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Some new results on core partial order and strong core orthogonal matrices

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Abstract. Recently, Ferreyra and Malik (Some new results on the core partial order, Linear and Multilinear Algebra, DOI: 10.1080/03081087.2020.1841078) give an example to show that $A\overset{\textcircled{\#}}{\leq}B$ does not imply $(B-A)^{\textcircled{\#}} = B^{\textcircled{\#}} - A^{\textcircled{\#}}$, and put forward an open question: Let $A, B\in\mathbb{C}_n^{\mathrm{GM}}$, can $A\overset{-}{\leq}B$ and $(B-A)^{\textcircled{\#}} = B^{\textcircled{\#}} - A^{\textcircled{\#}} \Rightarrow A\overset{\textcircled{\#}}{\leq}B$ be true? In this paper, the above problem will be completely solved. We also give some necessary and sufficient conditions for core partial order, and we give some new characterizations of strong core orthogonality.

1. Introduction

In this paper, we use the following notations. We denote the set of all $m\times n$ complex matrices by $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, symbols A^*, $R(A)$, $\mathrm{rk}(A)$, $\mathrm{Ind}(A)$, A^{-1} stand for the conjugate transpose, range, rank, index and the inverse ($m = n$) of A, respectively. I_n refers to the $n\times n$ identity matrix.

We first review definitions of some well-known generalized inverses. Let $A \in \mathbb{C}^{m\times n}$, then the Moore-Penrose inverse $A^\dagger$ [15] of A is the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying the equations: (1)$AXA = A$, (2)$XAX = X$, (3)$(AX)^* = AX$ and (4)$(XA)^* = XA$. The symbol $A\{i,\dots,j\}$ is the set of matrices $X\in\mathbb{C}^{m\times n}$ which satisfy the conditions $(i),\dots,(j)$ from the Eqs.(1)-(4). A matrix $X\in A\{i,\dots,j\}$ is called an $\{i,\dots,j\}$-inverse of A, and denoted by $A^{\{i,\dots,j\}}$.

For $A \in \mathbb{C}^{n\times n}$, when $\mathrm{rk}(A) = \mathrm{rk}(A^2)$, we denote $\mathrm{Ind}(A) = 1$, and call A as a group matrix (or a core matrix). The symbol $\mathbb{C}_n^{\mathrm{GM}}$ stands for the subset of $\mathbb{C}^{n\times n}$ consisting of group matrices. The group inverse $A^\#$ [15] of a group matrix A is the unique matrix $X \in \mathbb{C}^{n\times n}$, which satisfies the following equations: $AXA = A$, $XAX = X$ and $AX = XA$.

Moreover, for the square matrix A with index 1, Baksalary and Trenkler [1] give the definition of core inverse, which refers to the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the conditions $AX = AA^\dagger$, $R(X)\subseteq R(A)$, and denoted by $A^{\textcircled{\#}}$.

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They also put forward the definition of core partial order in [1], which has attracted extensive attention from the fields of generalized inverse theory and matrix partial order. Subsequently, a series of articles on the core partial order appeared [5, 9, 11, 13, 14, 18].

A binary relation is called a partial order if it is reflexive, transitive and anti-symmetric on a non-empty set. In general, we can characterize the corresponding partial order by using the known generalized inverse. The star, minus and core partial orders are defined as follows [1–3, 6–8, 12]:

$$(1)\ A,B\in\mathbb{C}^{m\times n}, A\overset{*}{\leq}B \Leftrightarrow AA^*=BA^*,\ A^*A=A^*B, \quad \Leftrightarrow AA^\dagger=BA^\dagger,\ A^\dagger A=A^\dagger B;$$

$$(2)\ A,B\in\mathbb{C}^{m\times n}, A\overset{-}{\leq}B \Leftrightarrow AA^= = BA^=,\ A^-A=A^-B,\ \text{ for some } A^-,A^=\in A\{1\}, \quad \Leftrightarrow \mathrm{rk}(B)-\mathrm{rk}(A)=\mathrm{rk}(B-A);$$

$$(3)\ A,B\in\mathbb{C}_n^{\mathrm{GM}},\ A\overset{\#\!\!\!\bigcirc}{\leq}B \Leftrightarrow AA^{\#\!\!\!\bigcirc}=BA^{\#\!\!\!\bigcirc}, A^{\#\!\!\!\bigcirc}A=A^{\#\!\!\!\bigcirc}B.$$

The other well-known partial orders are the sharp, Löwner, C-N, GL, CL partial orders, etc. For more details, see [2, 3, 6, 12, 17, 19].

