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Kenmotsu 3−manifold admitting gradient Ricci-Yamabe solitons and $* - \eta-$Ricci-Yamabe solitons

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Abstract. In this paper, we classify Kenmotsu manifolds admitting gradient Ricci-Yamabe solitons and $* - \eta-$Ricci-Yamabe solitons. We find conditions of Kenmotsu manifold about when it shrink, expand and steady. It is shown that Kenmotsu 3-manifold endowed with gradient Ricci-Yamabe soliton and with constant scalar curvature becomes an Einstein manifold. We, also study Kenmotsu manifold admitting $* - \eta-$ Ricci-Yamabe solitons becomes generalized $\eta-$Einstein manifold and the curvature condition $R.S = 0$. Finally, we provide two examples which proves existence of gradient Ricci-Yamabe soliton and $* - \eta-$ Ricci-Yamabe soliton in Kenmotsu manifolds.

1. Introduction

In Riemannian geometry, geometric flows has an important role to analyzing the geometric structures. In Riemannian geometry, the assumption of geometric flows is a useful mathematical tool for describing geometric structures. A specific section of solutions on which the metric evolves by diffeomorphisms has a signifcant impact in the investigation of singularities of the flows which are called as soliton solutions. R.S. Hamilton ([11], [12]) introduced a geometric flow which is called as Ricci flow and is defined by

$$\frac{\partial}{\partial t} g(t) = -2S(t), t \geq 0 \quad g(0) = g, \tag{1}$$

where g and r represents the Riemannian metric and the scalar curvature[11] respectively. With the help of it, Hamilton proved three-dimensional sphere theorem [12].

A Riemannian manifold is called as Ricci soliton, if there exists a vector field V and a constant λ such that

$$\mathcal{L}_V g + 2S = 2\lambda g, \tag{2}$$

where $\mathcal{L}_V$ is the Lie derivative along the vector field V and $\lambda \in \mathbb{R}$.
After Ricci flow, Hamilton gave the notion of Yamabe flow defined by

$$\frac{\partial}{\partial t} g(t) = -2S(t), t \geq 0 \quad g(0) = g, \tag{3}$$

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where, S denotes the $(0,2)-$ symmetric Ricci tensor. Same as Ricci soliton, Yamabe soliton is a self-similar solution to Yamabe flow [5] and is defined as follows:

$$\mathcal{L}_V g = 2(\lambda - r)g. \tag{4}$$

There are several applications of Ricci flow and Yamabe flow especially in mathematics and physics. S. Gular and M. Crasmareanu [3] defined a map in 2019 known as Ricci-Yamabe flow which is a linear combination of Ricci flow and Yamabe flow given as follows:

$$\frac{\partial}{\partial t}g(t) = c_2 r(t)g(t) - 2S(t), t \geq 0 \quad g(0) = g, \tag{5}$$

where $c_2 \in \mathbb{R}$.

The solutions of Ricci flow and Yamabe flow are called Ricci soliton and Yamabe soliton respectively. Both solitons are equivalent for dimension $n = 2$, although for the dimension $n > 2$, they are not equal. A Ricci-Yamabe soliton on $(\mathcal{M}^n, g)$ is a structure $(g, V, \lambda, c_1, c_2)$ fulfilling

$$\mathcal{L}_V g = -2c_1 S - (2\lambda - c_2 r)g, \tag{6}$$

where $\lambda, c_1, c_2 \in \mathbb{R}$. If V is a gradient of a smooth function f on the manifold $(\mathcal{M}^n, g)$, then the foregoing notion is called gradient Ricci-Yamabe soliton and (6) takes the form

$$\widetilde{\nabla}^2 f = -c_1 S - (\lambda - \frac{c_2 r}{2})g, \tag{7}$$

where $\widetilde{\nabla}^2 f$ indicates the Hession of f.

The Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be expanding, steady and shrinking when $\lambda > 0, = 0, < 0$. Moreover, if λ, c_1 and c_2 are smooth functions on $(\mathcal{M}^n, g)$, then Ricci-Yamabe soliton is called an almost Ricci-Yamabe soliton. Ricci-Yamabe soliton turns into Ricci soliton (or gradient Ricci soliton) if $c_2 = 0, c_1 = 1$. Similarly, for $c_2 = 1, c_1 = 0$ it becomes Yamabe soliton (or gradient Yamabe soliton). Also, it reduces to an Einstein soliton (or gradient Einstein soliton) when $c_2 = -1, c_1 = 1$. The Ricci-Yamabe soliton is said to be proper if $c_1 \neq 0, 1$. For the study of gradient Ricci-Yamabe soliton please refer to [7],[8],[25].

The present paper is organized in following ways: In Section 2 we present the basic results and definitions of Kenmotsu manifolds. Section 3 and 4, deals with the study of Ricci-Yamabe and gradient Ricci-Yamabe solitons in Kenmotsu manifolds. Moreover in section 5 and 6, we study Kenmotsu manifold admitting $* - \eta -$ Ricci -Yamabe solitons with curvature condition $R.S = 0$. Finally we give two examples of such manifold.

2. Preliminaries

Let $(\mathcal{M}^n, g)$ be an n-dimensional smooth manifold endowed with structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is the vector field, η is 1- form and g is the Riemannian metric. It is well known that the structure (ϕ, ξ, η, g) satisfies the following conditions

$$\phi\xi = 0, \quad \eta(\phi\mathcal{X}_1) = 0, \quad \eta(\xi) = 1 \tag{8}$$

$$\phi^2\mathcal{X}_1 = -\mathcal{X}_1 + \eta(\mathcal{X}_1)\xi, \quad g(\mathcal{X}_1, \xi) = \eta(\mathcal{X}_1), \tag{9}$$

$$g(\phi\mathcal{X}_1, \phi\mathcal{X}_2) = g(\mathcal{X}_1, \mathcal{X}_2) - \eta(\mathcal{X}_1)\eta(\mathcal{X}_2), \tag{10}$$

for vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ on $\mathcal{X}(\mathcal{M})$.
Further,

$$(\widetilde{\nabla}_{\mathcal{X}_1}\phi)\mathcal{X}_2 = -g(\mathcal{X}_1, \phi\mathcal{X}_2)\xi - \eta(\mathcal{X}_2)\phi\mathcal{X}_1, \tag{11}$$

$$\widetilde{\nabla}_{\mathcal{X}_1}\xi = \mathcal{X}_1 - \eta(\mathcal{X}_1)\xi, \tag{12}$$

where $\widetilde{\nabla}$ denotes the Riemannian connection g, then (ϕ, ξ, η, g) is called a Kenmotsu manifold.

