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Maps preserving the local spectral subspace of Jordan product of operators

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Abstract. Let X be a complex Banach space with $\dim X \geq 4$, $\mathcal{B}(X)$ the algebra of all bounded linear operators on X, and let λ_0 be a fixed complex scalar. Let $X_T(\{\lambda_0\})$ denote the local spectral subspace of an operator $T \in \mathcal{B}(X)$ associated with $\{\lambda_0\}$. In the present paper, we characterize all maps ϕ on $\mathcal{B}(X)$ with range contains the operators of rank at most six that satisfy

$$X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{\lambda_0\}) = X_{AT+TA}(\{\lambda_0\})$$

for all $A, T \in \mathcal{B}(X)$.

1. Introduction

Throughout this paper, X denote a complex Banach space and $\mathcal{B}(X)$ denote the algebra of all bounded linear operators on X. The local resolvent set of an operator $T \in \mathcal{B}(X)$ at a vector $x \in X$, $\rho_T(x)$, is the union of all open $U \subset \mathbb{C}$ for which there exists an analytic function $\varphi : U \longrightarrow X$ such that $(T - \lambda)\varphi(\lambda) = x$ for all $\lambda \in U$. The local spectrum of T at x is defined by

$$\sigma_T(x) = \mathbb{C} \backslash \rho_T(x),$$

and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. We say that T has the single valued extension property (SVEP), if for every open set $U \subset \mathbb{C}$, the only analytic solution $\varphi : U \to X$ of the equation $(T - \mu)\,\varphi(\mu) = 0$ for all $\mu \in U$, is the null function on U.
For a subset $F \subseteq \mathbb{C}$, the local spectral subspace of T associated with F, denoted by $X_T(F)$, is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $X_T(F_1) \subseteq X_T(F_2)$.
For more information on general local spectral theory, the interested reader may consult the books [1] and [11].

In the last years, the study of additive and linear local spectra preserver problems attracted the attention of many authors; see for example [3–10] and their references. In the context of local spectral subspace, M.

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Elhodaibi and A. Jaatit in [7], were the first ones to consider the type of preserver problems. They proved that if the additive map ϕ on $\mathcal{B}(X)$ satisfies

$$X_{\phi(T)}(\{\lambda\}) = X_T(\{\lambda\})$$

for all $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$, then ϕ is the identity map on $\mathcal{B}(X)$. This result was treated in [3] by removing the additive condition on ϕ. The authors gave the form of the surjective maps $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ satisfying

$$X_{\phi(T)-\phi(A)}(\{\lambda\}) = X_{T-A}(\{\lambda\})$$

for all $\lambda \in \mathbb{C}$ and $A, T \in \mathcal{B}(X)$.

For a fixed scalar $\lambda_0 \in \mathbb{C}$, in [10], the author characterized maps ϕ on $\mathcal{B}(X)$ that preserve the local spectral subspace of the sum and difference of operators associated with $\{\lambda_0\}$. Additionally, in [4], the authors determined the form of maps preserving the product or Jordan triple product of operators associated with $\{\lambda_0\}$.

Let λ_0 be a fixed scalar in $\mathbb{C}\backslash\{0\}$. The purpose of this paper is to show that a map ϕ on $\mathcal{B}(X)$ with range contains the operators of rank at most six and satisfies

$$X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{\lambda_0\}) = X_{AT+TA}(\{\lambda_0\})$$

for all $A, T \in \mathcal{B}(X)$, if and only if there exists $\alpha \in \mathbb{C}$ with $\alpha^2 = 1$ such that $\phi(T) = \alpha T$ for all $T \in \mathcal{B}(X)$.

After the preceding section, the second section presents the basic properties of the local spectrum and the local spectral subspace. In the last section, we determine the structure of all maps on $\mathcal{B}(X)$ that preserve the local spectral subspace of the Jordan product of operators associated with a fixed singleton $\{\lambda_0\}$ with $\lambda_0 \in \mathbb{C}\backslash\{0\}$, and also explore the case when $\lambda_0 = 0$.

2. Preliminaries

In this section, we collect some lemmas that are needed for the proof of our main results. Let x be a nonzero vector in X and f be a nonzero linear functional in the topological dual X^* of X. We denote, as usual, by $x \otimes f$ the rank one operator given by $(x \otimes f)z = f(z)x$ for $z \in X$. Let $\mathcal{F}_n(X)$ be the set of operators of rank at most $n \in \mathbb{N}\backslash\{0\}$.
For any operator $T \in \mathcal{B}(X)$, let $N(T)$ be the kernel of T and $R(T)$ be its range.

The first lemma summarizes some known basic properties of the local spectrum.

Lemma 2.1. *Let $T \in \mathcal{B}(X)$, $x, y \in X$ and a scalar $\alpha \in \mathbb{C}\backslash\{0\}$. The following statements hold.*

1. $\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ *for every* $n \geq 1$.
2. $\sigma_T(\alpha x) = \sigma_T(x)$ *and* $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.
3. *If* $Tx = \lambda x$ *for some* $\lambda \in \mathbb{C}$, *then* $\sigma_T(x) \subseteq \{\lambda\}$. *Furthermore, if T has SVEP and* $x \neq 0$, *then* $\sigma_T(x) = \{\lambda\}$.
4. *If T has SVEP and* $Tx = \alpha y$, *then* $\sigma_T(y) \subset \sigma_T(x) \subset \sigma_T(y) \cup \{0\}$.

Proof. See [1, 11]. □

In the next lemma we collect some basic properties of the local spectral subspace of an operator $T \in \mathcal{B}(X)$ associated with a singleton $\{\lambda\}$ where $\lambda \in \mathbb{C}$.

Lemma 2.2. *Let $T \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$ and $c \in \mathbb{C}\backslash\{0\}$. The following statements hold.*

1. $(T - \mu)X_T(\{\lambda\}) = X_T(\{\lambda\})$ *for every scalar* $\mu \in \mathbb{C}$ *such that* $\mu \neq \lambda$.
2. $X_{T-\lambda}(\{0\}) = X_T(\{\lambda\})$ *and* $X_{cT}(\{\lambda\}) = X_T(\{\frac{\lambda}{c}\})$.
3. $N((T - \lambda I)^n) \subseteq X_T(\{\lambda\})$ *for all* $n \in \mathbb{N}$.

