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New mixed Herz-Hardy spaces and their applications

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Abstract. In this paper, Herz-Hardy spaces with mixed-norm are introduced, and some properties of these spaces are established, such as the characterization of various maximal operators, including property and some inequalities. Furthermore, we investigate atomic decomposition and molecular decomposition of mixed-norm Herz-Hardy spaces. As an application, the authors obtain the boundedness of some operators on these spaces by atomic decomposition.

1. Introduction

The study of Herz spaces originated from the work of Beurling [1]. Later, Herz spaces were systematically studied by Herz [2] to study the Fourier series and Fourier transform. In the 1990s, Lu and Yang [3] introduced the homogeneous Herz spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and non-homogeneous Herz spaces $K_q^{\alpha,p}(\mathbb{R}^n)$ and extended the boundedness of a large class of operators to these spaces.

As is well known, Hardy spaces are proper substitutes of Lebesgue spaces in some situations. For example, the Riesz transforms are bounded on the Hardy space $H^p(\mathbb{R}^n)$, but not bounded on the corresponding Lebesgue space $L^p(\mathbb{R}^n)$ when $0 < p \leqslant 1$. The theory of classical Hardy spaces was originally studied by Stein and Weiss [4] and then systematically developed in [5]. Hardy spaces also have various real variable characterizations, such as different maximal operators characterizations, atomic characterization [6] and molecular characterization [7], and so on. These characterizations greatly facilitate the researchers to derive the dual spaces of Hardy spaces and establish the boundedness of operators on these spaces, see [8] for more details. As a variant of the classical real Hardy spaces, the Hardy spaces associated with the Beurling algebras on the real line were first introduced by Chen and Lau [9], in which the dual spaces and the maximal function characterizations of these spaces were established. Later, García-Cuerva [10] extended the results of Chen and Lau [9] to higher-dimensional case, and García-Cuerva and Herrero [11] further studied their maximal function and Littlewood-Paley function characterizations. In 1995, Lu and Yang [12] systematically studied Herz-Hardy spaces with general indices and established their atomic and molecular characterizations. For more studies of Herz-Hardy type spaces, the readers can refer to [13–17].

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Recently, mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, as natural extensions of the classical Lebesgue spaces, have attracted widespread attention. The theory of mixed-norm function spaces can be traced back to the work of Benedek and Panzone [18], in which it was proved that $L^{\vec{p}}(\mathbb{R}^n)$ also possesses some basic properties similar to the classical Lebesgue spaces, such as completeness, Hölder's inequality, Minkowski's inequality, and so on. These properties provide the possibility to solve a series of subsequence problems. On the other hand, since the study of partial differential equations (for example the heat equation and the wave equation) always involves both space and time variables, mixed-norm spaces possess better structures than classical spaces in the time-space estimates for PDEs. For these reasons, many researchers renewed the interest in mixed-norm Lebesgue spaces and extended them to other mixed-norm function spaces. For instance, the real variable characterizations and the atom characterization of mixed-norm Hardy spaces were established in [19, 20]. In 2019, Nogayama [21] introduced mixed-norm Morrey spaces and gave some applications in the operator theory.

Note that recently Herz spaces were also extended to the mixed-norm situation by Wei [22]. By extending the extrapolation theory to mixed Herz spaces, Wei [22] established the boundedness of some classical operators in harmonic analysis on these spaces. Moreover, the extrapolation theory can further give boundedness results of some classical operators on mixed Herz spaces.

In this paper, we introduce mixed Herz-Hardy spaces and investigate some basic properties of these spaces in this paper. Moreover, we will establish the atomic decomposition and molecular decomposition for these spaces. As applications, the boundedness for a wide class of sublinear operators on mixed Herz-Hardy spaces is obtained.

The organization of the remainder of this article is as follows. Some necessary definitions and lemmas are given in section 2. The atomic decomposition and the molecular decomposition of mixed Herz-Hardy spaces will be given in section 3 and section 4, respectively. Some applications of the atomic and molecular decomposition are presented in section 5.

2. Preliminary

Throughout this paper, we use the following notations. The letter $\vec{q}$ will denote n-tuples of the numbers in $(0,\infty]$ $(n \geqslant 1)$, $\vec{q} = (q_1, q_2, \ldots, q_n)$. By definition, the inequality $0 < \vec{q} < \infty$ means that $0 < q_i < \infty$ for all i. For $\vec{q} = (q_1, q_2, \ldots, q_n)$, we write $1/\vec{q} = (1/q_1, 1/q_2, \ldots, 1/q_n)$. In addition, if $\vec{q} \in [1,\infty]^n$, we denote by $\vec{q}' = (q_1', q_2', \ldots, q_n')$, where $q_i' = q_i/(q_i - 1)$ is conjugate exponent of q_i. $|B|$ denotes the volume of the ball B, χ_E is the characteristic function of a set E. The notation $A \lesssim B$ means that $A \leqslant CB$ with some positive constant C independent of appropriate quantities, and, if $A \lesssim B \lesssim A$, then we write $A \sim B$. $[a]$ denotes take the integer number for a. Let $B_k = \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$ and $A_k = B_k \backslash B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{A_k}$ for any $k \in \mathbb{Z}$, and $\widetilde{\chi}_k = \chi_k$ for any $k \in \mathbb{N}$, $\widetilde{\chi}_0 = \chi_{B_0}$.

Definition 2.1. *(Mixed Lebesgue spaces)([18]) Let $\vec{p} = (p_1, p_2, \ldots, p_n) \in (0,\infty]^n$. Then the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that*

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} | f(x_1, x_2, \ldots, x_n) |^{p_1} \, dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \ldots dx_n \right)^{\frac{1}{p_n}} < \infty,$$

If $p_j = \infty$, then we have to make appropriate modifications.

Definition 2.2. *([22]) Let $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < \vec{q} \leqslant \infty$. The mixed homogeneous Herz space $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n) := \left\{ f \in L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n/\{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} = \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p} \|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \right)^{1/p} < \infty \right\}.$$

Definition 2.3. *([22]) Let $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < \vec{q} \leqslant \infty$. The non-homogeneous mixed Herz space $K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) := \left\{ f \in L_{\mathrm{loc}}^{\vec{q}}(\mathbb{R}^n) : \|f\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{k\alpha p} \|f \widetilde{\chi}_k\|_{L^{\vec{q}}(\mathbb{R}^n)}^p \right)^{1/p} < \infty \right\}.$$

Remark 2.4. (i) *If $0 < \vec{q} = (q_1, q_2, \ldots, q_n) \leqslant \infty$ and $q_1 = q_2 = \cdots = q_n = q$, then $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = K_q^{\alpha,p}(\mathbb{R}^n)$, where $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ are classical Herz spaces.*
(ii) *The mixed homogeneous Herz space $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ and the non-homogeneous mixed Herz space $K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ are quasi-Banach spaces. But, if $\vec{q}, p \geqslant 1$, they are Banach spaces. These can be inferred from definitions of mixed Lebesgue spaces and classical Herz spaces.*

To give the definition of mixed Herz-Hardy spaces $H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ and $HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$, We first introduce some maximal-type operators. Denote $\mathcal{S}(\mathbb{R}^n)$ by the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ by the dual space of $\mathcal{S}(\mathbb{R}^n)$.
Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi_t(x) = t^{-n}\phi(x/t)$, such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For $t > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$, define the smooth maximal operator $M(f;\phi)$ by

$$M(f;\phi)(x) = \sup_{t>0} \left| (f * \phi_t)(x) \right|.$$

Also, we define the non-tangential maximal operator $M_a^*(f;\phi)$ (with $a > 0$) and the auxiliary maximal operator $M_b^{**}(f;\phi)$ (with $b > 0$) by

$$M_a^*(f;\phi)(x) = \sup_{t>0} \sup_{|x-y|<at} \left| f * \phi_t(x) \right|$$

and

$$M_b^{**}(f;\phi)(x) = \sup_{(y,t)\in\mathbb{R}_+^{n+1}} \frac{\left| f * \phi_t(y) \right|}{\left(\frac{|x-y|+t}{t} \right)^b}.$$

Let $\mathcal{M}_N f(x)$ be the grand maximal function of $f(x)$ defined by

$$\mathcal{M}_N f(x) = \sup_{\phi \in \mathcal{A}_N} M_1^*(f;\phi)(x),$$

where $\mathcal{A}_N = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leqslant N} \left| x^\alpha D^\beta \phi(x) \right| \leqslant 1 \right\}$.

In 2017, anisotropic mixed-norm Hardy spaces are introduced by Cleanthous et al., particularly, who also investigate isotropic mixed-norm Hardy spaces and a crucial theorem as follows.

Let $\vec{p} \in (0,\infty)^n$ we denote by $p_- := \min(1, p_1, \ldots, p_n)$ and

$$N_{\vec{p}} := \left[n\left(1 + \frac{1}{p_-}\right) + n + 2 \right] + 1$$

We will say that a distribution $f \in \mathcal{S}'$ belongs to the isotropic mixed-norm Hardy space $H^{\vec{p}}(\mathbb{R}^n)$ when $\mathcal{M}_{N_{\vec{p}}} f \in L^{\vec{p}}(\mathbb{R}^n)$. The map $\|f\|_{H^{\vec{p}}(\mathbb{R}^n)} := \left\| \mathcal{M}_{N_{\vec{p}}} f \right\|_{L^{\vec{p}}(\mathbb{R}^n)}$ is the quasi-norm of $H^{\vec{p}}(\mathbb{R}^n)$.

