Filomat 38:13 (2024), 4597–4609
https://doi.org/10.2298/FIL2413597C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Precise large deviations for sums of two-dimensional random vectors with dependent and real-valued components

**Yao Cheng[a], Linmin Kan[a], Qingwu Gao[a,b,*]**

[a]School of Statistics and Data Science, Nanjing Audit University, Nanjing, China
[b]School of Mathematics, Nanjing Audit University, Nanjing, China

**Abstract.** In this paper, we study the precise large deviations of sums of two-dimensional random vectors with two dependent and real-valued components. In the presence of heavy tails, we obtain some uniformly asymptotic results for the bivariate case, which can provide novel insights into dependence structure between two marginal components.

## 1. Introduction

Let $\{\vec{\xi}_i = (\xi_i^{(1)}, \xi_i^{(2)})^T, i \geq 1\}$ be a sequence of independent, identically distributed and real-valued copies of a generic random vector $\vec{\xi} = (\xi^{(1)}, \xi^{(2)})^T$ with mean vector $\vec{\mu} = E\vec{\xi} = (\mu_1, \mu_2)^T$, and common marginal distributions $F_1$ and $F_2$, respectively. Denote the $n$-th partial sums of two-dimensional random vectors by

$$\vec{S}_n =: \sum_{i=1}^{n} \vec{\xi}_i = \left( \sum_{i=1}^{n} \xi_i^{(1)}, \sum_{i=1}^{n} \xi_i^{(2)} \right)^T =: \left( S_n^{(1)}, S_n^{(2)} \right)^T, \quad n \geq 1. \tag{1}$$

In this paper, we consider the precise large deviations for the partial sums of random vectors with two dependent components of heavy tails.

An important class of heavy-tailed distributions is the long-tailed class $\mathcal{L}$. Say that a distribution $F$ belongs to the class $\mathcal{L}$, denoted by $F \in \mathcal{L}$, if $\overline{F}(x) > 0$ for all $x > 0$, and

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1, \quad \text{for all } \ y \in (-\infty, \infty).$$

2020 *Mathematics Subject Classification*. Primary 60F10; Secondary 60E05.
*Keywords*. Precise large deviations; Uniform asymptotics; Sums of random vectors; Dependent components; Heavy-tails
Received: 19 January 2023; Revised: 17 April 2023; Accepted: 16 November 2023
Communicated by Dora Selesi
Research supported by the National Natural Science Foundation of China (Nos.12371471, 12371267), the Humanities and Social Sciences Foundation of the Ministry of Education of China (No.20YJCZH034), the Natural Science Foundation of Jiangsu Province of China (No. BK20241905), the Natural Science Foundation of the Jiangsu Higher Education Institutions (No.22KJA110001), the Open Project of Jiangsu Key Laboratory of Financial Engineering (No.NSK2021-05), and the Qing-Lan Project of Jiangsu Province.
* Corresponding author: Qingwu Gao
*Email addresses:* chengyao@nau.edu.cn (Yao Cheng), klinmin425@163.com (Linmin Kan), qwgao@aliyun.com,qwgao@nau.edu.cn (Qingwu Gao)

Another important class of heavy-tailed distribution is the dominated variation class $\mathcal{D}$. Say that a distribution $F$ belongs to the class $\mathcal{D}$, denoted by $F \in \mathcal{D}$, if $\overline{F}(x) > 0$ for all $x > 0$, and

$$\lim_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} < \infty, \quad \text{for all} \quad 0 < v < 1.$$

A slightly smaller class of $\mathcal{L} \cap \mathcal{D}$ is the consistent variation class $C$. Say that a distribution $F$ belongs to the class $C$, if

$$\lim_{v\searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = 1, \quad \text{or equivalently,} \quad \lim_{v\nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} = 1.$$

Finally, a distribution $F$ is said to belong to the extended regular variation class, if for some $0 < \alpha \le \beta < \infty$,

$$v^{-\beta} \le \liminf_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \le \limsup_{x\to\infty} \frac{\overline{F}(vx)}{\overline{F}(x)} \le v^{-\alpha}, \quad \text{for all} \;\; v \ge 1,$$

where we denote $F \in ERV(-\alpha, -\beta)$. It is well-known that the inclusions below are proper, that is,

$$ERV(-\alpha, -\beta) \subset C \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{L}.$$

For more details on the heavy-tailed distributions and their applications, we refer to Bingham et al. (1987) and Embrechts et al. (1997).

The precise large deviation is a classical research issue in probability theory, and also play a critical role in many applied areas such as insurance and finance. The study of precise large deviations for sums of heavy-tailed random variables was initiated from the pioneer works of Heyde (1967), A.V. Nagaev (1969a,b), and S.V. Nagaev (1979), and revisited by other researchers afterwards. See, for example, Cline and Hsing (1991), Rozovskiĭ (1993), Klüppelberg and Mikosch (1997), Mikosch and Nagaev (1998), Ng et al. (2004), Tang (2006), and Liu (2009). For recent works, especially those with applications in risk theory, we refer to Tang et al. (2001), Kaas and Tang (2005), Wang and Wang (2007), Konstantinides and Loukissas (2010), Chen and Yuen (2012), Yang and Wang (2013), He et al. (2013), Lu et al. (2014), Jiang et al. (2015), Shen et al. (2016), Liu et al. (2017), Wang and Chen (2019), Gao et al. (2020), Chen et al. (2021), Gao and Pan (2023), and references therein.

To our best knowledge, there is a relative dearth of asymptotic results in bivariate case, namely the precise large deviations for sums of two-dimensional random vectors. However, Shen and Tian (2016) studied the precise large deviations for sums of nonnegative random vectors with two dependent components using copulas for operational risk measurement, and then obtained the corresponding results for aggregate claims in a two-dimensional risk model. Fu et al. (2022) further extended Shen and Tian's results to the case in which the claim-size vectors and their waiting times are arbitrarily dependent.

In the present paper, we aim to extend the study in bivariate case from nonnegative random vectors to real-valued ones, where the main difficulty that we will encounter in the proof is due to the two-sided support of distributions $F_k$, $k = 1, 2$. As the marginal distributions of each vectors are long-tailed or consistent-varying-tailed, we obtain some asymptotic results of the precise large deviations for sums of two-dimensional random vectors. Note that our study is among the initial efforts to study the precise large deviations for sums of real-valued random vectors, and can provide invaluable insights into dependence structure between two marginal components.

The remaining part of this paper is organized as follows: we state the main results with their motivation in Section 2, and prove them in Sections 3 and 4, respectively.