Proof. See [1, 11]. □

The following result gives an explicit identification of local spectral subspace of rank one operators.

Lemma 2.3. *Let $F \in \mathcal{F}_1(X)$ be a non-nilpotent rank one operator and let λ be a nonzero eigenvalue of F. Then*

$$X_F(\{0\}) = N(F) \ \text{and} \ X_F(\{\lambda\}) = R(F).$$

Proof. See [5]. □

3. Maps preserving the local spectral subspace of Jordan product of operators

We begin this section with the main lemma that's gives necessary and sufficient condition for two operators to be equal.

Lemma 3.1. *Let λ_0 be a non-zero fixed scalar in $\mathbb{C}$. Let X be a complex Banach space such that $\dim X \geq 3$ and $A, B \in \mathcal{B}(X)$. Then the following statements are equivalent.*

1. $A = B$.
2. $X_{AT+TA}(\{\lambda_0\}) = X_{BT+TB}(\{\lambda_0\})$ *for all* $T \in \mathcal{F}_1(X)$.

Proof. We only need to prove that the implication $2 \Longrightarrow 1$ holds. So, let $A, B \in \mathcal{B}(X)$ be two operators such that

$$X_{AT+TA}(\{\lambda_0\}) = X_{BT+TB}(\{\lambda_0\}) \ \text{for all} \ T \in \mathcal{B}(X).$$

Without loss of generality and by Lemma 2.2, we can choose $\lambda_0 = 1$. We may and shall assume that both A and B are nonzero operators. Assume that there exists a nonzero vector $x \in X$ such that x, Ax and Bx are linearly independent. Then there exists a linear functional $f \in X^*$ such that $f(x) = f(Ax) = 0$ and $f(Bx) = 1$. Let $T = x \otimes f$ and take $S_1 = AT + TA$ and $S_2 = BT + TB$. We have

$$\begin{cases} S_1 x &= 0 \\ S_1 A x &= f(A^2 x)x. \end{cases}$$

Then S_1 is nilpotent and $X_{S_1}(\{1\}) = \{0\}$.
On the other hand, we have $S_2 x = x$ which implies that $x \in X_{S_1}(\{1\}) = X_{S_2}(\{1\}) = \{0\}$. This contradiction shows that $\{x, Ax, Bx\}$ is linearly dependent for all $x \in X$. Now, we shall discuss two situations.

Case 1. If $A = \mu I$, $\mu \in \mathbb{C}\backslash\{0\}$ and if there exists $x \in X$ such that x and Bx are linearly independent. Pick $f \in X^*$ such that $f(x) = \frac{1}{2\mu}$, $f(Bx) = 0$ and take $T = x \otimes f$, $S_1 = AT + TA$ and $S_2 = BT + TB$. We obtain that

$$S_1 x = x \Longrightarrow x \in X_{S_1}(\{1\}) = X_{S_2}(\{1\}).$$

Which implies that $\sigma_{S_2}(x) \subseteq \{1\}$.
On the other hand, we have $S_2 x = \frac{1}{2\mu} Bx$. Since S_2 has the SVEP, by Lemma 2.1, we get

$$\sigma_{S_2}(Bx) \subset \sigma_{S_2}(x) \subset \sigma_{S_2}(Bx) \cup \{0\} \Longrightarrow \sigma_{S_2}(Bx) \subseteq \{1\}.$$

Consequently, $Bx \in X_{S_2}(\{1\}) = X_{S_1}(\{1\}) = \text{span}\{x\}$, contradiction. Thus, $B = \nu I$ for some non-zero scalar $\nu \in \mathbb{C}$.

Case 2. If $A \notin \mathbb{C}I$, by [12, Lemma 2.4], we have $B = \alpha I + \mu A$ for some $(\alpha, \mu) \in \mathbb{C} \times \mathbb{C}\backslash\{0\}$.

Case 2.1 There exists a nonzero vector $x \in X$ such that $\{x, Ax, A^2x\}$ is linearly independent. Pick a linear functional $f \in X^*$ such that $f(Ax) = f(A^2x) = 0$ and $f(x) \neq 0$. Take $T = x \otimes f$, $S_1 = AT + TA$ and $S_2 = BT + TB$. Then we have $S_2 = \mu S_1 + 2\alpha T$ and

$$\begin{aligned} S_1^2 &= (Ax \otimes f + x \otimes A^* f)(Ax \otimes f + x \otimes A^* f) \\ &= f(Ax)Ax \otimes f + f(x)Ax \otimes A^* f + f(A^2x)x \otimes f + f(Ax)x \otimes A^* f \\ &= f(x)Ax \otimes A^* f. \end{aligned}$$

For $y = Ax + 2\frac{\alpha}{\mu}x$, it easy to see that $S_1^2 y = 0$. Then by Lemma 2.1, $(\sigma_{S_1}(y))^2 = \sigma_{S_1^2}(y) = \{0\}$. Thus $\sigma_{S_1}(y) = \{0\}$. On the other hand, we have

$$S_2 y = 2\alpha f(x) y.$$

If $\alpha \neq 0$, we can also choose $f \in X^*$ such that $2\alpha f(x) = 1$, then $\sigma_{S_2}(y) = \{1\}$, contradiction. Hence $\alpha = 0$ and $B = \mu A$.

Let $x \in X$ be a non-zero vector such that x and Ax are linearly independent. Let $f \in X^*$ such that $f(x) = 0$ and $f(Ax) = 1$, take $T = x \otimes f$, $S_1 = AT + TA$ and $S_2 = BT + TB$. Then $S_1 x = x$ and $X_{S_1}(\{1\}) \neq \{0\}$. We have

$$X_{S_1}(\{1\}) = X_{S_2}(\{1\}) = X_{\mu S_1}(\{1\}) = X_{S_1}(\{\frac{1}{\mu}\}),$$

since S_1 has SVEP then $\mu = 1$. Which shows that $A = B$.

