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On the stability of the linearized compressible Adiabatic flow through porous media

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Abstract. In this paper, we studied the stability of the Linearized compressible Adiabatic flow System (LAS) in one dimension. First, by studying the semigroup of the LAS, we proved the existence and uniqueness of a solution to LAS. Then, by using a deep analysis of the spectrum of the associated linear operator to the LAS, we were able to determine some new exponential stability results for the solution to the LAS in two different Hilbert spaces. Finally, some numerical experiments are given to confirm the theoretical results.

1. Introduction

The study of the stability of partial differential equations has interested several researchers in the field of mathematics [1–9, 11, 13, 14]. In particular, the study of the exponential stability of unstable systems or those which have a slow stability is a problem which represents a great challenge for many researchers in the world. In [1], the author studied the exponential stability of the Saint-Venant system linearized around a steady state. In fact, the linearized Saint-Venant system had a very slow stability and the aim of this study was to accelerate this stability and in this case some results were established in different Hilbert spaces. Also, in [15, 16], some results of exponential stability for the linearized Navier-Stokes system were demonstrated.

In this work, we are interested in the study of the exponential stability of the Adiabatic system linearized around a given steady state. In this case, we have obtained extremely large exponential stability results in different Hilbert spaces such as $L^2(\Omega) \times L^2(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$, ($\Omega$ is a bounded domain in $\mathbb{R}$). In fact, thanks to an in-depth study of the spectrum of the linear operator associated with the linear system LAS, we were able to determine an explicit exponential decay rate able to stabilize the solution very fast. Also, we have confirmed the theoretical results obtained by numerical results given at the end of this paper.

Note that the results obtained will be used for the study of several problems such as the controllability problem and the study of the stabilizability problems by control localized on a part of the space domain for example.

Let's summarize the novelties of this work. In fact, we have developed several goals which are as follows:

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- We proved that the unbounded operator associated to LAS is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(\Omega) \times L^2(\Omega)$, (Lemma 2.1).
- We proved an exponential stability result, for the LAS, on $L^2(\Omega) \times L^2(\Omega)$, (Theorem 3.1).
- We proved an exponential stability result, for the LAS, on $H^1(\Omega) \times H^1(\Omega)$, (Theorem 3.2).
- Presentation of some numerical experiments to validate and confirm the theoretical results.

The paper is organized as follows. in section 2, we studied the semigroup generated by the linear operator associated with LAS. Also, we performed a full spectrum analysis of the linear operator of the LAS. Section 3 was devoted to the presentation of the main results of this paper. In fact, we have established two results of exponential stability of the solution in Theorem 3.1 and Theorem 3.2. Finally and in section 4, we have studied numerically the stabilization problem where we have implemented and presented numerical results which confirm well the theoretical results obtained.

The motion of the Adiabatic gas flow through porous media can be modeled by the following damped hyperbolic system [10]:

$$\vartheta_t(x,t) - \varrho_x(x,t) = 0, \tag{1}$$
$$\varrho_t(x,t) + (p(\vartheta,s))_x(x,t) + \alpha\varrho(x,t) = 0, \tag{2}$$
$$s_t = 0, \tag{3}$$

where $\varrho(x,t)$ is the velocity of the flow, $\vartheta(x,t)$ is the specific volume for every $(x,t) \in \mathbb{R} \times (0,\infty)$. In this model p represents the pressure of the gas with $(p(\vartheta,s))_\vartheta < 0$ for $\vartheta > 0$, the parameter $\alpha > 0$ and s is the entropy. We assume that the solutions of the system (1)-(3) are smooth. Here, the pressure satisfies the following law:

$$p(\vartheta,s) = (\sigma-1)\vartheta^{-\sigma} g^s, \quad \text{for} \quad \sigma > 1,$$

and g is the specific internal energy for which one has $g_s \neq 0$, $g_\vartheta + p = 0$ (this is a consequence of the second law of thermodynamics).

