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Some characterizations of weakly parallel sum

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Abstract. For bounded linear operators A and B acting in a Hilbert space, the basic properties of weakly parallel sum $A : B$ are developed. Many explicit expressions of weakly parallel sum are given and some equations and inequalities involving the parallel sum and the weakly parallel sum are obtained.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For an operator A, denote by $\mathcal{R}(A)$ and $\mathcal{N}(A)$ the range and the null space of A, respectively. The closure of $\mathcal{R}(A)$ is denoted by $\overline{\mathcal{R}(A)}$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For a positive operator A, the unique positive square root is denoted by $A^{\frac{1}{2}}$. Denote by $|A| = (A^*A)^{\frac{1}{2}}$. Let $\mathcal{B}^+(\mathcal{H})$ be the set of all positive operators and $P_{\mathcal{K}}$ be the orthogonal projection on the closed subspace $\mathcal{K} \subseteq \mathcal{H}$.

Some authors, in particular Anderson and Trapp [1]-[3], Ando [6], Arias, Corach and Maestripieri [8], Fillmore and Williams [13], as well as Mitra and Puri [20] studied the parallel sum within the class of nonnegative definite matrices or bounded nonnegative hermitian operators. Xu et al. [14, 18, 19] obtained the perturbation estimation of the parallel sum and extended some properties of parallel sums to adjoint operators on Hilbert C^* modules. The basic properties of the parallel sum for bounded linear operators in Hilbert spaces were developed recently. Some extensions of the parallel sum and many different equivalent definitions and the properties were also studied in [3, 9, 11, 13, 19].

Recently, F. Hansen obtains a number of results for the parallel sum of positive definite operators in [15]. The purpose of this paper is to generalise some of them to parallel summable operators and weakly parallel summable operators. The notion of weakly parallel sum of two operators was introduced by Antezana, Corach and Stojanoff in [5]. We characterize the weakly parallel sum in terms of the reduced solutions to particular operator equations and obtain new insight into the theory of weakly parallel sum. We obtain many results concerning parallel sum and weakly parallel sum. Our goal however is to prove as many results as possible under no additional hypothesis on the operators involved.

2. Some lemmas

In this section, we begin with some lemmas which play important roles in the sequel. The following lemma is a standard result.

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Lemma 2.1. ([12, 17], [13, Theorem 2.2]) *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{R}(A)+\mathcal{R}(B) = \mathcal{R}\Big((AA^*+BB^*)^{\frac{1}{2}}\Big)$. $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A) = \mathcal{R}(AA^*)$ if and only if $\mathcal{R}(A^*)$ is closed. If $A \geq 0$, then $\overline{\mathcal{R}(A^{\frac{1}{2}})} = \overline{\mathcal{R}(A)}$.*

Lemma 2.2. *For $A \in \mathcal{B}(\mathcal{H})$ with closed range, let $A = UP$ be the polar decomposition of A, where U is a unique partial isometry such that $\mathcal{N}(U) = \mathcal{N}(A)$ and $P = (A^*A)^{\frac{1}{2}}$. Let $P^\dagger$ be the Moore-Penrose inverse of P. Then $A^\dagger = P^\dagger U^*$ satisfying $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$,*

$$AA^\dagger = P_{\mathcal{R}(A)} = P_{\mathcal{N}(A^*)^\perp}, \quad A^\dagger A = P^\dagger P = PP^\dagger = U^*U = P_{\mathcal{R}(A^*)} = P_{\mathcal{N}(A)^\perp}.$$

By Lemma 2.1, $\mathcal{R}(A)$ is closed $\Longleftrightarrow \mathcal{R}(P)$ is closed $\Longleftrightarrow 0$ is not an accumulation point of $\sigma(P)$. Note that $P^\dagger$ and $A^\dagger \in \mathcal{B}(\mathcal{H})$ if $\mathcal{R}(A)$ is closed, where $A^\dagger \in \mathcal{B}(\mathcal{H})$ is called the Moore-Penrose inverse of A. It is well known that the Moore-Prose inverse $A^\dagger \in \mathcal{B}(\mathcal{H})$ if and only if $\mathcal{R}(A)$ is closed and the Moore-Penrose inverse of A is unique (see [16]). In general, $A^\dagger$ is a closed densely defined operator if $\mathcal{R}(A)$ is not closed [10, 21]. Let C be a positive operator. The case of a non-invertible term can also be handled by taking the limit of a sequence of invertible approximations. From spectral theory and the monotone convergence theorem, the sequence $\{(C+\frac{1}{n}I)^{-1}C\}_{n=1}^{\infty}$ converges in the strong operator topology (SOT) monotonically up to $C^\dagger C$, i.e., $(C+\frac{1}{n}I)^{-1}C \longrightarrow C^\dagger C$. If $A, B \in \mathcal{B}(\mathcal{H})$ with $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then $A^\dagger B \in \mathcal{B}(\mathcal{H})$ even if A has a non-closed range [7, Lemma 2.1]. We include the proof of this fact here for completeness.

Lemma 2.3. [7, Lemma 2.1] *Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Then $A^\dagger B$ is bounded.*

Proof. If $(x_n, A^\dagger Bx_n) \longrightarrow (x, y)$ for $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ and $x, y \in \mathcal{H}$, then $x_n \longrightarrow x$, $A^\dagger Bx_n \longrightarrow y$. We get $Bx_n \longrightarrow Bx$ and$y \in \overline{\mathcal{R}(A^\dagger B)}$. Since $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, one has $Bx_n = AA^\dagger Bx_n \longrightarrow Ay$. We get $Bx = Ay$ and $A^\dagger Bx = A^\dagger Ay = y$. Hence, $A^\dagger Bx_n \longrightarrow A^\dagger Bx$. By the closed graph theorem, we know $A^\dagger B \in \mathcal{B}(\mathcal{H})$. □

The following well-known criteria concerning the range inclusions and factorization of operators are given by Douglas [12] and Fillmore-Williams [13].

Lemma 2.4. [12, 13] *If $A, B \in \mathcal{B}(\mathcal{H})$, then the following results are equivalent:*

(i) *$A = BC$ for some operator $C \in \mathcal{B}(\mathcal{H})$.*

(ii) *$AA^* \leq kBB^*$ for some $k > 0$.*

(iii) *$\mathcal{R}(A) \subseteq \mathcal{R}(B)$.*

If one of these conditions holds then there exists a unique solution $C_0 \in \mathcal{B}(\mathcal{H})$ of the equation $BX = A$ such that $R(C_0) \subset \overline{\mathcal{R}(B^*)}$ and $\mathcal{N}(C_0) = \mathcal{N}(A)$. This solution is called the Douglas reduced solution. Moreover, $\|C_0\|^2 = \inf\{\lambda > 0 : AA^* \leq \lambda BB^*\}$. By Lemma 2.3, Douglas reduced solution is $C_0 = B^\dagger A \in \mathcal{B}(\mathcal{H})$.

It is worth pointing out that, if $0 \leq A \leq B$, then $\mathcal{R}(A^{\frac{1}{2}}) \subseteq \mathcal{R}(B^{\frac{1}{2}})$ and $\|A\| \leq \|B\|$. Moreover, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ if $\mathcal{R}(B)$ is closed. As we known, $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are parallel summable (p.s) if $\mathcal{R}(A) \subseteq \mathcal{R}(A+B)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(A^*+B^*)$. These conditions imply that $\mathcal{R}(B) \subseteq \mathcal{R}(A+B)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*+B^*)$. We recall the definition of the weakly parallel sum of operators introduced in [5].