## 2. Main results and their motivations

Henceforth, all limit relationships are taken as $n \to \infty$ unless otherwise stated. For two bivariate positive functions $f$ and $g$, we write $f = O(1)g$ if $\limsup f/g < \infty$, write $f = o(1)g$ if $\lim f/g = 0$, write $f \lesssim g$ or $g \gtrsim f$ if $\limsup f/g \le 1$, write $f \sim g$ if $f \lesssim g$ and $f \gtrsim g$.

Tang (2006) studied the asymptotic behavior of precise large deviations for sums $\sum_{i=1}^{n}\xi_i$ of a sequence of negatively dependent random variables $\{\xi_i, i \geq 1\}$, namely that for each $n = 1, 2, \cdots$, and all $x_1, \cdots, x_n$, the two inequalities below hold simultaneously,

$$P(\xi_1 \leq x_1, \cdots, \xi_n \leq x_n) \leq \prod_{i=1}^{n} P(\xi_i \leq x_i)$$

and

$$P(\xi_1 > x_1, \cdots, \xi_n > x_n) \leq \prod_{i=1}^{n} P(\xi_i > x_i).$$

See Lehmann (1966), Ebrahimi and Ghosh (1981), and Block et al. (1982).

**Theorem A.** *Let $\{\xi_i, i \geq 1\}$ be a sequence of negatively dependent random variables with common distribution $F \in \mathcal{C}$ and mean 0, satisfying $x\overline{F}(-x) = o(1)\overline{F}(x)$ as $x \to \infty$. Then for each fixed $\gamma > 0$, it holds uniformly for all $x \geq \gamma n$ that*

$$P(S_n > x) \sim n\overline{F}(x),$$

*namely,*

$$\lim_{n\to\infty} \sup_{x\geq\gamma n} \left| \frac{P(S_n > x)}{n\overline{F}(x)} - 1 \right| = 0.$$

For the bivariate case, Shen and Tian (2016) investigated the precise large deviations for sums $\vec{S}_n$ of two-dimensional random vectors under the following assumptions.

**$A_1$**. *Let $\{\vec{\xi}_i, i \geq 1\}$ be a sequence of independent and nonnegative random vectors with finite mean vector $\vec{\mu}$ and common marginal distributions $F_1$ and $F_2$, respectively. For every $i \geq 1$, $(\xi_i^{(1)}, \xi_i^{(2)})$ has a survival copula $\hat{C}(\cdot,\cdot)$ satisfying*

$$\hat{C}\left(\overline{F}_1(x_1), \overline{F}_2(x_2)\right) \leq M\overline{F}_1(x_1)\overline{F}_2(x_2) \tag{2}$$

*where $M$ is a positive constant.*

**$A_2$**. *$F_1 \in ERV(-\alpha, -\beta)$ for some $1 < \alpha \leq \beta < \infty$, and for some constant $c > 0$,*

$$\overline{F}_2(x) \sim c\overline{F}_1(x), \quad as \quad x \to \infty. \tag{3}$$

**$A_3$**. *There exists $1 \leq r < \infty$ such that*

$$\lim_{x\to\infty} \frac{C\,(vf(x), vg(x))}{C(f(x), g(x))} = v^r, \quad v > 0,$$

*for all $f(x) > 0$ and $g(x) > 0$ such that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.*

**Theorem B.** *Let $\{\vec{\xi}_i, i \geq 1\}$ be a sequence of independent and nonnegative random vectors with Assumptions $A_1$, $A_2$ and $A_3$, it holds uniformly for all $\vec{x} \geq \vec{\gamma}n$ that*

$$P(\vec{S}_n - n\vec{\mu} > \vec{x}) \sim \sum_{i=1}^{n}\sum_{j=1}^{n} P(\xi_i^{(1)} > x_1, \xi_j^{(2)} > x_2),$$

*namely,*

$$\lim_{n\to\infty} \sup_{\vec{x}\geq\vec{\gamma}n} \left| \frac{P(\vec{S}_n - n\vec{\mu} > \vec{x})}{\sum_{i=1}^{n}\sum_{j=1}^{n} P(\xi_i^{(1)} > x_1, \xi_j^{(2)} > x_2)} - 1 \right| = 0,$$

*where* $\vec{x} = (x_1, x_2)^T$ *and* $\vec{\gamma} = (\gamma_1, \gamma_2)^T > \vec{0} = (0, 0)^T$.

**Remark 2.1.** *As was pointed out by Shen and Tian (2016), it follows from Sklar's Theorem that* $P(\xi^{(1)} > x_1, \xi^{(2)} > x_2) = \hat{C}\left(\overline{F}_1(x_1), \overline{F}_2(x_2)\right)$. *Hence, the inequality (2) can cover not only a negative dependence structure but also a positive one.*

More recently, Fu et al. (2022) further gave the weakly asymptotic formula for precise large deviations for sums $\vec{S}_n$ of two-dimensional random vectors with dominatedly-varying-tailed components.

**Theorem C.** *Let* $\{\vec{\xi}_i, i \geq 1\}$ *be a sequence of independent and nonnegative random vectors with Assumption* $A_1$ *with the positive constant M replaced by a positive finite function. If* $F_k \in \mathcal{D}$, $k = 1, 2$, *then for any* $\vec{\gamma} > \vec{0}$, *relation*

$$L_{F_1} L_{F_2} n^2 \overline{F}_1(x_1) \overline{F}_2(x_2) \lesssim P(\vec{S}_n - n\vec{\mu} > \vec{x}) \lesssim (L_{F_1} L_{F_2})^{-1} n^2 \overline{F}_1(x_1) \overline{F}_2(x_2)$$

*holds uniformly for all* $\vec{x} \geq \vec{\gamma} n$.

Inspired by the above-referenced results, we will further in this paper consider the following issues.
(1) We will discuss the sums of a sequence of real-valued random vectors rather than nonnegative ones considered by Shen and Tian (2016) as well as Fu et al. (2022), and so the main difficulty in this paper is due to the two-sided support of distributions $F_k$, $k = 1, 2$.
(2) We will extend the class of marginal distributions of random vectors from the class ERV of Shen and Tian (2016) to the class $\mathcal{L}$ or $\mathcal{C}$.
(3) We will remove the interrelationship (3) between $F_1$ and $F_2$ and Assumption $A_3$ of Shen and Tian (2016).

In the following, we state our main results of this paper, among which the first one establishes the asymptotic lower bound of precise large deviations for sums $\vec{S}_n$ of two-dimensional random vectors with long-tailed components.