In the case of an isentropic flow, $s = c_0$ is a constant, the system (1)-(3) becomes as follows:

$$\vartheta_t(x,t) - \varrho_x(x,t) = 0, \tag{4}$$
$$\varrho_t(x,t) + (p(\vartheta,c_0))_x(x,t) + \alpha\varrho(x,t) = 0. \tag{5}$$

The initial conditions are as follows:

$$\vartheta(x,0) = \vartheta_0(x), \quad \varrho(x,0) = \varrho_0(x), \quad x \in \mathbb{R}.$$

The boundary conditions are given by:

$$\varrho(-\infty,t) = q_-(t), \quad \varrho(+\infty,t) = q_+(t), \quad t > 0.$$

Let (ϑ_c, ϱ_c) be a constant state, $\vartheta_c > 0$ and $\varrho_c > 0$. The linearized compressible Adiabatic flow system around the state (ϑ_c, ϱ_c) is given as follows:

$$z_t - y_x = 0, \tag{6}$$
$$y_t - \eta z_x + \alpha y = 0, \tag{7}$$

where $\alpha > 0$ and the constant η is given by:

$$\eta = \frac{g^{c_0}\sigma(\sigma-1)}{\vartheta_c^{\sigma+1}}, \quad \sigma > 1.$$

For the simplicity of the analysis, we consider a bounded spatial domain $\Omega = (a,b)$, $a,b \in \mathbb{R}$, $(a < b)$. Then, the initial conditions are as follows:

$$z(x,0) = z_0(x), \quad y(x,0) = y_0(x), \quad x \in \Omega.$$

The boundary conditions are given by:

$$y(a,t) = d_-(t), \quad y(b,t) = d_+(t), \quad t > 0.$$

Let's introduce some Hilbert spaces that will be used later.

1. Let $\tilde{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ with the inner product:

$$(V,U)_{\tilde{L}^2(\Omega)} = \eta \int_\Omega v_1 u_1 + \int_\Omega v_2 u_2,$$

   where $V = (v_1, v_2)$, $U = (u_1, u_2)$.
2. Let $\tilde{H}^1(\Omega) = H^1(\Omega) \times H^1(\Omega)$, with the inner product:

$$(V,U)_{\tilde{H}^1(\Omega)} = \eta \int_\Omega v_1' u_1' + \int_\Omega v_2' u_2',$$

   where $V = (v_1, v_2)$, $U = (u_1, u_2)$.

2. Analysis of the linearized system

Let $\Omega = (0,\pi)$. For all $(x,t) \in \Omega \times [0,\infty)$, consider the linear homogeneous system:

$$z_t - y_x = 0, \tag{8}$$
$$y_t - \eta z_x + \alpha y = 0, \tag{9}$$
$$z(x,0) = z_0(x), \quad y(x,0) = y_0(x), \quad x \in \Omega, \tag{10}$$
$$y(0,t) = y(\pi,t) = 0, \quad t > 0. \tag{11}$$

Let $U = (z,y)$, then the system (8)-(11) can be written as follows:

$$U'(t) = AU(t), \quad \forall t > 0 \quad \text{and} \quad U(0) = U_0, \tag{12}$$

where the unbounded operator A is given by:

$$A = \begin{pmatrix} 0 & \frac{d}{dx} \\ \eta \frac{d}{dx} & -\alpha \end{pmatrix},$$

and $D(A)$ the domain of A is defined by:

$$D(A) = H^1(\Omega) \times H_0^1(\Omega).$$

Lemma 2.1. *The operator A is maximal dissipative in $\tilde{L}^2(\Omega)$ and we have:*

$$ker(A) = \{(c,0)^T, c \in \mathbb{R}\}.$$

Consequently, $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $\tilde{L}^2(\Omega)$ denoted by $S(t) = e^{-tA}$, $t \geq 0$. Moreover, for any $U_0 \in \tilde{L}^2(\Omega)$, there exists a unique solution U of (12) in $C(0,\infty; \tilde{L}^2(\Omega))$ such that:

$$\|U(t)\|_{\tilde{L}^2(\Omega)} \leq \|U_0\|_{\tilde{L}^2(\Omega)}, \quad \forall t \geq 0.$$

Proof. We can verify the following properties for the operator A:

◇ A is maximal. Indeed, for $U \in D(A)$

$$(AU,U)_{\tilde{L}^2(\Omega)} = \eta \int_\Omega \partial_x y z dx + \int_\Omega (\eta \partial_x z - \alpha y) y dx = -\alpha \|y\|^2_{L^2(\Omega)} \leq 0.$$