Kenmotsu manifold have been studied by several authors such as U.C. De and G. Pathak[9], K. Kenmotsu [21] and many others. In Kenmotsu manifold the following relations hold[21]

$$(\widetilde{\nabla}_{\mathcal{X}_1}\eta)\mathcal{X}_2 = g(\mathcal{X}_1, \mathcal{X}_2) - \eta(\mathcal{X}_1)\eta(\mathcal{X}_2), \tag{13}$$

$$\eta(R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3) = \eta(\mathcal{X}_2)g(\mathcal{X}_1, \mathcal{X}_3) - \eta(\mathcal{X}_1)g(\mathcal{X}_2, \mathcal{X}_3), \tag{14}$$

$$R(\mathcal{X}_1, \mathcal{X}_2)\xi = \eta(\mathcal{X}_1)\mathcal{X}_2 - \eta(\mathcal{X}_2)\mathcal{X}_1, \tag{15}$$

$$R(\xi, \mathcal{X}_1)\mathcal{X}_2 = \eta(\mathcal{X}_2)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_2)\xi, \tag{16}$$

$$S(\mathcal{X}_1, \xi) = -(n-1)\eta(\mathcal{X}_1), \tag{17}$$

$$(\widetilde{\nabla}_Z R)(\mathcal{X}_1, \mathcal{X}_2)\xi = g(\mathcal{X}_3, \mathcal{X}_1)\mathcal{X}_2 - g(\mathcal{X}_3, \mathcal{X}_2)\mathcal{X}_1 - R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3, \tag{18}$$

$$R(\mathcal{X}_1, \xi)\mathcal{X}_2 = g(\mathcal{X}_1, \mathcal{X}_2)\xi - \eta(\mathcal{X}_2)\mathcal{X}_1, \tag{19}$$

$$R(\mathcal{X}_1, \xi)\xi = \eta(\mathcal{X}_1)\xi - \mathcal{X}_1, \tag{20}$$

for every vector fields $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \in \mathfrak{X}(\mathcal{M})$, where S and R are the Ricci tensor and Riemannian curvature tensor respectively.

Definition 2.1. *Let $(\mathcal{M}^n, g)$ be a Riemannian manifold of dimension n. Since 3-dimensional Riemannian manifold is conformally flat then the Riemannian curvature tensor R of $(\mathcal{M}^3, g)$ is written as*

$$R(\mathcal{X}_1, W)\mathcal{X}_3 = \{S(W, \mathcal{X}_3)\mathcal{X}_1 - S(\mathcal{X}_1, \mathcal{X}_3)W + g(W, \mathcal{X}_3)Q\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)QW\} - \frac{r}{2}\{g(W, \mathcal{X}_3)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)W\} \tag{21}$$

for any vector fields $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \in \mathfrak{X}(\mathcal{M})$, where Q denotes the Ricci operator defined by $S(\mathcal{X}_1, \mathcal{X}_2) = g(Q\mathcal{X}_1, \mathcal{X}_2)$ and r is the scalar curvature defined by $r = \sum_{i=1}^n S(u_i, u_i) = \sum_{i=1}^n g(Qu_i.u_i)$ for any orthonormal basis $\{u_i\}_{i=1}^n$ of the tangent space at any point of $(\mathcal{M}^n, g)$.

Lemma 2.2. *The Ricci tensor S of a 3-dimensional Kenmotsu manifold is given by*

$$S(\mathcal{X}_1, \mathcal{X}_2) = (1 + \frac{r}{2})g(\mathcal{X}_1, \mathcal{X}_2) - (3 + \frac{r}{2})\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{22}$$

Proof. *Taking inner product of* (21) *with $\mathcal{X}_2$, we have*

$$g(R(\mathcal{X}_1, W)\mathcal{X}_3, \mathcal{X}_2) = S(W, \mathcal{X}_3)g(\mathcal{X}_1, \mathcal{X}_2) - S(\mathcal{X}_1, \mathcal{X}_3)g(W, \mathcal{X}_2) + g(W, \mathcal{X}_3)g(Q\mathcal{X}_1, \mathcal{X}_2) - g(\mathcal{X}_1, \mathcal{X}_3)g(QW, \mathcal{X}_2) - \frac{r}{2}\{g(W, \mathcal{X}_3)g(\mathcal{X}_1, \mathcal{X}_2) - g(\mathcal{X}_1, \mathcal{X}_3)g(W, \mathcal{X}_2). \tag{23}$$

Putting $W = \mathcal{X}_3 = \xi$ in (23) *and using* (15) (17), *we get desired result* (22).
□

Now equation (22) *can be rewritten as*

$$g(Q\mathcal{X}_1, \mathcal{X}_2) = (1 + \frac{r}{2})g(\mathcal{X}_1, \mathcal{X}_2) - (3 + \frac{r}{2})\eta(\mathcal{X}_1)\eta(\mathcal{X}_2).$$

Removing $\mathcal{X}_2$ on both sides, we get

$$Q\mathcal{X}_1 = (1 + \frac{r}{2})\mathcal{X}_1 - (3 + \frac{r}{2})\eta(\mathcal{X}_1)\xi. \tag{24}$$

Lemma 2.3. *In 3-dimensional Kenmotsu manifold, we have*

$$2(r+6)=\xi r. \tag{25}$$

Proof. *Taking covariant derivative of* (24) *along vector field* $\mathcal{X}_3$, *we have*

$$(\widetilde{\nabla}_{\mathcal{X}_3}Q)\mathcal{X}_1=\frac{1}{2}(\mathcal{X}_3 r)\mathcal{X}_1-\frac{1}{2}(\mathcal{X}_3 r)\eta(\mathcal{X}_1)\xi-(3+\frac{r}{2})\{g(\mathcal{X}_1,\mathcal{X}_3)\xi-2\eta(\mathcal{X}_1)\eta(\mathcal{X}_3)\xi+\mathcal{X}_3\eta(\mathcal{X}_1)\}. \tag{26}$$

On contracting with respect to $\mathcal{X}_3$, *we get*

$$(\xi r+2(r+6))\eta(\mathcal{X}_1)=0$$

which implies equation (25) *and hence it completes the proof.* □

3. Ricci-Yamabe solitons in Kenmotsu manifolds

Let us assume that 3-dimensional Kenmotsu manifold $(\mathcal{M}^3,g)$ admits Ricci-Yamabe soliton, then equation (6) yields

$$\mathcal{L}_V g(\mathcal{X}_1,\mathcal{X}_2)=-2c_1 S(\mathcal{X}_1,\mathcal{X}_2)-(2\lambda-c_2 r)g(\mathcal{X}_1,\mathcal{X}_2). \tag{27}$$

Taking covariant derivative of (27), we obtain

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{L}_V g)(\mathcal{X}_1,\mathcal{X}_2)&+(\mathcal{L}_V g)(\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{X}_1,\mathcal{X}_2)\\ +(\mathcal{L}_V g)(\mathcal{X}_1,\widetilde{\nabla}_{\mathcal{X}_3}Y)&=-2c_1\{(\widetilde{\nabla}_{\mathcal{X}_3}S)(\mathcal{X}_1,\mathcal{X}_2)+S(\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{X}_1,\mathcal{X}_2)+S(\mathcal{X}_1,\widetilde{\nabla}_{\mathcal{X}_3}Y)\}\\ &\quad+c_2(\mathcal{X}_3 r)g(\mathcal{X}_1,\mathcal{X}_2)+(c_2 r-2\lambda)\{(\widetilde{\nabla}_{\mathcal{X}_3}g)(\mathcal{X}_1,\mathcal{X}_2)+g(\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{X}_1,\mathcal{X}_2)+g(\mathcal{X}_1,\widetilde{\nabla}_{\mathcal{X}_3}Y)\}\end{aligned} \tag{28}$$

Now, using (27) in (28), we get

$$(\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{L}_V g)(\mathcal{X}_1,\mathcal{X}_2)=-2c_1(\widetilde{\nabla}_{\mathcal{X}_3}S)(\mathcal{X}_1,\mathcal{X}_2)+c_2(\mathcal{X}_3 r)g(\mathcal{X}_1,\mathcal{X}_2). \tag{29}$$

From equation (22), we get

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{X}_3}S)(\mathcal{X}_1,\mathcal{X}_2)&=\frac{1}{2}(\mathcal{X}_3 r)g(\mathcal{X}_1,\mathcal{X}_2)-\frac{1}{2}(\mathcal{X}_3 r)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\\ &\quad-(3+\frac{r}{2})\{g(\mathcal{X}_1,\mathcal{X}_3)\eta(\mathcal{X}_2)+g(\mathcal{X}_2,\mathcal{X}_3)\eta(\mathcal{X}_1)-2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)\},\end{aligned} \tag{30}$$