**Theorem 2.1.** *Let* $\{\vec{\xi}_i, i \geq 1\}$ *be a sequence of independent and real-valued random vectors with mean vector* $\vec{0}$. *If* $F_k \in \mathcal{L}$, $k = 1, 2$, *and Assumption* $A_1$ *is satisfied, then for any* $\vec{\gamma} > \vec{0}$, *it holds uniformly for all* $\vec{x} \geq \vec{\gamma} n$ *that*

$$P(\vec{S}_n > \vec{x}) \gtrsim \sum_{i=1}^{n} \sum_{j=1}^{n} P(\xi_i^{(1)} > x_1, \xi_j^{(2)} > x_2) \sim n^2 \overline{F}_1(x_1) \overline{F}_2(x_2).$$

The second theorem provides an asymptotic formula of the precise large deviations for sums $\vec{S}_n$ of two-dimensional random vectors with consistent-varying-tailed components.

**Theorem 2.2.** *Under the conditions of Theorem 2.1 except that* $F_k \in \mathcal{C}$, $k = 1, 2$, *it holds uniformly for all* $\vec{x} \geq \vec{\gamma} n$ *that*

$$P(\vec{S}_n > \vec{x}) \sim \sum_{i=1}^{n} \sum_{j=1}^{n} P(\xi_i^{(1)} > x_1, \xi_j^{(2)} > x_2) \sim n^2 \overline{F}_1(x_1) \overline{F}_2(x_2).$$

## 3. Proof of Theorem 2.1

By (2) and Remark 2.1, it follows that

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1}^{n} P\left(\xi_i^{(1)} > x_1, \xi_j^{(2)} > x_2\right) \\
&= \sum_{i=1}^{n} P\left(\xi_i^{(1)} > x_1, \xi_i^{(2)} > x_2\right) + \sum_{1 \leq i \neq j \leq n} P\left(\xi_i^{(1)} > x_1, \xi_j^{(2)} > x_2\right) \\
&\sim n^2 \overline{F}_1(x_1) \overline{F}_2(x_2).
\end{aligned} \tag{4}$$

So it suffices to show that, uniformly for all $\vec{x} \geq \vec{\gamma} n$,

$$P(\vec{S}_n > \vec{x}) \gtrsim n^2 \overline{F}_1(x_1)\overline{F}_2(x_2). \tag{5}$$

Consider a distribution $F \in \mathcal{L}$, there exists a function $h$: $(0,\infty) \mapsto [0,\infty)$ such that $h(x) \to \infty$, $h(x) = o(1)x$ and

$$\overline{F}(x + h(x)) \sim \overline{F}(x), \quad \text{as} \quad x \to \infty. \tag{6}$$

See, for example, Embrechts et al. (1997). Then for the function $h(x)$ as above, we have

$$\begin{aligned}
& P(\vec{S}_n > \vec{x}) \\
& \geq P\left(\vec{S}_n > \vec{x}, \max_{1\leq i\leq n} \xi_i^{(1)} > x_1 + h(x_1), \max_{1\leq j\leq n} \xi_j^{(2)} > x_2 + h(x_2)\right) \\
& \geq \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(\vec{S}_n > \vec{x}, \xi_i^{(1)} > x_1 + h(x_1), \xi_j^{(2)} > x_2 + h(x_2)\right) \\
& \quad - \sum_{i=1}^{n} \sum_{1\leq j_1\neq j_2\leq n} P\left(\xi_i^{(1)} > x_1 + h(x_1), \xi_{j_1}^{(2)} > x_2 + h(x_2), \xi_{j_2}^{(2)} > x_2 + h(x_2)\right) \\
& \quad - \sum_{1\leq i_1\neq i_2\leq n} \sum_{j=1}^{n} P\left(\xi_{i_1}^{(1)} > x_1 + h(x_1), \xi_{i_2}^{(1)} > x_1 + h(x_1), \xi_j^{(2)} > x_2 + h(x_2)\right) \\
& = I_1(\vec{x}, n) - I_2(\vec{x}, n) - I_3(\vec{x}, n).
\end{aligned} \tag{7}$$

Firstly, we consider $I_1(\vec{x}, n)$, which is written as

$$\begin{aligned}
& I_1(\vec{x}, n) \\
& \geq \sum_{i=1}^{n}\sum_{j=1}^{n} P\begin{pmatrix} \sum_{k=1,k\neq i}^{n} \xi_k^{(1)} > -h(x_1), & \sum_{k=1,k\neq j}^{n} \xi_k^{(2)} > -h(x_2), \\ \xi_i^{(1)} > x_1 + h(x_1), & \xi_j^{(2)} > x_2 + h(x_2) \end{pmatrix} \\
& \geq \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(\xi_i^{(1)} > x_1 + h(x_1), \xi_j^{(2)} > x_2 + h(x_2)\right) \\
& \quad - \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(\xi_i^{(1)} > x_1 + h(x_1), \xi_j^{(2)} > x_2 + h(x_2), \sum_{k=1,k\neq i}^{n} \xi_k^{(1)} < -h(x_1)\right) \\
& \quad - \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(\xi_i^{(1)} > x_1 + h(x_1), \xi_j^{(2)} > x_2 + h(x_2), \sum_{k=1,k\neq j}^{n} \xi_k^{(2)} < -h(x_2)\right) \\
& = I_{11}(\vec{x}, n) - I_{12}(\vec{x}, n) - I_{13}(\vec{x}, n),
\end{aligned} \tag{8}$$

where the second step is due to an elementary inequality $P(ABC) \geq P(A) - P(A\overline{B}) - P(A\overline{C})$. For $I_{12}(\vec{x}, n)$, one has

$$\begin{aligned}
I_{12}(\vec{x}, n) &= \sum_{i=1}^{n} P\left(\xi_i^{(1)} > x_1 + h(x_1), \xi_i^{(2)} > x_2 + h(x_2)\right) P\left(\sum_{k=1,k\neq i}^{n} \xi_k^{(1)} < -h(x_1)\right) \\
& \quad + \sum_{1\leq i\neq j\leq n} P\left(\xi_i^{(1)} > x_1 + h(x_1)\right) P\left(\xi_j^{(2)} > x_2 + h(x_2), \sum_{k=1,k\neq i}^{n} \xi_k^{(1)} < -h(x_1)\right) \\
&= I_{121}(\vec{x}, n) + I_{122}(\vec{x}, n).
\end{aligned} \tag{9}$$

By the law of large numbers, it holds that

$$\lim_{n\to\infty}\sup_{x_1\geq\gamma_1 n} P\left(\sum_{k=1,k\neq i}^{n}\xi_k^{(1)} < -h(x_1)\right) = 0,$$

which leads to

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{I_{121}(\vec{x},n)}{I_{11}(\vec{x},n)}\\
\leq\ &\lim_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{\sum_{i=1}^{n}P(\xi_i^{(1)}>x_1+h(x_1),\xi_i^{(2)}>x_2+h(x_2))P\left(\sum_{k=1,k\neq i}^{n}\xi_k^{(1)}<-h(x_1)\right)}{\sum_{i=1}^{n}P(\xi_i^{(1)}>x_1+h(x_1),\xi_i^{(2)}>x_2+h(x_2))}\\
=\ &0
\end{aligned}\tag{10}$$