Lemma 2.5. *([19]) Let $\vec{p} \in (0,\infty)^n$ and $0 < \theta < \min(1, p_1, \ldots, p_n)$. Then for every $a > 0, b > n/\theta, N > 2n + b + 2$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$ we have for all $f \in \mathcal{S}'$*

$$\|M(f;\phi)\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\| M_a^*(f;\phi) \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\| M_b^{**}(f;\phi) \right\|_{L^{\vec{p}}(\mathbb{R}^n)} \sim \left\| \mathcal{M}_N f \right\|_{L^{\vec{p}}(\mathbb{R}^n)}.$$

Definition 2.6. *Let $\alpha \in \mathbb{R}, 0 < p < \infty, 1 < \vec{q} \leqslant \infty$, and $N > N_{\vec{q}}$. The mixed homogeneous Herz-type Hardy space $H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} = \left\| \mathcal{M}_N f \right\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}.$$

Definition 2.7. *Let $\alpha \in \mathbb{R}, 0 < p < \infty, 1 < \vec{q} \leqslant \infty$, and $N > N_{\vec{q}}$. The mixed non-homogeneous Herz-type Hardy space $HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ is defined by*

$$HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} = \left\| \mathcal{M}_N f \right\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}.$$

Remark 2.8. (i) *Notice that if $p = 1, 1 < \vec{q} < \infty$, and $\alpha = n - \sum_{i=1}^{n} \frac{1}{q_i}$, then the space $H\dot{K}_{\vec{q}}^{\alpha,1}(\mathbb{R}^n)$ is the space $HA^{\vec{q}}(\mathbb{R}^n)$ defined in [23], which can be inferred from the atom decomposition of mixed Herz-Hardy spaces (see section 3).*

(ii) *By the sublinearity of $\mathcal{M}_N$ and the definition of mixed Herz-Hardy spaces, we can conclude that mixed Herz-Hardy spaces are quasi-Banach spaces.*

Now, we will give some necessary lemmas and basic properties of mixed Herz-Hardy spaces. The following theorem tells us that besides the grand maximal operator $\mathcal{M}_N$, the space $H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ can also be characterized by some other maximal-type operators.

Theorem 2.9. *Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 < \vec{q} < \infty$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, the following statements are equivalent:*

(i) $f \in H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$.

(ii) *There exists a function $\phi \in \mathcal{S}(\mathbb{R}^n)$, such that for some $a \geqslant 1$, $M_a^*\left(f;\phi\right) \in \dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$.*

(iii) *There exists a function $\phi \in \mathcal{S}(\mathbb{R}^n)$, such that $M_b^{**}(f;\phi) \in \dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$.*

(iv) *There exists a function $\phi \in \mathcal{S}(\mathbb{R}^n)$, such that $M\left(f;\phi\right) \in \dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$.*

Proof. The proof of Theorem 2.9. By Lemma 2.5 and the definitions of mixed-norm Herz-Hardy spaces can directly gain. □

Lemma 2.10. *Let $0 < p \leqslant \infty$, $1 < \vec{q} < \infty$, and $-\sum_{i=1}^{n} \frac{1}{q_i} < \alpha < n(1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i})$. Suppose a sublinear operators T satisfied that*

(i) *T is bounded on $L^{\vec{q}}(\mathbb{R}^n)$;*

(ii) *for any $f \in L^1(\mathbb{R}^n)$ with compact support have*

$$\left|Tf(x)\right| \leqslant C \int_{\mathbb{R}} \frac{\left|f(x)\right|}{\left|x-y\right|^n}, \qquad x \notin \operatorname{supp} f. \tag{1}$$

Then T is bounded on $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$.

Proof. This proof is similar to that of ([22], Theorem 4.2), and the only difference is that we should take the size condition into consideration. For simplification, we omit the details. □

Proposition 2.11. *Let $1 < \vec{q} < \infty$, $0 < \theta < \min(1, q_1, \ldots, q_n)$, $\beta > n/\theta$, and $N > 2n + \beta + 2$. Then $\mathcal{M}_N$ is bounded on $L^{\vec{q}}(\mathbb{R}^n)$.*

Proof. By Lemma 2.5, $\|M(f;\phi)\|_{L^{\vec{q}}(\mathbb{R}^n)} \sim \left\|\mathcal{M}_N f\right\|_{L^{\vec{q}}(\mathbb{R}^n)}$ when $\int_{\mathbb{R}^n}\phi(x)dx = 1$. We only need to consider the boundedness of operators $M(f;\phi)$ on $L^{\vec{q}}(\mathbb{R}^n)$. It is well known that $\phi \in L^1(\mathbb{R}^n)$, then

$$M(f;\phi)(x) \leqslant C\|\phi\|_{L^1(\mathbb{R}^n)}Mf(x) \leqslant CMf(x),$$

where Mf is Hardy-Littlewood maximal operators of f. Moreover, since the Hardy-Littlewood maximal operator is bounded on $L^{\vec{q}}(\mathbb{R}^n)$ (see [24]), we have

$$\left\|\mathcal{M}_N f\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \sim \|M(f;\phi)\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{\vec{q}}(\mathbb{R}^n)}.$$

It implies the result of this proposition. □

Proposition 2.12. *Let $0 < p < \infty, 1 < \vec{q} = (q_1, q_2, \ldots, q_n) < \infty$ and $N > N_{\vec{q}}$. If $-\sum_{i=1}^m \frac{1}{q_i} < \alpha < n(1 - \frac{1}{n}\sum_{i=1}^m \frac{1}{q_i})$. Then*

$$H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)\bigcap L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) = \dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$$

and

$$HK^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)\bigcap L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n) = K^{\alpha,p}_{\vec{q}}(\mathbb{R}^n).$$

Proof. It suffices to prove homogeneous spaces. By using the trivial inequality $|f(x)| \leqslant C\mathcal{M}_N(f)(x)$, we get

$$\|f\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} \lesssim \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|\mathcal{M}_N(f)(x)\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)}\right)^p = \|\mathcal{M}_N(f)\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} = \|f\|_{H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)},$$

which yields $H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)\bigcap L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) \subset \dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$.

From the boundedness of sublinear operators on mixed Herz spaces $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$, we can check that $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n) \subset H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)\bigcap L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$. We claim that when $\phi \in \mathcal{A}_N$, the grand maximal function $\mathcal{M}_N(f)$ satisfies

$$|\mathcal{M}_N(f)(x)| \leqslant C\int_{\mathbb{R}^n}\frac{\left|f(x)\right|}{|x-y|^n}dy. \tag{2}$$

In fact,

$$\begin{aligned}|\mathcal{M}_N(f)(x)| &= \sup_{t>0}\sup_{|x-y|<t}\left|\phi_t * f(y)\right| \\ &\lesssim \sup_{t>0}\sup_{|x-y|<t}\int_{\mathbb{R}^n}\left|t^{-n}\left(\frac{t}{y-x}\right)^n\right|\left|f(y)\right|dy \\ &\lesssim \int_{\mathbb{R}^n}\frac{\left|f(x)\right|}{\left|y-x\right|^n}dy.\end{aligned}$$

It follows from (2) and Lemma 2.10 that

$$\left\|f\right\|_{H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} = \left\|\mathcal{M}_N(f)\right\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} \leqslant C\left\|f\right\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)}.$$

As a consequence, $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n) \subset H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)\bigcap L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$.

In addition, we can obtain $\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n) \subset L^{\vec{q}}_{\mathrm{loc}}(\mathbb{R}^n)$ from the definition of mixed Herz spaces. □

Remark 2.13. *In view of Proposition 2.12, we will only consider the characterizations of the spaces $H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ in terms of central atom and molecular decomposition for $\alpha \geqslant n(1 - \frac{1}{n}\sum_{i=1}^m \frac{1}{q_i})$.*

Proposition 2.14. *Let $\alpha \in \mathbb{R}, 0 < p < \infty, 1 < \vec{q} < \infty$. Then*

$$HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = L^{\vec{q}}(\mathbb{R}^n) \bigcap H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n).$$

Proof. *To prove this proposition we need the following assertion. For $0 < p, \vec{q} \leqslant \infty$, we have*

$$K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) = \dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) \bigcap L^{\vec{q}}(\mathbb{R}^n), \tag{3}$$

and for all $f \in \dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n) \bigcap L^{\vec{q}}(\mathbb{R}^n)$, there holds

$$\|f\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} + \|f\|_{L^{\vec{q}}(\mathbb{R}^n)}.$$

Note that

$$\begin{aligned}\|f\|^p_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} &= \sum_{k=0}^{\infty} 2^{k\alpha p}\|f\widetilde{\chi}_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \\ &= \|f\widetilde{\chi}_0\|^p_{L^{\vec{q}}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant \|f\|^p_{L^{\vec{q}}(\mathbb{R}^n)} + \|f\|^p_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} - \sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)}.\end{aligned}$$

We claim that the following estimation is correct:

$$\sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant C\|f\|^p_{L^{\vec{q}}(|x|\leqslant 1)}.$$

When $p \leqslant q_n$, by using the Hölder inequality and

$$\left(\sum_{k=1}^{\infty}|a_k|\right)^r \leqslant \sum_{k=1}^{\infty}|a_k|^r \qquad (0 < r \leqslant 1),$$

we obtain

$$\sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant \left(\sum_{k=-\infty}^{0} 2^{k\alpha p(\frac{q_n}{p})'}\right)^{\frac{1}{(\frac{q_n}{p})'}} \left(\sum_{k=-\infty}^{0} \|f\chi_k\|^{p\frac{q_n}{p}}_{L^{\vec{q}}(\mathbb{R}^n)}\right)^{\frac{p}{q_n}} \leqslant C\sum_{k=-\infty}^{0} \|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant C\|f\|^p_{L^{\vec{q}}(|x|\leqslant 1)}.$$

When $q_n < p$,

$$\sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} = \sum_{k=-\infty}^{0}\left(2^{k\alpha q_n}\|f\chi_k\|^{q_n}_{L^{\vec{q}}(\mathbb{R}^n)}\right)^{\frac{p}{q_n}} \leqslant \left(\sum_{k=-\infty}^{0} 2^{k\alpha q_n}\|f\chi_k\|^{q_n}_{L^{\vec{q}}(\mathbb{R}^n)}\right)^{\frac{p}{q_n}} \leqslant C\|f\|^p_{L^{\vec{q}}(|x|\leqslant 1)},$$

we also obtain $\|f\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \leqslant \|f\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} + \|f\|_{L^{\vec{q}}(\mathbb{R}^n)}$.