⋄ A is dissipative. Indeed, for $X = (u, v) \in D(A)$ and $F = (f, \ell) \in \tilde{L}^2(\Omega)$

$$(I - A)X = F,$$

is equivalent to the following problem:

$$\begin{array}{l} \text{Find} \quad (u, v) \in D(A) : \\ u - v' = f, \\ -\eta u' + (1 + \alpha)v = \ell, \\ v(0) = v(\pi) = 0. \end{array}$$

This problem has a unique solution in D(A). Moreover, we have:

$$\|U(t)\|_{D(A)} \leq C\|F(t)\|_{\tilde{L}^2(\Omega)},$$

where $C > 0$ is a positive constant.

Let $U = (z, y) \in ker(A)$. So, we get:

$$\begin{array}{rcl} y' &=& 0, \\ \eta z' - \alpha y &=& 0. \end{array}$$

We have $y' = 0$ and $y \in H_0^1(\Omega)$. Thus, $y \equiv 0$ and $\eta z' - \alpha y = 0$ implies that $z \equiv c$, $c \in \mathbb{R}$. From the semigroup theory [12], for any $U_0 \in \tilde{L}^2(\Omega)$, there exists a unique solution U of (8)-(11) in $C(0, \infty; \tilde{L}^2(\Omega))$ and the estimation holds.

□

Let $(\mathcal{B}_k)_{k\geq 0} = \{\psi_0\} \cup \{\psi_k^m,\ m = 1, 2\}_{k\geq 1}$ be the family in $\tilde{L}^2(\Omega)$ defined by:

$$\begin{array}{rcl} \psi_0 &=& \sqrt{\dfrac{2}{\pi\eta}}\,(1, 0), \\ \psi_k^1(x) &=& \sqrt{\dfrac{2}{\pi\eta}}\,(\cos(\dfrac{kx}{2}), 0),\ k \geq 1, \\ \psi_k^2(x) &=& \sqrt{\dfrac{2}{\pi}}\,(0, \sin(\dfrac{kx}{2})),\ k \geq 1. \end{array}$$

We can easily prove that $(\mathcal{B}_k)_{k\geq 0}$ is an orthonormal basis in $\tilde{L}^2(\Omega)$. Moreover, we have:

$$\begin{array}{rcl} \tilde{L}^2(\Omega) &=& \bigoplus\limits_{k\geq 0} V_k, \\ \tilde{H}^1(\Omega) &=& \bigoplus\limits_{k\geq 0} V_k, \end{array}$$

where the subspaces $(V_k)_{k\geq 0}$ are defined by:

$$\begin{array}{rcl} V_0 &=& span\{\psi_0\}, \\ V_k &=& span\{\psi_k^m,\ m = 1, 2\}, \quad \text{for} \quad k \geq 1. \end{array}$$

Lemma 2.2. *For all $k \in \mathbb{N}$, V_k is invariant under A, and $A_k = A/V_k$ has the following matrix representation:*

$$A_k = \begin{pmatrix} 0 & -\frac{k\sqrt{\eta}}{2} \\ \frac{k\sqrt{\eta}}{2} & -\alpha \end{pmatrix}. \tag{13}$$

Proof. For all $k \in \mathbb{N}$, we have:

$$\begin{aligned} A\psi_k^1(x) &= -\frac{k\sqrt{\eta}}{2}\psi_k^2(x), \\ A\psi_k^2(x) &= \frac{k\sqrt{\eta}}{2}\psi_k^1(x) - \alpha\psi_k^2(x), \end{aligned}$$

we deduce that $A\psi_k^i(x) \in V_k$ for $i = 1,2$. So, V_k is invariant under A. Moreover, $A|V_k$ is given by (13).
□

Lemma 2.3. *The spectrum of A_k is given as follows:*

- *If $k \in [1, \frac{\alpha}{\sqrt{\eta}}[$, the eigenvalues are real:*

$$\lambda_k^1 = -\frac{1}{2}\Big[\alpha - \sqrt{\alpha^2 - \eta k^2}\Big] < 0,$$
$$\lambda_k^2 = -\frac{1}{2}\Big[\alpha + \sqrt{\alpha^2 - \eta k^2}\Big] < 0.$$