Clearly, one knows that

$$\begin{aligned}
&I_{122}(\vec{x},n)\\
&=\sum_{1\leq i\neq j\leq n}P\left(\xi_i^{(1)}>x_1+h(x_1)\right)P\left(\xi_j^{(2)}>x_2+h(x_2),\sum_{k=1,k\neq i}^{n}(-\xi_k^{(1)})>h(x)\right)\\
&\leq\sum_{1\leq i\neq j\leq n}P\left(\xi_i^{(1)}>x_1+h(x_1)\right)P\left(\xi_j^{(2)}>x_2+h(x_2),\sum_{k=1,k\neq i}^{n}\xi_k^{(1)-}>h(x)\right)
\end{aligned}$$

where $\xi_k^{(1)-}=\max\{-\xi_k^{(1)},0\}, k=1,2,\cdots,n$. Now we construct two independent nonnegative random variables $\eta^{(1)}$ and $\eta^{(2)}$ with their tails as

$$\overline{G}_1(x_1)=\min\{1,KF_1(-x_1)\},$$

and

$$\overline{G}_2(x_2)=\min\{1,K\overline{F}_2(x_2)\}\sim K\overline{F}_2(x_2),$$

for some $K>1$ such that $(\xi_j^{(1)-},\xi_j^{(2)})\leq_{\rm st}(\eta^{(1)},\eta^{(2)})$. Let $(\eta^{(1)},\eta^{(2)})$ be independent of the other sources of randomness. Then, it follows that

$$\begin{aligned}
&I_{122}(\vec{x},n)\\
&\leq\sum_{1\leq i\neq j\leq n}P\left(\xi_i^{(1)}>x_1+h(x_1)\right)P\left(\eta^{(2)}>x_2+h(x_2),\sum_{k=1,k\neq i,j}^{n}\xi_k^{(1)-}+\eta^{(1)}>h(x_1)\right)\\
&=\sum_{1\leq i\neq j\leq n}P\left(\xi_i^{(1)}>x_1+h(x_1)\right)P\left(\eta^{(2)}>x_2+h(x_2)\right)P\left(\sum_{k=1,k\neq i,j}^{n}\xi_k^{(1)-}+\eta^{(1)}>h(x_1)\right)
\end{aligned}$$

Again by the law of large numbers, one gets

$$\lim_{n\to\infty}\sup_{x_1\geq\gamma_1 n}P\left(\sum_{k=1,k\neq i,j}^{n}\xi_k^{(1)-}+\eta^{(1)}>h(x_1)\right)=0,$$

which yields that

$$\lim_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{I_{122}(\vec{x},n)}{I_{11}(\vec{x},n)}=0.\tag{11}$$

Hence by substituting (10) and (11) into (9), we obtain that

$$\limsup_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{I_{12}(\vec{x},n)}{I_{11}(\vec{x},n)}=0. \tag{12}$$

Similarly, we still obtain that

$$\limsup_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{I_{13}(\vec{x},n)}{I_{11}(\vec{x},n)}=0. \tag{13}$$

So by (8), (12) and (13), it holds uniformly for all $\vec{x}\geq\vec{\gamma}n$ that

$$I_1(\vec{x},n)\gtrsim I_{11}(\vec{x},n). \tag{14}$$

Subsequently, we turn to consider $I_2(\vec{x},n)$, which is formulated as

$$\begin{aligned}
I_2(\vec{x},n) &= \sum_{1\leq j_1\neq j_2\leq n} P\left(\xi_{j_1}^{(1)}>x_1+h(x_1),\xi_{j_1}^{(2)}>x_2+h(x_2)\right)P\left(\xi_{j_2}^{(2)}>x_2+h(x_2)\right)\\
&\quad+\sum_{i=1,i\neq j_1}^{n}\sum_{1\leq j_1\neq j_2\leq n} P\left(\xi_{i}^{(1)}>x_1+h(x_1),\xi_{j_2}^{(2)}>x_2+h(x_2)\right)P\left(\xi_{j_1}^{(2)}>x_2+h(x_2)\right)\\
&\leq n\overline{F}_2(x_2+h(x_2))\sum_{j_1=1}^{n}P\left(\xi_{j_1}^{(1)}>x_1+h(x_1),\xi_{j_1}^{(2)}>x_2+h(x_2)\right)\\
&\quad+n\overline{F}_2(x_2+h(x_2))\sum_{i=1}^{n}\sum_{j_2=1}^{n}P\left(\xi_{i}^{(1)}>x_1+h(x_1),\xi_{j_2}^{(2)}>x_2+h(x_2)\right)\\
&\leq 2n\overline{F}_2(x_2+h(x_2))I_{11}(\vec{x},n),
\end{aligned}$$

which implies that

$$\begin{aligned}
\limsup_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{I_2(\vec{x},n)}{I_{11}(\vec{x},n)} &\leq \lim_{n\to\infty}\sup_{x_2\geq\gamma_2 n}2n\overline{F}_2(x_2+h(x_2)).\\
&\leq \lim_{x_2\to\infty}\frac{2x_2}{\gamma_2}\overline{F}_2(x_2+h(x_2))\\
&= 0. 
\end{aligned}\tag{15}$$

Similarly, it still holds that

$$\limsup_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{I_3(\vec{x},n)}{I_{11}(\vec{x},n)}=0. \tag{16}$$

Therefore, we substitute (14)-(16) into (7) to obtain that, uniformly for all $\vec{x}\geq\vec{\gamma}n$,

$$P(\vec{S}_n>\vec{x})\gtrsim I_{11}(\vec{x},n). \tag{17}$$

Note that by $F_k\in\mathcal{L}$, $k=1,2$, (2), (6) and Remark 2.1, it holds uniformly for all $\vec{x}\geq\vec{\gamma}n$ that

$$\begin{aligned}
I_{11}(\vec{x},n) &= nP(\xi^{(1)}>x_1+h(x_1),\xi^{(2)}>x_2+h(x_2)\\
&\quad+(n^2-n)\bar{F}_1(x_1+h(x_1))\bar{F}_2(x_2+h(x_2))\\
&\sim n^2\overline{F}_1(x_1)\overline{F}_2(x_2),
\end{aligned}$$

which, along with (17), proves that relation (5) holds uniformly for all $\vec{x}\geq\vec{\gamma}n$, and then completes the proof of Theorem 2.1.