Many scholars have discussed the relationship among various partial orders. While proposing the definition of core partial order, Baksalary and Trenkler [1] also consider the relationship between the core partial order and the minus partial order. Let $A,B\in\mathbb{C}_n^{\mathrm{GM}}$. They also point out that if $A\overset{\#\!\!\!\bigcirc}{\leq}B$, then $A\overset{-}{\leq}B$. However, $A\overset{-}{\leq}B$ does not mean $A\overset{\#\!\!\!\bigcirc}{\leq}B$.

In [7], Hartwig and Styan propose an important equivalent characterization of star partial order, which is related to the minus partial order and dagger-subtractivity. They also give some other characterizations of star partial order. For $A,B\in\mathbb{C}^{m\times n}$,

$$A\overset{*}{\leq}B \Leftrightarrow A\overset{-}{\leq}B \text{ and } (B-A)^\dagger=B^\dagger-A^\dagger;$$
$$\Leftrightarrow A\overset{-}{\leq}B \text{ and } A^*B, BA^* \text{ are both Hermitian};$$
$$\Leftrightarrow A\overset{-}{\leq}B \text{ and } A^\dagger B, BA^\dagger \text{ are both Hermitian};$$
$$\Leftrightarrow A\overset{-}{\leq}B \text{ and } B^\dagger-A^\dagger \text{ is a \{1,3\}-generalized inverse of } B-A.$$

It is natural to make some similar conjectures for core partial order. In [5], Ferreyra and Malik define the core-subtractivity property for two group invertible matrices. They give an example to show that $A\overset{\#\!\!\!\bigcirc}{\leq}B$ does not imply $(B-A)^{\#\!\!\!\bigcirc}=B^{\#\!\!\!\bigcirc}-A^{\#\!\!\!\bigcirc}$, and put forward an open question:

Problem 1.1. *Let $A,B\in\mathbb{C}_n^{\mathrm{GM}}$. Can*

$$A\overset{-}{\leq}B \text{ and } (B-A)^{\#\!\!\!\bigcirc}=B^{\#\!\!\!\bigcirc}-A^{\#\!\!\!\bigcirc} \Rightarrow A\overset{\#\!\!\!\bigcirc}{\leq}B \tag{1}$$

be true?

Based on core inverse, Ferreyra and Malik give concepts of core orthogonality and strongly core orthogonality [4] in $\mathbb{C}_n^{\mathrm{GM}}$. Let $A,B\in\mathbb{C}_n^{\mathrm{GM}}$. When $A^{\#\!\!\!\bigcirc}B=0$ and $BA^{\#\!\!\!\bigcirc}=0$, A is said to be core orthogonal to B (denoted by $A\perp_{\#\!\!\!\bigcirc}B$); when $A\perp_{\#\!\!\!\bigcirc}B$ and $B\perp_{\#\!\!\!\bigcirc}A$, A and B are said to be strongly core orthogonal (denoted by $A\perp_{\#\!\!\!\bigcirc,S}B$). Moreover, they point out that

$$A\perp_{\#\!\!\!\bigcirc}B \Leftrightarrow A\overset{\#\!\!\!\bigcirc}{\leq}(A+B); \tag{2}$$
$$A\perp_{\#\!\!\!\bigcirc,S}B \Leftrightarrow A\overset{\#\!\!\!\bigcirc}{\leq}(A+B) \text{ and } B\overset{\#\!\!\!\bigcirc}{\leq}(A+B); \tag{3}$$
$$A\perp_{\#\!\!\!\bigcirc,S}B \Leftrightarrow A^{\#\!\!\!\bigcirc}B=0, BA^{\#\!\!\!\bigcirc}=0 \text{ and } AB^{\#\!\!\!\bigcirc}=0. \tag{4}$$