With the help of (30), (29) becomes

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{L}_V g)(\mathcal{X}_1,\mathcal{X}_2)&=(c_2-c_1)(\mathcal{X}_3 r)g(\mathcal{X}_1,\mathcal{X}_2)+c_1(\mathcal{X}_3 r)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\\ &\quad+c_1(6+r)\{g(\mathcal{X}_1,\mathcal{X}_3)\eta(\mathcal{X}_2)+g(\mathcal{X}_2,\mathcal{X}_3)\eta(\mathcal{X}_1)-2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)\}.\end{aligned} \tag{31}$$

Following Yano [25], the following relation holds

$$(\mathcal{L}_V\widetilde{\nabla}_{\mathcal{X}_1}g-\mathcal{L}_{\mathcal{X}_1}\widetilde{\nabla}_V g-\widetilde{\nabla}_{[\mathcal{X}_1,V]}g)(\mathcal{X}_2,\mathcal{X}_3)=-g((\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1,\mathcal{X}_2),\mathcal{X}_3)-g((\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1,\mathcal{X}_3),\mathcal{X}_2), \tag{32}$$

which implies that

$$(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{L}_V g)(\mathcal{X}_2,\mathcal{X}_3)=g((\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1,\mathcal{X}_2),\mathcal{X}_3)+g((\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1,\mathcal{X}_3),\mathcal{X}_2). \tag{33}$$

Now, since $(\mathcal{L}_V\widetilde{\nabla})$ is a $(1,2)$ type symmetric tensor, i.e.

$(\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \mathcal{X}_2) = (\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_2, \mathcal{X}_1)$, then (33) turns into

$$g((\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \mathcal{X}_2), \mathcal{X}_3) = \frac{1}{2}(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{L}_V g)(\mathcal{X}_2, \mathcal{X}_3) + \frac{1}{2}(\widetilde{\nabla}_Y\mathcal{L}_V g)(\mathcal{X}_1, \mathcal{X}_3) - \frac{1}{2}(\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{L}_V g)(\mathcal{X}_1, \mathcal{X}_2). \tag{34}$$

Using (31) in (34) entails that

$$\begin{aligned}2g((\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \mathcal{X}_2), \mathcal{X}_3) &= (c_2 - c_1)\{(\mathcal{X}_1 r)g(\mathcal{X}_2, \mathcal{X}_3) + (Yr)g(\mathcal{X}_1, \mathcal{X}_3) - (\mathcal{X}_3 r)g(\mathcal{X}_1, \mathcal{X}_2)\}\\ &+ c_1\{(\mathcal{X}_1 r)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3) + (Yr)\eta(\mathcal{X}_1)\eta(\mathcal{X}_3) - (\mathcal{X}_3 r)\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\}\\ &+ 2c_1(6 + r)\{g(\mathcal{X}_1, \mathcal{X}_2)\eta(\mathcal{X}_3) - 2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\eta(\mathcal{X}_3)\}\end{aligned} \tag{35}$$

which implies that

$$\begin{aligned}2(\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \mathcal{X}_2) &= (c_2 - c_1)\{(\mathcal{X}_1 r)\mathcal{X}_2 + (\mathcal{X}_2 r)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_2)(Dr)\}\\ &+ c_1\{(\mathcal{X}_1 r)\eta(\mathcal{X}_2)\xi + (\mathcal{X}_2 r)\eta(\mathcal{X}_1)\xi - \eta(\mathcal{X}_1)\eta(\mathcal{X}_2)(Dr)\} + 2c_1(6 + r)\{g(\mathcal{X}_1, \mathcal{X}_2)\xi - 2\eta(\mathcal{X}_1)\eta(\mathcal{X}_2)\xi\}\end{aligned} \tag{36}$$

Putting $\mathcal{X}_2 = \xi$ in the foregoing equation, we get

$$\begin{aligned}2(\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \xi) &= (c_2 - c_1)\{(\mathcal{X}_1 r)\xi + (\xi r)\mathcal{X}_1 - \eta(\mathcal{X}_1)(Dr)\} + c_1\{(\mathcal{X}_1 r)\xi + (\xi r)\eta(\mathcal{X}_1)\xi - \eta(\mathcal{X}_1)(Dr)\}\\ &+ 2c_1(6 + r)\{\eta(\mathcal{X}_1)\xi - 2\eta(\mathcal{X}_1)\xi\}.\end{aligned} \tag{37}$$

From equation (25), if we take r = constant, we obtain $r = -6$.

Hence equation (37) implies

$$(\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \xi) = 0 \implies (\widetilde{\nabla}_{\mathcal{X}_3}\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \xi) = 0. \tag{38}$$

Next, we know that

$$(\mathcal{L}_V R)(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = (\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_2, \mathcal{X}_3) - (\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \mathcal{X}_3). \tag{39}$$

Replacing $\mathcal{X}_3 = \xi$ in (39), infers

$$(\mathcal{L}_V R)(\mathcal{X}_1, \mathcal{X}_2)\xi = (\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_2, \xi) - (\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{L}_V\widetilde{\nabla})(\mathcal{X}_1, \xi). \tag{40}$$

Using (38) in (40), we infer

$$(\mathcal{L}_V R)(\mathcal{X}_1, \mathcal{X}_2)\xi = 0 \tag{41}$$

Putting $\mathcal{X}_2 = \xi$ in above equation, we get

$$(\mathcal{L}_V R)(\mathcal{X}_1, \xi)\xi = 0. \tag{42}$$

Now equation (16) can be rewritten as

$$R(\mathcal{X}_1, \xi)\mathcal{X}_2 = g(\mathcal{X}_1, \mathcal{X}_2)\xi - \eta(\mathcal{X}_2)\mathcal{X}_1 \tag{43}$$

which implies

$$R(\mathcal{X}_1, \xi)\xi = \eta(\mathcal{X}_1)\xi - \mathcal{X}_1. \tag{44}$$

Taking covariant derivative of (44) and using (15), (43), we get

$$(\mathcal{L}_V R)(\mathcal{X}_1, \xi)\xi = 2\eta(\mathcal{L}_V\xi)\mathcal{X}_1 + (\mathcal{L}_V\eta)\mathcal{X}_1\xi - g(\mathcal{X}_1, \mathcal{L}_V\xi)\xi. \tag{45}$$

Substituting $\mathcal{X}_2 = \xi$ and using (17) (for $n = 3$), we get

$$(\mathcal{L}_V \eta)\mathcal{X}_1 = g(\mathcal{X}_1, \mathcal{L}_V \xi) + (4c_1 + c_2 r - 2\lambda)\eta(\mathcal{X}_1). \tag{46}$$

Putting $\mathcal{X}_1 = \xi$ in foregoing equation, we infers

$$\eta(\mathcal{L}_V \xi) = -\frac{1}{2}(4c_1 + c_2 r - 2\lambda). \tag{47}$$

With the help of (46) and (47), (45) yields

$$(\mathcal{L}_V R)(\mathcal{X}_1, \xi)\xi = -(4c_1 + c_2 r - 2\lambda)(\mathcal{X}_1 - \eta(\mathcal{X}_1)\xi). \tag{48}$$

Using (42) in (48), we get

$$4c_1 + c_2 r - 2\lambda = 0,$$

which by taking $r = -6$ gives

$$\lambda = (2c_1 - 3c_2). \tag{49}$$

Hence we have:

Theorem 3.1. *If 3-dimensional Kenmotsu manifold admits Ricci-Yamabe soliton, then the constant λ, c_1 and c_2 are related by the equation*

$$\lambda = (2c_1 - 3c_2), \tag{50}$$

provided that the scalar curvature r is constant. If we take $c_1 = 1$ and $c_2 = -1$, then from (50)*, we get*

$$\lambda = 5. \tag{51}$$

From which we get the following corollary:

Corollary 3.2. *If $(\mathcal{M}^3, g)$ Kenmotsu manifold with constant scalar curvature admits an Einstein soliton, then the soliton is expanding.*

Again, if we take $c_1 = 0$ and $c_2 = 1$, from (50) *we obtain*

$$\lambda = -3. \tag{52}$$

Thus we have:

Corollary 3.3. *If $(\mathcal{M}^3, g)$ Kenmotsu manifold with constant scalar curvature admits a Yamabe soliton, then the soliton is shrinking.*

Next when we take $c_1 = 1$ and $c_2 = 0$, equation (50)*, implies*

$$\lambda = 2. \tag{53}$$

Hence we have:

Corollary 3.4. *If $(\mathcal{M}^3, g)$ Kenmotsu manifold with constant scalar curvature admits a Ricci soliton, then the soliton is expanding.*

4. Gradient Ricci-Yamabe solitons in Kenmotsu manifolds

Let $(\mathcal{M}^3, g)$ Kenmotsu manifold admits gradient Ricci-Yamabe solitons, then (7) implies

$$\widetilde{\nabla}_{\mathcal{X}_1} Df = -c_1 Q\mathcal{X}_1 - (\lambda - \frac{c_2 r}{2})\mathcal{X}_1. \tag{54}$$

Taking covariant derivative of (54), we obtain

$$\widetilde{\nabla}_{\mathcal{X}_2}\widetilde{\nabla}_{\mathcal{X}_1} Df = -c_1(\widetilde{\nabla}_{\mathcal{X}_2} Q)\mathcal{X}_1 - c_1 Q(\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_1) + \frac{c_2}{2}(\mathcal{X}_2 r)\mathcal{X}_1 - (\lambda - \frac{c_2 r}{2})\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_1. \tag{55}$$

Simillarily from (54), we have

$$\widetilde{\nabla}_{\mathcal{X}_1}\widetilde{\nabla}_{\mathcal{X}_2} Df = -c_1(\widetilde{\nabla}_{\mathcal{X}_1} Q)\mathcal{X}_2 - c_1 Q(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2) + \frac{c_2}{2}(\mathcal{X}_1 r)\mathcal{X}_2 - (\lambda - \frac{c_2 r}{2})\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2 \tag{56}$$

and

$$\widetilde{\nabla}_{[\mathcal{X}_1,\mathcal{X}_2]} Df = -c_1 Q(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2) + c_1 Q(\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_1) - (\lambda - \frac{c_2 r}{2})\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2 + (\lambda - \frac{c_2 r}{2})\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_1. \tag{57}$$

We know that

$$R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = \widetilde{\nabla}_{\mathcal{X}_1}\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_3 - \widetilde{\nabla}_{\mathcal{X}_2}\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_3 - \widetilde{\nabla}_{[\mathcal{X}_1,\mathcal{X}_2]}. \tag{58}$$

Using equation (55) – (57), in equation (58), we have

$$R(\mathcal{X}_1, \mathcal{X}_2)Df = -c_1\{(\widetilde{\nabla}_{\mathcal{X}_1} Q)\mathcal{X}_2 - (\widetilde{\nabla}_{\mathcal{X}_2} Q)\mathcal{X}_1\} + \frac{c_2}{2}\{(\mathcal{X}_1 r)\mathcal{X}_2 - (\mathcal{X}_2 r)\mathcal{X}_1\}. \tag{59}$$

In veiw of (26), (59) becomes

$$R(\mathcal{X}_1, \mathcal{X}_2)Df = \frac{(c_2 - c_1)}{2}\{(\mathcal{X}_1 r)\mathcal{X}_2 - (\mathcal{X}_2 r)\mathcal{X}_1\} + \frac{c_1}{2}\{(\mathcal{X}_1 r)\eta(\mathcal{X}_2)\xi - (\mathcal{X}_2 r)\eta(\mathcal{X}_1)\xi\}$$
$$+ c_1(3 + \frac{r}{2})\{\mathcal{X}_1\eta(\mathcal{X}_2) - \mathcal{X}_2\eta(\mathcal{X}_1)\}. \tag{60}$$

On contracting the above equation with respect to $\mathcal{X}_1$, we obtain the following

$$S(\mathcal{X}_2, Df) = (\frac{c_1}{2} - c_2)(\mathcal{X}_2 r) + c_1\eta(\mathcal{X}_2)\{\frac{\xi r}{2} + (6 + r)\}. \tag{61}$$

Now from equation (22), we have

$$S(\mathcal{X}_2, Df) = (1 + \frac{r}{2})(\mathcal{X}_2 f) - (3 + \frac{r}{2})\eta(\mathcal{X}_2)\xi f. \tag{62}$$

Equations (61) and (62) implies that

$$(\frac{c_1}{2} - c_2)(\mathcal{X}_2 r) + c_1\eta(\mathcal{X}_2)\{\frac{\xi r}{2} + (6 + r)\} = (1 + \frac{r}{2})(\mathcal{X}_2 f) - (3 + \frac{r}{2})\eta(\mathcal{X}_2)\xi f. \tag{63}$$

Putting $\mathcal{X}_2 = \xi$ in (63) and using (25), above yields

$$\xi f = \frac{(2c_2 - 3c_1)(r + 6)}{2}. \tag{64}$$

Taking inner product of (64) with ξ and using (14), gives

$$\eta(\mathcal{X}_2)\mathcal{X}_1 f - \eta(\mathcal{X}_1)\mathcal{X}_2 f = \frac{c_2}{2}\{\xi r\eta(\mathcal{X}_2) - \mathcal{X}_2 r\}. \tag{65}$$

Substituting $\mathcal{X}_1 = \xi$ in (65) and using equation (64), (25) gives

$$\mathcal{X}_2 f = (r + 6)\{c_2 + \frac{2c_2 - 3c_1}{2}\}\eta(\mathcal{X}_2) - \frac{c_2}{2}(\mathcal{X}_2 r). \tag{66}$$

Now, if we take r = constant, then equation (25), gives $r = -6$ and hence equation (66) implies

$$\mathfrak{X}_2 f = 0, \tag{67}$$

which shows that f is constant. Thus the soliton is trivial. Hence the manifold becomes an Einstein manifold and being 3-dimensional, the manifold becomes a space of constant curvature.
Thus we have:

Theorem 4.1. *If 3-dimensional Kenmotsu manifold of constant scalar curvature admits gradient Ricci-Yamabe soliton, then the soliton is trivial and $(\mathcal{M}^n, g)$ is a space of constant curvature.*

5. $* - \eta-$ Ricci-Yamabe soliton on Kenmotsu manifolds

The notion of $*-$Ricci solitons was first established by Kaimakamis and Panagiotidou [20]. After this, the notion of $*-$Ricci tensor on almost Hermitian manifolds was introduced by S. Tachibana [24] and then by T. Hamada [13] in case of real hypersurface on non flat complex space forms. On the smooth manifold (M^n, g), the Riemannian metric g is known as $*-$ Ricci soliton if a smooth vector field V and a real number λ satisfies

$$\mathcal{L}_V g = -2S^* - 2\lambda g, \tag{68}$$

where

$$S^*(\mathfrak{X}_1, \mathfrak{X}_2) = g(Q^*\mathfrak{X}_1, \mathfrak{X}_2) = Trace\{\phi \circ R(\mathfrak{X}_1, \phi\mathfrak{X}_2)\}, \tag{69}$$

for every vector fields $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathfrak{X}(\mathcal{M})$ and S^* and Q^* are the $*-$ Ricci tensor and $*-$ Ricci operator respectively. For the study of $*-$ Ricci soliton and $\eta-$ Ricci solitons on contact Riemannian geometry please see the references ([1], [4], [5], [6], [7], [14], [17], [18], [19], [23]).
Extending the notion of Ricci soliton, J. T. Cho and M. Kimura [2] introduced $\eta-$Ricci soliton which is obtained by perturbing the equation (2) by multiple of a certain $(0, 2)$-tensor field $\eta \otimes \eta$.