To finish the proof, we need to check $\|f\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \leqslant C\|f\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}$ and $\|f\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant C\|f\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}$. By a direct computation, we have

$$\begin{aligned}\|f\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} &\leqslant \sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant C\|f\|^p_{L^{\vec{q}}(|x|\leqslant 1)} + \sum_{k=0}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant C\|f\|^p_{L^{\vec{q}}(|x|\leqslant 1)} + \|f\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

The result $\|f\|_{L^{\vec{q}}(|x|\leqslant 1)} \leqslant C\|f\|_{K^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)}$ *is evident.*

Furthermore, to prove the other estimate, it suffices to show that

$$\|f\|_{L^{\vec{q}}(|x|>1)} \leqslant C\|f\|_{K^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)}.$$

When $0 < q_n \leqslant p$, *as a consequence of the Hölder inequality, there holds*

$$\|f\|^{q_n}_{L^{\vec{q}}(|x|>1)} \leqslant \left(\sum_{k=-\infty}^{0} 2^{-k\alpha q_n(\frac{p}{q_n})'}\right)^{\frac{1}{(\frac{p}{q_n})'}} \left(\sum_{k=-\infty}^{0} 2^{k\alpha p}\|f\chi_k\|^{q_n\frac{p}{q_n}}_{L^{\vec{q}}(\mathbb{R}^n)}\right)^{\frac{q_n}{p}} \leqslant C\|f\|^{q_n}_{K^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)}.$$

When $q_n > p$, *we also have*

$$\|f\|^{q_n}_{L^{\vec{q}}(|x|>1)} \leqslant \sum_{k=1}^{\infty}\|f\chi_k\|^{q_n}_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant \left(\sum_{k=1}^{\infty}\|f\chi_k\|^{p}_{L^{\vec{q}}(\mathbb{R}^n)}\right)^{\frac{q_n}{p}} \leqslant C\|f\|^{q_n}_{K^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)}.$$

The above estimates indicate that $K^{\alpha,p}_{\vec{q}}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n) \bigcap L^{\vec{q}}(\mathbb{R}^n)$ *holds.*

By using (3) and the boundedness of $\mathcal{M}_N$ *on mixed spaces* $L^{\vec{q}}(\mathbb{R}^n)$ *from Proposition 2.11, we can immediately show this proposition.* □

3. Atom characterization of mixed Herz-Hardy spaces

We begin with the definition of central atoms.

Definition 3.1. *Let* $1 < \vec{q} < \infty$, $n(1 - \frac{1}{n}\sum_{i=1}^{m}\frac{1}{q_i}) \leqslant \alpha < \infty$, *and non-negative integer* $s \geqslant \left[\alpha - n(1 - \frac{1}{n}\sum_{i=1}^{m}\frac{1}{q_i})\right]$.

(i) *A function* a *on* $\mathbb{R}^n$ *is said to be a central* $(\alpha, \vec{q})$*-atom, if it satisfies*

(a) $\operatorname{supp} a \subset B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$.

(b) $\|a\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant |B(0, r)|^{-\alpha/n}$.

(c) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, |\beta| \leqslant s$.

(ii) *A function* a *on* $\mathbb{R}^n$ *is said to be a central* $(\alpha, \vec{q})$*-atom of restricted type, if it satisfies the conditions* (b), (c) *above and* (a)′ $\operatorname{supp} a \subset B(0, r), r \geqslant 1$.

Remark 3.2. *If* $\vec{q} = q$ *is a constant, we recover the classical* (α, q)*-atom since* $1 - \frac{1}{n}\sum_{i=1}^{m}\frac{1}{q_i} = 1 - \frac{1}{q}$.

Theorem 3.3. *Let* $1 < \vec{q} < \infty$, $0 < p < \infty$ *and* $n(1 - \frac{1}{n}\sum_{i=1}^{m}\frac{1}{q_i}) \leqslant \alpha < \infty$. *Then*

(i) $f \in H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ *if and only if*

$$f = \sum_{k=-\infty}^{\infty}\lambda_k a_k \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k *is a central* $(\alpha, \vec{q})$*-atom with support contained in* B_k *and* $\sum_{k=0}^{\infty}|\lambda_k|^p < \infty$. *Moreover,*

$$\|f\|_{H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} \approx \inf\left(\sum_{k=0}^{\infty}|\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

(ii) $f \in HK^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$ *if and only if*

$$f = \sum_{k=0}^{\infty}\lambda_k a_k \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha, \vec{q})$-atom of restricted type with support contained in B_k and $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$. Moreover,

$$\|f\|_{HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \approx \inf\left(\sum_{k=0}^{\infty} |\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

Proof. We just need to show (i), and (ii) can be proved in the similar way.
Necessity: choose $\phi \in C_0^{\infty}(\mathbb{R}^n)$ with $\phi \geqslant 0$, $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\operatorname{supp}\phi \subset \{x : |x| \leqslant 1\}$. For $j \in \mathbb{Z}_+$, let

$$\phi_{(j)}(x) = 2^{jn}\phi\left(2^j x\right).$$

For each $f \in \mathcal{S}'(\mathbb{R}^n)$, write

$$f^{(j)}(x) = f * \phi_{(j)}(x).$$

It is easy to see that $f^{(j)} \in C^{\infty}(\mathbb{R}^n)$ and $\lim_{j\to\infty} f^{(j)} = f$ in the sense of distribution.

Let ψ be a radial smooth function such that $\operatorname{supp}\psi \subset \{x : 1/2 - \varepsilon \leqslant |x| \leqslant 1 + \varepsilon\}$ with $0 < \varepsilon < 1/4$, $\psi(x) = 1$ for $1/2 \leqslant |x| \leqslant 1$. Set $\psi_k(x) = \psi\left(2^{-k}x\right)$ for $k \in \mathbb{Z}$ and

$$\tilde{A}_{k,\varepsilon} = \left\{x : 2^{k-1} - 2^k\varepsilon \leqslant |x| \leqslant 2^k + 2^k\varepsilon\right\}.$$

From the conditions of ψ, we know that $\operatorname{supp}\psi_k \subset \tilde{A}_{k,\varepsilon}$ and $\psi_k(x) = 1$ for $x \in A_k = \left\{x : 2^{k-1} < |x| \leqslant 2^k\right\}$.

Obviously, $1 \leqslant \sum_{k=-\infty}^{\infty} \psi_k(x) \leqslant 2, |x| > 0$. Write

$$\Phi_k(x) = \begin{cases} \psi_k(x)/\sum_{l=-\infty}^{\infty} \psi_l(x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then, $\sum_{k=-\infty}^{\infty} \Phi_k(x) = 1$ for $x \neq 0$. For some $m \in \mathbb{N}$, we denote the class of all the real polynomials with the degree less than m by $\mathcal{P}_m(\mathbb{R}^n)$. Let $P_k^{(j)}(x) = P_{\tilde{A}_{k,\varepsilon}}\left(f^{(j)}\Phi_k\right)(x)\chi_{\tilde{A}_{k,\varepsilon}}(x) \in \mathcal{P}_m(\mathbb{R}^n)$ be the unique polynomial satisfying

$$\int_{\tilde{A}_{k,\varepsilon}} \left(f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)\right)x^\beta dx = 0, \quad |\beta| \leqslant m = \left[\alpha - n\left(1 - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}\right)\right].$$

Denote by

$$f^{(j)}(x) = \sum_{k=-\infty}^{\infty}\left(f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)\right) + \sum_{k=-\infty}^{\infty} P_k^{(j)}(x) = \sum_{I}^{(j)} + \sum_{II}^{(j)}.$$

For the term $\sum_I^{(j)}$, let $g_k^{(j)}(x) = f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)$. Now we deal with $\left\|g_k^{(j)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)}$. To do this, by the estimate in [17, pp.1369], we know that

$$\left|P_k^{(j)}(x)\right| \leqslant \frac{C}{\left|\tilde{A}_{k,\varepsilon}\right|}\int_{\tilde{A}_{k,\varepsilon}}\left|f^{(j)}(x)\Phi_k(x)\right|dx.$$

Moreover, by the Hölder inequality on mixed norm spaces, we have

$$\left|P_k^{(j)}(x)\right| \leqslant \frac{C}{\left|\tilde{A}_{k,\varepsilon}\right|}\left\|f^{(j)}\Phi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\left\|\chi_{\tilde{A}_{k,\varepsilon}}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)}.$$

Therefore, we have

$$\begin{aligned}\left\|g_k^{(j)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} &\leqslant \left\|f^{(j)}\Phi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)} + \left\|P_k^{(j)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant \left\|f^{(j)}\Phi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)} + C\left\|f^{(j)}\Phi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant C\left\|\left(f*\phi_{(j)}\right)\Phi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant C'\sum_{l=k-1}^{k+1}\left\|(\mathcal{M}_N f)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)}.\end{aligned}$$

Now write

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left(f^{(j)}(x)\Phi_k(x)-P_k^{(j)}(x)\right) &= \sum_{k=-\infty}^{\infty} C'\left|B_{k+1}\right|^{\frac{\alpha}{n}}\sum_{l=k-1}^{k+1}\left\|(\mathcal{M}_N f)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\quad\times\frac{f^{(j)}(x)\Phi_k(x)-P_k^{(j)}(x)}{C'\left|B_{k+1}\right|^{\frac{\alpha}{n}}\sum_{l=k-1}^{k+1}\left\|(\mathcal{M}_N f)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)}} \\ &= \sum_{k=-\infty}^{\infty}\lambda_k a_k^{(j)}.\end{aligned}$$

Then, $\left\|a_k^{(j)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant |B_{k+1}|^{-\alpha/n}$ *and each* $a_k^{(j)}$ *is a central* $(\alpha,\vec{q})$*-atom with support contained in* B_{k+1}.

$$\begin{aligned}\sum_{k=-\infty}^{\infty}|\lambda_k|^p &\leqslant C\sum_{k=-\infty}^{\infty}|B_{k+1}|^{\frac{\alpha p}{n}}\sum_{l=k-1}^{k+1}\left\|(\mathcal{M}_N(f)\chi_l)\right\|_{L^{\vec{q}}(\mathbb{R}^n)}^p \\ &\leqslant C\left\|\mathcal{M}_N f\right\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}^p \leqslant C\left\|f\right\|_{H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}^p.\end{aligned}$$

It remains to estimate $\sum_{II}^{(j)}$*. Let* $\left\{\psi_d^k : |d|\leqslant m\right\}$ *be the dual basis of* $\left\{x^\beta : |\beta|\leqslant m\right\}$ *with respect to the weight* $1/\left|\tilde{A}_{k,\varepsilon}\right|$ *on* $\tilde{A}_{k,\varepsilon}$*, that is*

$$\left\langle\psi_d^k, x^\beta\right\rangle = \frac{1}{\left|\tilde{A}_{k,\varepsilon}\right|}\int_{\tilde{A}_{k,\varepsilon}} x^\beta\psi_d^k(x)dx = \delta_{\beta d}.$$