Case 2.2 If $\{x, Ax, A^2x\}$ is linearly dependent for all $x \in X$. Then $A^2 x \in \mathrm{span}\{x, Ax\}$ for all $x \in X$. By [12, Lemma 2.4] there exist $a, b \in \mathbb{C}$ such that $A^2 x = aAx + bx$ for all $x \in X$.

If $b = 0$, then there exists $f \in X^*$ and $x \in X$ such that $f(x) = 1$ and $f(Ax) = 0$, a contradiction, see Case 2.1.

If $b \neq 0$, as A is not a scalar operator then there exists $x \in X$ such that x and Ax are linearly independent. Consequently, there exists a functional $f \in X^*$ satisfying $(f(Ax))^2 = f(A^2x)f(x)$ and $f(Ax) = \frac{1}{2} \neq 0$ which implies that $f(x) \neq 0$. Take $T = x \otimes f$, $S_1 = AT + TA$ and $S_2 = BT + TB$. We get

$$\begin{aligned} S_1^2 &= (Ax \otimes f + x \otimes fA)(Ax \otimes f + x \otimes fA) \\ &= f(Ax)Ax \otimes f + f(x)Ax \otimes A^* f + f(A^2x)x \otimes f + f(Ax)x \otimes A^* f \\ &= Ax \otimes (f(Ax)f + f(x)A^* f) + x \otimes (f(A^2x)f + f(Ax)A^* f) \\ &= (Ax + \frac{f(Ax)}{f(x)}x) \otimes (f(Ax)f + f(x)A^* f) \\ &= y \otimes g \end{aligned}$$

with $y = Ax + \frac{1}{2f(x)}x$, $g = \frac{1}{2}f + f(x)A^* f$. Note that $S_1 y = y$. If we substitute $Bx = \mu Ax + \alpha x$ then we find that $(f(Bx))^2 = f(B^2x)f(x)$. Again we also get $S_2^2 = z \otimes h$, with $z = Bx + \frac{f(Bx)}{f(x)}x = \mu Ax + (\frac{\mu}{2f(x)} + 2\alpha)x$ and $h = \frac{\mu}{2}f + \mu f(x)A^* f + 2\alpha f(x) f$. We have

$$y \in X_{S_1}(\{1\}) = X_{S_2}(\{1\}) \subseteq X_{S_2^2}(\{1\}) \subseteq \mathrm{span}\{z\}.$$

Therefore y and z are linearly dependent. In basis (x, Ax), we get

$$\det_{(x,Ax)}(y, z) = -2\alpha = 0.$$

Then $\alpha = 0$ and as before $B = A$. We conclude also that $A = 0$ if and only if $B = 0$.
□

The next lemma characterizes rank one operators $A \in \mathcal{B}(X)$ in terms of dimension of local spectral subspace of Jordan product of A with any operator $T \in \mathcal{B}(X)$.

Lemma 3.2. *Let λ_0 be a non-zero fixed scalar in $\mathbb{C}$. Let X be a complex Banach space such that $\dim X \geq 4$ and $A \in \mathcal{B}(X)$, then the following statements are equivalent.*

1. $\dim R(A) \leq 1$.
2. $\dim X_{AT+TA}(\{\lambda_0\}) \leq 2$ *for all* $T \in \mathcal{B}(X)$.
3. $\dim X_{AT+TA}(\{\lambda_0\}) \leq 2$ *for all* $T \in \mathcal{F}_6(X)$ *if* $\dim X \geq 6$.
4. $\dim X_{AT+TA}(\{\lambda_0\}) \leq 2$ *for all* $T \in \mathcal{F}_4(X)$ *if* $\dim X \in \{4, 5\}$.

Proof. By Lemma 2.2, we can reduce the proof for $\lambda_0 = 1$.
Assume that (1) holds and let $x \in X$ and $f \in X^*$, consider $A = x \otimes f$, then $AT + TA$ is an operator of rank at most two. We have $X_{AT+TA}(\{1\}) \subseteq \text{span}\{x, Tx\}$, which prove that $\dim X_{AT+TA}(\{1\}) \leq 2$.
Conversely, assume that A is not a rank one operator. If $A \in \mathbb{C}\backslash\{0\}I$, then $A = \alpha I$, with $\alpha \in \mathbb{C}\backslash\{0\}$ and let $T = \frac{1}{2\alpha}I$. So $AT + TA = I$ and $X_I(\{1\}) = X$, contradiction. As $\dim X \geq 4$, if there exists a vector u in X such that $\{u, Au, A^2u, A^3u\}$ is linearly independent. We can choose an operator $T \in \mathcal{B}(X)$ satisfying

$$Tu = 0, \;\; TAu = u, \;\; TA^2u = 0 \;\text{ and }\; TA^3u = A^2u.$$

Take $S = AT + TA$, then it easy to verify that

$$\begin{cases} Su &= u \\ SAu &= Au \\ SA^2u &= A^2u. \end{cases}$$

Hence, by 3 of Lemma 2.2, we have $\text{span}\{u, Au, A^2u\} \subseteq N(S - I) \subseteq X_S(\{1\})$, contradiction. Therefore, there exists a complex minimal polynomial P of A of degree at most 3 such that $P(A) = 0$; see [2]. We recall that in the following cases we can use 3 of Lemma 2.2.

Case 1. If $\deg(P) = 1$, then we get immediately that $A \in \mathbb{C}I$ and therefore $A = 0$.

Case 2. If $\deg(P) = 2$, then we discuss the following four cases.

Case 2.1. If P has two distinct non-zero roots $\lambda_1, \lambda_2 \in \mathbb{C}$, then $P(A) = (A - \lambda_1 I)(A - \lambda_2 I)$. We have necessary that $\dim N(A - \lambda_1 I) \geq 2$ or $\dim N(A - \lambda_2 I) \geq 2$. If $\dim N(A - \lambda_1 I) \geq 2$, then there exist $x_1, x_2, x_3 \in X$ linearly independent vectors such that $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_1 x_2$ and $Ax_3 = \lambda_2 x_3$. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{2\lambda_1}x_1, \; Tx_2 = \frac{1}{2\lambda_1}x_2 \;\text{ and }\; Tx_3 = \frac{1}{2\lambda_2}x_3.$$

We obtain that

$$\begin{cases} Sx_1 &= x_1 \\ Sx_2 &= x_2 \\ Sx_3 &= x_3. \end{cases}$$

Thus, $\text{span}\{x_1, x_2, x_3\} \subseteq N(S - I) \subseteq X_S(\{1\})$, which is a contradiction.