For $A, B \in \mathbb{C}^{m\times n}$, we have the following equivalent characterization of star orthogonality, which is related to dagger-additivity and rank-additivity [7, 12]:

$$A\perp_* B \Leftrightarrow (A+B)^\dagger = A^\dagger + B^\dagger \text{ and } \operatorname{rk}(A+B) = \operatorname{rk}(A) + \operatorname{rk}(B).$$

For strong core orthogonality, Ferreyra and Malik[4] prove that $A\perp_{\text{\#},S}B$ implies $(A+B)^{\text{\#}} = A^{\text{\#}} + B^{\text{\#}}$ and $\operatorname{rk}(A+B) = \operatorname{rk}(A) + \operatorname{rk}(B)$, and put forward an open question: Let $A, B \in \mathbb{C}_n^{\mathrm{GM}}$. Can $A\perp_{\text{\#},S}B \Leftrightarrow (A+B)^{\text{\#}} = A^{\text{\#}} + B^{\text{\#}}$ and $\operatorname{rk}(A+B) = \operatorname{rk}(A) + \operatorname{rk}(B)$ be true? In [10], we prove that the result is true and give some new characterizations of strong core orthogonality. Furthermore, based on the core-EP decomposition, we give forms of matrices A and B when $A \overset{\text{\#}}{\leq} B$ in [10]. This characterization is shown in Lemma 2.2, which is what we need to use in this paper.

In this paper, we will make some new contributions by revisiting the core partial order and strong core orthogonality. First, we consider the relationship between core partial order and core-subtractivity, and solve the above Problem (1). We derive some new equivalent conditions of core partial order, which are characterized by core inverse, group inverse and some Hermitian matrices. We slao get some new characterizations of strong core orthogonality, which are about {1,3}-generalized inverse.

2. Preliminaries

In order to get our conclusions, we need to use following results in this section.

Lemma 2.1 ([16], Core-EP decomposition). *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$ and $\operatorname{rk}(A^k) = p$. Then, it has $A = A_1 + A_2$, where $A_1 \in \mathbb{C}_n^{\mathrm{GM}}$, $A_2^k = 0$, $A_1^* A_2 = A_2 A_1 = 0$.*

Furthermore, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \quad A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \quad A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{5}$$

where $T \in \mathbb{C}^{p\times p}$ is nonsingular; $S \in \mathbb{C}^{p\times(n-p)}$; $N \in \mathbb{C}^{(n-p)\times(n-p)}$ is nilpotent of index k, i.e., $N^k = 0$.

Lemma 2.2 ([10]). *Let $B \in \mathbb{C}_n^{\mathrm{GM}}$, and $B = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$ be the core-EP decomposition of B, where $T \in \mathbb{C}^{p\times p}$ is nonsingular with $p = \operatorname{rk}(B)$ and U is unitary. If $A \overset{\text{\#}}{\leq} B$, then we have*

$$A = U\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}U^*, \tag{6}$$

where $A_{11} = A_{11}T^{-1}A_{11}$ and $A_{12} = A_{11}T^{-1}S$.

Lemma 2.3 ([16, 21]). *Let $A \in \mathbb{C}^{n\times n}$ be as in (5). Then $\operatorname{rk}(A) = \operatorname{rk}(A^2) \Leftrightarrow N = 0$, that is*

$$A = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*. \tag{7}$$

In that case, we have

$$A^{\text{\#}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \; A^{\#} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*. \tag{8}$$

Lemma 2.4 ([10]). *Let $A \in \mathbb{C}_n^{\mathrm{GM}}$ have the block form that is $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{11} \in \mathbb{C}^{t\times t}$ and t is any nonnegative number satisfying $0 \leq t \leq n$. Then*

(1) *$A_{21} = 0$ and $A_{22} = 0 \Leftrightarrow A_{11} \in \mathbb{C}_t^{\mathrm{GM}}$ and $A^{\text{\#}} = \begin{bmatrix} A_{11}^{\text{\#}} & 0 \\ 0 & 0 \end{bmatrix}$;*