Let

$$h_{k,d}^{(j)}(x) = \sum_{l=-\infty}^{k}\left(\frac{\psi_d^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{\left|\tilde{A}_{k,\varepsilon}\right|}-\frac{\psi_d^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{\left|\tilde{A}_{k+1,\varepsilon}\right|}\right)\int_{\mathbb{R}^n} f^{(j)}(x)\Phi_l(x)x^d dx.$$

We can write

$$\sum_{II}^{(j)} = \sum_{|d|\leqslant m}\sum_{k=-\infty}^{\infty}\frac{\alpha_{k,d}h_{k,d}^{(j)}(x)}{\alpha_{k,d}} = \sum_{|d|\leqslant m}\sum_{k=-\infty}^{\infty}\alpha_{k,d}a_{k,d}^{(j)}(x),$$

where

$$\alpha_{k,d} = C''\sum_{l=k-1}^{k+1}\left\|(\mathcal{M}_N f)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)}|B_{k+2}|^{\frac{\alpha}{n}}.$$

Note that

$$\int_{\mathbb{R}^n}\sum_{l=-\infty}^{k}\left|\Phi_l(x)x^d\right|dx = \sum_{l=-\infty}^{k}\int_{\tilde{A}_{k,\varepsilon}}\left|\Phi_l(x)x^d\right|dx \leqslant C2^{k(n+|d|)}.$$

Furthermore,

$$\left|\int_{\mathbb{R}^n} f^{(j)}(y)\sum_{l=-\infty}^{k}\Phi_l(y)y^d dy\right| \leqslant C2^{k(n+|d|)}\mathcal{M}_N f(x), \quad x\in B_{k+2}. \tag{4}$$

Inequality (4), together with the inequality that

$$\left(\frac{\psi_d^k(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{\left|\tilde{A}_{k,\varepsilon}\right|}-\frac{\psi_d^{k+1}(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{\left|\tilde{A}_{k+1,\varepsilon}\right|}\right)\leqslant C2^{-k(n+|d|)}\sum_{l=k-1}^{k+1}\chi_l(x),$$

shows that

$$\left\|h_{k,d}^{(j)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\leqslant C_1\sum_{l=k-1}^{k+1}\left\|(\mathcal{M}_N f)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)},$$

and

$$\left\|a_{k,d}^{(j)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\leqslant\left\|\frac{h_{k,d}^{(j)}}{C\sum\limits_{l=k-1}^{k+1}\left\|(\mathcal{M}_N f)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)}|B_{k+2}|^{-\frac{\alpha}{n}}}\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\leqslant C|B_{k+2}|^{-\frac{\alpha}{n}}.$$

It has been check that $a_{k,d}^{(j)}$ is a central $(\alpha,\vec{q})$-atom with support contained in $\tilde{A}_{k,\varepsilon}\cup\tilde{A}_{k+1,\varepsilon}\subset B_{k+2}$, and $C''=C_1$, $\alpha_{k,d}=C''\sum_{l=k-1}^{k+2}\left\|(Gf)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)}|B_{k+2}|^{\alpha/n}$. Moreover, we also can get

$$\sum_{k,d}\left|\alpha_{k,d}\right|^p\leqslant C\sum_{k=-\infty}^{\infty}|B_{k+2}|^{\frac{\alpha p}{n}}\left(\sum_{l=k-1}^{k+1}\left\|(\mathcal{M}_N f)\chi_l\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\right)^p\leqslant C\left\|\mathcal{M}_N f\right\|_{K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}^p<\infty.$$

Thus we obtain that

$$f^{(j)}(x)=\sum_{d=-\infty}^{\infty}\lambda_d a_d^{(j)}(x),$$

where each $a_d^{(j)}$ is a central $(\alpha,\vec{q})$-atom with support contained in $\tilde{A}_{d,\varepsilon}\cup\tilde{A}_{d+1,\varepsilon}\subset B_{d+2}$, and

$$\left(\sum_{d=-\infty}^{\infty}|\lambda_d|^p\right)^{1/p}\leqslant C\left\|\mathcal{M}_N f\right\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}^p<\infty.$$

Since $\sup_{j\in\mathbb{Z}_+}\left\|a_0^{(j)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\leqslant|B_2|^{-\alpha/n}$, by the Banach-Alaoglu theorem we can obtain a subsequence $\left\{a_0^{(j_{n_0})}\right\}$ of $\left\{a_0^{(j)}\right\}$ converging in the weak* *topology of $L^{\vec{q}}(\mathbb{R}^n)$ to some $a_0\in L^{\vec{q}}(\mathbb{R}^n)$ and a_0 is a central $(\alpha,\vec{q})$-atom supported on B_2. Repeating the above procedure for each $d\in\mathbb{Z}$, we can find a subsequence $\left\{a_d^{(j_{n_d})}\right\}$ of $\left\{a_d^{(j)}\right\}$ converging* weak* *in $L^{\vec{q}}(\mathbb{R}^n)$ to some $a_d\in L^{\vec{q}}(\mathbb{R}^n)$ which is a central $(\alpha,\vec{q})$-atom supported on B_{d+2}. By the usual diagonal method we obtain a subsequence $\{j_\nu\}$ of $\mathbb{Z}_+$ such that for each $d\in\mathbb{Z}$, $\lim_{\nu\to\infty}a_d^{(j_\nu)}=a_d$ in the* weak* *topology of $L^{\vec{q}}(\mathbb{R}^n)$ and in $\mathcal{S}'(\mathbb{R}^n)$. Now our proof is reduced to proving that*

$$f=\sum_{d=-\infty}^{\infty}\lambda_d a_d,\ \text{in the sense of } \mathcal{S}'(\mathbb{R}^n).$$

For each $\varphi\in\mathcal{S}(\mathbb{R}^n)$, noting that $\operatorname{supp} a_d^{(j_\nu)}\subset\left(\tilde{A}_{d,\varepsilon}\cup\tilde{A}_{d+1,\varepsilon}\right)\subset(A_{d-1}\cup A_d\cup A_{d+1}\cup A_{d+2})$, we have

$$\langle f,\varphi\rangle=\lim_{\nu\to\infty}\sum_{d=-\infty}^{\infty}\lambda_d\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx.$$

Recall that $m = \left[\alpha - \sum_{i=1}^{n} \frac{1}{q_i'}\right]$. *If* $d \leqslant 0$, *by using the Hölder inequality on mixed-norm spaces, then*

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)dx\right| &= \left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\left(\varphi(x) - \sum_{|\beta|\leqslant m} \frac{D^\beta\varphi(0)}{\beta!}x^\beta\right)dx\right| \\ &\leqslant C\int_{\mathbb{R}^n} \left|a_d^{(j_\nu)}(x)\right| \cdot |x|^{m+1}dx \\ &\leqslant C2^{d(m+1)} \left\|a_d^{(j_\nu)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \left\|\chi_{B_{d+2}}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)} \\ &\leqslant C2^{d\left(m+1-\alpha+\sum\limits_{i=1}^{n}\frac{1}{q_i'}\right)}.\end{aligned}$$

If $d > 0$, *let* $k_0 \in \mathbb{Z}_+$ *such that* $k_0 \geqslant \sum_{i=1}^{n} \frac{1}{q_i'} - \alpha$. *Again by using Hölder's inequality on mixed-norm spaces,*

$$\left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)dx\right| \leqslant C\int_{\mathbb{R}^n} \left|a_d^{(j_\nu)}(x)\right| |x|^{-k_0}\,dx \leqslant C2^{-dk_0}\left\|a_d^{(j_\nu)}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \left\|\chi_{B_{d+2}}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)} \leqslant C2^{-d\left(k_0+\alpha-\sum\limits_{i=1}^{n}\frac{1}{q_i'}\right)}.$$

Let

$$\mu_d = \begin{cases} |\lambda_d|\, 2^{d\left(m+1-\alpha+\sum\limits_{i=1}^{n}\frac{1}{q_i'}\right)}, & d \leqslant 0, \\ |\lambda_d|\, 2^{-d\left(k_0+\alpha-\sum\limits_{i=1}^{n}\frac{1}{q_i'}\right)}, & d > 0. \end{cases}$$

Then

$$\sum_{d=-\infty}^{\infty} \left|\mu_d\right| \leqslant C\left(\sum_{d=-\infty}^{\infty} |\lambda_d|^p\right)^{\frac{1}{p}} \leqslant C\left\|\mathcal{M}_N f\right\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} < \infty$$

and

$$|\lambda_d|\left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)dx\right| \leqslant C\left|\mu_d\right|,$$

which imply that

$$\langle f, \varphi\rangle = \sum_{d=-\infty}^{\infty} \lim_{\nu\to\infty} \lambda_d \int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)dx = \sum_{d=-\infty}^{\infty} \lambda_d \int_{\mathbb{R}^n} a_d(x)\varphi(x)dx. \tag{5}$$

This mean that $f = \sum_{d=-\infty}^{\infty} \lambda_d a_d$ *in the sense of* $\mathcal{S}'(\mathbb{R}^n)$.

Sufficiency: we will prove the conclusion for two different cases: $0 < p \leqslant 1$ *and* $1 < p < \infty$.

If $0 < p \leqslant 1$, *it suffices to show that for each central* $(\alpha, \vec{q})$*-atom* a,

$$\|\mathcal{M}_N a\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \leqslant C$$

with the constant $C > 0$ *independent of* a.

For a fixed central $(\alpha, \vec{q})$ *-atom* a, *with* $\operatorname{supp} a(x) \subset B\left(0, 2^{k_0}\right)$ *for some* $k_0 \in \mathbb{Z}$. *Write*

$$\|\mathcal{M}_N a\|_{\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}^p = \sum_{k=-\infty}^{k_0+3} 2^{k\alpha p}\left\|(\mathcal{M}_N a)\,\chi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)}^p + \sum_{k=k_0+4}^{\infty} 2^{k\alpha p}\left\|(\mathcal{M}_N a)\,\chi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)}^p = I + II.$$

By the $L^{\vec{q}}(\mathbb{R}^n)$ *boundedness of the grand maximal operator* $\mathcal{M}_N$ *from Proposition 2.11, we have*

$$I \leqslant \sum_{k=-\infty}^{k_0+3} 2^{k\alpha p}\left\|\mathcal{M}_N a\right\|_{L^{\vec{q}}(\mathbb{R}^n)}^p \leqslant C\|a\|_{L^{\vec{q}}(\mathbb{R}^n)}^p \sum_{k=-\infty}^{k_0+3} 2^{k\alpha p} \leqslant C.$$

The next step is to consider II. We need a pointwise estimate for $\mathcal{M}_N a(x)$ on A_k.