Case 2.2. If P has a double nonzero root, then $P(A) = (A - \lambda I)^2$ and we have necessary $N(A - \lambda I)^2 = X$ and so $\dim N(A - \lambda I) \geq 2$. Then there exist $x_1, x_2, x_3 \in X$ linearly independent vectors such that $Ax_1 = \lambda x_1$, $Ax_2 = x_1 + \lambda x_2$ and $Ax_3 = \lambda x_3$. Since $\dim X \geq 4$ then we can choose an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{2\lambda}x_1, \; Tx_2 = \frac{1}{2\lambda}x_2 \text{and } \; Tx_3 = \frac{1}{2\lambda}x_3.$$

Then we have

$$\begin{cases} Sx_1 &= x_1 \\ Sx_2 &= x_2 + \frac{1}{\lambda}x_1 \\ Sx_3 &= x_3. \end{cases}$$

Therefore, $\text{span}\{x_1, x_2, x_3\} \subseteq N((S - I)^2) \subseteq X_S(\{1\})$, contradiction.

Case 2.3. If P has a non-zero root and a zero root, then $P(A) = A(A - \lambda I)$. If $\dim N(A - \lambda I) = 1$, then A is rank one operator. If not, there exist $x_1, x_2 \in X$ linearly independent vectors such that $Ax_1 = \lambda x_1$ and $Ax_2 = \lambda x_2$. Let u be a non-zero vector in $N(A)$, then $\{x_1, x_2, u\}$ is linearly independent. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{\lambda}(x_1 + u), \; Tx_2 = \frac{1}{\lambda}x_2 \;\text{ and }\; T(x_1 + u) = 0.$$

It easy to verify that

$$\begin{cases} S(x_1 + u) &= x_1 + u \\ Sx_1 &= x_1 + (x_1 + u) \\ Sx_2 &= x_2. \end{cases}$$

Hence, span$\{x_1, x_2, u\} \subseteq N((S-I)^2) \subseteq X_S(\{1\})$, which is a contradiction.

Case 2.4. If zero is a double root of P, then $P(A) = A^2$. As $\dim X \geq 4$ so $\dim N(A) \geq 2$. If $\dim R(A) \geq 2$ then there exist $x_1, x_2, x_3, x_4 \in X$ linearly independent vectors such that $Ax_1 = 0$, $Ax_2 = 0$, $Ax_3 = x_1$ and $Ax_4 = x_2$. Let T be an operator in $\mathcal{B}(X)$ satisfying

$$Tx_1 = x_3,\ Tx_2 = x_4,\ Tx_3 = 0 \text{ and } Tx_4 = 0.$$

Therefore, span$\{x_1, x_2, x_3, x_4\} \subseteq N(S-I) \subseteq X_S(\{1\})$, contradiction.

Case 3. If $\deg(P) = 3$, then we discuss the following two steps.

Step 1. In this step we distinguish three cases.

Case 1. If P has three distinct non-zero roots $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, then $P(A) = (A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I)$. It follows that there exist $x_1, x_2, x_3 \in X$ linearly independent vectors such that $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$ and $Ax_3 = \lambda_3 x_3$. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{2\lambda_1}x_1,\ Tx_2 = \frac{1}{2\lambda_2}x_2 \text{ and } Tx_3 = \frac{1}{2\lambda_3}x_3.$$

As before, we get a contradiction.

Case 2. If P has a double non-zero root and a single non-zero root. It follows that $P(A) = (A-\lambda_1 I)^2(A-\lambda_2 I)$, then there exist $x_1, x_2, x_3 \in X$ linearly independent vectors such that $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_1 x_2 + x_1$ and $Ax_3 = \lambda_2 x_3$. We can choose an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{2\lambda_2}x_1,\ Tx_2 = \frac{1}{2\lambda_1}x_2 \text{ and } Tx_3 = \frac{1}{2\lambda_1}x_3.$$

Therefore, span$\{x_1, x_2, x_3\} \subseteq N((S-I)^2) \subseteq X_S(\{1\})$, contradiction.

Case 3. If P has a triple non-zero root, then $P(A) = (A - \lambda I)^3$. It follows that there exist $x_1, x_2, x_3 \in X$ linearly independent vectors such that $Ax_1 = \lambda x_1$, $Ax_2 = x_1 + \lambda x_2$ and $Ax_3 = x_2 + \lambda x_3$. Let $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{2\lambda}x_1,\ Tx_2 = \frac{1}{2\lambda}x_2 \text{ and } Tx_3 = \frac{1}{2\lambda}x_3.$$

As before, span$\{x_1, x_2, x_3\} \subseteq N((S-I)^3)$ and we get a contradiction.

Step 2. In this step we distinguish four cases.

Case 1. If P has zero as a single root and $\lambda \neq 0$ as a double root, then $P(A) = A(A - \lambda I)^2$. It follows that there exist $x_1, x_2, x_3 \in X$ linearly independent vectors such that $Ax_1 = 0$, $Ax_2 = \lambda x_2$ and $Ax_3 = x_2 + \lambda x_3$. Set $T \in \mathcal{B}(X)$ satisfying

$$Tx_2 = \frac{1}{\lambda}(x_1 + x_2),\ T(x_1 + x_2) = 0 \text{ and } Tx_3 = \frac{1}{2\lambda}x_3.$$

As before, we find a contradiction.

Case 2. If P has zero as a double root and $\lambda \neq 0$ as a single root, then $P(A) = A^2(A - \lambda I)$. There exist $u, x_1 \in X$ such that $Ax_1 = \lambda x_1$ and $A(A - \lambda I)u \neq 0$. Put $x_2 = (A - \lambda I)u$ and $x_3 = Ax_2$, we have $Ax_3 = 0$, then $\{x_1, x_2, x_3\}$ is linearly independent. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{2\lambda}x_1,\ Tx_2 = 0 \text{ and } Tx_3 = x_2.$$

As before, we get a contradiction.