(2) *$A_{11} = 0$ and $A_{12} = 0 \Leftrightarrow A_{22} \in \mathbb{C}_{n-t}^{\mathrm{GM}}$ and $A^{\text{\#}} = \begin{bmatrix} 0 & 0 \\ 0 & A_{22}^{\text{\#}} \end{bmatrix}$.*

Lemma 2.5 ([1, 20]). *Let* $A = U\begin{bmatrix} T & S \\ 0 & Z \end{bmatrix}U^*\in\mathbb{C}^{n\times n}$, *where* $T\in\mathbb{C}^{p\times p}$ *is nonsingular and* $Z\in\mathbb{C}_{n-p}^{\mathrm{GM}}$. *Then*

$$A^{\#} = U\begin{bmatrix} T^{-1} & -T^{-1}SZ^{\#} \\ 0 & Z^{\#} \end{bmatrix}U^*. \tag{9}$$

Lemma 2.6 ([4]). *Let* $A, B\in\mathbb{C}_n^{\mathrm{GM}}$ *with* $\mathrm{rk}(A) = t$ *and* $\mathrm{rk}(B) = p$. *Then the following statements are equivalent:*

(1) $A\perp_{\#,S}B$;

(2) *There exist nonsingular matrices* $T_1 \in \mathbb{C}^{t\times t}$ *and* $T_2 \in \mathbb{C}^{p\times p}$, *a unitary matrix* $U \in \mathbb{C}^{n\times n}$, $S \in \mathbb{C}^{t\times(n-p-t)}$ *and* $S_2 \in \mathbb{C}^{p\times(n-p-t)}$ *such that*

$$A = U\begin{bmatrix} T_1 & 0 & S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}U^*,\ B = U\begin{bmatrix} 0 & 0 & 0 \\ 0 & T_2 & S_2 \\ 0 & 0 & 0 \end{bmatrix}U^*. \tag{10}$$

3. The equivalent conditions for core partial order

According to Lemma 3 in Section 3 of [1], we give forms of matrices $A, B\in\mathbb{C}_n^{\mathrm{GM}}$ when they satisfy the core partial order under the core-EP decomposition.

Theorem 3.1. *Let* $A, B\in\mathbb{C}_n^{\mathrm{GM}}$, *then the following statements are equivalent:*

(1) $A\overset{\#}{\leq}B$;

(2) *There exist nonsingular matrix* $T\in\mathbb{C}^{p\times p}$ *and a unitary matrix* U *such that*

$$A = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*,\ B = U\begin{bmatrix} T & S \\ 0 & B_{22} \end{bmatrix}U^*, \tag{11}$$

where $B_{22}\in\mathbb{C}_{n-p}^{\mathrm{GM}}$.

In [7], the authors gave an equivalent characterization of star partial order, which is $A\overset{*}{\leq}B\Leftrightarrow A\overset{-}{\leq}B$ and $B^\dagger - A^\dagger$ is a {1,3}-generalized inverse of $B - A$. Next, we consider similar conditions of core partial order.

Theorem 3.2. *Let* $A, B\in\mathbb{C}_n^{\mathrm{GM}}$, *then the following statements are equivalent:*

(1) $A\overset{\#}{\leq}B$;

(2) $A\overset{-}{\leq}B$ *and* $B^{\#} - A^{\#}\in(B - A)\{1, 3\}$.

Proof. (1)⇒(2) If $A\overset{\#}{\leq}B$, then from Lemma 2.5 and Theorem 3.1, we can obtain that A and B have decomposition forms as in (11), and

$$A^{\#} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*,\ B^{\#} = U\begin{bmatrix} T^{-1} & -T^{-1}SB_{22}^{\#} \\ 0 & B_{22}^{\#} \end{bmatrix}U^*,$$
$$B - A = U\begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix}U^*,\ B^{\#} - A^{\#} = \begin{bmatrix} 0 & -T^{-1}SB_{22}^{\#} \\ 0 & B_{22}^{\#} \end{bmatrix}.$$