Definition 2.5. *Operators $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ are said to be weakly parallel summable (w.p.s) if the following range inclusion relations hold:*

$$\mathcal{R}(A) \subseteq \mathcal{R}(|A^*+B^*|^{\frac{1}{2}}), \quad \mathcal{R}(B) \subseteq \mathcal{R}(|A^*+B^*|^{\frac{1}{2}}), \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(|A+B|^{\frac{1}{2}}), \quad \mathcal{R}(B^*) \subseteq \mathcal{R}(|A+B|^{\frac{1}{2}}). \tag{1}$$

In this case, the reduced solutions of operator equations

$$A = |A^*+B^*|^{\frac{1}{2}} UX, \quad B = |A^*+B^*|^{\frac{1}{2}} UX, \quad A^* = |A+B|^{\frac{1}{2}} X, \quad B^* = |A+B|^{\frac{1}{2}} X \tag{2}$$

are denoted by E_A, E_B, F_A and F_B, respectively, where U is the partial isometry of the polar decomposition of $A+B$ and $\mathcal{N}(U) = \mathcal{N}(A+B)$.

Definition 2.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be w.p.s and let E_A, F_A be the reduced solutions of the operator equations*

$$A = | A^* + B^* |^{\frac{1}{2}} UX, \quad A^* = | A + B |^{\frac{1}{2}} X,$$

respectively, where U is the partial isometry of the polar decomposition of $A + B = U|A + B|$ and $\mathcal{N}(U) = \mathcal{N}(A + B)$. The weakly parallel sum of operators A and B, denoted by $A : B \in \mathcal{B}(\mathcal{H})$, is

$$A : B = A - F_A^* E_A. \tag{3}$$

We recall the famous polar decomposition theorem [10, 21]. For every operators $A, B \in \mathcal{B}(\mathcal{H})$, the polar decomposition $A + B = U|A + B| = |A^* + B^*|U$, where U is a partial isometry such that $U^*U = P_{\overline{\mathcal{R}(A^*+B^*)}}$ and $UU^* = P_{\overline{\mathcal{R}(A+B)}}$. By Lemma 2.2, if $\mathcal{R}(A + B)$ is closed, $(A + B)^\dagger = |A + B|^\dagger U^* = U^*|A^* + B^*|^\dagger$. Moreover, $U|A + B|^{\frac{1}{2}} = |A^* + B^*|^{\frac{1}{2}} U$ and $U|A + B|^{\frac{\dagger}{2}} = |A^* + B^*|^{\frac{\dagger}{2}} U$, where $|T|^{\frac{\dagger}{2}} = (|T|^{\frac{1}{2}})^\dagger$.

Lemma 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be w.p.s. The reduced solutions E_A, E_B, F_A and F_B of the equations in (2) can be denoted by*

$$E_A = U^* \mid A^* + B^* \mid^{\frac{\dagger}{2}} A, \quad F_A = \mid A + B \mid^{\frac{\dagger}{2}} A^*, \quad E_B = U^* \mid A^* + B^* \mid^{\frac{\dagger}{2}} B, \quad F_B = \mid A + B \mid^{\frac{\dagger}{2}} B^*, \tag{4}$$

respectively.

3. Further properties of the p.s and w.p.s operators

In this section, we review the relevant materials and present our characterization theorems. The weakly parallel sum and the reduced solutions satisfy several useful properties which are given in the following theorem. The partly results and their proofs can be found in [5, 11].

Theorem 3.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be w.p.s and let E_A, E_B, F_A and F_B be the reduced solutions of the equations in (2), respectively. Then*

(i) *$E_A + E_B$ and $F_A + F_B$ are the reduced solution of the equation $A + B = |A^* + B^*|^{\frac{1}{2}} UX$ and $A^* + B^* = |A + B|^{\frac{1}{2}} X$, respectively. Moreover,*

$$E_A + E_B = |A + B|^{\frac{1}{2}} \quad \text{and} \quad F_A + F_B = |A + B|^{\frac{1}{2}} U^*. \tag{5}$$

(ii) *λA and λB are w.p.s, $A : B = B : A = F_B^* E_A = F_A^* E_B$ and $\lambda A : \lambda B = \lambda(A : B) = \lambda B : \lambda A$ for all $\lambda \in \mathbb{C}$. If $\mathcal{R}(A + B)$ is closed, then*

$$\begin{aligned} A : B &= A - F_A^* E_A = B - F_B^* E_B = F_A^* E_B = F_B^* E_A = A - A(A + B)^\dagger A = B - B(A + B)^\dagger B \\ &= A(A + B)^\dagger B = B(A + B)^\dagger A = B : A. \end{aligned} \tag{6}$$

(iii) *A^* and B^* are w.p.s and $(A : B)^* = A^* : B^* = B^* : A^*$.*

(iv) *$\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \mathcal{R}(A : B) \subseteq \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. If $\mathcal{R}(A + B)$ is closed, then*

$$\mathcal{R}(A : B) = \mathcal{R}(A) \cap \mathcal{R}(B) \quad \text{and} \quad \mathcal{N}(A : B) = \left[\mathcal{R}(A^*) \cap \mathcal{R}(B^*)\right]^\perp.$$

(v) *$\mathcal{N}(A : B) = \overline{\mathcal{N}(A) + \mathcal{N}(B)}$ and $A(\mathcal{N}(A : B)) \subseteq \overline{\mathcal{R}(A + B)}$ when $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed.*

Proof. (i) By Lemma 2.7,

$$E_A + E_B = U^* \mid A^* + B^* \mid^{\frac{\dagger}{2}} (A + B) = U^* \mid A^* + B^* \mid^{\frac{1}{2}} U = |A + B|^{\frac{1}{2}}$$

and

$$F_A + F_B = |A + B|^{\frac{\dagger}{2}} (A^* + B^*) = |A + B|^{\frac{\dagger}{2}} |A + B| U^* = |A + B|^{\frac{1}{2}} U^*.$$

Since $A+B=|A^*+B^*|^{\frac{1}{2}}U(E_A+E_B)$ and

$$\mathcal{R}(E_A+E_B)=\mathcal{R}(|A+B|^{\frac{1}{2}})=\mathcal{R}(|A+B|^{\frac{1}{2}}U^*)\subseteq N(|A^*+B^*|^{\frac{1}{2}}U)^{\perp}. \tag{7}$$

By Lemma 2.4, E_A+E_B is the unique reduced solution of the equation $A+B=|A^*+B^*|^{\frac{1}{2}}UX$. Similarly, F_A+F_B is the unique reduced solution of the equation $A^*+B^*=|A+B|^{\frac{1}{2}}X$.

(ii) By Lemma 2.7 and item (i), $F_B^*E_A=(|A^*+B^*|^{\frac{1}{2}}U-F_A^*)E_A=A-F_A^*E_A=F_A^*(|A+B|^{\frac{1}{2}}-E_A)=F_A^*E_B$. These lead to the following two properties of weakly parallel sum:

$$A:B=B:A=F_B^*E_A=F_A^*E_B \quad \text{and} \quad \lambda A:\lambda B=\lambda(A:B)=\lambda B:\lambda A.$$

If $\mathcal{R}(A+B)$ is closed, then

$$F_A^*E_B=A|A+B|^{\frac{\dagger}{2}}|A+B|^{\frac{\dagger}{2}}U^*B=A|A+B|^{\dagger}U^*B=A(A+B)^{\dagger}B,$$

$$F_B^*E_A=B|A+B|^{\frac{\dagger}{2}}|A+B|^{\frac{\dagger}{2}}U^*A=B|A+B|^{\dagger}U^*A=B(A+B)^{\dagger}A,$$

$$F_A^*E_A=A(A+B)^{\dagger}A \quad \text{and} \quad F_B^*E_B=B(A+B)^{\dagger}B.$$

By the definition of weakly parallel sum in (3),

$$A:B=A-F_A^*E_A=A-A(A+B)^{\dagger}A=A-\Big[(A+B)-B\Big](A+B)^{\dagger}A=B(A+B)^{\dagger}A=F_B^*E_A.$$

Similarly, $B:A=B-F_B^*E_B=B-B(A+B)^{\dagger}B=A(A+B)^{\dagger}B=F_A^*E_B$.

(iii) By (ii) the result is obvious.