The concept of $* - \eta-$ Ricci soliton was presented by S. Dey and S. Roy [22]. A Riemannian manifold $(\mathcal{M}^n, g)$ is called $* - \eta-$ Ricci soliton fulfilling

$$\mathcal{L}_V g = -2S^* - 2\lambda g - 2\mu\eta \otimes \eta \tag{70}$$

A Riemannian manifold $(\mathcal{M}^n, g)$ is named $* - \eta-$ Ricci-Yamabe soliton of type (c_1, c_2) satisfying

$$\mathcal{L}_V g = -2c_1 S^* - (2\lambda - c_2 r)g - 2\mu\eta \otimes \eta, \tag{71}$$

where $c_1, c_2, \lambda, \mu \in \mathbb{R}$.

Lemma 5.1. *On a Kenmotsu manifold, we have*

$$\begin{aligned}\bar{R}(U, V, \phi W, \phi E) = &-g(V, \phi W)g(U, \phi E) - g(U, W)g(V, E)\\ &+ g(U, \phi W)g(V, \phi E) + g(V, W)g(U, E) + \eta(R(U, V)W, E).\end{aligned} \tag{72}$$

for any U, V, W, E on $\mathfrak{X}(\mathcal{M})$, where $\bar{R}(U, V, W, E) = g(R(U, V)W, E)$.

Proof. By the notion of well known definition of curvature tensor, we have

$$\bar{R}(U, V, \phi W, \phi E) = g(\widetilde{\nabla}_U \widetilde{\nabla}_V \phi W, \phi E) - g(\widetilde{\nabla}_V \widetilde{\nabla}_U \phi W, \phi E) - g(\widetilde{\nabla}_{[U,V]} \phi W, \phi E), \tag{73}$$

which by using (8), (11) to (14) takes the form (72). □

Lemma 5.2. *The $*-$ Ricci tensor S^* of Kenmotsu manifold is given by*

$$S^*(V,W) = S(V,W) + (n-2)g(V,W) + \eta(V)\eta(W) - ag(V,\phi W) \tag{74}$$

Proof. Let $\{u_i\}_{i=1}^n$ be an orthonormal basis of the tangent space at each point of the manifold. By the definition of $*-$ Ricci tensor and from (72), we get following:

$$S^*(V,W) = \sum_{i=1}^n g(\bar{R}(u_i,V,\phi W),\phi u_i)$$

$$= \sum_{i=1}^n -g(V,\phi W)g(u_i,\phi u_i) - \sum_{i=1}^n g(u_i,W)g(V,u_i) + \sum_{i=1}^n g(u_i,\phi W)g(V,\phi u_i) + \sum_{i=1}^n g(V,W)g(u_i,u_i) + \sum_{i=1}^n g(R(u_i,V)W,u_i).$$

After simplification, above equation becomes

$$S^*(V,W) = S(V,W) + (n-2)g(V,W) + \eta(V)\eta(W) - ag(V,\phi W). \tag{75}$$

□

Now, let $n-$ dimensional Kenmotsu manifold admitting $*-\eta-$ Ricci-Yamabe soliton, then from equation (71), we have

$$\mathcal{L}_V g(\mathcal{X}_1,\mathcal{X}_2) = -2c_1S^*(\mathcal{X}_1,\mathcal{X}_2) - (2\lambda - c_2r)g(\mathcal{X}_1,\mathcal{X}_2) - 2\mu\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{76}$$

We know that $\mathcal{L}_V g(\mathcal{X}_1,\mathcal{X}_2) = g(\widetilde{\nabla}_{\mathcal{X}_1}\xi,\mathcal{X}_2) + g(\mathcal{X}_1,\widetilde{\nabla}_{\mathcal{X}_2}\xi)$. Using this and (12) in (76), we obtain

$$S^*(\mathcal{X}_1,\mathcal{X}_2) = (\frac{c_2r - 2\lambda - 2}{2c_1})g(\mathcal{X}_1,\mathcal{X}_2) + (\frac{1-\mu}{c_1})\eta(\mathcal{X}_1)\eta(\mathcal{X}_2). \tag{77}$$

Using lemma (5.2), equation (77) yields

$$S(\mathcal{X}_1,\mathcal{X}_2) = \Big\{\frac{c_2r - 2\lambda - 2}{2c_1} - (n-2)\Big\}g(\mathcal{X}_1,\mathcal{X}_2) + \Big\{\frac{1-\mu}{c_1} - 1\Big\}\eta(\mathcal{X}_1)\eta(\mathcal{X}_2) + ag(\mathcal{X}_1,\phi\mathcal{X}_2). \tag{78}$$

Substituting $\mathcal{X}_2 = \xi$ in (78), infers

$$S(\mathcal{X}_1,\xi) = \Big\{\frac{c_2r - 2\lambda - 2}{2c_1} - (n-2) + (\frac{1-\mu}{c_1} - 1)\Big\}\eta(\mathcal{X}_1). \tag{79}$$

Equation (79) together with (17), gives

$$(\lambda + \mu) = \frac{c_2r}{2}. \tag{80}$$

Hence we have

Theorem 5.3. *If $(\mathcal{M}^n, g)$ Kenmotsu manifold admits $*-\eta-$ Ricci-Yamabe soliton, then manifold become a generalized $\eta-$ Einstein manifold and the constant λ, μ and c_2 are related by the equation $(\lambda + \mu) = \frac{c_2r}{2}$.*

6. Kenmotsu manifold admitting $* - \eta -$ Ricci-Yamabe soliton satisfying $R(\xi, \mathcal{X}_1).S = 0$

Let an n-dimensional Kenmotsu manifold admitting $*-\eta-$ Ricci-Yamabe soliton and satisfying $R(\xi, \mathcal{X}_1).S = 0$, then $R(\xi, \mathcal{X}_1).S = 0$ implies that

$$(R(\xi, \mathcal{X}_1).S)(\mathcal{X}_2, \mathcal{X}_3) - S(R(\xi, \mathcal{X}_1)\mathcal{X}_2, \mathcal{X}_3) - (\mathcal{X}_2, R(\xi, \mathcal{X}_1)\mathcal{X}_3) = 0, \tag{81}$$

which by using (16) yields

$$\eta(\mathcal{X}_2)S(\mathcal{X}_1, \mathcal{X}_3) - g(\mathcal{X}_1, \mathcal{X}_2)S(\xi, \mathcal{X}_3) + \eta(\mathcal{X}_3)S(\mathcal{X}_2, \mathcal{X}_1) - g(\mathcal{X}_1, \mathcal{X}_3)S(\mathcal{X}_2, \xi) = 0. \tag{82}$$