It is obvious that $\mathrm{rk}(B) - \mathrm{rk}(A) = \mathrm{rk}(B_{22}) = \mathrm{rk}(B - A)$, that is $A\overset{-}{\leq}B$. And

$$(B - A)(B^{\#} - A^{\#})(B - A) = U\begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix}U^* = B - A,$$
$$((B - A)(B^{\#} - A^{\#}))^* = U\begin{bmatrix} 0 & 0 \\ 0 & B_{22}B_{22}^{\#} \end{bmatrix}U^* = (B - A)(B^{\#} - A^{\#}),$$

which implies $B^{\circleddash} - A^{\circleddash} \in (B-A)\{1,3\}$.

(2)⇒(1) Let $B = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$ be the core-EP decomposition of B, where $T \in \mathbb{C}^{p\times p}$ is nonsingular with $\mathrm{rk}(B) = p$ and U is unitary. From $A \overset{-}{\leq} B$ and Lemma 2.2, the decomposition form of A is shown as in (6). Then from Lemma 2.4, it follows that

$$B - A = U\begin{bmatrix} T - A_{11} & S - A_{12} \\ 0 & 0 \end{bmatrix}U^*,\ B^{\circleddash} - A^{\circleddash} = U\begin{bmatrix} T^{-1} - A_{11}^{\circleddash} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{12}$$

Using (12) and $B^{\circleddash} - A^{\circleddash} \in (B-A)\{1,3\}$, we have

$$(T - A_{11})(T^{-1} - A_{11}^{\circleddash})(T - A_{11}) = T - A_{11}, \tag{13}$$

$$((T - A_{11})(T^{-1} - A_{11}^{\circleddash}))^* = (T - A_{11})(T^{-1} - A_{11}^{\circleddash}). \tag{14}$$

Let $A_{11} = V\begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix}V^*$ be the core-EP decomposition of A_{11}, where $T_1 \in \mathbb{C}^{t\times t}$ is nonsingular with $t = \mathrm{rk}(A_{11})$ and V is unitary. Then, $A_{11}^{\circleddash} = V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*$. Partitioning T^{-1} in conformation with partition of A_{11}, gives $T^{-1} = V\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}V^*$.

From $A_{11} = A_{11}T^{-1}A_{11}$, we have

$$\begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T_1F_{11} + S_1F_{21})T_1 & (T_1F_{11} + S_1F_{21})S_1 \\ 0 & 0 \end{bmatrix}.$$

T_1 is nonsingular, so

$$T_1F_{11} + S_1F_{21} = I. \tag{15}$$

As (13), (14), $A_{11} = A_{11}T^{-1}A_{11}$, $A_{11}A_{11}^{\circleddash}A_{11} = A_{11}$ and T is nonsingular, we have

$$-TA_{11}^{\circleddash}T + A_{11}A_{11}^{\circleddash}T + TA_{11}^{\circleddash}A_{11} - A_{11} = 0 \text{ and } (TA_{11}^{\circleddash} + A_{11}T^{-1})^* = TA_{11}^{\circleddash} + A_{11}T^{-1}.$$

Next, write $X = T_1F_{12} + S_1F_{22}$. Pre-multiply and post-multiply the first equation by T^{-1} respectively, $-A_{11}^{\circleddash} + T^{-1}A_{11}A_{11}^{\circleddash} + A_{11}^{\circleddash}A_{11}T^{-1} - T^{-1}A_{11}T^{-1} = 0$. By combining equation (15), it can be calculated that

$$\begin{bmatrix} 0 & T_1^{-1}X - F_{11}X \\ 0 & F_{21}X \end{bmatrix} = 0,$$

which leads to

$$T_1^{-1}X = F_{11}X \text{ and } F_{21}X = 0. \tag{16}$$

Let's consider the second equation. $TA_{11}^{\circleddash} + A_{11}T^{-1} = T(A_{11}^{\circleddash} + T^{-1}A_{11}T^{-1})(T^{-1})^*T^*$ and

$$\left(A_{11}^{\circleddash} + T^{-1}A_{11}T^{-1}\right)\left(T^{-1}\right)^* = V\begin{bmatrix} T_1^{-1}F_{11}^* + F_{11}F_{11}^* + F_{11}XF_{12}^* & T_1^{-1}F_{21}^* + F_{11}F_{21}^* + F_{11}XF_{22}^* \\ F_{21}F_{11}^* & F_{21}F_{21}^* \end{bmatrix}V^*,$$

we get

$$T_1^{-1}F_{21}^* + F_{11}F_{21}^* + F_{11}XF_{22}^* = \left(F_{21}F_{11}^*\right)^* = F_{11}F_{21}^*.$$