**4. Proof of Theorem 2.2**

Consider a distribution $F \in \mathscr{D}$ with its upper Matuszewska index

$$J_F^+ = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log(y)},$$

where $\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}$ for $y > 0$. By Proposition 2.2.1 of Bingham et al. (1987), one knows that for any $p > J_F^+$, there exist positive constant $x_0$ and $C$ such that

$$\frac{\overline{F}(y)}{\overline{F}(x)} \le C\left(\frac{x}{y}\right)^p \tag{18}$$

holds for all $x \ge y \ge x_0$.

According to Theorem 2.1, it suffices to establish the asymptotic upper bound of $P(\vec{S}_n > \vec{x})$, namely that, uniformly for all $\vec{x} \ge \vec{\gamma} n$,

$$P(\vec{S}_n \ge \vec{x}) \lesssim n^2 \overline{F}_1(x_1)\overline{F}_2(x_2). \tag{19}$$

For any $0 < v < 1$, we apply a standard truncation argument to obtain that

$$\begin{aligned} P(\vec{S}_n > \vec{x}) &\le \sum_{i=1}^n \sum_{j=1}^n P\left(\xi_i^{(1)} > vx_1, \xi_j^{(2)} > vx_2\right) \\ &\quad + P\left(\vec{S}_n > \vec{x}, \max_{1\le i\le n} \xi_i^{(1)} \le vx_1, \max_{1\le j\le n} \xi_j^{(2)} \le vx_2\right) \\ &\quad + P\left(\vec{S}_n > \vec{x}, \max_{1\le i\le n} \xi_i^{(1)} \le vx_1, \max_{1\le j\le n} \xi_j^{(2)} > vx_2\right) \\ &\quad + P\left(\vec{S}_n > \vec{x}, \max_{1\le i\le n} \xi_i^{(1)} > vx_1, \max_{1\le j\le n} \xi_j^{(2)} \le vx_2\right) \\ &= \sum_{i=1}^4 K_i(\vec{x}, n) \end{aligned} \tag{20}$$

Clearly, by $F_k \in \mathcal{C}$, $k = 1, 2$, (2) and Remark 2.1, it follows that

$$\limsup_{n\to\infty} \sup_{\vec{x}\ge\vec{\gamma}n} \frac{K_1(\vec{x}, n)}{n^2\overline{F}_1(x_1)\overline{F}_2(x_2)} \le \lim_{v\nearrow 1} \limsup_{n\to\infty} \sup_{\vec{x}\ge\vec{\gamma}n} \frac{\overline{F}_1(vx_1)\overline{F}_2(vx_2)}{\overline{F}_1(x_1)\overline{F}_2(x_2)} = 1. \tag{21}$$

Now we deal with $K_2(\vec{x}, n)$, where the method we used is motivated by those of Tang (2006) and Shen and Tian (2016), but the main difficulty we encounter is due to the two-sided support of distributions $F_k$ of real-valued random variable $\xi^{(k)}$, $k = 1, 2$. Define $\widetilde{\xi}_i^{(1)} = \min\{\xi_i^{(1)}, vx_1\}$, $\widetilde{\xi}_i^{(2)} = \min\{\xi_i^{(2)}, vx_2\}$, $i = 1, 2, \cdots, n$, and $\widetilde{S}_n^{(1)} = \sum_{i=1}^n \widetilde{\xi}_i^{(1)}$, $\widetilde{S}_n^{(2)} = \sum_{i=1}^n \widetilde{\xi}_i^{(2)}$. Take $a_1 = \max\{-\log n\overline{F}_1(vx_1), 1\}$ and $a_2 = \max\{-\log n\overline{F}_2(vx_2), 1\}$, which both tend to $\infty$ uniformly for all $\vec{x} \ge \vec{\gamma}n$. For two arbitrarily fixed $h_1(x_1, n) > 0$ and $h_2(x_2, n) > 0$, it follows from Chebyshev's inequality that

$$\begin{aligned} K_2(\vec{x}, n) &\le P(\widetilde{S}_n^{(1)} > x_1, \widetilde{S}_n^{(2)} > x_2) \\ &\le e^{-h_1x_1 - h_2x_2}(Ee^{h_1\widetilde{\xi}_1^{(1)} + h_2\widetilde{\xi}_1^{(2)}})^n. \end{aligned}$$

The expectation $Ee^{h_1\tilde{\xi}_1^{(1)}+h_2\tilde{\xi}_1^{(2)}}$ is divided as

$$\begin{aligned}
& Ee^{h_1\tilde{\xi}_1^{(1)}+h_2\tilde{\xi}_1^{(2)}} \\
&= 1+\int_{-\infty}^{vx_2}\int_{-\infty}^{vx_1}(e^{h_1t_1+h_2t_2}-1)F(dt_1,dt_2)+\int_{-\infty}^{vx_2}\int_{vx_1}^{\infty}(e^{h_1vx_1+h_2t_2}-1)F(dt_1,dt_2) \\
&\quad +\int_{vx_2}^{\infty}\int_{-\infty}^{vx_1}(e^{h_1t_1+h_2vx_2}-1)F(dt_1,dt_2)+\int_{vx_2}^{\infty}\int_{vx_1}^{\infty}(e^{h_1vx_1+h_2vx_2}-1)F(dt_1,dt_2) \\
&= 1+\sum_{i=1}^{4}H_i(\vec{x}),
\end{aligned}$$

where $F(t_1,t_2)$ is the joint distribution of random vector $(\xi^{(1)},\xi^{(2)})$. Hence, we have

$$\begin{aligned}
\frac{K_2(\vec{x},n)}{K_1(\vec{x},n)} &\leq \frac{e^{-h_1x_1-h_2x_2}}{n(n-1)\overline{F}_1(vx_1)\overline{F}_2(vx_2)}\cdot\left(1+\sum_{i=1}^{4}H_i(\vec{x})\right)^n \\
&\leq \frac{n}{n-1}\exp\left\{n\sum_{i=1}^{4}H_i(\vec{x})-h_1x_1-h_2x_2+a_1+a_2\right\},
\end{aligned}\tag{22}$$

where in the last step we used an elementary inequality $1+s\leq e^s$ for all $s$. For $H_1(\vec{x})$, it is divided as

$$\begin{aligned}
H_1(\vec{x}) &\leq \Bigg(\int_{-\infty}^{0}\int_{-\infty}^{0}+\int_{0}^{vx_2/a_2^2}\int_{-\infty}^{vx_1/a_1^2}+\int_{-\infty}^{vx_2/a_2^2}\int_{0}^{vx_1/a_1^2}+\int_{-\infty}^{vx_2/a_2^2}\int_{vx_1/a_1^2}^{vx_1} \\
&\qquad +\int_{vx_2/a_2^2}^{vx_2}\int_{-\infty}^{vx_1/a_1^2}+\int_{vx_2/a_2^2}^{vx_2}\int_{vx_1/a_1^2}^{vx_1}\Bigg)(e^{h_1t_1+h_2t_2}-1)F(dt_1,dt_2) \\
&=: \sum_{i=1}^{6}H_{1i}(\vec{x})
\end{aligned}\tag{23}$$