(iv) For every $z\in\mathcal{R}(A)\cap\mathcal{R}(B)$, there exist x,y such that $z=Ax=By$, i.e., $z=|A^*+B^*|^{\frac{1}{2}}UE_Ax=|A^*+B^*|^{\frac{1}{2}}UE_By$. The relation $|A^*+B^*|^{\frac{1}{2}}U(E_Ax-E_By)=0$ implies that $E_Ax=E_By$ since E_A and E_B are the reduced solutions of the equations in (2), respectively. By items (i) and (ii),

$$(A:B)(x+y) \;=\; F_B^*E_Ax+F_A^*E_By=F_B^*E_Ax+F_A^*E_Ax=(F_A^*+F_B^*)E_Ax=|A^*+B^*|^{\frac{1}{2}}UE_Ax=z, \tag{8}$$

i.e., $\mathcal{R}(A)\cap\mathcal{R}(B)\subseteq\mathcal{R}(A:B)$.

Note that $A=F_A^*|A+B|^{\frac{1}{2}}$, $B=|A^*+B^*|^{\frac{1}{2}}UE_B$ and $\mathcal{R}(E_B)\subseteq\mathcal{N}(|A^*+B^*|^{\frac{1}{2}}U)^{\perp}\subseteq\overline{\mathcal{R}(|A+B|^{\frac{1}{2}})}$. For any $x\in\mathcal{H}$, there exists a sequence $\{y_n\}\in\mathcal{H}$ such that $|A+B|^{\frac{1}{2}}y_n\to E_Bx$ and $Ay_n=F_A^*|A+B|^{\frac{1}{2}}y_n\to F_A^*E_Bx=(A:B)x$. Hence, $\mathcal{R}(A:B)\subseteq\overline{\mathcal{R}(A)}$. Similarly, one has $\mathcal{R}(A:B)\subseteq\overline{\mathcal{R}(B)}$ and $\mathcal{R}(A:B)\subseteq\overline{\mathcal{R}(A)}\cap\overline{\mathcal{R}(B)}$.

If $\mathcal{R}(A+B)$ is closed, by item (ii) and Lemma 2.4 we have, $\mathcal{R}(A:B)=\mathcal{R}\left(A(A+B)^{\dagger}B\right)=\mathcal{R}\left(B(A+B)^{\dagger}A\right)\subseteq\mathcal{R}(A)\cap\mathcal{R}(B)$. Hence, $\mathcal{R}(A:B)=\mathcal{R}(A)\cap\mathcal{R}(B)$. Moreover, by item (iii), $\mathcal{N}(A:B)=\mathcal{R}[(A:B)^*]^{\perp}=\mathcal{R}(A^*:B^*)^{\perp}=\left[\mathcal{R}(A^*)\cap\mathcal{R}(B^*)\right]^{\perp}$.

(v) If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, then $\mathcal{N}(A:B)=\left[\mathcal{R}(A^*)\cap\mathcal{R}(B^*)\right]^{\perp}=\overline{\mathcal{N}(A)+\mathcal{N}(B)}$. Hence, for every $x\in\mathcal{N}(A:B)$, there exist $x_n\in\mathcal{N}(A)$ and $y_n\in\mathcal{N}(B)$ such that $x_n+y_n\to x$ as $n\to\infty$. One has $Ax=A\lim_{n\to\infty}(x_n+y_n)=\lim_{n\to\infty}Ay_n=\lim_{n\to\infty}(A+B)y_n$. Thus $Ax\subseteq\overline{\mathcal{R}(A+B)}$ for every $x\in\mathcal{N}(A:B)$. □

In case that the ranges of A, B and $A+B$ are closed, for every $y\in\mathcal{R}(A)\cap\mathcal{R}(B)$, there exists $x=(A^{\dagger}+B^{\dagger})y\in\mathcal{H}$ such that

$$\begin{aligned}(A:B)x &= F_A^*E_BB^{\dagger}y+F_B^*E_AA^{\dagger}y=F_A^*|A+B|^{\frac{\dagger}{2}}U^*y+F_B^*|A+B|^{\frac{\dagger}{2}}U^*y=(F_A^*+F_B^*)|A+B|^{\frac{\dagger}{2}}U^*y\\ &= |A^*+B^*|^{\frac{1}{2}}UU^*|A^*+B^*|^{\frac{\dagger}{2}}y=P_{\mathcal{R}(A+B)}y=y.\end{aligned} \tag{9}$$

It is worth noting that, if $A,B\in\mathcal{B}(\mathcal{H})$ are P.S, then $\mathcal{R}(A)\subseteq\mathcal{R}(A+B)$ and $\mathcal{R}(B)=\mathcal{R}(A+B-A)\subseteq\mathcal{R}(A+B)+\mathcal{R}(A)=\mathcal{R}(A+B)$. By Lemma 2.3, it has $(A+B)^{\dagger}A$ and $(A+B)^{\dagger}B\in\mathcal{B}(\mathcal{H})$. $A^{\dagger}(A:B)$ and $B^{\dagger}(A:B)\in\mathcal{B}(\mathcal{H})$ if $\mathcal{R}(A+B)$ is closed by Theorem 3.1. Some constructions for the weakly parallel sum are given below.

Theorem 3.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be w.p.s. If the ranges A, B and $A+B$ are closed, then*

$$\begin{aligned}(i)\quad A:B &= (A:B)(A^\dagger + B^\dagger)(A:B).\\ (ii)\quad A:B &= \Big[P_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}(A^\dagger + B^\dagger)P_{\mathcal{R}(A)\cap\mathcal{R}(B)}\Big]^\dagger.\\ (iii)\quad A:B &= \tfrac{1}{2}\Big[F_A^*E_A + F_B^*E_B - (A-B)(A+B)^\dagger(A-B)\Big].\end{aligned}$$

Proof. If $A, B \in \mathcal{B}(\mathcal{H})$ are w.p.s, then

$$\begin{aligned}\mathcal{R}(A) + \mathcal{R}(B) &\subseteq \mathcal{R}(|A^* + B^*|^{\frac{1}{2}}) \subseteq \overline{\mathcal{R}(|A^* + B^*|)} = \overline{\mathcal{R}(A+B)},\\ \mathcal{R}(A^*) + \mathcal{R}(B^*) &\subseteq \mathcal{R}(|A + B|^{\frac{1}{2}}) \subseteq \overline{\mathcal{R}(|A + B|)} = \overline{\mathcal{R}(A^*+B^*)}.\end{aligned} \tag{10}$$

So, $P_{\overline{\mathcal{R}(A+B)}}x = x$, $\forall x \in \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}$. If A, B and $A+B$ are closed, then the following (i) $\sim$ (iii) hold.

(i) For every $z \in \mathcal{H}$, by Theorem 3.1 and (9), $y := (A:B)z \in \mathcal{R}(A) \cap \mathcal{R}(B)$ and $(A:B)(A^\dagger + B^\dagger)y = y$. Hence, $(A:B)(A^\dagger + B^\dagger)(A:B)z = (A:B)z$ for all $z \in \mathcal{H}$, i.e., $(A:B)(A^\dagger + B^\dagger)(A:B) = A:B$.

(ii) Note that $(A:B)^\dagger(A:B) = P_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}$ and $(A:B)(A:B)^\dagger = P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$. By multiplying $(A:B)^\dagger$ from left and right on both sides of item (i), we have $(A:B)^\dagger = P_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}(A^\dagger + B^\dagger)P_{\mathcal{R}(A)\cap\mathcal{R}(B)}$, i.e., $A:B = \Big[P_{\mathcal{R}(A^*)\cap\mathcal{R}(B^*)}(A^\dagger + B^\dagger)P_{\mathcal{R}(A)\cap\mathcal{R}(B)}\Big]^\dagger$.

(iii) By Theorem 3.1, item (ii),

$$\begin{aligned}(A-B)(A+B)^\dagger(A-B) &= A(A+B)^\dagger A + B(A+B)^\dagger B - A(A+B)^\dagger B - B(A+B)^\dagger A\\ &= F_A^*E_A + F_B^*E_B - 2A:B,\end{aligned}$$

i.e., $A:B = \frac{1}{2}\Big[F_A^*E_A + F_B^*E_B - (A-B)(A+B)^\dagger(A-B)\Big]$. This completes the proof. □

Let $A, B \in \mathcal{B}(H)$ be closed range operators. For every $x \in \mathcal{H}$, denote by $y = (A:B)x$, $z = (A^*:B^*)x$. By Theorem 3.2, item (i), it is obvious that $\langle (A:B)x, x\rangle = \langle A^\dagger y, z\rangle + \langle B^\dagger y, z\rangle$. This implies that $A:B \geq 0$ if A, $B \in \mathcal{B}^+(\mathcal{H})$ in Theorem 3.2.