Case 3. If P has two single non-zero roots and a zero root, then $P(A) = A(A - \lambda_1 I)(A - \lambda_2 I)$. It follows that there exist $x_1, x_2, x_3 \in X$ linearly independent vectors such that $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$ and $Ax_3 = 0$. We choose an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx_1 = \frac{1}{\lambda_1}(x_1 + x_3),\ T(x_1 + x_3) = 0 \text{ and } Tx_2 = \frac{1}{2\lambda_2}x_2.$$

As before, we get a contradiction.

Case 4. If zero is a triple root of P, then $P(A) = A^3$. We discuss two points.

• If $\dim R(A) = 2$, so we have $\dim R(A^2) = 1$ and there exist $u, v, w \in X$ linearly independent vectors such that $R(A) = \text{span}\{v, w\}$ and

$$Au = v,\ Av = w,\ Aw = 0.$$

We have $\dim X \geq 4$. Let $x \in N(A)$ such that $\{x, w\}$ is linearly independent, put $y = v + x$ then $\{u, v, w, y\}$ is linearly independent and $Ay = w$. Take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tv = u,\ Tw = y \ \text{ and } \ Ty = 0.$$

Hence,

$$\begin{cases} Sv &= v + y \\ Sw &= w \\ Sy &= y \\ Su &= u + ATu. \end{cases}$$

As $ATu \in \text{span}\{v, w\}$ therefore, $\text{span}\{u, v, w, y\} \subseteq N((S - I)^3) \subseteq X_S(\{1\})$, contradiction.

• If $\dim R(A) \geq 3$, we have two points.

• If $\dim R(A^2) = 1$, then there exist $x, y \in X$ such that $\{Ax, Ay, A^2x\}$ is linearly independent so $\{x, Ax, Ay, A^2x\}$ is linearly independent. Since $\dim X \geq 4$ then we take an operator $T \in \mathcal{B}(X)$ satisfying

$$Tx = 0,\ TAx = x,\ TAy = y \ \text{ and } \ TA^2x = 0.$$

This is contradiction.

• If $\dim R(A^2) \geq 2$, then necessary $\dim X \geq 6$ and there exist $u, v \in X$ such that $\{A^2u, A^2v\}$ is linearly independent so $\{u, v, Au, Av, A^2u, A^2v\}$ is linearly independent. Take $T \in \mathcal{B}(X)$ satisfying

$$Tu = Tv = 0,\ TAu = u,\ TAv = v \ \text{ and } \ TA^2u = TA^2v = 0.$$

As before, we get a contradiction.

The proof of the lemma is thus complete. □

Now, we are ready to state the first main result.

Theorem 3.3. *Let λ_0 be a non-zero fixed scalar in $\mathbb{C}$ and X be a complex Banach space such that* $\dim X \geq 6$. *Let $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ be a map such that $\mathcal{F}_6(X) \subset \phi(\mathcal{B}(X))$. Then ϕ satisfies*

$$X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{\lambda_0\}) = X_{AT+TA}(\{\lambda_0\}) \ \textit{for every} \ \ A, T \in \mathcal{B}(X),$$

if and only if there exists $\alpha \in \mathbb{C}$ with $\alpha^2 = 1$ such that $\phi(T) = \alpha T$ for all $T \in \mathcal{B}(X)$.

Proof. Let $\mu_0 \in \mathbb{C}\backslash\{0\}$ such that $\mu_0^2 = \lambda_0$ and let $\psi(A) = \frac{1}{\mu_0}\phi(\mu_0 A)$ for all $A \in \mathcal{B}(X)$. Then

$$X_{\psi(A)\psi(T)+\psi(T)\psi(A)}(\{1\}) = X_{AT+TA}(\{1\}) \ \text{ for every } \ A, T \in \mathcal{B}(X),$$

and we reduce the proof for $\lambda_0 = 1$.

The 'if' part is easily verified, so we need only to prove the 'only if' part. Indeed, assume that ϕ is a map from $\mathcal{B}(X)$ into itself such that for all $A, T \in \mathcal{B}(X)$, we have

$$X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{1\}) = X_{AT+TA}(\{1\}).$$

We divide the proof into five claims.

Claim 1. ϕ is injective and $\phi(0) = 0$.

Let $A, B \in \mathcal{B}(X)$ such that $\phi(A) = \phi(B)$. Then for every $F \in \mathcal{F}_1(X)$ we have

$$\begin{aligned} X_{AF+FA}(\{1\}) &= X_{\phi(A)\phi(F)+\phi(F)\phi(A)}(\{1\}) \\ &= X_{\phi(B)\phi(F)+\phi(F)\phi(B)}(\{1\}) \\ &= X_{BF+FB}(\{1\}). \end{aligned}$$

By Lemma 3.1, $A = B$, and so ϕ is injective.
In a similar way, we show that $\phi(0) = 0$. Indeed,

$$X_{\phi(A)\phi(0)+\phi(0)\phi(A)}(\{1\}) = X_{A0+0A}(\{1\}) = X_0(\{1\}) = X_{\phi(A)0+0\phi(A)}(\{1\})$$

for all $A \in \mathcal{B}(X)$. Since the range of ϕ contains all rank one operators, it follows by Lemma 3.1 that $\phi(0) = 0$.

Claim 2. ϕ preserves rank-one operators in both directions.
Let $A \in \mathcal{B}(X)$ be a rank one operator, then by Claim 1 we have $\phi(A) \neq 0$. By Lemma 3.2, we have $\dim X_{AT+TA}(\{1\}) \leq 2$ for every $T \in \mathcal{B}(X)$. Since

$$X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{1\}) = X_{AT+TA}(\{1\}).$$

Then

$$\dim X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{1\}) \leq 2$$

for all $T \in \mathcal{B}(X)$. Since the range of ϕ contains $\mathcal{F}_6(X)$, we reduce that $\phi(A)$ is a rank one operator.
Let $\phi(A)$ be a rank one operator. Similarly, as above, we establish that A is a rank one operator.