By the elementary inequality $s\leq e^s-1\leq se^s$ for all $s$, we get that for all $t_1\leq 0, t_2\leq 0$,

$$\begin{aligned}
0\leq\frac{e^{h_1t_1+h_2t_2}-1-(h_1t_1+h_2t_2)}{h_1+h_2} &\leq \frac{(h_1t_1+h_2t_2)(e^{h_1t_1+h_2t_2}-1)}{h_1+h_2} \\
&\leq -\frac{h_1t_1+h_2t_2}{h_1+h_2} \\
&\leq -(t_1+t_2).
\end{aligned}\tag{24}$$

Considering the arbitrariness of $h_1$ and $h_2$, we take $(h_1,h_2)\searrow(0,0)$, and $h_1=O(1)h_2$, $h_2=O(1)h_1$, which means that $0<\liminf h_1/h_2\leq\limsup h_1/h_2<\infty$, and so for all large $n$, there exist constants $B_1>0$ and $B_2>0$ such that $B_1<h_1/h_2<B_2$. Then by combining (24) and the dominated convergence theorem, we derive that

$$\begin{aligned}
&\lim_{(h_1,h_2)\searrow(0,0)}\frac{\int_{-\infty}^{0}\int_{-\infty}^{0}(e^{h_1t_1+h_2t_2}-1)F(dt_1,dt_2)}{h_1+h_2} \\
\leq\ &\lim_{(h_1,h_2)\searrow(0,0)}\Bigg\{\int_{-\infty}^{0}\int_{-\infty}^{0}\frac{e^{h_1t_1+h_2t_2}-1-(h_1t_1+h_2t_2)}{h_1+h_2}F(dt_1,dt_2) \\
&\qquad +\int_{-\infty}^{\infty}\int_{-\infty}^{0}\frac{h_1t_1}{h_1+h_2}F(dt_1,dt_2)+\int_{-\infty}^{0}\int_{-\infty}^{\infty}\frac{h_2t_2}{h_1+h_2}F(dt_1,dt_2)\Bigg\} \\
\leq\ & B(\mu_{1-}+\mu_{2-}),
\end{aligned}$$

where $B = \{\frac{B_1}{1+B_1}, \frac{1}{1+B_2}\}$, $\mu_{1^-} = E\xi^{(1)}1_{\{\xi^{(1)}\le 0\}}$, and $\mu_{2^-} = E\xi^{(2)}1_{\{\xi^{(2)}\le 0\}}$. So there exists a bivariate real function $\varepsilon(\cdot,\cdot)$ with $\varepsilon(h_1,h_2)\to 0$ as $(h_1,h_2)\searrow(0,0)$, such that

$$H_{11}(\vec{x}) \le (1+\varepsilon(h_1,h_2))B(h_1+h_2)(\mu_{1^-}+\mu_{2^-}).$$

Again by the elementary inequality $e^s - 1 \le se^s$ for all $s$, we have

$$\begin{aligned} H_{12}(\vec{x}) &\le \int_0^{vx_2/a_2^2}\int_{-\infty}^{vx_1/a_1^2}(h_1t_1+h_2t_2)e^{h_1t_1+h_2t_2}F(dt_1,dt_2) \\ &\le e^{h_1vx_1/a_1^2+h_2vx_2/a_2^2}\int_0^{vx_2/a_2^2}\int_{-\infty}^{vx_1/a_1^2}(h_1t_1+h_2t_2)F(dt_1,dt_2) \\ &\le e^{h_1vx_1/a_1^2+h_2vx_2/a_2^2}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_1t_1F(dt_1,dt_2)+\int_0^{\infty}\int_{-\infty}^{\infty}h_2t_2F(dt_1,dt_2)\right\} \\ &\le e^{h_1vx_1/a_1^2+h_2vx_2/a_2^2}h_2\mu_{2^+}, \end{aligned}$$

where $\mu_{2^+} = E\xi^{(2)}1_{\{\xi^{(2)}>0\}}$, and the last step is also due to $\mu_1 = 0$. Similarly, we still have

$$H_{13}(\vec{x}) \le e^{h_1vx_1/a_1^2+h_2vx_2/a_2^2}h_1\mu_{1^+},$$

where $\mu_{1^+} = E\xi^{(1)}1_{\{\xi^{(1)}>0\}}$. For $H_{14}(\vec{x})$, $H_{15}(\vec{x})$ and $H_{16}(\vec{x})$, we obtain that

$$\begin{aligned} H_{14}(\vec{x}) &\le \int_{-\infty}^{vx_2/a_2^2}\int_{vx_1/a_1^2}^{vx_1}e^{h_1t_1+h_2t_2}F(dt_1,dt_2) \\ &\le e^{h_1vx_1+h_2vx_2/a_2^2}\overline{F}_1(vx_1/a_1^2), \end{aligned}$$

$$H_{15}(\vec{x}) \le e^{h_1vx_1/a_1^2+h_2vx_2}\overline{F}_2(vx_2/a_2^2),$$

and

$$H_{16}(\vec{x}) \le e^{h_1vx_1+h_2vx_2}\overline{F}(vx_1/a_1^2, vx_2/a_2^2).$$

Therefore by (23) and the above derivations for $H_{1i}(\vec{x})$, $i = 1,2,\cdots,6$, we derive that

$$\begin{aligned} H_1(\vec{x}) \le\ & (1+\varepsilon(h_1,h_2))(h_1+h_2)(\mu_{1^-}+\mu_{2^-}) + e^{h_1vx_1/a_1^2+h_2vx_2/a_2^2}(h_1\mu_{1^+}+h_2\mu_{2^+}) \\ & + e^{h_1vx_1+h_2vx_2/a_2^2}\overline{F}_1(vx_1/a_1^2) + e^{h_1vx_1/a_1^2+h_2vx_2}\overline{F}_2(vx_2/a_2^2) \\ & + e^{h_1vx_1+h_2vx_2}\overline{F}(vx_1/a_1^2, vx_2/a_2^2). \end{aligned} \tag{25}$$