Moreover, we have:

$$-\frac{\alpha}{2} \leq \lambda_k^1 \leq 0 \quad and \quad -\alpha \leq \lambda_k^2 \leq -\frac{\alpha}{2}.$$

- *If $k = \frac{\alpha}{\sqrt{\eta}}$, we obtained:*

$$\lambda_k^1 = \lambda_k^2 = -\frac{\alpha}{2}.$$

- *If $k \in]\frac{\alpha}{\sqrt{\eta}}, +\infty[$; the eigenvalues are complex:*

$$\lambda_k^1 = -\frac{1}{2}\Big[\alpha - i\sqrt{\eta k^2 - \alpha^2}\Big], \quad \lambda_k^2 = \overline{\lambda_k^1}.$$

Moreover, we have:

$$\Re(\lambda_k^1) = \Re(\lambda_k^2) = -\frac{1}{2}\alpha,$$
$$|\Im(\lambda_k^1)| = |\Im(\lambda_k^2)| = \sqrt{\eta k^2 - \alpha^2}.$$

Let $U = (z, y) \in \tilde{L}^2(\Omega)$. The expression of U in the basis $(\mathcal{B}_k)_{k\geq 0}$ is given as follows:

$$\begin{aligned} U(t) &= \sum_{k\geq 1}(a_k(t)\psi_k^1 + b_k(t)\psi_k^2), \\ U_0 &= \sum_{k\geq 1}(a_{0k}\psi_k^1 + b_{0k}\psi_k^2). \end{aligned}$$

Then, the projection of the equation:

$$\begin{aligned} U'(t) &= AU(t), \\ U(0) &= U_0 \end{aligned} \tag{14}$$

on the subspaces $(V_k)_{k\geq 1}$ is given by

$$(S)\begin{cases} \begin{pmatrix} a_k \\ b_k \end{pmatrix}'(t) = A_k\begin{pmatrix} a_k \\ b_k \end{pmatrix}(t), \quad t \geq 0, \\ \\ \begin{pmatrix} a_k \\ b_k \end{pmatrix}(0) = \begin{pmatrix} a_{0k} \\ b_{0k} \end{pmatrix}, \end{cases}$$

where $U|V_k = U_k = (a_k, b_k)$, for all $k \geq 1$.

Proposition 2.4. *For all $k \geq 1$, the system (S) has a unique solution given as follows:*

- *If $k \in [1, \frac{\alpha}{\sqrt{\eta}}[$, then:*

$$\begin{aligned} a_k(t) &= \frac{-1}{\lambda_k^1 - \lambda_k^2}\Big\{\Big(\lambda_k^2 e^{\lambda_k^1 t} - \lambda_k^1 e^{\lambda_k^2 t}\Big)a_{0k} + \frac{k\sqrt{\eta}}{2}\Big(e^{\lambda_k^1 t} - e^{\lambda_k^2 t}\Big)b_{0k}\Big\},\\ b_k(t) &= \frac{1}{(\lambda_k^1 - \lambda_k^2)}\Big\{\frac{k\sqrt{\eta}}{2}\Big(e^{\lambda_k^1 t} - e^{\lambda_k^2 t}\Big)a_{0k} + \Big(\lambda_k^1 e^{\lambda_k^1 t} - \lambda_k^2 e^{\lambda_k^2 t}\Big)b_{0k}\Big\}. \end{aligned}$$

- *If $k \in]\frac{\alpha}{\sqrt{\eta}}, +\infty[$, then:*

$$\begin{aligned} a_k(t) &= \frac{e^{-\frac{\alpha}{2}t}}{m_k}\Big\{\Big(m_k\cos(m_k t) - \frac{\alpha}{2}\sin(m_k t)\Big)a_{0k} + \frac{k\sqrt{\eta}}{2}\sin(m_k t)b_{0k}\Big\},\\ b_k(t) &= \frac{2e^{-\frac{\alpha}{2}t}}{m_k}\Big\{k\sqrt{\eta}\sin(m_k t)a_{0k} + \frac{1}{2}\Big(m_k\cos(m_k t) - \frac{\alpha}{2}\sin(m_k t)\Big)b_{0k}\Big\}, \end{aligned}$$

  where $m_k = \sqrt{\eta k^2 - \alpha^2}$.

