l(\tau) = -1,\ A_0 \in \mathbb{R}.$
We see that

$$\begin{aligned}
|f(\tau, y)| &\leq \left|\cos\left(\frac{1}{3+\sin^2\tau}\right)\right| + |2y|\\
&\leq 1 + 2|y|. \ \textit{i.e., assumption (i) of Theorem 4.2 holds good.}
\end{aligned}$$

We take any continuous L-function ψ, *then* $\frac{1}{T}\psi(\chi(Q(\tau))) > 0$ *for each and every* $t \in \mathcal{L}$.
Now we take $Q = [0.5, 1]$, *then*

$$\begin{aligned}
e_l(\tau,\sigma(u)) f(\tau, y) &< 5\, e_l(\tau,\sigma(u))\\
&= \frac{5}{e_l(\sigma(u), \tau)}
\end{aligned}$$

$$= \frac{5}{[1+\mu(u)\,l(u)]\,e_l(u,\tau)}$$
$$= \frac{5\,e_{\ominus l}(u,\tau)}{[1+\mu(u)\,l(u)]}$$
$$= \frac{5\,e_{\ominus(-1)}(u,\tau)}{[1+\mu(u)\,l(u)]}$$

for $t \in \mathcal{L}, u \in \mathcal{I}$ *and* $y \in Q$ *as in that case* $f(u,y) < 5$. *But* $\chi\left(e_{\ominus(-1)}(s,\tau)\right) = 0$ *and hence* $\chi\left(e_{\ominus(-1)}(\mathcal{L},\sigma(\mathcal{I}))\,f(\mathcal{I}\times D)\right) = 0$. *So assumption (ii) of Theorem 4.2 is also satisfied, hence Theorem 4.2 ensures a solution for the problem (14).*

6. Conclusion

In this manuscript we focused on finding a continuous solution of the dynamic equation (1), but taking few restrictions on the functions involved as well as on the domain. For examples we took the function l to be both regressive and rd-continuous. But there occurs several occasions where the function l does not behave to be a regressive one. Also the function f instead on depending on the current situations sometimes becomes vulnerable to some previous state. In that case we need to deal those problem taking delay factor into consideration. In future we keep an scope of analyzing such dynamic equation taking those above mentioned situations under consideration and likely using the concept of Meir-Keeler condensing operator.

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