Claim 3. There exists $\alpha \in \mathbb{C}$ such that $\alpha^2 = 1$ and $\phi(x \otimes f) = \alpha x \otimes f$ for all $x \in X$ and $f \in X^*$ verifying $f(x) = \frac{1}{\sqrt{2}}$.
Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$. By Claim 2, there exist $y \in X$ and $g \in X^*$ such that $\phi(\frac{1}{\sqrt{2}}x \otimes f) = y \otimes g$. It's evident that $\frac{1}{\sqrt{2}}(x \otimes f)\frac{1}{\sqrt{2}}(x \otimes f) + \frac{1}{\sqrt{2}}(x \otimes f)\frac{1}{\sqrt{2}}(x \otimes f) = x \otimes f$.Then

$$X_{\frac{1}{\sqrt{2}}(x\otimes f)\frac{1}{\sqrt{2}}(x\otimes f)+\frac{1}{\sqrt{2}}(x\otimes f)\frac{1}{\sqrt{2}}(x\otimes f)}(\{1\}) = \text{span}\{x\}.$$

On the other hand, we obtain that

$$\phi(\frac{1}{\sqrt{2}}x \otimes f)\phi(\frac{1}{\sqrt{2}}x \otimes f) + \phi(\frac{1}{\sqrt{2}}x \otimes f)\phi(\frac{1}{\sqrt{2}}x \otimes f) = 2g(y)y \otimes g.$$

Then we get

$$X_{\phi(\frac{1}{\sqrt{2}}x\otimes f)\phi(\frac{1}{\sqrt{2}}x\otimes f)+\phi(\frac{1}{\sqrt{2}}x\otimes f)\phi(\frac{1}{\sqrt{2}}x\otimes f)}(\{1\}) \subseteq \text{span}\{y\}.$$

Therefore, $\text{span}\{x\} \subseteq \text{span}\{y\}$. Consequently, x and y are linearly dependent and we assume that for all $x \in X$ and $f \in X^*$, with $f(x) = 1$, there exists $g_{x,f} \in X^*$ such that $\phi(\frac{1}{\sqrt{2}}x \otimes f) = \frac{1}{\sqrt{2}}x \otimes g_{x,f}$.
Now, suppose that f and $g_{x,f}$ are linearly independent. So, let $z \in X$ be a nonzero vector such that $f(z) = 1$ and $g_{x,f}(z) = 0$ and let $h \in X^*$ such that $h(x) = h(z) = 1$. Let $\phi(\frac{1}{\sqrt{2}}z \otimes h) = \frac{1}{\sqrt{2}}z \otimes g_{z,h}$, therefore

$$\frac{1}{\sqrt{2}}(x \otimes f)\frac{1}{\sqrt{2}}(z \otimes h) + \frac{1}{\sqrt{2}}(z \otimes h)\frac{1}{\sqrt{2}}(x \otimes f)(x + z) = x + z,$$

it follows that $x + z \in X_{\frac{1}{\sqrt{2}}(x\otimes f)\frac{1}{\sqrt{2}}(z\otimes h)+\frac{1}{\sqrt{2}}(z\otimes h)\frac{1}{\sqrt{2}}(x\otimes f)}(\{1\})$ and
$x + z \notin X_{\frac{1}{\sqrt{2}}(x\otimes g_{x,f})\frac{1}{\sqrt{2}}(z\otimes g_{z,h})+\frac{1}{\sqrt{2}}(z\otimes g_{z,h})\frac{1}{\sqrt{2}}(x\otimes g_{x,f})}(\{1\}) = \{0\}$, contradiction. Hence f and $g_{x,f}$ are linearly dependent, thus $\phi(\frac{1}{\sqrt{2}}x \otimes f) = \frac{\alpha_{x,f}}{\sqrt{2}}x \otimes f$ for some nonzero scalar $\alpha_{x,f} \in \mathbb{C}$.
Let $x \in X$ and $f \in X^*$ such that $f(x) = 1$, then

$$\begin{aligned}
\{0\} \neq X_{x\otimes f}(\{1\}) &= X_{\frac{1}{\sqrt{2}}(x\otimes f)\frac{1}{\sqrt{2}}(x\otimes f)+\frac{1}{\sqrt{2}}(x\otimes f)\frac{1}{\sqrt{2}}(x\otimes f)}(\{1\}) \\
&= X_{\phi(\frac{1}{\sqrt{2}}x\otimes f)\phi(\frac{1}{\sqrt{2}}x\otimes f)+\phi(\frac{1}{\sqrt{2}}x\otimes f)\phi(\frac{1}{\sqrt{2}}x\otimes f)}(\{1\}) \\
&= X_{\alpha_{x,f}^2 x\otimes f}(\{1\}).
\end{aligned}$$

Consequently, $\alpha_{x,f}^2 = 1$.
Let $(x, y) \in X^2$ and $(f, g) \in X^{*2}$ such that $\{x, y\}$ and $\{f, g\}$ are linearly independent and $f(x) = g(y) = \frac{1}{\sqrt{2}}$. We

can choose $z \in X$ such that $\{x, y, z\}$ is linearly independent, $z \in N(f - g)$ and $f(z) = \frac{1}{\sqrt{2}}$.
We can choose $h \in X^*$ such that $\{x - z, y - z\} \subseteq N(h)$ and $h(z) = \frac{1}{\sqrt{2}}$, it is clear that

$$(x \otimes f)(z \otimes h) + (z \otimes h)(x \otimes f)(x + z) = x + z.$$

Since $x + z \in X_{(x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f)}(\{1\})$ then $X_{(x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f)}(\{1\}) \neq \{0\}$. Consequently, we get

$$\begin{aligned} \{0\} \neq X_{(x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f)}(\{1\}) &= X_{\phi(x\otimes f)\phi(z\otimes h)+\phi(z\otimes h)\phi(x\otimes f)}(\{1\}) \\ &= X_{\alpha_{x,f}\alpha_{z,h}((x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f))}(\{1\}). \end{aligned}$$

Therefore, $\alpha_{x,f}\alpha_{z,h} = 1$ and $\alpha_{x,f} = \alpha_{z,h}$.
Similarly, we find that $\alpha_{y,g}\alpha_{z,h} = 1$. Consequently, $\alpha_{x,f} = \alpha_{y,g}$.
If $\{x, y\}$ is linearly dependent,we reduce to the case $x = y$ and $f(x) = g(x) = \frac{1}{\sqrt{2}}$. Then $((x \otimes f)(x \otimes f) + (x \otimes f)(x \otimes f))x = x$ and similarly, we find $\alpha_{x,f} = \alpha_{y,g}$.
If $\{x, y\}$ is linearly independent and $f = g$, then $((x \otimes f)(y \otimes f) + (y \otimes f)(x \otimes f))(x + y) = x + y$ and we find $\alpha_{x,f} = \alpha_{y,f}$.