- *If $k = \frac{\alpha}{\sqrt{\eta}}$, then we get:*

$$\begin{aligned} a_k(t) &= e^{-\frac{\alpha}{2}t}\Big\{(1 + \frac{\alpha}{2}t)a_{0k} - \frac{\alpha}{2}tb_{0k}\Big\},\\ b_k(t) &= e^{-\frac{\alpha}{2}t}\Big\{\frac{\alpha}{2}ta_{0k} + (1 - \frac{\alpha}{2}t)b_{0k}\Big\}. \end{aligned}$$

Proof. From the general solution of linear differential systems, we can establish the proof of the proposition. □

3. Main results

Theorem 3.1. *For any $U_0 \in \tilde{L}^2(\Omega)$, there exists a strictly positive $t_0 > 0$ such that the solution U to the problem (12) satisfies the following exponential stability identity:*

$$\|U(t)\|_{\tilde{L}^2(\Omega)} \leq C(1 + \frac{\alpha}{2}t)e^{-\frac{\alpha}{2}t}\|U_0\|_{\tilde{L}^2(\Omega)}, \qquad \forall t \geq t_0, \tag{15}$$

where C is a strictly positive constant dependent only on α, η.

Proof. The existence and the uniqueness of a solution to the problem (12) follows Lemma 2.1. Now, from Proposition 2.4, we have:

◇ If $k \in [1, \frac{\alpha}{\sqrt{\eta}}[$.

$$a_k(t) = \Big(\frac{\lambda_k^1 e^{\lambda_k^2 t} - \lambda_k^2 e^{\lambda_k^1 t}}{\lambda_k^1 - \lambda_k^2}\Big)a_{0k} + \frac{k\sqrt{\eta}}{2}\Big(\frac{e^{\lambda_k^2 t} - e^{\lambda_k^1 t}}{\lambda_k^1 - \lambda_k^2}\Big)b_{0k}.$$

Remark that we have:

$$e^{\lambda_k^1 t} = e^{-\frac{\alpha}{2}}e^{\frac{1}{2}\sqrt{\alpha^2 - \eta k^2}t}, \quad \forall t \geq 0, \tag{16}$$

and the function

$$k \mapsto e^{\frac{1}{2}\sqrt{\alpha^2 - \eta k^2}t} - \frac{\eta k^2}{4}t, \quad t \geq 0,$$

is decreasing with respect to $k \in [1, \frac{\alpha}{\sqrt{\eta}}]$. Now, it's easy to show that there exists $t_0 > 0$ such that:

$$e^{\frac{1}{2}\sqrt{\alpha^2-\eta k^2 t}} \leq \frac{\eta k^2}{4} t, \quad \forall t \geq t_0. \tag{17}$$

Thus:

$$|a_k(t)| \leq \frac{1}{|\lambda_k^1 - \lambda_k^2|}\Big\{\Big(|\lambda_k^1| e^{\lambda_k^2 t} + |\lambda_k^2| e^{\lambda_k^1 t}\Big)|a_{0k}| + \frac{k\sqrt{\eta}}{2}\Big(e^{\lambda_k^2 t} + e^{\lambda_k^1 t}\Big)|b_{0k}|\Big\}.$$

From Lemma 2.3, we deduce the following identities:

$$|\lambda_k^1| \leq \frac{\alpha}{2}, \quad |\lambda_k^2| \leq \alpha, \quad \forall k \in [1, \frac{\alpha}{\sqrt{\eta}}], \tag{18}$$

$$\exists\, C > 0 : \frac{1}{|\lambda_k^1 - \lambda_k^2|} \leq C, \quad \forall k \in [1, \frac{\alpha}{\sqrt{\eta}}]. \tag{19}$$