Let $\phi \in \mathcal{A}_N$, $m \in \mathbb{N}$ such that $\alpha - \sum_{i=1}^{n} \frac{1}{q_i'} < m+1$. Denote by P_m the m-th order Taylor series expansion. If $|x-y| < t$, then from the vanishing moment condition of a we can get

$$\begin{aligned}\left|a * \phi_t(y)\right| &= t^{-n}\left|\int_{\mathbb{R}^n} a(z)\left(\phi\left(\frac{y-z}{t}\right) - P_m\left(\frac{y}{t}\right)\right)dz\right| \\ &\leqslant Ct^{-n}\int_{\mathbb{R}^n} |a(z)|\left|\frac{z}{t}\right|^{m+1}\left(1+\frac{|y-\theta z|}{t}\right)^{-(n+m+1)} dz \\ &\leqslant C\int_{\mathbb{R}^n} |a(z)||z|^{m+1}(t+|y-\theta z|)^{-(n+m+1)}dz,\end{aligned}$$

where $0<\theta<1$.

Since $x \in A_k$ for $k \geqslant k_0+4$, we have $|x| \geqslant 2\cdot 2^{k_0+1}$. From $|x-y|<t$ and $|z|<2^{k_0+1}$, we have

$$t+|y-\theta z| \geqslant |x-y|+|y-\theta z| \geqslant |x|-|z| \geqslant \frac{|x|}{2}.$$

Thus, by the Hölder inequality, we have

$$\begin{aligned}\left|a * \phi_t(y)\right| &\leqslant C\int_{\mathbb{R}^n} |a(z)||z|^{m+1}(|x-y|+|y-\theta z|)^{-(n+m+1)}dz \\ &\leqslant C2^{k_0(m+1)}|x|^{-(n+m+1)}\int_{\mathbb{R}^n}|a(z)|dz \\ &\leqslant C2^{k_0(m+1)}|x|^{-(n+m+1)}\left|B_{k_0}\right|^{-\frac{\alpha}{n}}\left\|\chi_{B_{k_0}}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, we have

$$\mathcal{M}_N a(x) \leqslant C2^{k_0(m+1)-k(n+m+1)}\left|B_{k_0}\right|^{-\frac{\alpha}{n}}\left\|\chi_{B_{k_0}}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)}, \quad x\in A_k \quad k\geqslant k_0+4.$$

As a consequence,

$$II \leqslant C\sum_{k=k_0+4}^{\infty} 2^{k\alpha p}2^{k_0(m+1)-k(n+m+1)}2^{k_0\alpha p}\left\|\chi_{B_{k_0}}\right\|^p_{L^{\vec{q}'}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant C\sum_{k=k_0+4}^{\infty} 2^{(k_0-k)(m+1-\alpha+\sum\limits_{i=1}^{n}\frac{1}{q_i'})} \leqslant C.$$

If $1<p<\infty$, *write*

$$\left\|\mathcal{M}_N f\right\|^p_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} \lesssim \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\,\|a_l\|_{L^{\vec{q}}(\mathbb{R}^n)}\right)^p + \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=-\infty}^{k-2}|\lambda_l|\,\|(\mathcal{M}_N a_l)\,\chi_k\|_{L^{\vec{q}}(\mathbb{R}^n)}\right)^p = III+IV.$$

Using the Hölder inequality on mixed-norm spaces, we get

$$\begin{aligned}III &\leqslant C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=k-1}^{\infty}|\lambda_l|\,|B_l|^{\frac{\alpha}{n}}\right)^p \\ &\leqslant C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{l=k-1}^{\infty}|\lambda_l|^p\,|B_l|^{\frac{-\alpha p}{2n}}\right)\left(\sum_{l=k-1}^{\infty}|B_l|^{\frac{-\alpha p'}{2n}}\right)^{\frac{p}{p'}} \\ &\leqslant C\sum_{k=-\infty}^{\infty} 2^{k\alpha p/2}\sum_{l=k-1}^{\infty}|\lambda_l|^p\,|B_l|^{\frac{-\alpha p}{2n}} \\ &\leqslant C\sum_{l=-\infty}^{\infty}|\lambda_l|^p.\end{aligned}$$

Now suppose $\alpha - \sum_{i=1}^{n} \frac{1}{q_i'} < m+1$. *As in the argument for II, we can obtain that*

$$\begin{aligned} IV &\leqslant C \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \, 2^{l(m+1)-k(n+m+1)} \left(\frac{|B_k|}{|B_l|} \right)^{\frac{\alpha}{n}} \left\| \chi_{B_l} \right\|_{L^{\vec{q}'}(\mathbb{R}^n)} \left\| \chi_{B_k} \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \right)^p \\ &\leqslant C \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l| \, 2^{(l-k)\left(m+1-\alpha+\sum\limits_{i=1}^{n} \frac{1}{q_i'}\right)} \right)^p \\ &\leqslant C \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-2} |\lambda_l|^p \, 2^{(l-k)\left(m+1-\alpha+\sum\limits_{i=1}^{n} \frac{1}{q_i'}\right)\frac{p}{2}} \right) \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)\left(m+1-\alpha+\sum\limits_{i=1}^{n} \frac{1}{q_i'}\right)\frac{p'}{2}} \right)^{\frac{p}{q'}} \\ &\leqslant C \sum_{l=-\infty}^{\infty} |\lambda_l|^p . \end{aligned}$$

The proof is finished. □

4. Molecular characterization of mixed Herz-Hardy spaces

In this section, we will obtain the molecular decomposition of mixed Herz-type Hardy spaces. We first give the notation of central $(\alpha, \vec{q}; s, \varepsilon)_l$-molecule.

Definition 4.1. *Let* $n - \sum_{i=1}^{n} \frac{1}{q_i} \leqslant \alpha < \infty$, $0 < p < \infty$, $1 < \vec{q} < \infty$, *and* $s \geqslant [\alpha - n + \sum_{i=1}^{n} \frac{1}{q_i}]$ *be a non-negative integer. Set* $\varepsilon > \max\{\frac{s}{n}, \frac{\alpha}{n} + \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} - 1\}$, $a = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} - \frac{\alpha}{n} + \varepsilon$ *and* $b = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} + \varepsilon$. *A function* $M_l \in L^{\vec{q}}(\mathbb{R}^n)$ *with* $l \in \mathbb{Z}$ *(or* $l \in \mathbb{N}$ *) is said to be a dyadic central* $(\alpha, \vec{q}; s, \varepsilon)_l$*-molecule (or dyadic central* $(\alpha, \vec{q}; s, \varepsilon)_l$*)-molecule of restricted type) if it satisfies*

(d) $\|M_l\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant 2^{-l\alpha}$.

(e) $\mathcal{R}_{\vec{q}}(M_l) = \|M_l\|_{L^{\vec{q}}(\mathbb{R}^n)}^{a/b} \left\| |\cdot|^{nb} M_l(\cdot) \right\|_{L^{\vec{q}}(\mathbb{R}^n)}^{1-a/b} < \infty$.

(f) $\int_{\mathbb{R}^n} M_l(x) x^\beta dx = 0$, *for any* β *with* $|\beta| \leqslant s$.

Definition 4.2. *Let* $n - \sum_{i=1}^{n} \frac{1}{q_i} \leqslant \alpha < \infty$, $0 < p < \infty$, $1 < \vec{q} < \infty$, *and* $s \geqslant [\alpha - n + \sum_{i=1}^{n} \frac{1}{q_i}]$ *be a non-negative integer. Set* $\varepsilon > \max\{\frac{s}{n}, \frac{\alpha}{n} + \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} - 1\}$, $a = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} - \frac{\alpha}{n} + \varepsilon$ *and* $b = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} + \varepsilon$.

(i) *A function* $M \in L^{\vec{q}}(\mathbb{R}^n)$ *is said to be a central* $(\alpha, \vec{q}; s, \varepsilon)$*-molecule if it satisfies*

(g) $\mathcal{R}_{\vec{q}}(M) = \|M\|_{L^{\vec{q}}(\mathbb{R}^n)}^{a/b} \left\| |\cdot|^{nb} M(\cdot) \right\|_{L^{\vec{q}}(\mathbb{R}^n)}^{1-a/b} < \infty$.

(h) $\int_{\mathbb{R}^n} M(x) x^\beta dx = 0$, *for any* β *with* $|\beta| \leqslant s$.

(ii) *A function* $M \in L^{\vec{q}}(\mathbb{R}^n)$ *is said to be a central* $(\alpha, \vec{q}; s, \varepsilon)$*-molecule of restricted type if it satisfies* (g),(h) *in* (i) *and* $(g)'$ $\|M\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant 1$.

The following lemma implies that the molecule is a generalization of atom.

Lemma 4.3. *Let* $n - \sum_{i=1}^{n} \frac{1}{q_i} \leqslant \alpha < \infty$, $0 < p < \infty$, $1 < \vec{q} < \infty$, *and* $s \geqslant [\alpha - n + \sum_{i=1}^{n} \frac{1}{q_i}]$ *be a non-negative integer. Set* $\varepsilon > \max\{\frac{s}{n}, \frac{\alpha}{n} + \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} - 1\}$, $a = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} - \frac{\alpha}{n} + \varepsilon$ *and* $b = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{q_i} + \varepsilon$. *If* M *is a central* $(\alpha, \vec{q})$*-atom (or* $(\alpha, \vec{q})$*-atom of restricted type), then* M *is also a central* $(\alpha, \vec{q}; s, \varepsilon)$*-molecule (or* $(\alpha, \vec{q}; s, \varepsilon)$*-molecule of restricted type) such that* $\mathcal{R}_{\vec{q}}(M) \leqslant C$ *with* C *independent of* M.

Proof. *We only need consider the case that* a *is a* $(\alpha, \vec{q})$*-atom with support on a ball* $B(0, r)$. *A straightforward computation leads to that*

$$\|M\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant 2^{-\alpha l}$$

Furthermore,

$$\|M\|_{L^{\vec{q}}(\mathbb{R}^n)}^{a/b} \left\||\cdot|^{nb}M(\cdot)\right\|_{L^{\vec{q}}(\mathbb{R}^n)}^{1-a/b} \leqslant r^{nb(1-a/b)}\|M\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant Cr^{\alpha}r^{-\alpha} \leqslant C.$$

This finishes the proof. □

Now we give the molecular decomposition of mixed Herz-type Hardy spaces.