Therefore,

$$F_{21} = -F_{22}X^*F_{11}^*T_1^*. \tag{17}$$

Using (16) and (17), we can obtain that

$$0 = -F_{21}X = F_{22}X^*F_{11}^*T_1^*X = F_{22}(T_1^{-1}X)^*T_1^*X = F_{22}X^*X.$$

Since $F_{21} = -F_{22}X^*F_{11}^*T_1^*$ and $T^{-1} = V\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}V^*$ is nonsingular, we get that F_{22} is nonsingular. Furthermore, we get $X^*X = 0$, i.e., $X = 0$; and $F_{21} = 0$. Then, from (15) it is simple to observe that $F_{11} = T_1^{-1}$. So,

$$T^{-1} = V\begin{bmatrix} T_1^{-1} & F_{12} \\ 0 & F_{22} \end{bmatrix}V^*.$$

Then

$$T^{-1}A_{11}A_{11}^{\#} = V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* = A_{11}^{\#},\ A_{11}^{\#}A_{11}T^{-1} = V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* = A_{11}^{\#},$$

which imply

$$A_{11}A_{11}^{\#} = TA_{11}^{\#} \text{ and } A_{11}^{\#}A_{11} = A_{11}^{\#}T. \tag{18}$$

From $A_{12} = A_{11}T^{-1}S$, we have

$$A_{11}^{\#}A_{12} = A_{11}^{\#}A_{11}T^{-1}S = A_{11}^{\#}S. \tag{19}$$

By applying (18) and (19), we have

$$AA^{\#} = V\begin{bmatrix} A_{11}A_{11}^{\#} & 0 \\ 0 & 0 \end{bmatrix}V^* = V\begin{bmatrix} TA_{11}^{\#} & 0 \\ 0 & 0 \end{bmatrix}V^* = BA^{\#},$$

$$A^{\#}A = V\begin{bmatrix} A_{11}^{\#}A_{11} & A_{11}^{\#}A_{12} \\ 0 & 0 \end{bmatrix}V^* = V\begin{bmatrix} A_{11}^{\#}T & A_{11}^{\#}S \\ 0 & 0 \end{bmatrix}V^* = A^{\#}B,$$

that is $A\overset{\#}{\leq}B$. □

Example 3.3. *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *Then*

$$A^{\#} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B^{\#} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\ B - A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ B^{\#} - A^{\#} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

It is obvious that $\mathrm{rk}(B) - \mathrm{rk}(A) = \mathrm{rk}(B - A) = 1$, *i.e.* $A\overset{-}{\leq}B$.

$$(B - A)(B^{\#} - A^{\#})(B - A) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(B - A)(B^{\#} - A^{\#}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which imply $B^{\#} - A^{\#} \in (B - A)\{1, 3\}$.

In that case,

$$AA^{\#} = BA^{\#} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A^{\#}A = A^{\#}B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

that is, $A\overset{\#}{\leq}B$.

Remark 3.4. *Let* $A, B \in \mathbb{C}_n^{\rm GM}$, *it is obvious that*

$$A \overset{-}{\leq} B,\ (B-A)^{\#} = B^{\#} - A^{\#} \Rightarrow\ A \overset{-}{\leq} B,\ B^{\#} - A^{\#} \in (B-A)\{1,3\}.$$

Therefore, by Theorem 3.2, we can give the conclusion to Problem 1, that is

$$A \overset{-}{\leq} B \text{ and } (B-A)^{\#} = B^{\#} - A^{\#} \Rightarrow A \overset{\#}{\leq} B$$

is true.