Claim 4. Let α be a complex scalar in Claim 3, for every rank one operator $A \in \mathcal{B}(X)$, we have $\phi(A) = \alpha A$. We distinguish two cases.

Case 1. If A is a rank one non-nilpotent operator.
Let $x \in X$ and $f \in X^*$, with $f(x) \neq 0$. Claim 2 shows that there exist $y \in X$ and $g \in X^*$ such that $\phi(x\otimes f) = y\otimes g$.
Let a scalar $\lambda \in \mathbb{C}^*$ satisfying

$$\sqrt{2}\lambda^2 + \lambda f(x) - f(x) = 0.$$

Suppose that x and y are linearly independent, then there exist $h \in X^*$ and $z \in X$ such that

$$h(y) = 0, h(x) = f(x) = \lambda, h(z) = \frac{1}{\sqrt{2}} \text{ and } f(z) = \sqrt{2}\lambda.$$

Then

$$\begin{cases} f(z)h(x) + \frac{1}{\sqrt{2}}f(z) = 1 \\ h(x)f(x) + h(x)f(z) = 1. \end{cases}$$

By Claim 3, we obtain that $\phi(z \otimes h) = \alpha z \otimes h$.
It is clear that

$$(x \otimes f)(z \otimes h) + (z \otimes h)(x \otimes f)(x + z) = x + z.$$

Since $x + z \in X_{(x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f)}(\{1\})$ and $X_{(y\otimes g)\alpha(z\otimes h)+\alpha(z\otimes h)(y\otimes g)}(\{1\}) = \{0\}$, contradiction. Thus, x and y are linearly dependent. We assume that for all $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$, there exists $g_{x,f} \in X^*$ such that $\phi(x \otimes f) = x \otimes g_{x,f}$.
Now, suppose that f and $g_{x,f}$ are linearly independent. So, let $z \in X$ be a nonzero vector such that $\{x, z\}$ is linearly independent, $f(z) = f(x) = \frac{1}{2}, g_{x,f}(z) = 0$ and let $h \in X^*$ a linear functional such that $h(x) = h(z) = 1$.
Let $\phi(z \otimes h) = z \otimes g_{z,h}$, therefore

$$(x \otimes f)(z \otimes h) + (z \otimes h)(x \otimes f)(x + z) = x + z,$$

it follows that $x + z \in X_{(x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f)}(\{1\})$ and $x + z \notin X_{(x\otimes g_{x,f})(z\otimes g_{z,h})+(z\otimes g_{z,h})(x\otimes g_{x,f})}(\{1\}) = \{0\}$, contradiction.
Hence, f and $g_{x,f}$ are linearly dependent, thus $\phi(x \otimes f) = \lambda_{x,f} x \otimes f$ for some nonzero scalar $\lambda_{x,f} \in \mathbb{C}$.
Now, let $x \in X$ and $f \in X^*$ such that $f(x) \neq 0$. As before, we get that $\phi(A) = \lambda_A A$, with $\lambda_A \in \mathbb{C}^*$ and $A = x \otimes f$. We can choose as before $B = z \otimes h$ with $z \in X, h \in X^*$, $\{x, z\}$ is linearly independent, $h(z) = \frac{1}{\sqrt{2}}$ and $(AB + BA)(x + z) = x + z$, hence $X_{AB+BA}(\{1\}) \neq \{0\}$ and $\phi(B) = \alpha B$, see Claim 3. We obtain that

$$\begin{aligned} X_{AB+BA}(\{1\}) &= X_{\phi(A)\phi(B)+\phi(B)\phi(A)}(\{1\}) \\ &= X_{\alpha\lambda_A(AB+BA)}(\{1\}) \neq \{0\}. \end{aligned}$$

Consequently, $\alpha\lambda_A = 1$. Since $\alpha^2 = 1$, therefore, $\alpha = \lambda_A$. Thus $\phi(x \otimes f) = \alpha x \otimes f$.

Case 2. If A is a rank one nilpotent operator.
Let $x \in X$ and $f \in X^*$, with $f(x) = 0$. Claim 2 shows that there exist $y \in X$ and $g \in X^*$ such that $\phi(x \otimes f) = y \otimes g$. Suppose that f and g are linearly independent. Let $z \in X$ such that $f(z) = 1, g(z) = 0$ and let $h \in X^*$ such that $h(z) = h(x) = 1$. It easy to verify that

$$((x \otimes f)(z \otimes h) + (z \otimes h)(x \otimes f))x = x.$$

Then

$$x \in X_{(x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f)}(\{1\}) = X_{(y\otimes g)(z\otimes h)+(z\otimes h)(y\otimes g)}(\{1\}) = \{0\}.$$

This contradiction asserts that $g = \beta f$ for a nonzero scalar $\beta \in \mathbb{C}$. So there exists $y_{x,f} \in X$ such that $\phi(x \otimes f) = y_{x,f} \otimes f$.
We claim that x and $y_{x,f}$ are linearly dependent. If not, let respectively, $z \in X$ and $h \in X^*$ such that $f(z) = h(z) = h(x) = 1$ and $h(y_{x,f}) = 0$. We have $((x \otimes f)(z \otimes h) + (z \otimes h)(x \otimes f))x = x$, therefore

$$x \in X_{(x\otimes f)(z\otimes h)+(z\otimes h)(x\otimes f)}(\{1\}) = X_{(y_{x,f}\otimes f)(z\otimes h)+(z\otimes h)(y_{x,f}\otimes f)}(\{1\}) = \{0\}.$$

This contradiction shows that x and $y_{x,f}$ are linearly dependent, and
$\phi(x \otimes f) = \alpha_{x,f} x \otimes f$ for certain nonzero $\alpha_{x,f} \in \mathbb{C}$.
Let $z \in X$ and $h \in X^*$ such that $h(z) = f(z) = 1$ and $h(x) = 1$, then we have $x \in X_{x\otimes f z\otimes h+z\otimes h x\otimes f}(\{1\}) \neq \{0\}$. As before we get that $\alpha_{x,f} = \alpha$ and therefore, $\phi(x \otimes f) = \alpha x \otimes f$.