Now, using the relations (16)-(19), we obtain:

$$|a_k(t)| \leq C e^{-\frac{\alpha}{2}t}\Big\{\Big(\frac{\alpha^3}{4}t + \frac{\alpha}{2}\Big)|a_{0k}| + \frac{\alpha}{2}\Big(\frac{\alpha^2}{4}t + 1\Big)|b_{0k}|\Big\},$$

and then we get:

$$|a_k(t)| \leq C_0(1 + \frac{\alpha}{2}t)e^{-\frac{\alpha}{2}t}\Big(|a_{0k}| + |b_{0k}|\Big), \tag{20}$$

where $C_0 > 0$ is a constant dependent only on α and η. With the same calculation, we can prove easily that:

$$|b_k(t)| \leq C_1(1 + \frac{\alpha}{2}t)e^{-\frac{\alpha}{2}t}\Big(|a_{0k}| + |b_{0k}|\Big), \tag{21}$$

where $C_1 > 0$ is a constant dependent only on α and η.

- ◇ If $k = \frac{\alpha}{\sqrt{\eta}}$

$$|a_k(t)|^2 \leq 4e^{-\alpha t}\Big\{(1 + \frac{\alpha^2}{4}t^2)a_{0k}^2 + \frac{\alpha^2}{4}t^2 b_{0k}^2\Big\}, \tag{22}$$

$$|b_k(t)|^2 \leq 4e^{-\alpha t}\Big\{\frac{\alpha^2}{4}t^2 a_{0k}^2 + (1 + \frac{\alpha^2}{4}t^2)b_{0k}^2\Big\}, \tag{23}$$

- ◇ If $k \in]\frac{\alpha}{\sqrt{\eta}}, +\infty[$. Using the following identity

$$\frac{\sin(cx)}{c} \leq x, \quad \forall x \geq 0, \quad \forall c > 0,$$

we proved that:

$$|a_k(t)|^2 \leq 4e^{-\alpha t}\Big\{(1 + (\frac{\alpha}{2})^2 t^2)a_{0k}^2 + \frac{k^2\eta}{4}t^2 b_{0k}^2\Big\}, \tag{24}$$

$$|b_k(t)|^2 \leq 4e^{-\alpha t}\Big\{\frac{k^2\eta}{4}t^2 a_{0k}^2 + (1 + (\frac{\alpha}{2})^2 t^2)b_{0k}^2\Big\}. \tag{25}$$

So from identities (20)-(25) we can deduce the identity (15). □

Theorem 3.2. *For any $U_0 \in \tilde{H}^1_0(\Omega)$, there exists a strictly positive $t_0 > 0$ such that the solution U to the problem (12) is exponentially stable on $\tilde{H}^1(\Omega)$ and we have:*

$$\|U(t)\|_{\tilde{H}^1(\Omega)} \leq C(1 + \frac{\alpha}{2}t)e^{-\frac{\alpha}{2}t}\|U_0\|_{\tilde{H}^1(\Omega)} \quad \forall t \geq 0, \tag{26}$$

where C is a strictly positive constant dependent only on α, η.

Proof. The existence and the uniqueness of a solution to the problem (12) follows Lemma 2.1. Knowing that

$$\begin{aligned} \|U(t)\|^2_{\tilde{H}^1(\Omega)} &= \sum_{k\geq 1} \|U_k(t)\|^2_{\tilde{H}^1(\Omega)} = \sum_{k\geq 1} k^2 \|U_k(t)\|^2_{\tilde{L}^2(\Omega)}, \\ \|U_0\|^2_{\tilde{H}^1(\Omega)} &= \sum_{k\geq 1} k^2 \|U_0\|^2_{\tilde{L}^2(\Omega)}, \end{aligned}$$

and we use the same calculation as in Theorem (3.1), we can establish the identity (26). □

Remark 3.3. *It is very important to point out that the results obtained in this paper, (Lemma 2.1, Theorem 3.1, Theorem 3.2), are extremely important for other future works. Indeed, these results constitute a solid basis for the study of several problems such as the control problem for the LAS with boundary control (for example), or the study of the controllability problem for the LAS.*

4. Numerical simulation and interpretations

Recall that we are interested in the study of the following system:

$$\begin{aligned} z_t - y_x &= 0, && (27) \\ y_t - \eta z_x + \alpha y &= 0, && (28) \\ z(x,0) &= z_0(x), && (29) \\ y(x,0) &= y_0(x). && (30) \end{aligned}$$