Now, Problem 1 has been solved completely. More importantly, we get an equivalent characterization of core partial order with the minus partial order and {1,3}-generalized inverse.

Theorem 3.5. *Let* $A, B \in \mathbb{C}_n^{\rm GM}$, *then the following statements are equivalent:*

(1) $A \overset{\#}{\leq} B$;

(2) $A \overset{-}{\leq} B, BA^{\#}$ *is Hermitian and* $AA^{\#}B = A$;

(3) $A \overset{-}{\leq} B, BA^{\#}$ *and* A^*B *are Hermitian.*

Proof. (1)⇒(2), (3) If $A \overset{\#}{\leq} B$, then A and B have decomposition forms as in (11), so $A^{\#} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*$. Then,

$$B - A = U\begin{bmatrix} 0 & 0 \\ 0 & B_{22} \end{bmatrix}U^*,\ BA^{\#} = U\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}U^*,$$
$$AA^{\#}B = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^* = A,\ A^*B = U\begin{bmatrix} T^*T & T^*S \\ S^*T & S^*S \end{bmatrix}U^* = A^*A.$$

It is obvious that $\mathrm{rk}(B) - \mathrm{rk}(A) = \mathrm{rk}(B-A) = \mathrm{rk}(B_{22})$, that is, $A \overset{-}{\leq} B$. And (2), (3) are true.

Let $B = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$ be the core-EP decomposition of B, where $T \in \mathbb{C}^{p\times p}$ is nonsingular with $p = \mathrm{rk}(B)$ and U is unitary. From $A \overset{-}{\leq} B$ and Lemma 2.2, we can get that A has the decomposition form as in (6). Then

$$A^{\#} = U\begin{bmatrix} A_{11}^{\#} & 0 \\ 0 & 0 \end{bmatrix}U^*,\ B^{\#} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*, \tag{20}$$

$$AA^{\#} = U\begin{bmatrix} A_{11}A_{11}^{\#} & 0 \\ 0 & 0 \end{bmatrix}U^*,\ A^{\#}A = U\begin{bmatrix} A_{11}^{\#}A_{11} & A_{11}^{\#}A_{12} \\ 0 & 0 \end{bmatrix}U^*, \tag{21}$$

$$BA^{\#} = U\begin{bmatrix} TA_{11}^{\#} & 0 \\ 0 & 0 \end{bmatrix}U^*,\ A^{\#}B = U\begin{bmatrix} A_{11}^{\#}T & A_{11}^{\#}S \\ 0 & 0 \end{bmatrix}U^*, \tag{22}$$

$$AA^{\#}B = U\begin{bmatrix} A_{11}A_{11}^{\#}T & A_{11}A_{11}^{\#}S \\ 0 & 0 \end{bmatrix}U^*,\ A^*B = U\begin{bmatrix} A_{11}^*T & A_{11}^*S \\ A_{12}^*T & A_{12}^*S \end{bmatrix}U^*. \tag{23}$$

Let $A_{11} = V\begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix}V^*$ be the core-EP decomposition of A_{11}, where $T_1 \in \mathbb{C}^{t\times t}$ is nonsingular with $t = \mathrm{rk}(A_{11})$ and V is unitary. Then, $A_{11}^{\#} = V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^*$. Partitioning T^{-1} in conformation with partition of A_{11}, gives $T^{-1} = V\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}V^*$.

(2)⇒(1) Because $BA^{\#}$ is Hermitian, it is obvious that $TA_{11}^{\#}$ is Hermitian. Then, by T_1 is nonsingular we have

$$\begin{aligned}(TA_{11}^{\#})^* = TA_{11}^{\#} \quad &\Rightarrow \quad T^{-1}(A_{11}^{\#})^* = A_{11}^{\#}(T^{-1})^* \\ &\Rightarrow \quad \begin{bmatrix} F_{11}(T_1^{-1})^* & 0 \\ F_{21}(T_1^{-1})^* & 0 \end{bmatrix} = \begin{bmatrix} T_1^{-1}F_{11}^* & T_1^{-1}F_{21}^* \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow \quad F_{21}(T_1^{-1})^* = 0 \\ &\Rightarrow \quad F_{21} = 0.\end{aligned}$$