Theorem 4.4. *Let $n-\sum_{i=1}^{n}\frac{1}{q_i} \leqslant \alpha < \infty$, $0<p<\infty$, $1<\vec{q}<\infty$, and $s \geqslant [\alpha - n + \sum_{i=1}^{n}\frac{1}{q_i}]$ be a non-negative integer. Set $\varepsilon > \max\{\frac{s}{n}, \frac{\alpha}{n}+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}-1\}$, $a = 1-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}-\frac{\alpha}{n}+\varepsilon$ and $b = 1-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}+\varepsilon$. Then we have*

(i) *$f \in H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ if and only if f can be represented as*

$$f = \sum_{k=-\infty}^{\infty}\lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each M_k is a dyadic central $(\alpha,\vec{q};s,\varepsilon)_k$-molecule, and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$. Moreover,

$$\|f\|_{H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \approx \inf\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

(ii) *$f \in HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ if and only if*

$$f = \sum_{k=0}^{\infty}\lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each M_k is a dyadic central $(\alpha,\vec{q};s,\varepsilon)_k$-molecule of restricted type, and $\sum_{k=0}^{\infty}|\lambda_k|^p<\infty$. Moreover,

$$\|f\|_{HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \approx \inf\left(\sum_{k=0}^{\infty}|\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

Theorem 4.5. *Let $n-\sum_{i=1}^{n}\frac{1}{q_i} \leqslant \alpha < \infty$, $0<p<\infty$, $1<\vec{q}<\infty$, and $s \geqslant [\alpha - n + \sum_{i=1}^{n}\frac{1}{q_i}]$ be a non-negative integer. Set $\varepsilon > \max\{\frac{s}{n}, \frac{\alpha}{n}+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}-1\}$, $a = 1-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}-\frac{\alpha}{n}+\varepsilon$ and $b = 1-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}+\varepsilon$. Then we have*

(i) *$f \in H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ if and only if f can be represented as*

$$f = \sum_{k=1}^{\infty}\lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each M_k is a central $(\alpha,\vec{q};s,\varepsilon)$-molecule, and $\sum_{k=1}^{\infty}|\lambda_k|^p<\infty$. Moreover,

$$\|f\|_{H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \approx \inf\left(\sum_{k=1}^{\infty}|\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

(ii) *$f \in HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ if and only if*

$$f = \sum_{k=1}^{\infty}\lambda_k M_k, \quad \text{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each M_k is a central $(\alpha,\vec{q};s,\varepsilon)$-molecule of restricted type, and $\sum_{k=1}^{\infty}|\lambda_k|^p<\infty$. Moreover,

$$\|f\|_{HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}\approx\inf\left(\sum_{k=1}^{\infty}|\lambda_k|^p\right)^{1/p},$$

where the infimum is taken over all above decompositions of f.

Lemma 4.6. *Let $n-\sum_{i=1}^{n}\frac{1}{q_i}\leqslant\alpha<\infty$, $0<p<\infty$, $1<\vec{q}<\infty$, and $s\geqslant[\alpha-n+\sum_{i=1}^{n}\frac{1}{q_i}]$ be a non-negative integer. Set $\varepsilon>\max\{\frac{s}{n},\frac{\alpha}{n}+\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}-1\}$, $a=1-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}-\frac{\alpha}{n}+\varepsilon$ and $b=1-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i}+\varepsilon$.*

(i) *If $0<p\leqslant1$, there exists a constant C such that for any central $(\alpha,\vec{q};s,\varepsilon)$ molecule (or $(\alpha,\vec{q};s,\varepsilon)$-molecule of restricted type) M,*

$$\|M\|_{H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}\leqslant C\left(\text{ or }\|M\|_{HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}\leqslant C\right).$$

(ii) *There exists a constant C such that for any $l\in\mathbb{Z}$(or $l\in\mathbb{N}$) and dyadic central $(\alpha,\vec{q};s,\varepsilon)_l$-molecule (or dyadic central $(\alpha,\vec{q};s,\varepsilon)_l$-molecule of restricted type) M_l,*

$$\|M_l\|_{H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}\leqslant C\left(\text{ or }\ \|M_l\|_{HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)}\leqslant C\right).$$

Proof. *We just prove (i) for the homogeneous case, since the non-homogeneous case and the proof of (ii) can be proved similarly.*

Let M be a central $(\alpha,\vec{q};s,\varepsilon)$-molecule. Write

$$\sigma=\|M\|_{L^{\vec{q}}(\mathbb{R}^n)}^{-1/\alpha},\quad E_0=\{x:|x|\leqslant\sigma\}$$

and

$$E_{k,\sigma}=\left\{x:2^{k-1}\sigma<|x|\leqslant2^k\sigma\right\},k\in\mathbb{Z}_+.$$

Set $B_{k,\sigma}=\left\{x:|x|\leqslant2^k\sigma\right\}$ and denote by $\chi_{k,\sigma}$ the characteristic function of $E_{k,\sigma}$. Immediately, we have

$$M(x)=\sum_{k=0}^{\infty}M(x)\chi_{k,\sigma}(x).$$

Let $M_k(x)=M(x)\chi_{k,\sigma}(x)$, $\mathcal{P}_m$ be the class of all real polynomials of degree m, and $P_{E_{k,\sigma}}M_k\in\mathcal{P}_m$ be the unique polynomial satisfying

$$\int_{E_{k,\sigma}}\left(M_k(x)-P_{E_{k,\sigma}}M_k(x)\right)x^\beta dx=0,\quad|\beta|\leqslant s.$$

Set $Q_k(x)=\left(P_{E_{k,\sigma}}M_k\right)(x)\chi_{k,\sigma}(x)$. If we can prove that

(a) there is a constant $C>0$ and a sequences of numbers $\{\lambda_k\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty}|\lambda_k|^p<\infty$, we have

$$M_k-Q_k=\lambda_k a_k,$$

where each a_k is a $(\alpha,\vec{q})$-atom;

(b) $\sum_{k=0}^{\infty}Q_k$ has a $(\alpha,\vec{q})$-atom decomposition; then our desired conclusion can be deduced directly.

We first show (a). Without loss of generality, we suppose that $\mathcal{R}_{\vec{q}}(M)=1$, which implies

$$\left\||\cdot|^{nb}M(\cdot)\right\|_{L^{\vec{q}}(\mathbb{R}^n)}=\|M\|_{L^{\vec{q}}(\mathbb{R}^n)}^{-a/(b-a)}=\sigma^{na}.$$

Let $\left\{\varphi_l^k:|l|\leqslant s\right\}\subset\mathcal{P}_s(\mathbb{R}^n)$ such that

$$\left\langle\varphi_\mu^k,\varphi_\nu^k\right\rangle_{E_{k,\sigma}}=\frac{1}{\left|E_{k,\sigma}\right|}\int_{E_{k,\sigma}}\varphi_\mu^k(x)\varphi_\nu^k(x)dx=\delta_{\mu\nu}.$$

It is easy to deduce that

$$Q_k(x) = \sum_{|l|\leqslant s} \left\langle M_k, \varphi_l^k \right\rangle_{E_{k,\sigma}} \varphi_l^k(x), \text{ if } x \in E_{k,\sigma}$$

and

$$|Q_k(x)| \leqslant \left| \sum_{|l|\leqslant s} \frac{1}{\left|E_{k,\sigma}\right|} \int_{E_{k,\sigma}} M_k(x)\varphi_l^k(x)dx\varphi_l^k(x) \right| \leqslant \frac{C}{\left|E_{k,\sigma}\right|} \int_{E_{k,\sigma}} |M_k(x)|\, dx.$$

Thus for any $k \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\|M_k - Q_k\|_{L^{\vec{q}}(\mathbb{R}^n)} &\leqslant \|M_k\|_{L^{\vec{q}}(\mathbb{R}^n)} + \|Q_k\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\leqslant \|M_k\|_{L^{\vec{q}}(\mathbb{R}^n)} + \frac{C}{\left|E_{k,\sigma}\right|} \|M_k\|_{L^{\vec{q}}(\mathbb{R}^n)} \left\|\chi_{E_{k,\sigma}}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)} \left\|\chi_{E_{k,\sigma}}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\leqslant C \left\| |\cdot|^{nb} M(\cdot) \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \left|2^k\sigma\right|^{-nb} \\
&= C \left|2^k\sigma\right|^{-nb} \sigma^{na} = C2^{-kna} \left|B_{k,\sigma}\right|^{-\alpha}.
\end{aligned}$$

We see that $M_k - Q_k = \lambda_k a_k$, with $\lambda_k = C2^{-kn\alpha}$ and a_k a central $(\alpha, \vec{q})$-atom supported in $B_{k,\sigma}$. Moreover, one can easily get $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$.

Next we will show (b). Let $\left\{\psi_l^k : |l| \leqslant s\right\} \subset \mathcal{P}_s(\mathbb{R}^n)$ be the dual basis of $\{x^\alpha : |\alpha| \leqslant s\}$ with respect to the weight $1/\left|E_{k,\sigma}\right|$ on $E_{k,\sigma}$. First, by the same way in [17, pp.1384-1385], we conclude that there is a constant $C > 0$ such that

$$\left|\psi_l^k(x)\right| \leqslant C\left(2^{k-1}\sigma\right)^{-|l|}. \tag{6}$$

Futhermore, setting

$$N_l^k = \sum_{j=k}^{\infty} \left|E_{j,\sigma}\right| \left\langle M_j, x^l \right\rangle_{E_{j,\sigma}}, \quad k \in \mathbb{N},$$

it is readily to see that

$$N_l^0 = \sum_{j=0}^{\infty} \left|E_{j,\sigma}\right| \left\langle M_j, x^l \right\rangle_{E_{j,\sigma}} = \sum_{j=0}^{\infty} \int_{E_{j,\sigma}} M(x)x^l dx = \int_{\mathbb{R}^n} M(x)x^l dx = 0,$$

and for $k \in \mathbb{Z}_+$, there exists $E_\sigma \subset E_{j,\sigma}$ such that $|E_\sigma| = \min\left\{1, \left|E_{j,\sigma}\right|\right\}$. Therefore,

$$\begin{aligned}
\left|N_l^k\right| &\leqslant \sum_{j=k}^{\infty} \int_{E_{j,\sigma}} \left|M_j(x)x^l\right| dx \\
&\leqslant C \sum_{j=k}^{\infty} \left\|M_j(\cdot)|\cdot|^l\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \left\|\chi_{E_{j,\sigma}}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)}^{\infty} \\
&\leqslant C \sum_{j=k}^{\infty} \left(2^j\sigma\right)^{|l|-nb} \left\| |\cdot|^{nb} M_j(\cdot)\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \left(\frac{\left|E_{j,\sigma}\right|}{|E_\sigma|}\right)^{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i'}} \left\|\chi_{E_\sigma}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)} \\
&\leqslant C\sigma^{|l|-\alpha+\sum_{i=1}^{n}\frac{1}{q_i'}} 2^{k\left(|l|-nb+\sum_{i=1}^{n}\frac{1}{q_i'}\right)}.
\end{aligned}$$