Claim 5. $\phi(T) = \alpha T$ for all $T \in \mathcal{B}(X)$.
For every rank one operator $A \in \mathcal{B}(X)$ and every $T \in \mathcal{B}(X)\backslash\{0\}$. We have

$$\begin{aligned} X_{TA+AT}(\{1\}) &= X_{\phi(T)\phi(A)+\phi(A)\phi(T)}(\{1\}) \\ &= X_{\alpha\phi(T)A+\alpha A\phi(T)}(\{1\}). \end{aligned}$$

By Claim 1 and Lemma 3.1 we obtain that $\phi(T) = \alpha T$ for all $T \in \mathcal{B}(X)$. □

As remark the Theorem 3.3 is valid if $\dim X \in \{4, 5\}$, then we need only that $\mathcal{F}_4(X) \subset \phi(\mathcal{B}(X))$.
The following corollary is a direct consequence of the previous Theorem.

Corollary 3.4. *Let X be a complex Banach space such that* $\dim X \geq 6$ *and* $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ *be a map such that* $\mathcal{F}_6(X) \subset \phi(\mathcal{B}(X))$. *Then* ϕ *satisfies*

$$X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{\lambda\}) = X_{AT+TA}(\{\lambda\})$$

for every $\lambda \in \mathbb{C}$ *and* $A, T \in \mathcal{B}(X)$, *if and only if there exists* $\alpha \in \mathbb{C}$ *with* $\alpha^2 = 1$ *such that* $\phi(T) = \alpha T$ *for all* $T \in \mathcal{B}(X)$.

Proof. It suffices to choose $\lambda = 1$ and we applied the Theorem 3.3. □

Now we discuss the case where $\lambda_0 = 0$. Recall that the quasi-nilpotent part of an operator $T \in \mathcal{B}(X)$ is defined by

$$H_0(T) = \{x \in X : \limsup_{n\to+\infty} \| T^n x \|^{\frac{1}{n}} = 0\}.$$

It is known that for all $A, T \in \mathcal{B}(X)$, $H_0(AT + TA) \subseteq X_{AT+TA}(\{0\})$. If $AT + TA$ is of finite rank (has a SVEP) then $X_{AT+TA}(\{0\}) = H_0(AT + TA)$. The techniques we used to prove Theorem 3.3 are not valid for the next Theorem. Then we gave some lemmas that are needed for the proof of the Theorem.

Lemma 3.5. *Let* A, B *in* $\mathcal{B}(X)$, *then the following statements are equivalent.*

1. $B = \alpha A$ *for some non-zero scalar* $\alpha \in \mathbb{C}$.
2. $X_{AT+TA}(\{0\}) = X_{BT+TB}(\{0\})$ *for all* $T \in \mathcal{F}_2(X)$.

Proof. Let $A, B \in \mathcal{B}(X)$ and $T \in \mathcal{F}_2(X)$ then $AT + TA$ and $BT + TB$ are finite rank operators. Therefore $X_{AT+TA}(\{0\}) = H_0(AT + TA)$, $X_{BT+TB}(\{0\}) = H_0(BT + TB)$, and we applied the Lemma 2.3 in [9]. □

Lemma 3.6. *Let A in $\mathcal{B}(X)$, then the following statements are equivalent.*

1. *A is at most rank one operator.*
2. $\operatorname{codim} X_{AT+TA}(\{0\}) \leq 2$ *for all* $T \in \mathcal{B}(X)$.
3. $\operatorname{codim} X_{AT+TA}(\{0\}) \leq 2$ *for all* $T \in \mathcal{F}_4(X)$.

Proof. If $A = x \otimes f$ is a rank one operator with $x \in X$ and $f \in X^*$, then for all $T \in \mathcal{B}(X)$

$$N(f) \cap N(T^* f) \subseteq H_0(AT + TA) = X_{AT+TA}(\{0\})$$

this prove that $1 \Rightarrow 2$. It's evident that $2 \Rightarrow 3$. Let $A \in \mathcal{B}(X)$ and $T \in \mathcal{F}_4(X)$ then $AT + TA$ is a finite rank operator. Therefore $X_{AT+TA}(\{0\}) = H_0(AT + TA)$ and we applied the Lemma 2.2 in [9] and $3 \Rightarrow 1$. $\square$

Theorem 3.7. *Let X be a complex Banach space such that* $\dim X \geq 4$ *and $\phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ be a map such that $\mathcal{F}_4(X) \subset \phi(\mathcal{B}(X))$. Then ϕ satisfies*

$$X_{\phi(A)\phi(T)+\phi(T)\phi(A)}(\{0\}) = X_{AT+TA}(\{0\})$$

for every $A, T \in \mathcal{B}(X)$, if and only if there exists a functional $\gamma : \mathcal{B}(X) \longrightarrow \mathbb{C}\backslash\{0\}$ such that $\phi(T) = \gamma(T)T$ for all $T \in \mathcal{B}(X)$.

Proof. Using the lemmas 3.5 and 3.6, the proof of the Theorem 3.7 is similar to the proof of Theorem 3.1 in [9], but the only difference is in Step 2, we use the fact that $\mathcal{F}_4(X) \subset \phi(\mathcal{B}(X))$. $\square$

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