Theorem 3.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be w.p.s and $u, v \in \mathcal{R}(A) \cap \mathcal{R}(B)$ be such that $u = Ax = By$ and $v = Az = Bw$ for some $x, y, z, w \in \mathcal{H}$. If $(x+y) - (z+w) \in \mathcal{N}(A) + \mathcal{N}(B)$, then $u = v$.*

Proof. Since $(x+y)-(z+w) \in \mathcal{N}(A)+\mathcal{N}(B)$, there exist $n_A \in \mathcal{N}(A)$ and $n_B \in \mathcal{N}(B)$ such that $(x+y)-(z+w) = n_A + n_B$. Since $A, B \in \mathcal{B}(\mathcal{H})$ is w.p.s, by (8), the relations

$$u = |A^*+B^*|^{\frac{1}{2}}UE_Ax = |A^*+B^*|^{\frac{1}{2}}UE_By \quad \text{and} \quad v = |A^*+B^*|^{\frac{1}{2}}UE_Az = |A^*+B^*|^{\frac{1}{2}}UE_Bw$$

imply that $u = (A:B)(x+y)$ and $v = (A:B)(z+w)$. In virtue of $A:B = B:A$ and Lemma 2.7, we get $\mathcal{N}(A) \subseteq N(A:B)$ and $\mathcal{N}(B) \subseteq N(A:B)$. Hence,

$$u \;=\; A:B(x+y) = A:B(z+w+n_A+n_B) = A:B(z+n_A) + A:B(w+n_B) = A:B(z+w) = v.$$

□

By Theorem 3.1, it is easy to get that $A+B = U|A+B| = U|A+B|^{\frac{1}{2}}(E_A+E_B) = (F_A^*+F_B^*)|A+B|^{\frac{1}{2}}$. Moreover,

$$\mathcal{R}(A)\cap\mathcal{R}(B)\cap\mathcal{R}(C) \subseteq \mathcal{R}\big(A:(B:C)\big) \subseteq \overline{\mathcal{R}(A)}\cap\overline{\mathcal{R}(B)}\cap\overline{\mathcal{R}(C)}$$

and

$$\mathcal{R}(A)\cap\mathcal{R}(B)\cap\mathcal{R}(C) \subseteq \mathcal{R}\big((A:B):C\big) \subseteq \overline{\mathcal{R}(A)}\cap\overline{\mathcal{R}(B)}\cap\overline{\mathcal{R}(C)}$$

if all operators are w.p.s. Similar to the proof of the parallel sum results in [18, Proposition 4.4], the w.p.s. operators have the following similar property.

Theorem 3.4. *Let A, B, $C \in \mathcal{B}(\mathcal{H})$ be such that all weakly parallel sums $A : B$, $B : C$, $(A : B) : C$ and $A : (B : C)$ exist. If the ranges of $A + B$, $B + C$, $(A : B) + C$ and $A + (B : C)$ are closed. Then $(A : B) : C = A : (B : C)$.*

Proof. Since A and B are w.p.s, one has $\mathcal{R}(A) \subseteq \mathcal{R}(|A^* + B^*|^{\frac{1}{2}}) = \mathcal{R}(A + B)$. Since B and C are w.p.s, one has $\mathcal{R}(C^*) \subseteq \mathcal{R}(|B+C|^{\frac{1}{2}}) = \mathcal{R}(B^*+C^*)$. Since C and $A : B$ are w.p.s, one has $\mathcal{R}(C) \subseteq \mathcal{R}(|(A^* : B^*)+C^*|^{\frac{1}{2}}) = \mathcal{R}((A : B)+C)$. Hence,

$$(A + B)(A + B)^\dagger A = A, \quad C(B + C)^\dagger(B + C) = C \quad \text{and} \quad \Big[(A : B) + C\Big]\Big[(A : B) + C\Big]^\dagger C = C.$$

For every $x \in \mathcal{H}$, denote by

$$\begin{aligned} u &= x - \Big(A + (B : C)\Big)^\dagger Ax, \\ v &= \Big[I - (B + C)^\dagger B\Big]\Big(A + (B : C)\Big)^\dagger Ax, \\ w &= \Big[I - (B + C)^\dagger C\Big]\Big(A + (B : C)\Big)^\dagger Ax. \end{aligned}$$

Then, $u+v+w = x+\Big[I-(B+C)^\dagger(B+C)\Big]\Big(A+(B : C)\Big)^\dagger Ax$. By Theorem 3.1, item (ii), $\Big[A : (B : C)\Big]x = Au = Bv = Cw$ and

$$\begin{aligned} \Big[(A : B) : C\Big](u + v + w) &= \Big[(A : B) : C\Big]x + (A : B)\Big[(A : B) + C\Big]^\dagger C\Big[I - (B + C)^\dagger(B + C)\Big]\Big[A + (B : C)\Big]^\dagger Ax \\ &= \Big[(A : B) : C\Big]x. \end{aligned}$$

Moreover,

$$\begin{aligned} &\Big[(A : B) : C\Big](u + v + w) = C\Big[(A : B) + C\Big]^\dagger (A : B)(u + v) + (A : B)\Big[(A : B) + C\Big]^\dagger Cw \\ &= C\Big[(A : B) + C\Big]^\dagger \Big[B(A + B)^\dagger Au + A(A + B)^\dagger Au\Big] + (A : B)\Big[(A : B) + C\Big]^\dagger Cw \\ &= C\Big[(A : B) + C\Big]^\dagger Cw + (A : B)\Big[(A : B) + C\Big]^\dagger Cw = Cw = \Big[A : (B : C)\Big]x. \end{aligned}$$

Hence, $(A : B) : C = A : (B : C)$. □

The following results extend a formula connecting the weakly parallel sum of operators which need not to be positive, using the concept of certain Douglas reduced solutions.

Theorem 3.5. *Let A, $B \in \mathcal{B}(\mathcal{H})$ be w.p.s, E_B and F_B be the reduced solutions of the equations in (2). For every $C \in \mathcal{B}(\mathcal{H})$, if $\mathcal{R}(A + B)$ is closed, then*

$$C^*AC + (I - C)^*B(I - C) - A : B = \Big[C^* - F_B^*|A + B|^{\frac{\dagger}{2}} U^*\Big](A + B)\Big[C - |A + B|^{\frac{\dagger}{2}} E_B\Big].$$

Moreover, if $(A + B)C = B$ or $C^*(A + B) = B$, then $A : B = C^*AC + (I - C)^*B(I - C)$.