Taking $\mathcal{X}_3 = \xi$ and using (78) for the value of $S(\mathcal{X}_1, \xi)$ and $S(\xi, \xi)$, we get

$$S(\mathcal{X}_1, \mathcal{X}_2) = \left\{ \frac{c_2 r - 2\lambda - 2 - 2c_1\mu - 2c_1 n + 4c_1}{2c_1} \right\} g(\mathcal{X}_1, \mathcal{X}_2). \tag{83}$$

From equations (78) and (83), we have

$$\left\{ \mu + \frac{2 - 2\mu - 2c_1}{2c_1} \right\} \eta(\mathcal{X}_1) = 0. \tag{84}$$

Which implies $\mu = 1$,(since $c_1 \neq 0$) then from (80) we get $\lambda = \frac{c_2 r}{2} - 1$.
Using these values in (83), we have

$$S(\mathcal{X}_1, \mathcal{X}_2) = (1 - n)g(\mathcal{X}_1, \mathcal{X}_2). \tag{85}$$

Thus we have:

Theorem 6.1. *If an n-dimensional Kenmotsu manifold admits $*-\eta-$ Ricci-Yamabe soliton and satisfying $R(\xi, \mathcal{X}_1).S = 0$, then $\mu = 1$ and $\lambda = \frac{c_2 r}{2} - 1$.*

Corollary 6.2. *If an n-dimensional Kenmotsu manifold admits $*-\eta-$ Ricci-Yamabe soliton and satisfying $R(\xi, \mathcal{X}_1).S = 0$, then the manifold become an Einstein manifold.*

7. Examples

Example 7.1. *We consider the 3-dimensional manifold*

$$\mathcal{M} = \{(x, y, z) \in \mathbb{R}^3, z > 0\},$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let u_1, u_2 and u_3 be the vector fields on $\mathcal{M}$ given by

$$u_1 = e^z \frac{\partial}{\partial x}, u_2 = e^z \frac{\partial}{\partial y}, u_3 = -e^z \frac{\partial}{\partial z},$$

which are linearly independent at each point p of $\mathcal{M}$ and hence form a basis of $T_p\mathcal{M}$. Define a Lorentzian metric g on $\mathcal{M}$ such that

$$g(u_i, u_j) = \begin{cases} 1, & \text{if } i = j \leq 3 \\ 0, & \text{if } i \neq j \end{cases}$$

Let η be the 1-form on $\mathcal{M}$ defined by $\eta(\mathcal{X}_1) = g(\mathcal{X}_1, \xi)$, for all $\mathcal{X}_1 \in \mathfrak{X}(\mathcal{M})$. Let ϕ be the (1,1)-tensor field on $\mathcal{M}$ defined as
$\phi u_1 = -u_2$, $\phi u_2 = -u_1$, $\phi u_3 = 0$.

Using the linearty of ϕ *and* g*, we have*
$\eta(u_3) = 1$, $\phi^2\mathcal{X}_1 = \mathcal{X}_1 + \eta(\mathcal{X}_1)u_3$, $g(\phi\mathcal{X}_1, \phi\mathcal{X}_2) = g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2)$

for all $\mathcal{X}_1$, $\mathcal{X}_2 \in \mathfrak{X}(\mathcal{M})$. *Thus for* $u_3 = \xi$, (ϕ, ξ, η, g) *defines an almost contact metric structure on* $\mathcal{M}$.
Let $\widetilde{\nabla}$ *be the Levi Civita connection with respect to the Lorentzian metric* g*. Then, we have*

$[u_1, u_2]=[u_2, u_1]=0$, $[u_1, u_3]= u_1$, $[u_3, u_1]= -u_1$, $[u_2, u_3]= -u_2$, $[u_3, u_2]= u_2$.

and

$$2g(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{X}_3) = \mathcal{X}_1 g(\mathcal{X}_2, \mathcal{X}_3) + \mathcal{X}_2 g(\mathcal{X}_3, \mathcal{X}_1) - \mathcal{X}_3 g(\mathcal{X}_1, \mathcal{X}_2) - g(\mathcal{X}_1, [\mathcal{X}_2, \mathcal{X}_3]) + g(\mathcal{X}_2, [\mathcal{X}_3, \mathcal{X}_1]) + g(\mathcal{X}_3, [\mathcal{X}_1, \mathcal{X}_2]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\widetilde{\nabla}_{u_1}u_1 = -u_3, \widetilde{\nabla}_{u_1}u_2 = 0, \widetilde{\nabla}_{u_1}u_3 = u_1, \widetilde{\nabla}_{u_2}u_1 = 0, \tag{86}$$

$$\widetilde{\nabla}_{u_2}u_2 = -u_3, \widetilde{\nabla}_{u_2}u_3 = u_2, \widetilde{\nabla}_{u_3}u_1 = 0, \widetilde{\nabla}_{u_3}u_2 = 0, \widetilde{\nabla}_{u_3}u_3 = 0.$$

From the above equation, it can be easily verify that the manifold satisfies

$$\widetilde{\nabla}_{\mathcal{X}_1}\xi = \mathcal{X}_1 - \eta(\mathcal{X}_1)\xi, \textit{for } \xi = u_3.$$

Hence the manifold is a Kenmotsu manifold.
Now, let

$$X = \sum_{i=1}^{3} X^i u_i = X^1 u_1 + X^2 u_2 + X^3 u_3,$$

$$Y = \sum_{i=1}^{3} Y^i u_i = Y^1 u_1 + Y^2 u_2 + Y^3 u_3 +,$$

and

$$Z = \sum_{i=1}^{3} Z^i u_i = Z^1 u_1 + Z^2 u_2 + Z^3 u_3.$$

As we know that

$$R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_2 = \widetilde{\nabla}_{\mathcal{X}_1}\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_2 - \widetilde{\nabla}_{\mathcal{X}_2}\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2 - \widetilde{\nabla}_{[\mathcal{X}_1, \mathcal{X}_2]}\mathcal{X}_2. \tag{87}$$

From equation (86) *and* (87)*, we can verify the following results*

$$R(u_1, u_2)u_2 = -u_1, R(u_1, u_3)u_3 = -u_1, R(u_2, u_1)u_1 = -u_2, \tag{88}$$

$$R(u_2, u_3)u_3 = -u_2, R(u_3, u_1)u_1 = -u_3, R(u_3, u_2)u_2 = -u_3,$$

and

$$S(u_1, u_1) = S(u_2, u_2) = S(u_3, u_3) = -2.$$

With the help of above equations, we obtain $r = -6$.