For the simplicity of the numerical experiments, we consider a finite spatial domain $\Omega = [0,1]$. Then, the boundary conditions are as follows:

$$\begin{aligned} y(0,t) &= d_0(t), && (31) \\ y(1,t) &= d_1(t). && (32) \end{aligned}$$

Let $\nu = \frac{1}{\mathcal{M}}$ be a spatial step and $t^n = n\tau$, $n \geq 0$, where τ is a time step. So, a point x_i, $i = 0, \cdots, \mathcal{M}$ in Ω is given by: $x_i = i\nu$, $i = 0, \cdots, \mathcal{M}$. For the discretization of the problem (27)-(32), we use the finite differences $\theta-$scheme [1]. Thus, at the grid point (x_j, t^n), we have:

$$\begin{aligned} &\frac{z_j^{n+1} - z_j^n}{\tau} - \theta\frac{y_{j+1}^{n+1} - y_j^{n+1}}{\nu} - (1-\theta)\frac{y_{j+1}^n - y_j^n}{\nu} = 0, \\ &\frac{y_j^{n+1} - y_j^n}{\tau} - \eta\theta\frac{z_{j+1}^{n+1} - z_j^{n+1}}{\nu} - \eta(1-\theta)\frac{z_{j+1}^n - z_j^n}{\nu} + \alpha y_j^{n+1} = 0. \end{aligned}$$

Note that the $\theta-$scheme is unconditionally stable for $\theta \in [0.5, 1]$. In fact, this scheme is well adapted for the hyperbolic systems.

For the below examples, we consider the following data:

$$\begin{aligned} &z_0(x) = 0, \quad x \in \Omega, \\ &y_0(x) = 2\sin(11\pi x)\cos(13\pi x)e^x, \quad x \in \Omega, \\ &d_0(t) = d_1(t) = 0, \quad t \geq 0, \\ &\alpha = 8, \quad \eta = 2. \end{aligned}$$

We consider here an initial state with an important perturbation see Figure 1. In fact, this choice is taken to demonstrate the exponential stability of the solution (z, y). In Figure 2, we have plotted the eigenvalues of the system. Remark that there are 10 real eigenvalues for $\alpha = 8$ in the interval $[1, \frac{\alpha}{\sqrt{\eta}}[= [1, 5.65[$ and for $\alpha = 20$ there are 28 eigenvalues in $[1, \frac{\alpha}{\sqrt{\eta}}[= [1, 14.142[$, and infinite number of complex eigenvalues. Figure 3 confirms the theoretical results of the exponential stability obtained in Theorem 3.1. Indeed, at time $t = 1.5$ we have a good stabilization of the solution (z, y) around $(0, 0)$. Also, in figure 4 we have plotted the evolution in time of the solution (z, y) a long the time interval $[0, 1.5]$. Moreover, we studied numerically the evolution of the solution with respect to the values of the parameter α. Notes that when α increases the stabilization becomes very fast. This explains well that the real eigenvalues are responsible for a fast stablization of the solution. Indeed, from Lemma 2.3, for any value of α the real eigenvalues are λ_k^1 and λ_k^2, with $k \in [1, \frac{\alpha}{\sqrt{\eta}}[$. Thus, when $\alpha = 3, 8, 20$, $(\eta = 2)$, the number of the real eigenvalues are 4, 10, 28, respectively.

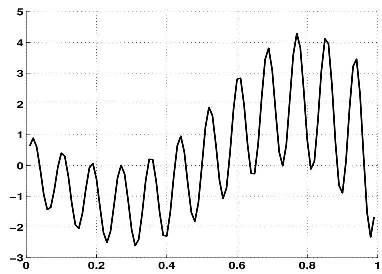

Figure 1: The initial state $y_0(x)$, $\quad x \in \Omega$.