Proof. Since E_B and F_B are the reduced solutions of the equations in (2), it has $B = |A^*+B^*|^{\frac{1}{2}}UE_B = U|A+B|^{\frac{1}{2}}E_B$ and $B^* = |A+B|^{\frac{1}{2}}F_B$. If $\mathcal{R}(A+B)$ is closed,

$$\begin{aligned}
& \left[C^* - F_B^*|A+B|^{\frac{\dagger}{2}}U^*\right](A+B)\left[C - |A+B|^{\frac{\dagger}{2}}E_B\right] \\
= \;& C^*(A+B)C - C^*(A+B)|A+B|^{\frac{\dagger}{2}}E_B - F_B^*|A+B|^{\frac{\dagger}{2}}U^*(A+B)C + F_B^*|A+B|^{\frac{\dagger}{2}}U^*(A+B)|A+B|^{\frac{\dagger}{2}}E_B \\
= \;& C^*(A+B)C - C^*U|A+B||A+B|^{\frac{\dagger}{2}}E_B - F_B^*|A+B|^{\frac{\dagger}{2}}U^*U|A+B|C + F_B^*|A+B|^{\frac{\dagger}{2}}U^*U|A+B||A+B|^{\frac{\dagger}{2}}E_B \\
= \;& C^*(A+B)C - C^*U|A+B|^{\frac{1}{2}}E_B - F_B^*|A+B|^{\frac{1}{2}}C + F_B^*E_B = C^*(A+B)C - C^*B - BC + F_B^*E_B \\
= \;& C^*AC + (I-C)^*B(I-C) - (B - F_B^*E_B) = C^*AC + (I-C)^*B(I-C) - A:B.
\end{aligned}$$

Moreover, if $C^*(A+B) = B$, then

$$\begin{aligned}
& \left[C^* - F_B^*|A+B|^{\frac{\dagger}{2}}U^*\right](A+B) = C^*(A+B) - F_B^*|A+B|^{\frac{\dagger}{2}}U^*U|A+B| \\
= \;& C^*(A+B) - F_B^*|A+B|^{\frac{1}{2}} = C^*(A+B) - B = 0.
\end{aligned}$$

The result holds. Similarly, if $(A+B)C = B$, then

$$(A+B)\left[C - |A+B|^{\frac{\dagger}{2}}E_B\right] = (A+B)C - U|A+B|^{\frac{1}{2}}E_B = (A+B)C - B = 0.$$

We obtain the result. □

For every A and $B \in \mathcal{B}^+(\mathcal{H})$, by Lemma 2.1, $\mathcal{R}(A) \subseteq \mathcal{R}(A^{\frac{1}{2}}) \subseteq \mathcal{R}(A^{\frac{1}{2}}) + \mathcal{R}(B^{\frac{1}{2}}) = \mathcal{R}((A+B)^{\frac{1}{2}}) = \mathcal{R}(|A+B|^{\frac{1}{2}})$ and $\mathcal{R}(B) \subseteq \mathcal{R}(|A+B|^{\frac{1}{2}})$, arbitrary positive operators are w.p.s. In this case, $A+B$ has polar decomposition $A+B = U|A+B| = P_{\overline{\mathcal{R}(A+B)}}(A+B)$. The reduced solutions are reduced as $E_A = F_A = (A+B)^{\frac{\dagger}{2}}A$ and $E_B = F_B = (A+B)^{\frac{\dagger}{2}}B$. The weakly parallel sum can be denoted as $A:B = A - E_A^*E_A = B - E_B^*E_B = E_A^*E_B = E_B^*E_A$. If $\mathcal{R}(A+B)$ is closed,

$$(A+B)^\dagger = (A+B)^\dagger(A+B)(A+B)^\dagger = \left((A+B)^\dagger(A+B)^{\frac{1}{2}}\right)\left((A+B)^\dagger(A+B)^{\frac{1}{2}}\right)^* \geq 0$$

and

$$\|A^{\frac{1}{2}}(A+B)^\dagger A^{\frac{1}{2}}\| = \|(A+B)^{\frac{\dagger}{2}}A(A+B)^{\frac{\dagger}{2}}\| \leq \|(A+B)^{\frac{\dagger}{2}}(A+B)(A+B)^{\frac{\dagger}{2}}\| \leq 1$$

we know that $A^{\frac{1}{2}}(A+B)^\dagger A^{\frac{1}{2}}$ is a contractive positive operator and

$$A:B = A - E_A^*E_A = A^{\frac{1}{2}}\left(I - A^{\frac{1}{2}}(A+B)^\dagger A^{\frac{1}{2}}\right)A^{\frac{1}{2}} \geq 0. \tag{11}$$

Theorem 3.5 has many useful consequences. For example, if A and B are two positive operators, then Theorem 3.5 reduces as the following corollary, which is given by F. Hansen in [15, Theorem 2.1].

Corollary 3.6. [15, Theorem 2.1] *Let E_A and E_B be the reduced solutions of the equations in (2). For every A, $B \in \mathcal{B}^+(\mathcal{H})$ with $\mathcal{R}(A+B)$ closed and every $C \in \mathcal{B}(\mathcal{H})$, the weakly parallel sum satisfies*

$$0 \leq A:B \leq C^*AC + (I-C)^*B(I-C)$$

and $E_A^*E_B + E_B^*E_A \leq \frac{A+B}{2} \leq E_A^*E_A + E_B^*E_B$.

Proof. By Theorem 3.5, $\left[C^* - F_B^*|A+B|^{\frac{\dagger}{2}}U^*\right]^* = C - |A+B|^{\frac{\dagger}{2}}E_B$ for any $A, B \in \mathcal{B}^+(\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{H})$. So,

$$\begin{aligned}
0 \;\leq\; & \left[C - |A+B|^{\frac{\dagger}{2}}E_B\right]^*(A+B)\left[C - |A+B|^{\frac{\dagger}{2}}E_B\right] = \left[C^* - F_B^*|A+B|^{\frac{\dagger}{2}}U^*\right](A+B)\left[C - |A+B|^{\frac{\dagger}{2}}E_B\right] \\
= \;& C^*AC + (I-C)^*B(I-C) - A:B.
\end{aligned}$$

Put $C = \frac{1}{2}I$ and by Theorem 3.1, $A : B = F_A^* E_B \leq \frac{A+B}{4}$, $A : B = F_B^* E_A \leq \frac{A+B}{4}$ and $2A : B = A+B-(F_A^* E_A + F_B^* E_B) \leq \frac{A+B}{2}$. It follows that $F_A^* E_B + F_B^* E_A \leq \frac{A+B}{2} \leq F_A^* E_A + F_B^* E_B$. Note that $E_A = F_A$ and $E_B = F_B$. It follows that

$$E_A^* E_B + E_B^* E_A \leq \frac{A+B}{2} \leq E_A^* E_A + E_B^* E_B.$$

□

Theorem 3.5 and Corollary 3.6 have many useful inequalities consequences. Especially, the following items (ii) and (iii) are also the special cases in [15, Theorem 3.1 (i)] when the orthogonal projection $P = P_{\overline{\mathcal{R}(B)}}$ and $P = I - P_{\overline{\mathcal{R}(A)}}$, respectively.

Corollary 3.7. *For every $A, B \in \mathcal{B}^+(\mathcal{H})$ with $\mathcal{R}(A+B)$ closed, the following weakly parallel sum inequalities hold.*

(i) $A : B \leq |\lambda|^2 A + |1-\lambda|^2 B$. *($C = \lambda I, \lambda \in \mathbb{C}$ in Corollary 3.6)*

(ii) $A : B \leq P_{\overline{\mathcal{R}(B)}} A P_{\overline{\mathcal{R}(B)}}$. *($C = P_{\overline{\mathcal{R}(B)}}$ in Corollary 3.6)*

(iii) $A : B \leq P_{\overline{\mathcal{R}(A)}} B P_{\overline{\mathcal{R}(A)}}$. *($C = I - P_{\overline{\mathcal{R}(A)}}$ in Corollary 3.6)*

(iv) $A : B \leq P_{\mathcal{N}(B)} A P_{\mathcal{N}(B)} + P_{\overline{\mathcal{R}(A)}} B P_{\overline{\mathcal{R}(A)}}$. *($C = P_{\overline{\mathcal{R}(B)}} - P_{\overline{\mathcal{R}(A)}}$ in Corollary 3.6)*

(v) $A : B \leq P_{\overline{\mathcal{R}(B)}} A P_{\overline{\mathcal{R}(B)}} + P_{\mathcal{N}(A)} B P_{\mathcal{N}(A)}$. *($C = P_{\overline{\mathcal{N}(A)}} + P_{\overline{\mathcal{R}(B)}}$ in Corollary 3.6)*

(vi) $A : B \leq \frac{A+B}{4}$. *($C = \frac{1}{2}I$ in Corollary 3.6)*

(vii) $A : B \leq \frac{1}{4}\Big[P_{\overline{\mathcal{R}(B)}} A P_{\overline{\mathcal{R}(B)}} + P_{\overline{\mathcal{R}(A)}} B P_{\overline{\mathcal{R}(A)}}\Big]$. *($C = \frac{1}{2}(P_{\overline{\mathcal{R}(B)}} + P_{\mathcal{N}(A)})$ in Corollary 3.6)*