For $H_2(\vec{x})$, it holds that

$$\begin{aligned} H_2(\vec{x}) &= \left(\int_{-\infty}^{vx_2/a_2^2}\int_{vx_1}^{\infty}+\int_{vx_2/a_2^2}^{vx_2}\int_{vx_1}^{\infty}\right)(e^{h_1vx_1+h_2t_2}-1)F(dt_1,dt_2) \\ &\le e^{h_1vx_1+h_2vx_2/a_2^2}\overline{F}_1(vx_1) + e^{h_1vx_1+h_2vx_2/a_2^2}\overline{F}(vx_1, vx_2/a_2^2). \end{aligned} \tag{26}$$

Similarly, it still holds that

$$\begin{aligned} H_3(\vec{x}) &= \left(\int_{vx_2}^{\infty}\int_{-\infty}^{vx_1/a_1^2}+\int_{vx_2}^{\infty}\int_{vx_1/a_1^2}^{vx_1}\right)(e^{h_1t_1+h_2vx_2}-1)F(dt_1,dt_2) \\ &\le e^{h_1vx_1/a_1^2+h_2vx_2}\overline{F}_2(vx_2) + e^{h_1vx_1/a_1^2+h_2vx_2}\overline{F}(vx_1/a_1^2, vx_2), \end{aligned} \tag{27}$$

and

$$H_4(\vec{x}) \le e^{h_1vx_1+h_2vx_2}\overline{F}(vx_1, vx_2). \tag{28}$$

Take $h_1(x_1, n) = \frac{a_1 - 2p_1 \log a_1}{vx_1}$ and $h_2(x_2, n) = \frac{a_2 - 2p_2 \log a_2}{vx_2}$, where $p_1 > J^+_{F_1}$ and $p_2 > J^+_{F_2}$. Hence, by substituting (25)-(28) into (22), and using (18) and (2), we conclude that for all large $n$, there exist constants $C_1$ and $C_2$, depending on $F_1$ and $F_2$, respectively, such that

$$\begin{aligned}
&\frac{K_2(\vec{x}, n)}{K_1(\vec{x}, n)} \\
\leq\ & \frac{n}{n-1} \exp\Bigg\{\Big[(1 + \varepsilon(h_1, h_2))(\mu_{1^-} + \mu_{2^-}) + e^{\frac{1}{a_1} + \frac{1}{a_2}}(\mu_{1^+} + \mu_{2^+})\Big] n(h_1 + h_2) \\
& + C_1 e^{\frac{1}{a_2}} + C_2 e^{\frac{1}{a_1}} + \frac{MC_1C_2}{n} + o(1) e^{\frac{1}{a_2}} + o(1) e^{\frac{1}{a_1}} + o(1)\frac{MC_1}{n} \\
& + o(1)\frac{MC_2}{n} + o(1)\frac{M}{n} - h_1 x_1 - h_2 x_2 + a_1 + a_2\Bigg\} \\
\leq\ & O(1) \exp\left\{o(1) n(h_1 + h_2) + \left(1 - \frac{1}{v}\right)(a_1 + a_2)\right\} \\
=\ & O(1) \exp\left\{o(1) a_1 + o(1) a_2 + \left(1 - \frac{1}{v}\right)(a_1 + a_2)\right\},
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \sup_{\vec{x} \geq \vec{\gamma} n} \frac{K_2(\vec{x}, n)}{K_1(\vec{x}, n)} = 0. \tag{29}$$

Subsequently, we turn to estimate $K_3(\vec{x}, n)$. By (2), it holds uniformly for all $\vec{x} \geq \vec{\gamma} n$ that

$$\begin{aligned}
& K_3(\vec{x}, n) \\
& \leq \sum_{j=1}^{n} P(\tilde{S}_n^{(1)} > x_1, \xi_j^{(2)} > vx_2) \\
& = \sum_{j=1}^{n} \int_0^{vx_1} \cdots \int_0^{vx_1} P\left(\xi_j^{(1)} > x_1 - x_1^{(1)} - \cdots - x_{j-1}^{(1)} - x_{j+1}^{(1)} - \cdots - x_n^{(1)}, \xi_j^{(2)} > vx_2\right) \\
& \quad G_j(dx_1^{(1)}, \cdots, dx_{j-1}^{(1)}, dx_{j+1}^{(1)}, \cdots, dx_n^{(1)}) \\
& \leq M \sum_{j=1}^{n} \int_0^{vx_1} \cdots \int_0^{vx_1} P\left(\xi_j^{(1)} > x_1 - x_1^{(1)} - \cdots - x_{j-1}^{(1)} - x_{j+1}^{(1)} - \cdots - x_n^{(1)}\right) \\
& \quad \cdot P(\xi_j^{(2)} > vx_2) G_j(dx_1^{(1)}, \cdots, dx_{j-1}^{(1)}, dx_{j+1}^{(1)}, \cdots, dx_n^{(1)}) \\
& = Mn\overline{F}_2(vx_2) P(\tilde{S}_n^{(1)} > x_1),
\end{aligned}$$

where $G_j$ is the joint distribution of $(\xi_1^{(1)}, \cdots, \xi_{j-1}^{(1)}, \xi_{j+1}^{(1)}, \cdots, \xi_n^{(1)})$. Thus we obtain that, uniformly for all $\vec{x} \geq \vec{\gamma} n$,

$$\begin{aligned}
\frac{K_3(\vec{x}, n)}{K_1(\vec{x}, n)} &\leq \frac{Mn\overline{F}_2(vx_2) P(\tilde{S}_n^{(1)} > x_1)}{n(n-1)\overline{F}_1(vx_1)\overline{F}_2(vx_2)} \\
&= \frac{MP(\tilde{S}_n^{(1)} > x_1)}{(n-1)\overline{F}_1(vx_1)}.
\end{aligned}$$

By mimicking the proof of Theorem 3.1 of Tang (2006), and taking $a = \max\{1, -\log(n-1)\overline{F}_1(vx_1)\}$ and

$h = \frac{a-2p\log a}{vx_1}$, we derive that for all large $n$,

$$\begin{aligned}
\frac{P(\tilde{S}_n^{(1)} > x_1)}{(n-1)\overline{F}_1(vx_1)} &\leq e^{-hx+a}\left((1+\varepsilon(h))hu_{1^-} + e^{\frac{1}{a}}hu_{1^+} + O(1)e^a\overline{F}_1(vx_1) + 1\right)^n \\
&\leq \exp\left\{[1+\varepsilon(h))u_{1^-} + e^{\frac{1}{a}}u_{1^+}]nh + O(1)e^a\overline{F}_1(vx_1) - hx + a\right\} \\
&= O(1)\exp\{o(1)nh - hx + a\} \\
&= O(1)\exp\left\{o(a) + \left(1 - \frac{1}{v}\right)a\right\} \\
&= o(1),
\end{aligned}$$

which leads to

$$\limsup_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{K_3(\vec{x},n)}{K_1(\vec{x},n)} = 0. \tag{30}$$

Similarly, we also get that

$$\limsup_{n\to\infty}\sup_{\vec{x}\geq\vec{\gamma}n}\frac{K_4(\vec{x},n)}{K_1(\vec{x},n)} = 0. \tag{31}$$

Consequently, we substitute (21), (29)-(31) into (20) to show that relation (19) holds uniformly for all $\vec{x} \geq \vec{\gamma}n$, and then give the proof of this theorem.