This, together with (6) shows that

$$\left|E_{k,\sigma}\right|^{-1} \left|N_l^k \psi_l^k(x)\chi_{k,\sigma}(x)\right| \leqslant C\sigma^{\sum_{i=1}^{n}\frac{1}{q_i'}-n-\alpha} 2^{-kn\left(b+1-\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i'}\right)} \to 0, \text{ if } k \to \infty. \tag{7}$$

By using (7) and Abel transform, it yields

$$\sum_{k=0}^{\infty} Q_k(x) = \sum_{|l|\leqslant s}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\left|E_{j,\sigma}\right|\left\langle M_j, x^l\right\rangle_{E_{j,\sigma}}\right)\times\left\{\left|E_{k,\sigma}\right|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-\left|E_{k+1,\sigma}\right|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}$$
$$= \sum_{|l|\leqslant s}\sum_{k=0}^{\infty}\left(-N_l^{k+1}\right)\left\{\left|E_{k,\sigma}\right|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-\left|E_{k+1,\sigma}\right|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}.$$

Meanwhile,

$$\begin{aligned}&\left|N_l^{k+1}\left\{\left|E_{k,\sigma}\right|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-\left|E_{k+1,\sigma}\right|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}\right|\\ &\leqslant C\left|N_l^{k+1}\right|\left|E_{k+1,\sigma}\right|^{-1}\left|\psi_l^{k+1}(x)\right|\\ &\leqslant C2^{-kna}\left|E_{k+1,\sigma}\right|^{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{q_i'}-1-\alpha/n}.\end{aligned}$$

Set $\lambda_{lk} = C2^{-kna}$ *and*

$$a_{lk} = \lambda_{lk}^{-1}\left(-N_l^{k+1}\right)\left\{\left|E_{k,\sigma}\right|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-\left|E_{k+1,\sigma}\right|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\right\}.$$

Then

$$\sum_{k=0}^{\infty} Q_k(x) = \sum_{|l|\leqslant s}\sum_{k=0}^{\infty}\lambda_{lk}a_{lk}$$

with a_{lk} *a* $(\alpha,\vec{q})$*-atom, and* $\sum_{|l|\leqslant s}\sum_{k=0}^{\infty}|\lambda_{lk}|^p < \infty$. *The conclusion (b) then holds.* □

5. Application

In this section, we will give two applications for atom decomposition and molecular decomposition of mixed Herz-Hardy spaces. This applications are devoted to building a boundedness criterion for certain sublinear operators from $H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ (from $HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ to $K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$). The result can be stated as follows.

Theorem 5.1. *Let* $\sum_{i=1}^{n}\frac{1}{q_i'} \leqslant \alpha < \infty, 0 < p < \infty, 1 < \vec{q} < \infty$ *and the integer* $s = \left[\alpha - \sum_{i=1}^{n}\frac{1}{q_i'}\right]$. *Suppose T is a sublinear operator satisfying:*

(i) *T is bounded on* $L^{\vec{q}}(\mathbb{R}^n)$*;*

(ii) *there exists a constant* $\delta > 0$ *such that* $s+\delta > \alpha - \sum_{i=1}^{n}\frac{1}{q_i'}$*, and for any compact support function f with*

$$\int_{\mathbb{R}^n} f(x)x^\beta dx = 0, \quad |\beta| \leqslant s,$$

Tf satisfies the size condition

$$|Tf(x)| \leqslant C(\mathrm{diam}(\mathrm{supp}\, f))^{s+\delta}|x|^{-(n+s+\delta)}\|f\|_{L^1(\mathbb{R}^n)}, \quad \textit{if } \mathrm{dist}(x,\mathrm{supp}\, f) \geqslant |x|/2.$$

Then T can be extended to be a bounded operator from $H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ *to* $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ *(from* $HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ *to* $K_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$*).*

Proof. It suffices to prove the homogeneous case. Suppose $f \in H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)$. By Theorem 3.3, we may rewrite f as $f = \sum_{j=-\infty}^{\infty} \lambda_j b_j$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each b_j is a central $(\alpha, \vec{q})$-atom with support contained in B_j and

$$\|f\|_{H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} \approx \inf\left(\sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p\right)^{1/p}.$$

Then, we get

$$\|Tf\|^p_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} \leqslant C\left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \left\|\left(Tb_j\right)\chi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\right)^p + \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{\infty} \left|\lambda_j\right| \left\|\left(Tb_j\right)\chi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\right)^p\right) = C\left(I_1 + I_2\right).$$

Let us first estimate I_1. By the size condition of T and the Hölder inequality, it yields

$$\begin{aligned}
\left|Tb_j(x)\right| &\leqslant C\left(\operatorname{diam}(\operatorname{supp} b_j)\right)^{s+\delta} |x|^{-(n+s+\delta)} \|b_j\|_{L^1(\mathbb{R}^n)} \\
&\leqslant C|x|^{-(n+s+\delta)} 2^{j(s+\delta)} \left\|b_j\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \left\|\chi_{B_j}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)} \\
&\leqslant C 2^{j(s+\delta-\alpha)-k(s+\delta+n)} \left\|\chi_{B_j}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)}.
\end{aligned}$$

As a consequence,

$$\left\|\left(Tb_j\right)\chi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant C 2^{j(s+\delta-\alpha)-k(s+\delta+n)} \left\|\chi_{B_j}\right\|_{L^{\vec{q}'}(\mathbb{R}^n)} \left\|\chi_{B_k}\right\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant C 2^{(j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}\right)-j\alpha}.$$

Therefore, when $0 < p \leqslant 1$, by $\sum_{i=1}^n \frac{1}{q_i'} \leqslant \alpha < s + \delta + \sum_{i=1}^{n} \frac{1}{q_i'}$, we get

$$\begin{aligned}
I_1 &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| \left\|\left(Tb_j\right)\chi_k\right\|_{L^{\vec{q}}(\mathbb{R}^n)}\right)^p \\
&\leqslant C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right|^p 2^{\left((j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}\right)-j\alpha\right)p}\right) \\
&= C\sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \sum_{k=j+2}^{\infty} 2^{(j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}-\alpha\right)p} \leqslant C\sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p.
\end{aligned}$$

For $1 < p < \infty$, noting that $\sum_{i=1}^n \frac{1}{q_i'} \leqslant \alpha < s + \delta + \sum_{i=1}^n \frac{1}{q_i'}$, by the estimate of $|Tb_j(x)|$ and the Hölder inequality, we have

$$\begin{aligned}
I_1 &\leqslant C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right| 2^{(j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}\right)-j\alpha}\right)^p \\
&\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right|^p 2^{(j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}-\alpha\right)\frac{p}{2}}\right) \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}-\alpha\right)\frac{p'}{2}}\right)^{\frac{p}{p'}} \\
&\leqslant C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} \left|\lambda_j\right|^p 2^{(j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}-\alpha\right)\frac{p}{2}}\right) \\
&\leqslant C\sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p \sum_{k=j+2}^{\infty} 2^{(j-k)\left(s+\delta+\sum_{i=1}^{n}\frac{1}{q_i'}-\alpha\right)\frac{p}{2}} \leqslant C\sum_{j=-\infty}^{\infty} \left|\lambda_j\right|^p.
\end{aligned}$$

Let us now estimate I_2. When $0 < p \leqslant 1$, by the $L^{\vec{q}}(\mathbb{R}^n)$ boundedness of T, there holds

$$\begin{aligned}
I_2 &= \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \left\| \left(Tb_j\right) \chi_k \right\|_{L^{\vec{q}}(\mathbb{R}^n)} \right)^p \\
&\leqslant C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p \|b_j\|^p_{L^{\vec{q}}(\mathbb{R}^n)} \right) \\
&\leqslant C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p |B_j|^{\frac{-\alpha p}{n}} \right) \\
&= C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p} \leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p .
\end{aligned}$$

When $1 < p < \infty$, again by the $L^{\vec{q}}(\mathbb{R}^n)$ boundedness of T and the Hölder inequality, we have

$$\begin{aligned}
I_2 &\leqslant C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p \left\| \left(Tb_j\right) \chi_k \right\|^{\frac{p}{2}}_{L^{\vec{q}}(\mathbb{R}^n)} \right) \left(\sum_{j=k-1}^{\infty} \left\| \left(Tb_j\right) \chi_k \right\|^{\frac{p'}{2}}_{L^{\vec{q}}(\mathbb{R}^n)} \right)^{\frac{p}{p'}} \\
&\leqslant C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p \|b_j\|^{\frac{p}{2}}_{L^{\vec{q}}(\mathbb{R}^n)} \right) \left(\sum_{j=k-1}^{\infty} \|b_j\|^{\frac{p'}{2}}_{L^{\vec{q}}(\mathbb{R}^n)} \right)^{\frac{p}{p'}} \\
&\leqslant C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p |B_j|^{\frac{-\alpha p}{2n}} \right) \left(\sum_{j=k-1}^{\infty} |B_j|^{\frac{-\alpha p'}{2n}} \right)^{\frac{p}{p'}} \\
&\leqslant C \sum_{k=-\infty}^{\infty} 2^{k\alpha p/2} \left(\sum_{j=k-1}^{\infty} |\lambda_j|^p |B_j|^{\frac{-\alpha p}{2n}} \right) \leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j|^p .
\end{aligned}$$

Combining all the estimates above, we arrive at

$$\|Tf\|_{\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)} \leqslant C \|f\|_{H\dot{K}^{\alpha,p}_{\vec{q}}(\mathbb{R}^n)}.$$

Thus, the proof is completed. □

Now we give some concrete operators satisfying the conditions of Theorem 5.1.