(viii) $A : B \leq \frac{1}{4}\Big[P_{\overline{\mathcal{R}(B)}} A P_{\overline{\mathcal{R}(B)}} + (I + P_{\mathcal{N}(A)}) B (I + P_{\mathcal{N}(A)})\Big]$. *($C = \frac{1}{2}(P_{\overline{\mathcal{R}(B)}} - P_{\mathcal{N}(A)})$ in Corollary 3.6)*

Let weakly parallel sums $A : B$ and $C : D$ be well defined. As we know,

$$A : B = B - F_B^* E_B, \quad B = \mid A^* + B^* \mid^{\frac{1}{2}} U E_B \quad \text{and} \quad B^* = \mid A + B \mid^{\frac{1}{2}} F_B,$$

where U is the partial isometry of the polar decomposition of $A+B$. Let E_D and F_D be the reduced solutions of operator equations $D = \mid C^* + D^* \mid^{\frac{1}{2}} U_0 X$ and $D^* = \mid C + D \mid^{\frac{1}{2}} X$, respectively, where U_0 is the partial isometry of the polar decomposition $C + D = U_0|C+D| = |C^* + B^*|U_0$. Then,

$$C : D = D - F_D^* E_D, \quad D = \mid C^* + D^* \mid^{\frac{1}{2}} U_0 E_D \quad \text{and} \quad D^* = \mid C + D \mid^{\frac{1}{2}} F_D.$$

Using a similar approach, the next results extended [14, Theorem 6.1] on arbitrary Hilbert spaces setting under the assumptions that the operators, which is not necessary to be positive, are w.p.s. If all the weakly parallel sums $A : B$, $C : D$ and $(A+B) : (C+D)$ exist, we give the following result. The aim is to show that the w.p.s operators have the property $(A+C) : (B+D) = A : B + C : D$ if certain Douglas reduced solutions satisfy $|A+B|^{\frac{\dagger}{2}} E_B = |C+D|^{\frac{\dagger}{2}} E_D$.

Theorem 3.8. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(A+B)$, $\mathcal{R}(C+D)$ and $\mathcal{R}(A+B+C+D)$ are closed. Then*

$$\begin{aligned} & (A+C) : (B+D) - A : B - C : D \\ = \; & \Big[F_B^* |A+B|^{\frac{\dagger}{2}} U^* - F_D^* |C+D|^{\frac{\dagger}{2}} U_0^*\Big]\Big[(A+B) : (C+D)\Big]\Big[|A+B|^{\frac{\dagger}{2}} E_B - |C+D|^{\frac{\dagger}{2}} E_D\Big] \end{aligned}$$

if all the w.p.s operations are well defined. Moreover, if $|A+B|^{\frac{\dagger}{2}} E_B = |C+D|^{\frac{\dagger}{2}} E_D$, then $(A+C) : (B+D) = A : B + C : D$.

Proof. By Theorem 3.1,

$$\begin{aligned}(A+C):(B+D)-A:B-C:D &= (B+D)-(B+D)(A+B+C+D)^{\dagger}(B+D)-B+F_B^*E_B-D+F_D^*E_D \\ &= F_B^*E_B+F_D^*E_D-(B+D)(A+B+C+D)^{\dagger}(B+D).\end{aligned} \tag{12}$$

Since A and B are w.p.s, one has $F_B^*|A+B|^{\frac{\dagger}{2}}U^*(A+B)=F_B^*|A+B|^{\frac{\dagger}{2}}U^*U|A+B|=F_B^*|A+B|^{\frac{1}{2}}=B$ and $(A+B)|A+B|^{\frac{\dagger}{2}}E_B=U|A+B||A+B|^{\frac{\dagger}{2}}E_B=U|A+B|^{\frac{1}{2}}E_B=|A^*+B^*|^{\frac{1}{2}}UE_B=B$. Similarly, since C and D are w.p.s, one has $F_D^*|C+D|^{\frac{\dagger}{2}}U_0^*(C+D)=D$ and $(C+D)|C+D|^{\frac{\dagger}{2}}E_D=D$. Moreover, since $A+B$ and $C+D$ are w.p.s, $\mathcal{R}(B^*)\subseteq\mathcal{R}(A^*+B^*)\subseteq\mathcal{R}(A^*+B^*+C^*+D^*)$ and $\mathcal{R}(D^*)\subseteq\mathcal{R}(C^*+D^*)\subseteq\mathcal{R}(A^*+B^*+C^*+D^*)$. Hence, $B(A+B+C+D)^{\dagger}(A+B+C+D)=B$ and $D(A+B+C+D)^{\dagger}(A+B+C+D)=D$. These follow that

$$\begin{aligned}
&\left[F_B^*|A+B|^{\frac{\dagger}{2}}U^*-F_D^*|C+D|^{\frac{\dagger}{2}}U_0^*\right]\left[(A+B):(C+D)\right]\left[|A+B|^{\frac{\dagger}{2}}E_B-|C+D|^{\frac{\dagger}{2}}E_D\right]\\
=&\left[F_B^*|A+B|^{\frac{\dagger}{2}}U^*(A+B)(A+B+C+D)^{\dagger}(C+D)-F_D^*|C+D|^{\frac{\dagger}{2}}U_0^*(C+D)(A+B+C+D)^{\dagger}(A+B)\right]\\
&\times\left[|A+B|^{\frac{\dagger}{2}}E_B-|C+D|^{\frac{\dagger}{2}}E_D\right]\\
=&\left[B(A+B+C+D)^{\dagger}(C+D)-D(A+B+C+D)^{\dagger}(A+B)\right]\left[|A+B|^{\frac{\dagger}{2}}E_B-|C+D|^{\frac{\dagger}{2}}E_D\right]\\
=&\ B(A+B+C+D)^{\dagger}(C+D)|A+B|^{\frac{\dagger}{2}}E_B-D(A+B+C+D)^{\dagger}B\\
&-B(A+B+C+D)^{\dagger}D+D(A+B+C+D)^{\dagger}(A+B)|C+D|^{\frac{\dagger}{2}}E_D\\
=&\ B(A+B+C+D)^{\dagger}\left[(A+B+C+D)-(A+B)\right]|A+B|^{\frac{\dagger}{2}}E_B-D(A+B+C+D)^{\dagger}B\\
&-B(A+B+C+D)^{\dagger}D+D(A+B+C+D)^{\dagger}\left[(A+B+C+D)-(C+D)\right]|C+D|^{\frac{\dagger}{2}}E_D\\
=&\ B|A+B|^{\frac{\dagger}{2}}E_B-B(A+B+C+D)^{\dagger}B-D(A+B+C+D)^{\dagger}B\\
&-B(A+B+C+D)^{\dagger}D+D|C+D|^{\frac{\dagger}{2}}E_D-D(A+B+C+D)^{\dagger}D\\
=&\ F_B^*E_B+F_D^*E_D-(B+D)(A+B+C+D)^{\dagger}(B+D).
\end{aligned}$$

By (12), we obtain the first result. Moreover, if $|A+B|^{\frac{\dagger}{2}}E_B=|C+D|^{\frac{\dagger}{2}}E_D$, it is obvious that $(A+C):(B+D)=A:B+C:D$. □

Similar to Corollary 3.6, if $A,B,C,D\in\mathcal{B}^+(\mathcal{H})$, then $(A+B):(C+D)\geq 0$ and $\left[F_B^*|A+B|^{\frac{\dagger}{2}}U^*-F_D^*|C+D|^{\frac{\dagger}{2}}U_0^*\right]^*=|A+B|^{\frac{\dagger}{2}}E_B-|C+D|^{\frac{\dagger}{2}}E_D$. As an application of the preceding theorem, we get a corollary as follows. The well known series-parallel inequality $(A+C):(B+D)\geq A:B+C:D$ is first given in [1, Lemma 20] when A, B, C and D are Hermitian semidejfinite.