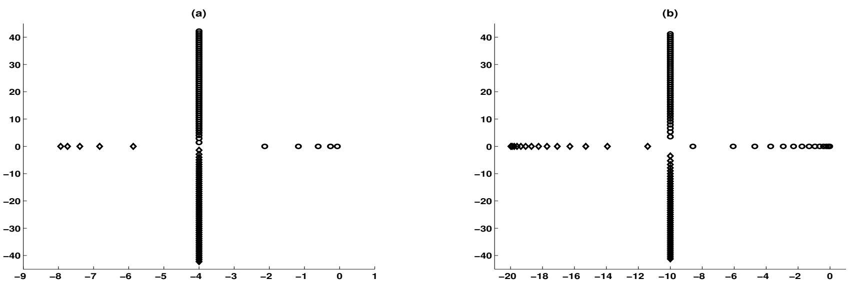

Figure 2: The eigenvalues of the system: $\alpha = 8$ in the left (a), $\alpha = 20$ in the right (b).

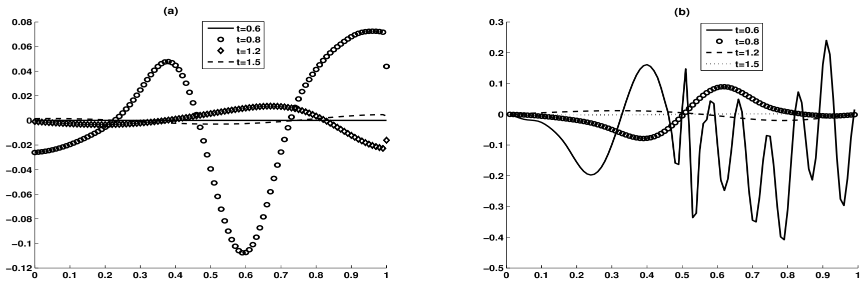

Figure 3: Numerical solutions: z in the left (a), y in the right (b).

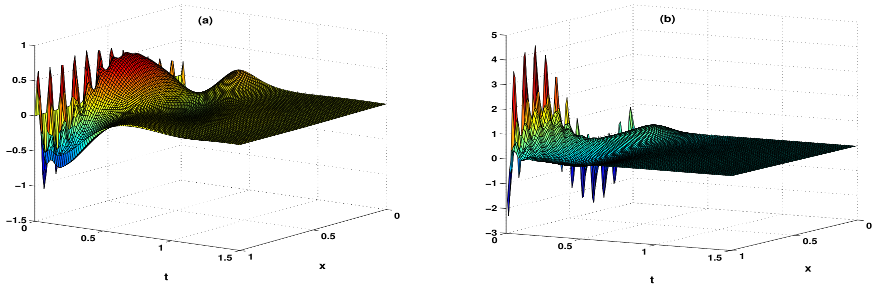

Figure 4: Numerical solutions for $t \in [0, 1.5]$: z in the left (a), y in the right (b).

5. Conclusion

In this paper, we have succeeded in determining an exponential stability result, with a decay rate $\frac{\alpha}{2} > 0$, of the solution to the linearized compressible Adiabatic flow system in one dimension. This result is obtained in two different Hilbert spaces, (see Theorem 3.1 and Theorem 3.2). Numerical experiments are given to confirm the theoretical results obtained.
This very important results can serve as a starting point for the study of several very interesting problems such as, for example, the study of the controllability problem, also the problem of stabilizability by boundary Dirichlet control. In fact, this last problem is the one I am focusing on at the moment.

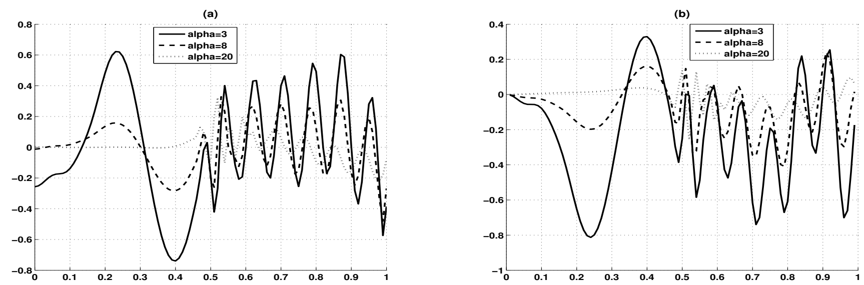

Figure 5: Numerical solutions for $t = 0.6$: z in the left (a), y in the right (b).

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