The classical Calderón-Zygmund operator T is a initially $L^2(\mathbb{R}^n)$ bounded operator with the associated standard kernel K, that is to say, functions $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$ satisfying

(i) the size condition:

$$|K(x, y)| \leqslant \frac{A}{|x - y|^n},$$

for some constant $A > 0$;

(ii) the regularity conditions: for some $\delta > 0$,

$$|K(x, y) - K(x', y)| \leqslant \frac{A |x - x'|^\delta}{\left(|x - y| + |x' - y|\right)^{n+\delta}}$$

holds whenever $|x - x'| \leqslant \frac{1}{2} \max\left(|x - y|, |x' - y|\right)$ and

$$|K(x, y) - K(x, y')| \leqslant \frac{A |y - y'|^\delta}{\left(|x - y| + |x - y'|\right)^{n+\delta}}$$

holds whenever $\left|y-y'\right| \leqslant \frac{1}{2}\max\left(|x-y|, \left|x-y'\right|\right)$.

Then T can be represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \notin \operatorname{supp} f,$$

which is obvious a $L^{\vec{q}}(\mathbb{R}^n)$-bounded operator.

Theorem 5.2. *Suppose that T is an above Calderón-Zygmund operator, and that* $0 < \delta \leqslant 1$ *is the constant associated with the standard kernel K, then for* $\sum_{i=1}^n \frac{1}{q_i'} \leqslant \alpha < \sum_{i=1}^n \frac{1}{q_i'} + \delta$ *and* $0 < p < \infty$, *T is bounded from* $H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$ *to* $\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)$.

Proof. Noting that $\sum_{i=1}^n \frac{1}{q_i'} \leqslant \alpha < \sum_{i=1}^n \frac{1}{q_i'} + \delta$ implies that $s = [\alpha - \sum_{i=1}^n \frac{1}{q_i'}] = 0$, the operator T satisfies the conditions of Theorem 5.1 with $s = 0$. The desired conclusion follows directly. □

Theorem 5.3. *For any central* $(\alpha,\vec{q})$*-atom* f, *let*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \ x \notin \operatorname{supp} f$$

satisfying $\int_{\mathbb{R}^n} Tf(x)dx = 0$ *be a bounded operator on* $L^{\vec{q}}(\mathbb{R}^n)$ *for some* $1 < \vec{q} < \infty$, *and the kernel K satisfies that there are constants* $C' > 0$ *and* $0 < \delta \leqslant 1$ *such that*

$$|K(x,y) - K(x,0)| \leqslant C' \frac{|y|^\delta}{|x-y|^{n+\delta}}, \quad |x| \geqslant 2|y|.$$

Then for any α *and* p *with* $\sum_{i=1}^n \frac{1}{q_i'} \leqslant \alpha < \sum_{i=1}^n \frac{1}{q_i'} + \delta$ *and* $0 < p < \infty$, *there exists a constant C such that* $\|Tf\|_{H\dot{K}_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \leqslant C \left(\text{ or } \|Tf\|_{HK_{\vec{q}}^{\alpha,p}(\mathbb{R}^n)} \leqslant C\right)$.

Proof. We only prove the homogeneous case. Let f be a central $(\alpha,\vec{q})$*-atom supporting in* $B(0,r)(r>0)$. *It suffices to show Tf is a central* $(\alpha,\vec{q};0,\varepsilon)$*-molecule for some* $1 + \frac{\delta}{n} - \frac{1}{n}\sum_{i=1}^n \frac{1}{q_i'} \geqslant \varepsilon > \frac{\alpha}{n} - \frac{1}{n}\sum_{i=1}^n \frac{1}{q_i'}$. *Let* $a = \frac{1}{n}\sum_{i=1}^n \frac{1}{q_i'} - \frac{\alpha}{n} + \varepsilon, b = \frac{1}{n}\sum_{i=1}^n \frac{1}{q_i'} + \varepsilon$. *Next, we will verify the size condition for molecules, that is*

$$\mathcal{R}_{\vec{q}}(Tf) = \|Tf\|_{L^{\vec{q}}(\mathbb{R}^n)}^{a/b} \left\| |\cdot|^{nb}(Tf)(\cdot)\right\|_{L^{\vec{q}}(\mathbb{R}^n)}^{1-a/b} \leqslant C,$$

with C independent of f. To do this, we first estimate $\left\||\cdot|^{nb}(Tf)(\cdot)\right\|_{L^{\vec{q}}(\mathbb{R}^n)}$ *In fact, we have*

$$\left\||\cdot|^{nb}(Tf)(\cdot)\right\|_{L^{\vec{q}}(|\cdot|\leqslant 2r)} \leqslant Cr^{nb}\|Tf\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant Cr^{nb-\alpha}.$$

Moreover, the vanishing moment of f and the regularity of K give us that for x with $|x| > 2r$,

$$\begin{aligned}
|Tf(x)| &= \left|\int_{\mathbb{R}^n} K(x,y)f(y)dy\right| \\
&= \left|\int_{\mathbb{R}^n} (K(x,y) - K(x,0))f(y)dy\right| \\
&\leqslant C\int_{\mathbb{R}^n} \frac{|y|^\delta}{|x-y|^{n+\delta}}|f(y)|dy \\
&\leqslant Cr^{n+\delta}|x|^{-(n+\delta)}\frac{1}{\left|B_{(0,r)}\right|}\int_{B_{(0,r)}} |f(y)|dy \\
&\leqslant Cr^{n+\delta}|x|^{-(n+\delta)}Mf(x).
\end{aligned}$$

Therefore, by the fact $nb - n - \delta \leqslant 0$, we further get

$$\begin{aligned}\left\| |\cdot|^{nb}(Tf)(\cdot) \right\|_{L^{\vec{q}}(|\cdot|>2r)} &\leqslant Cr^{n+\delta} \left\| |\cdot|^{nb-n-\delta} Mf(\cdot) \right\|_{L^{\vec{q}}(|\cdot|>2r)} \\ &\leqslant Cr^{n+\delta+nb-n-\delta} \|Mf\|_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant Cr^{nb} \|f\|_{L^{\vec{q}}(\mathbb{R}^n)} \leqslant Cr^{nb-\alpha}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{R}_{\vec{q}}(Tf) &= \|Tf\|^{a/b}_{L^{\vec{q}}(\mathbb{R}^n)} \left\| |\cdot|^{nb}(Tf)(\cdot) \right\|^{1-a/b}_{L^{\vec{q}}(\mathbb{R}^n)} \\ &\leqslant C\|f\|^{a/b}_{L^{\vec{q}}(\mathbb{R}^n)} r^{(nb-\alpha)(1-a/b)} \\ &\leqslant Cr^{-\alpha a/b+(nb-\alpha)(1-a/b)} \leqslant C.\end{aligned}$$

This completes the proof. □

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References

[1] A. Beurling, *Construction and analysis of some convolution algebras*, Ann. I. Fourier **14** (1964), 1-32.
[2] C.S. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mechanics **18** (1968), 283-323.
[3] S.Z. Lu, D.C. Yang, *The Littlewood-Paley function and φ transform characterizations of a new Hardy space HK_2 associated with the Herz space*, Studia Math. **101** (1992), 285-298.
[4] E.M. Stein, G. Weiss, *On the theory of Harmonic functions of several variables: I. The theory of H^p spaces*, Acta Math. **103** (1960), 25-62.
[5] C.L. Fefferman, E.M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137-193.
[6] A. Grossmann, J. Morlet, *Decomposition of Hardy functions into square integrable Wavelets of constant shape*, Siam J. Math. Anal. **15** (1984), 723-736.
[7] M.Y. Lee, C.C. Lin, *The molecular characterization of weighted Hardy spaces*, J. Funct. Anal. **188** (2002), 442-460.
[8] L. Grafakos, *Modern fourier analysis*, (3rd edition), McGraw-Hill, New York, 1986.
[9] Y. Chen, K. Lau, *Some new classes of Hardy spaces*, J. Funct. Anal. **84** (1989), 255-278.
[10] J. García-Cuerva, *Hardy spaces and Beurling algebras*, J. Lond.Math. Soc. **2** (1989), 499-513.
[11] J. García-Cuerva, M.J.L. Herrero, *A theory of Hardy spaces associated to the Herz spaces*, Proc. Lond. Math. Soc. **3** (1994), 605-628.
[12] S.Z. Lu, D.C. Yang, *The local versions of $H^p(\mathbb{R}^n)$ spaces at the origin*, Studia Math. **2** (1995), 103-131.
[13] S.Z. Lu, D.C. Yang, *Some characterizations of weighted Herz-type Hardy spaces and their applications*, Acta Math. Sin. **13** (1997), 47-60.
[14] A. Miyachi, *Remarks on Herz-type Hardy spaces*, Acta Math. Sin. **17** (2001), 332-360.
[15] H.B. Wang, Z.G. Liu, *Some characterizations of Herz-type Hardy spaces with variable exponent*, Ann. Funct. Anal. **1** (2015), 128-141.
[16] H. Zhao, J. Zhou, *Anisotropic Herz-type Hardy spaces with variable exponent and their applications*, Acta Math. Hung. **156** (2018), 309-335.
[17] H.B. Wang, Z.G. Liu, *The Herz-type Hardy spaces with variable exponent and their applications*, Taiwanese J. Math. **16** (2012), 1363-1389.
[18] A. Benedek, R. Panzone, *The space L_p, with mixed norm*, Duke Math. J. **28** (1961), 301-324.
[19] G. Cleanthous, A.G. Georgiadis, M. Nielsen, *Anisotropic mixed-norm Hardy spaces*, J. Geom. Anal. **27** (2017), 2758-2787.
[20] L. Huang, J. Liu, D.C. Yang, W. Yuan, *Atomic and Littlewood–Paley characterizations of anisotropic mixed-norm Hardy spaces and their applications*, J. Geom. Anal. **29** (2019), 1991-2067.
[21] T. Nogayama, *Mixed Morrey spaces*, Positivity **23** (2019), 961-1000.
[22] M.Q. Wei, *Extrapolation to mixed Herz spaces and its applications*, submitted.
[23] T. Nogayama, *A characterization of via the commutator of Hardy-type operators on mixed Herz spaces*, Appl. Anal. (2021), 1-16.
[24] D. S. Kurtz, *Classical operators on mixed-normed spaces with product weights*, The Rocky Mountain J. Math. (2007), 269-283.