Corollary 3.9. [1, Lemma 20] *Let A, B, C, $D\in\mathcal{B}^+(\mathcal{H})$ be such that $\mathcal{R}(A+B)$, $\mathcal{R}(C+D)$ and $\mathcal{R}(A+B+C+D)$ are closed. Then $(A+C):(B+D)\geq A:B+C:D$.*

Corollary 3.9 has many useful consequences.

Corollary 3.10. *If A, A', B, B', C and D are finite nonnegative matrix, then*

(i) *If $A'\geq A$, then $A':B\geq A:B$.* *($A'=A+C, D=0$ in Corollary 3.9)*

(ii) *If $B'\geq B$, then $A:B'\geq A:B$.* *($B'=B+D, C=0$ in Corollary 3.9)*

(iii) *If* $B \geq A$ *and* $C \geq A$ *then* $B : C \geq \frac{1}{2}A$. *(by (i), (ii) and* $\frac{1}{2}A = A : A$*)*

The weakly parallel sum is not distributive with the usual operator product. In [2, Theorem 5], Anderson shows that, if A, B are positive semi-definite operators, and Z is an operator, then $Z^*(A : B)Z \leq (Z^*AZ) : (Z^*BZ)$. As for bounded operators, we give the following result. This is a generalization of [2, Theorem 5].

Theorem 3.11. *Let* $A, B, Z \in \mathcal{B}(\mathcal{H})$ *and* $\widehat{Z} = Z^* P_{\mathcal{R}(A^*+B^*)}$ *be such that* A *and* B, $\widehat{Z}AZ$ *and* $\widehat{Z}BZ$ *are w.p.s, respectively. If* $\mathcal{R}(A+B)$ *and* $\mathcal{R}(\widehat{Z}AZ + \widehat{Z}BZ)$ *are closed,* $B^*(A^*+B^*)^\dagger A = A^*(A^*+B^*)^\dagger B$ *and* $\mathcal{R}(\widehat{Z}) \subseteq \mathcal{R}(\widehat{Z}AZ + \widehat{Z}BZ)$*, then*

$$
\begin{aligned}
&(\widehat{Z}AZ) : (\widehat{Z}BZ) = \widehat{Z}(A : B)Z + T^*AT + T^*BT \\
= \;& \left[|A+B|^{\frac{\dagger}{2}} E_B Z + T\right]^* A \left[|A+B|^{\frac{\dagger}{2}} E_B Z + T\right] + \left[|A+B|^{\frac{\dagger}{2}} E_A Z - T\right]^* B \left[|A+B|^{\frac{\dagger}{2}} E_A Z - T\right],
\end{aligned}
$$

where $T = \widehat{Z}^* \left[(\widehat{Z}AZ + \widehat{Z}BZ)^*\right]^\dagger (\widehat{Z}BZ)^* - (A+B)^\dagger BZ$.

Proof. Denote by $X_0 = |A+B|^{\frac{\dagger}{2}} E_B Z$ and $Y_0 = |A+B|^{\frac{\dagger}{2}} E_A Z$. Since A and B are w.p.s, by (2), (4) and (10),

$$
\begin{aligned}
&X_0 + Y_0 = \widehat{Z}^*, \quad AX_0 = F_A^* E_B Z = (A : B)Z, \quad BY_0 = F_B^* E_A Z = (A : B)Z, \quad X_0^* A X_0 + Y_0^* B Y_0 = \widehat{Z}(A : B)Z, \\
&(A+B)X_0 = U|A+B|^{\frac{1}{2}} E_B Z = BZ = B\widehat{Z}^*, \quad (A+B)\widehat{Z}^* = (A+B)Z, \quad BX_0 = F_B^* E_B Z = (B - A : B)Z.
\end{aligned} \tag{13}
$$

Since $B^*(A^*+B^*)^\dagger A = A^*(A^*+B^*)^\dagger B$, one has

$$
X_0^* A = Y_0^* B \quad \text{and} \quad X_0^*(A+B) = Y_0^* B + X_0^* B = \widehat{Z}B. \tag{14}
$$

Denote by $S = \widehat{Z}AZ + \widehat{Z}BZ$, $X = \widehat{Z}^*\left(S^*\right)^\dagger (\widehat{Z}BZ)^*$, $Y = \widehat{Z}^*\left(S^*\right)^\dagger (\widehat{Z}AZ)^*$ and $T = X - X_0$. Note that $\mathcal{R}(\widehat{Z}) = \mathcal{R}(S)$. Since $\widehat{Z}AZ$ and $\widehat{Z}BZ$ are w.p.s, one has $\mathcal{R}((\widehat{Z}BZ)^*) \subseteq \mathcal{R}(S^*)$. Then

$$
\widehat{Z} = SS^\dagger \widehat{Z}, \quad \widehat{Z}^* = \widehat{Z}^*\left(S^*\right)^\dagger S^* \quad \text{and} \quad \widehat{Z}BZ = \widehat{Z}BZS^\dagger S. \tag{15}
$$

By (13) and (15), $Y_0 - T = X_0 + Y_0 - X = \widehat{Z}^*\left(S^*\right)^\dagger S^* - \widehat{Z}^*\left(S^*\right)^\dagger (\widehat{Z}BZ)^* = Y$. Hence, $X = X_0 + T$, $Y = Y_0 - T$ and $X + Y = \widehat{Z}^*$. By (13), (14) and (15),

$$
\begin{aligned}
&T^*AT + T^*BT = \left[\widehat{Z}^*\left(S^*\right)^\dagger (\widehat{Z}BZ)^* - X_0\right]^* (A+B) \left[\widehat{Z}^*\left(S^*\right)^\dagger (\widehat{Z}BZ)^* - X_0\right] \\
= \;& (\widehat{Z}BZ)S^\dagger \widehat{Z}(A+B)\widehat{Z}^*\left(S^*\right)^\dagger (\widehat{Z}BZ)^* - (\widehat{Z}BZ)S^\dagger \widehat{Z}(A+B)X_0 - X_0^*(A+B)\widehat{Z}^*\left(S^*\right)^\dagger (\widehat{Z}BZ)^* + X_0^*(A+B)X_0 \\
= \;& (\widehat{Z}BZ)\left(S^*\right)^\dagger (\widehat{Z}BZ)^* - (\widehat{Z}BZ)S^\dagger \widehat{Z}BZ - (\widehat{Z}BZ)\left(S^*\right)^\dagger (\widehat{Z}BZ)^* + \widehat{Z}BX_0 \\
= \;& -(\widehat{Z}BZ)S^\dagger \widehat{Z}BZ + \widehat{Z}(B - A : B)Z.
\end{aligned} \tag{16}
$$

By (13), (14) and (16), it follows that

$$
\begin{aligned}
&X^*AX + Y^*BY = (X_0 + T)^* A (X_0 + T) + (Y_0 - T)^* B (Y_0 - T) \\
= \;& X_0^* A X_0 + Y_0^* B Y_0 + T^*AT + T^*BT + T^*AX_0 - T^*BY_0 + X_0^*AT - Y_0^*BT \\
= \;& \widehat{Z}(A : B)Z + T^*AT + T^*BT = \widehat{Z}(A : B)Z + \widehat{Z}(B - A : B)Z - (\widehat{Z}BZ)S^\dagger \widehat{Z}BZ \\
= \;& \widehat{Z}BZ - (\widehat{Z}BZ)S^\dagger(\widehat{Z}BZ) = (\widehat{Z}AZ) : (\widehat{Z}BZ).
\end{aligned}
$$

This completes the proof. □

Note that

$$T^*AT + T^*BT = -(\widehat{Z}BZ)S^\dagger\widehat{Z}BZ + \widehat{Z}(B - A : B)Z = \widehat{Z}B\Big[(A+B)^\dagger - Z\Big(\widehat{Z}(A+B)Z\Big)^\dagger\widehat{Z}\Big]BZ.$$

Theorem 3.11 includes the following special case.