Now suppose that there exist a function f on $(\mathcal{M}^3, g)$ such that

$$\widetilde{\nabla}_{\mathcal{X}_1} Df = -c_1 Q\mathcal{X}_1 - (\lambda - \frac{c_2 r}{2})\mathcal{X}_1. \tag{89}$$

Since $Df = (u_1 f)u_1 + (u_2 f)u_2 + (u_3 f)u_3$, then we have

$$\widetilde{\nabla}_{u_1} Df = \{u_1(u_1 f) + (u_3 f)\}u_1 + \{(u_1(u_3 f) - (u_3 f)\}u_3 + u_1(u_2 f)u_2,$$

$$\widetilde{\nabla}_{u_2} Df = u_2(u_1 f)u_1 + \{u_2(u_2 f) + (u_3 f)\}u_2 + \{u_2(u_3 f) - (u_2 f)u_3\}$$

and

$$\widetilde{\nabla}_{u_3} Df = u_3(u_1 f)u_1 + u_3(u_2 f)u_2 + (u_3(u_3 f) - (u_3 f)u_3.$$

Thus from (24) *and* (54)*, f satisfies the following equations:*

$$u_1(u_1 f) + u_3 f = (2c_1 - 3c_2 - \lambda),$$

$$u_2(u_2 f) + u_3 f = (2c_1 - 3c_2 - \lambda),$$

$$u_3(u_3) f = (2c_1 - 3c_2 - \lambda),$$

$$u_1(u_2 f) = 0,$$

$$u_2(u_1 f) = 0,$$

$$u_2(u_3 f) - u_2 f = 0,$$

Which again equivalent to

$$e^{2z}\frac{\partial^2 f}{\partial x^2} - e^z \frac{\partial f}{\partial z} = (2c_1 - 3c_2 - \lambda),$$

$$e^{2z}\frac{\partial^2 f}{\partial y^2} - e^z \frac{\partial f}{\partial z} = (2c_1 - 3c_2 - \lambda),$$

$$e^{2z}\frac{\partial^2 f}{\partial z^2} = (2c_1 - 3c_2 - \lambda),$$

$$e^{2z}\frac{\partial^2 f}{\partial x \partial y} = (2c_1 - 3c_2 - \lambda),$$

$$-e^{2z}\frac{\partial^2 f}{\partial y \partial z} - e^z \frac{\partial f}{\partial y} = (2c_1 - 3c_2 - \lambda).$$

These equations implies that f is constant when $\lambda = 2c_1 - 3c_2$. Thus the equation (54) *is satisfied. Hence g becomes gradient Ricci-Yamabe soliton with soliton vector field $V = Df$, where f = constant and $\lambda = 2c_1 - 3c_2$. Therefore, theorem* (4.1) *is verified.*

.

Example 7.2. *We consider the 5-dimensional manifold*

$$\mathcal{M} = \{(x_1, x_2, x_3, x_4, z) \in \mathbb{R}^5, z > 0\}, \tag{90}$$

where (x_1, x_2, x_3, x_4, z) *are the standard coordinates in* $\mathbb{R}^5$*. Let* u_1, u_2, u_3, u_4 *and* u_5 *be the vector fields on* $\mathcal{M}$ *given by*

$$u_1 = e^z \frac{\partial}{\partial x_1}, u_2 = e^z \frac{\partial}{\partial x_2}, u_3 = e^z \frac{\partial}{\partial x_3}$$

$$u_4 = e^z \frac{\partial}{\partial x_4}, u_5 = -e^z \frac{\partial}{\partial z}$$

which are linearly independent at each point p of $\mathcal{M}$*, and hence form a basis of* $T_p(\mathcal{M})$*. Define a metric g on* $\mathcal{M}$ *by*

$$g(u_i, u_j) = \begin{cases} 1, & \text{if } i = j \leq 5 \\ 0, & \text{if } i \neq j \end{cases}$$

Let η *be the 1-form on* $\mathcal{M}$ *defined by* $\eta(\mathcal{X}_1) = g(\mathcal{X}_1, u_5) = g(\mathcal{X}_1, \xi)$ *for all* $\mathcal{X}_1 \in \mathcal{X}(\mathcal{M})$ *and let* ϕ *be the (1,1)-tensor field on* $\mathcal{M}$ *defined as*
$\phi u_1 = -u_2$, $\phi u_2 = -u_1$, $\phi u_3 = -u_4$, $\phi u_4 = -u_3$, $\phi u_5 = 0$.
By applying linearty of ϕ *and* g*, we have*

$$\eta(\xi) = g(\xi, \xi) = 1, \phi^2\mathcal{X}_1 = \mathcal{X}_1 + \eta(\mathcal{X}_1)\xi, \eta(\phi\mathcal{X}_1) = 0,$$

$$g(\mathcal{X}_1, \xi) = \eta(\mathcal{X}_1), g(\phi\mathcal{X}_1, \phi\mathcal{X}_2) = g(\mathcal{X}_1, \mathcal{X}_2) + \eta(\mathcal{X}_1)\eta(\mathcal{X}_2),$$

for all $\mathcal{X}_1$, $\mathcal{X}_2 \in \mathcal{X}(\mathcal{M})$.

Let $\widetilde{\nabla}$ *be the Levi Civita connection with respect to the Lorentzian metric g. Hence we have*

$$[u_1, u_2]=[u_1, u_3]=[u_1, u_4]=[u_2, u_3]=[u_2, u_4]=[u_3, u_4]=0,$$

$$[u_1, u_5] = u_1, [u_2, u_5] = u_2, [u_3, u_5] = u_3, [u_4, u_5] = u_4.$$

Now, the Koszul's formula is defined as

$$2g(\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{X}_3) = \mathcal{X}_1 g(\mathcal{X}_2, \mathcal{X}_3) + \mathcal{X}_2 g(\mathcal{X}_3, \mathcal{X}_1) - \mathcal{X}_3 g(\mathcal{X}_1, \mathcal{X}_2) - g(\mathcal{X}_1, [\mathcal{X}_2, \mathcal{X}_3]) + g(\mathcal{X}_2, [\mathcal{X}_3, \mathcal{X}_1]) + g(\mathcal{X}_3, [\mathcal{X}_1, \mathcal{X}_2]),$$

Using Koszul's formula we easily calculate

$$\widetilde{\nabla}_{u_1}u_1 = -u_5, \widetilde{\nabla}_{u_1}u_2 = 0, \widetilde{\nabla}_{u_1}u_3 = 0, \widetilde{\nabla}_{u_1}u_4 = 0, \widetilde{\nabla}_{u_1}u_5 = u_1, \tag{91}$$

$$\widetilde{\nabla}_{u_2}u_1 = 0, \widetilde{\nabla}_{u_2}u_2 = -u_5, \widetilde{\nabla}_{u_2}u_3 = 0, \widetilde{\nabla}_{u_2}u_4 = 0, \widetilde{\nabla}_{u_2}u_5 = u_2,$$

$$\widetilde{\nabla}_{u_3}u_1 = 0, \widetilde{\nabla}_{u_3}u_2 = 0, \widetilde{\nabla}_{u_3}u_3 = -u_5, \widetilde{\nabla}_{u_3}u_4 = 0, \widetilde{\nabla}_{u_3}u_5 = u_3,$$

$$\widetilde{\nabla}_{u_4}u_1 = 0, \widetilde{\nabla}_{u_4}u_2 = 0, \widetilde{\nabla}_{u_4}u_3 = 0, \widetilde{\nabla}_{u_4}u_4 = -u_5, \widetilde{\nabla}_{u_4}u_5 = u_4,$$

$$\widetilde{\nabla}_{u_5}u_1 = 0, \widetilde{\nabla}_{u_5}u_2 = 0, \widetilde{\nabla}_{u_5}u_3 = 0, \widetilde{\nabla}_{u_5}u_4 = 0, \widetilde{\nabla}_{u_5}u_5 = 0.$$

Also one can easily verify that

$$\widetilde{\nabla}_{\mathcal{X}_1}\xi = -\mathcal{X}_1 - \eta(\mathcal{X}_1)\xi \text{ and } (\widetilde{\nabla}_{\mathcal{X}_1}\phi)\mathcal{X}_2 = -g(\phi\mathcal{X}_1, \mathcal{X}_2)\xi - \eta(\mathcal{X}_2)\phi\mathcal{X}_1.$$