**Acknowledgements.** The authors would like to thank the editors and the anonymous referee for their efforts on this paper.

## References


[1] N. Bingham, C. Goldie, J. Teugels, *Regular variation*, Cambridge University Press, Cambridge, 1987.
[2] H.W. Block, T.H. Savits, M. Shaked, *Some concepts of negative dependence*, Ann. Probab. **10** (1982), 765-772.
[3] Y. Chen, K. Yuen, *Precise large deviations of aggregate claims in a size-dependent renewal risk model*, Insur. Math. Econ. **51(2)** (2012), 457-461.
[4] Y. Chen, T. White, K. Yuen, *Precise large deviations of aggregate claims with arbitrary dependence between claim sizes and waiting times*, Insur. Math. Econ. **97** (2021), 1-6.
[5] D.B.H. Cline, T. Hsing, *Large deviation probabilities for sums and maxima of random variables with heavy or subexponential tails*, Preprint, Texas A&M University, 1991.
[6] N. Ebrahimi, M. Ghosh, *Multivariate negative dependence*, Comm. Statist. A-Theory Methods **10** (1981), 307-337.
[7] P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling extremal events for insurance and finance*, Springer, Berlin, 1997.
[8] K. Fu, Y. Liu, J. Wang, *Precise large deviations in a bidimensional risk model with arbitrary dependence between claim-size vectors and waiting times*, Statist. Probab. Lett **184** (2022), 109365.
[9] Q. Gao, X. Liu, C. Chai, *Asymptotic bounds for precise large deviations in a compound risk model under dependence structures*, J. Math. Inequal. **14(4)** (2020), 1067-1082.
[10] Q. Gao, W. Pan, *Precise large deviations of aggregate claims in a nonstandard risk model with arbitrary dependence between claim sizes and waiting times*, AIMS Mathematics **8(1)** (2023), 2191-2200.
[11] C.C. Heyde, *On large deviation problems for sums of random variables which are not attracted to the normal law*, Ann. Math. Statist. **38** (1967), 1575-1578.
[12] W. He, D. Cheng, Y. Wang, *Asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables*, Statist. Probab. Lett. **83** (2013), 331-338.
[13] T. Jiang, S. Cui, R. Ming, *Large deviations for the stochastic present value of aggregate claims in the renewal risk model*, Statist. Probab. Lett. **101** (2015), 83-91.
[14] R. Kaas, Q. Tang, *A large deviation result for aggregate claims with dependent claim occurencies*, Insur. Math. Econ. **36** (2005), 251-259.
[15] S. Kotz, N. Balakrishnan, N.I. Johnson, *Continuous multivariate distributions, Models and Applications*, (2nd edition), Willey, Newyork, 2000.
[16] C. Klüppelberg, T. Mikosch, *Large deviations of heavy-tailed random sums with applications in insurance and finance*, J. Appl. Probab. **34** (1997), 293-308.
[17] D. Konstantinides, F. Loukissas, *Precise large deviations for consistently varying-tailed distribution in the compound renewal risk model*, Lith. Math. J. **50(4)** (2010), 391-400.
[18] E.L. Lehmann, *Some concepts of dependence*, Ann. Math. Statist. **37** (1966), 1137-1153.

[19] J. Li, *On the joint tail behavior of randomly weighted sums of heavy-tailed random variables*, J. Multivariate Anal. **164** (2018), 40-53.
[20] L. Liu, *Precise large deviations for dependent variables with heavy tails*, Statist. Probab. Lett. **79(9)** (2009), 1290-1298.
[21] X. Liu, C. Yu, Q. Gao, *Precise large deviations of aggregate claim amount in a dependent renewal risk model*, Comm. Statist. Theory Methods **46** (2017), 2354-2363.
[22] D. Lu, L. Song, Y. Xu, *Precise large deviations for sums of independent random variables with consistently varying tails*, Comm. Statist. Theory Methods. **43** (2014), 28-43.
[23] T. Mikosch, A. Nagaev, *Large deviations of heavy-tailed sums with applications in insurance*, Extremes **1** (1998), 81-110.
[24] A.V. Nagaev, *Integral limit theorems with regard to large deviations when Cramér's condition is not satisfied*, I. Theory Probab. Appl. **14** (1969a), 51-64.
[25] A.V. Nagaev, *Integral limit theorems with regard to large deviations when Cramér's condition is not satisfied*, II. Theory Probab. Appl. **14** (1969b), 193-208.
[26] S.V. Nagaev, *Large deviations of sums of independent random variables*, Ann. Probab. **7** (1979), 745-789.
[27] K.W. Ng, Q. Tang, J. Yan, H. Yang, *Precise large deviations for sums of random variables with consistently varying tails*, J. Appl. Probab. **41** (2004), 93-107.
[28] L.V. Rozovskiĭ, *Probabilities of large deviations on the whole axis*, Theory Probab. Appl. **38** (1993), 53-79.
[29] X. Shen, M. Xu, E.F.E.A. Mills, *Precise large deviation results for sums of sub-exponential claims in a size-dependent renewal risk model*, Statist. Probab. Lett. **114** (2016), 6-13.
[30] X. Shen, H. Tian, *Precise large deviations for sums of two-dimensional random vectors with dependent components of heavy tails*, Comm. Statist. Theory Methods **45(21)** (2016), 6357-6368.
[31] Q. Tang, *Insensitivity to negative dependence of the asymptotic behavior of precise large deviations*, Electron. J. Probab. **11(4)** (2006), 107-120.
[32] Q. Tang, C. Su, T. Jiang, J. Zhang, *Large deviations for heavy-tailed random sums in compound renewal model*, Statist. Probab. Lett. **52(1)** (2001), 91-100.
[33] K. Wang, L. Chen, *Precise large deviations for the aggregate claims in a dependent compound renewal risk model*, J. Inequal. Appl. **257** (2019), 1-25.
[34] S. Wang, W. Wang, *Precise large deviations for sums of random variables with consistently varying tails in multi-risk models*, J. Appl. Probab. **44(4)** (2007), 889-900.
[35] Y. Yang, K. Wang, *Precise large deviations for dependent random variables with applications to the compound renewal risk model*, Rocky Mountain J. Math. **43(4)** (2013), 1395-1414.