Corollary 3.12. *Let $A, B, Z \in \mathcal{B}(\mathcal{H})$ and $\widehat{Z} = Z^*P_{\mathcal{R}(A^*+B^*)}$ be such that $\mathcal{R}(A+B)$ and $\mathcal{R}(\widehat{Z}AZ + \widehat{Z}BZ)$ are closed and A and B, $\widehat{Z}AZ$ and $\widehat{Z}BZ$ are w.p.s, respectively. If $B^*(A^*+B^*)^\dagger A = A^*(A^*+B^*)^\dagger B$, $\mathcal{R}(\widehat{Z}) \subseteq \mathcal{R}(\widehat{Z}AZ + \widehat{Z}BZ)$ and $(A+B)^\dagger = Z\Big(\widehat{Z}(A+B)Z\Big)^\dagger\widehat{Z}$, then $\widehat{Z}(A : B)Z = (\widehat{Z}AZ) : (\widehat{Z}BZ)$.*

If $A, B \in \mathcal{B}^+(\mathcal{H})$ and $Z \in \mathcal{B}(\mathcal{H})$, then $\widehat{Z}A = Z^*A$ and $\widehat{Z}B = Z^*B$. Hence, we have the following corollary.

Corollary 3.13. *Let $A, B \in \mathcal{B}^+(\mathcal{H})$ and $Z \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{R}(A+B)$ and $\mathcal{R}(Z^*AZ + Z^*BZ)$ are closed.*
(i) $(Z^*AZ) : (Z^*BZ) \geq Z^*(A : B)Z \geq 0$.
(ii)

$$\begin{aligned}(Z^*AZ) : (Z^*BZ) &= Z^*(A : B)Z + T^*AT + T^*BT \\ &= \Big[|A+B|^{\frac{\dagger}{2}}E_BZ + T\Big]^*A\Big[|A+B|^{\frac{\dagger}{2}}E_BZ + T\Big] + \Big[|A+B|^{\frac{\dagger}{2}}E_AZ - T\Big]^*B\Big[|A+B|^{\frac{\dagger}{2}}E_AZ - T\Big],\end{aligned}$$

where $T = P_{\mathcal{R}(A^*+B^*)}Z\Big[Z^*AZ + Z^*BZ\Big]^\dagger(Z^*BZ) - (A+B)^\dagger BZ$.

(iii) *If $(A+B)^\dagger = Z\Big(Z^*(A+B)Z\Big)^\dagger Z^*$, then $(Z^*AZ) : (Z^*BZ) = Z^*(A : B)Z$.*

Similar to the proof of Theorem 3.11, one has the following inner product properties.

Theorem 3.14. *Let A and $B \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{R}(A+B)$ is closed and $\widetilde{u} = P_{\mathcal{R}(A^*+B^*)}u$ for every $u \in \mathcal{H}$. Then for $x, y, z \in \mathcal{H}$ such that $x + y = z$,*

$$\langle Ax, \widetilde{x}\rangle + \langle By, \widetilde{y}\rangle = \langle (A : B)z, \widetilde{z}\rangle + \langle t, A^*x_0 - B^*y_0\rangle + \langle (A+B)t, t\rangle,$$

where $x_0 = |A+B|^{\frac{\dagger}{2}}E_B\widetilde{z}$, $y_0 = |A+B|^{\frac{\dagger}{2}}E_A\widetilde{z}$ and $t = \widetilde{x} - x_0$. Moreover, if A and B are self-adjoint,

$$\langle Ax, x\rangle + \langle By, y\rangle = \langle (A : B)z, z\rangle + \langle (A+B)t, t\rangle.$$

If A and B are positive (see [1, Lemma 18] for parallel sum case),

$$\langle Ax, x\rangle + \langle By, y\rangle \geq \langle (A : B)z, z\rangle, \quad \forall x + y = z.$$

Proof. Note that $\widetilde{x} = P_{\mathcal{R}(A^*+B^*)}x$, $\widetilde{y} = P_{\mathcal{R}(A^*+B^*)}y$ and $\widetilde{x} + \widetilde{y} = \widetilde{z}$. Denote by $x_0 = |A+B|^{\frac{\dagger}{2}}E_B\widetilde{z}$ and $y_0 = |A+B|^{\frac{\dagger}{2}}E_A\widetilde{z}$. Since A and B are w.p.s, by (5), (6) and (10),

$$\begin{aligned}&x_0 + y_0 = \widetilde{z}, \quad Ax_0 = F_A^*E_B\widetilde{z} = (A : B)\widetilde{z} = (A : B)z, \quad By_0 = F_B^*E_A\widetilde{z} = (A : B)z,\\ &(A+B)x_0 = U|A+B|^{\frac{1}{2}}E_B\widetilde{z} = B\widetilde{z} = Bz, \quad (A+B)y_0 = Az, \quad A\widetilde{x} = Ax, \quad B\widetilde{y} = By,\\ &Bx_0 = F_B^*E_B\widetilde{z} = (B - A : B)\widetilde{z} = (B - A : B)z, \quad Ay_0 = (A - A : B)z.\end{aligned}$$

Put $t = \widetilde{x} - x_0$. Then $\widetilde{x} = x_0 + t$, $\widetilde{y} = y_0 - t$,

$$\langle Ax, \widetilde{x}\rangle = \langle A\widetilde{x}, \widetilde{x}\rangle = \langle Ax_0 + At, x_0 + t\rangle = \langle (A : B)z, x_0\rangle + \langle (A : B)z, t\rangle + \langle At, x_0\rangle + \langle At, t\rangle$$

and

$$\langle By, \widetilde{y}\rangle = \langle B\widetilde{y}, \widetilde{y}\rangle = \langle By_0 - Bt, y_0 - t\rangle = \langle (A:B)z, y_0\rangle - \langle (A:B)z, t\rangle - \langle Bt, y_0\rangle + \langle Bt, t\rangle.$$

Hence, $\langle Ax, \widetilde{x}\rangle + \langle By, \widetilde{y}\rangle = \langle (A:B)z, \widetilde{z}\rangle + \langle t, A^*x_0 - B^*y_0\rangle + \langle (A+B)t, t\rangle$. Moreover, if A and B are self-adjoint, then $A^*x_0 = B^*y_0$ and $\langle Ax, x\rangle + \langle By, y\rangle = \langle (A:B)z, z\rangle + \langle (A+B)t, t\rangle$. It is obvious that $\langle Ax, x\rangle + \langle By, y\rangle \geq \langle (A:B)z, z\rangle$ if A and B are positive. □

4. Concluding remarks

If A and B are Hermitian positive semi-definite matrices, the parallel sum $A : B$ is defined by $A : B = A(A + B)^{\dagger}B$. Anderson and Duffin defined the parallel sum operation on Hermitian positive semi-definite matrices and investigated its most important properties [1]. The extensions of the theory to the positive operators in Hilbert space have been given by Anderson and Schreiber [4] and Fillmore and Williams [13]. Recently, Xu et al. [14, 18, 19] obtained the perturbation estimation of the parallel sum and extended some properties of parallel sums to adjoint operators on Hilbert C^* modules.

This paper considers a natural generalizations of parallel sum for bounded operators on infinite-dimensional spaces. Under the suitable range inclusion relations

$$\mathcal{R}(A) \subseteq \mathcal{R}(|\, A^* + B^* \,|^{\frac{1}{2}}), \quad \mathcal{R}(B) \subseteq \mathcal{R}(|\, A^* + B^* \,|^{\frac{1}{2}}), \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(|\, A + B \,|^{\frac{1}{2}}), \quad \mathcal{R}(B^*) \subseteq \mathcal{R}(|\, A + B \,|^{\frac{1}{2}}).$$

Definition 2.6 of weakly parallel sum $A : B = A - F_A^* E_A$ may be applied to any pair of linear operators. In particular, all the positive operators are w.p.s by Definition 2.5. The present article is devoted to the further development of the properties of weakly parallel sum for arbitrary bounded operators in a Hilbert space. By the spectral decomposition theorem and the closed graph theorem, the generalized inverse $A^{\dagger}$ is always defined (need not to be bounded) and $A^{\dagger}B$ is bounded if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Many rather natural extensions have been developed in this paper. When these results are restricted ro the positive operators, these results can be reduced as certain properties of the parallel sum.

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