Hence the manifold is Kenmotsu manifold of dimension 5.
Now let

$$X = \sum_{i=1}^{5} X^i u_i = X^1 u_1 + X^2 u_2 + X^3 u_3 + X^4 u_4 + X^5 u_5,$$

$$Y = \sum_{i=1}^{5} Y^i u_i = Y^1 u_1 + Y^2 u_2 + Y^3 u_3 + Y^4 u_4 + Y^5 u_5$$

and

$$Z = \sum_{i=1}^{5} Z^i u_i = Z^1 u_1 + Z^2 u_2 + Z^3 u_3 + Z^4 u_4 + Z^5 u_5.$$

It is known that

$$R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = \widetilde{\nabla}_{\mathcal{X}_1}\widetilde{\nabla}_{\mathcal{X}_2}\mathcal{X}_3 - \widetilde{\nabla}_{\mathcal{X}_2}\widetilde{\nabla}_{\mathcal{X}_1}\mathcal{X}_3 - \widetilde{\nabla}_{[\mathcal{X}_1,\mathcal{X}_2]}\mathcal{X}_3. \tag{92}$$

With the help of (91) *and* (92), *we get the components of the curvature tensor and Ricci tensor as*

$$R(u_1, u_2)u_1 = u_2, R(u_1, u_2)u_2 = -u_1, R(u_1, u_2)u_3 = 0, R(u_1, u_2)u_4 = 0, R(u_1, u_2)u_5 = 0, \tag{93}$$

$$R(u_2, u_3)u_1 = 0, R(u_2, u_3)u_2 = u_3, R(u_2, u_3)u_3 = -u_2, R(u_2, u_3)u_4 = 0, R(u_2, u_3)u_5 = 0$$

$$R(u_3, u_4)u_1 = 0, R(u_3, u_4)u_2 = 0, R(u_3, u_4)u_3 = u_4, R(u_3, u_4)u_4 = -u_3, R(u_3, u_4)u_5 = 0$$

$$R(u_4, u_5)u_1 = 0, R(u_4, u_5)u_2 = 0, R(u_4, u_5)u_3 = 0, R(u_4, u_5)u_4 = u_5, R(u_4, u_5)u_5 = -u_4$$

$$R(u_1, u_5)u_1 = u_5, R(u_1, u_5)u_2 = 0, R(u_1, u_5)u_3 = 0, R(u_1, u_5)u_4 = 0, R(u_1, u_5)u_5 = 0.$$

With the help of above expressions of the curvature tensors, it follows that

$$R(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3 = g(\mathcal{X}_2, \mathcal{X}_3)\mathcal{X}_1 - g(\mathcal{X}_1, \mathcal{X}_3)\mathcal{X}_2. \tag{94}$$

From (94), *we get*

$$S(\mathcal{X}_2, \mathcal{X}_3) = 4g(\mathcal{X}_2, \mathcal{X}_3). \tag{95}$$

On contracting (95), *we get* $r = 20$.
The Ricci tensor S is given by

$$S(u_1, u_1) = S(u_2, u_2) = S(u_3, u_3) = S(u_4, u_4) = S(u_5, u_5) = 4. \tag{96}$$

If this manifold admits $* - \eta-$ *Ricci-Yamabe soliton then from* (76) *and* (96), *we find* $\mu = -(7c_1 + 1)$ *and* $\lambda = (10c_2 + 7c_1 + 1)$, *which satisfies* $\lambda + \mu = \frac{c_2 r}{2}$.
Thus the example satisfies the theorem (5.2).

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References

[1] G. Catino and L. Mazzieri, Gradient Einstein solitons, Nonlinear Anal. **132** (2016), 66–94.
[2] J. T. Cho and M. Kimura, Ricci solitons and real hypetrsurfaces in a complex space form, Tohoku Math. J. **61** (2009), 205–212.
[3] S. Guler and M. Crasmareanu, Ricci-Yamabe maps for Riemannian flow and their volume variation and volume entropy, Turk. J. Math. **43**, 2631-–2641 (2019).
[4] B. Y. Chen and S. Deshmukh, Yamabe and quasi-Yamabe solitons on Euclidean submanifolds, Mediterr. J. Math. (2018) 15:194.
[5] S. Deshmukh, B. Y. Chen, A note on Yamabe solitons, Balkan J. Geom. Appl. **23** (2018), no. 1, 37–43.
[6] D. Dey, Almost Kenmotsu metric as Ricci-Yamabe solitons, https://arxiv.org/abs/2005.02322.
[7] U. C. De, A. Sardar, K. De, Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds, Turkish J. Math. **46** (2022), no. 3, 1078–1088.
[8] D. Dey, Sasakian 3-manifolds admitting a gradient Ricci-Yamabe soliton, Korean J. Math. **29**(2021), No. 3, 547–554.
[9] U. C. De and G. Pathak, On 3- dimensional Kenmotsu manifolds, Indian J. Pure Applied Math., **35**(2004) 159–165.
[10] A. Ghosh, D. S. Patra, $*$-Ricci soliton within the frame-work of Sasakian and (κ,μ)-contact manifold, Int. J. Geom. Methods Mod. Phys. **15** (2018), no. 7, 1850120, 21 pp.
[11] R. S. Hamilton, The Ricci flow on surfaces. Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.
[12] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differential Geometry **17** (1982), no. 2, 255–306.
[13] T. Hamada, Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor, Tokyo J. Math. **25** (2002), no. 2, 473–483.
[14] A. Haseeb, R. Prasad, $*$-conformal η-Ricci solitons in ϵ-Kenmotsu manifolds, Publ. Inst. Math. (Beograd) (N.S.) 108(122) (2020), 91–102.
[15] A. Haseeb, R. Prasad and F. Mofarreh, Sasakian manifolds admitting $*-\eta-$ Ricci-Yamabe solitons, Hindawi, Advances in Mathematical Physics, Volume 2022, Article ID 5718736, 7 pages.
[16] S. Hui, R. Prasad and D. Chakraborty, Ricci soliton on Kenmotsu manifolds with respect to quarter symmetric non-metric ϕ connection, GANITA, Vol. **67**(2), 2017, 195–204.
[17] A. Haseeb and U. C. De, η-Ricci solitons in ϵ-Kenmotsu manifolds, J. Geom. **110** (2019), no. 2, Paper No. 34, 12 pp.
[18] S. K. Hui, S. K. Yadav, and A. Patra, Almost conformal Ricci solitons on f-Kenmotsu manifolds, Khayyam J. Math. **5** (2019), no. 1, 89–104.
[19] S. Y. Hsu, A note on compact gradient Yamabe solitons, J. Math. Anal. Appl. 388 (2012), no. 2, 725–726.
[20] G. Kaimakamis and K. Panagiotidou, $*$-Ricci solitons of real hypersurfaces in non-flat complex space forms, J. Geom. Phys. **86** (2014), 408–413.
[21] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J. (2) **24** (1972), 93–103.
[22] S. Roy, S. Dey, A. Bhattacharyya, and S. K. Hui, $*$-conformal η-Ricci soliton on Sasakian manifold, Asian-Eur. J. Math. **15** (2022), no. 2, Paper No. 2250035, 17 pp.
[23] Y. J. Suh and U.C. De, Yamabe solitons and gradient Yamabe solitons on three-dimensional $N(k)$-contact manifolds, Int. J. Geom. Methods Mod. Phys. **17** (2020), no. 12, 2050177, 10 pp.
[24] S. Tachibana, On almost-analytic vectors in almost-K $\ddot{a}$ hlerian manifolds, Tohoku Math. J. (2) **11** (1959), 247–265.
[25] K. Yano, Integral formulas in Riemannian geometry, Pure and Applied Mathematics, No. 1. Marcel Dekker, Inc., New York, 1970. {ix}+156 pp.