From $AA^{\#}B = A$, we get $A_{11}A_{11}^{\#}T = A_{11}$ and $A_{11}A_{11}^{\#}S = A_{11}T^{-1}S$. $F_{21} = 0$, T and T_1 are nonsingular, so we have

$$\begin{aligned}A_{11}A_{11}^{\#} = A_{11}T^{-1} \quad &\Rightarrow \quad \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1F_{11} & T_1F_{12} + S_1F_{22} \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow \quad T_1F_{11} = I_t, T_1F_{12} + S_1F_{22} = 0 \\ &\Rightarrow \quad F_{11} = T_1^{-1}, F_{12} + T_1^{-1}S_1F_{22} = 0.\end{aligned}$$

In that case,

$$\begin{aligned}T^{-1}A_{11}A_{11}^{\#} &= V\begin{bmatrix} T_1^{-1} & F_{12} \\ 0 & F_{22} \end{bmatrix}\begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* = V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* = A_{11}^{\#}, \\ A_{11}^{\#}A_{11}T^{-1} &= V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^{-1} & F_{12} \\ 0 & F_{22} \end{bmatrix}V^* = V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}V^* = A_{11}^{\#},\end{aligned}$$

that is,

$$A_{11}A_{11}^{\#} = TA_{11}^{\#},\ A_{11}^{\#}A_{11} = A_{11}^{\#}T. \tag{24}$$

On the other hand, from $A_{11}A_{11}^{\#} = A_{11}T^{-1}$, we have

$$A_{11}^{\#}A_{12} = A_{11}^{\#}A_{11}T^{-1}S = A_{11}^{\#}A_{11}A_{11}^{\#}S = A_{11}^{\#}S. \tag{25}$$

Substituting equations (24), (25) into (21) and (22), it is obvious that $AA^{\#} = BA^{\#}$ and $A^{\#}A = A^{\#}B$ are true, i.e., $A\overset{\#}{\leq}B$.

(3)⇒(1) Substituting their decomposition forms into the equation $A_{11} = A_{11}T^{-1}A_{11}$ and because of the nonsingularity of T_1, we can obtain

$$T_1F_{11} + S_1F_{21} = I_t. \tag{26}$$

Because A^*B is Hermitian, we can get A_{11}^*T is Hermitian. Then by (26) and T, T_1 are nonsingular, we have

$$\begin{aligned}(A_{11}^*T)^* = A_{11}^*T \quad &\Rightarrow \quad A_{11}T^{-1} = (T^{-1})^*A_{11}^* \\ &\Rightarrow \quad \begin{bmatrix} I_t & T_1F_{12} + S_1F_{22} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_t & 0 \\ (T_1F_{12} + S_1F_{22})^* & 0 \end{bmatrix} \\ &\Rightarrow \quad T_1F_{12} + S_1F_{22} = 0.\end{aligned}$$

Because $BA^{\#}$ is Hermitian and by the proof in (2)⇒(1), we have $F_{21} = 0$. Then since T_1 are nonsingular and by (26), we can get $F_{11} = T_1^{-1}$.

Then, we can obtain $A\overset{\#}{\leq}B$. □

In [14], Rakić and Djordjević give some equivalent conditions for A and B to satisfy core partial order provided $A\overset{-}{\leq}B$, most of which are related to Moore-Penrose inverse and group inverse. In the following theorem, we give some equivalent conditions, which are characterized by core inverse and group inverse.

Theorem 3.6. *Let $A, B\in\mathbb{C}_n^{\mathrm{GM}}$, then the following statements are equivalent:*

(1) $A\overset{\#}{\leq}B$;
(2) $A^*A = A^*B$ *and* $BA^{\#}$ *is Hermitian;*
(3) $A^*A = A^*B$ *and* $B(A^\#)^2 = (A^{\#})^2B$;
(4) $A^*A = A^*B$ *and* $B(A^{\#})^2 = A^{\#}$;
(5) $A^*A = A^*B$ *and* $A^{\#}B = AA^\#$;
(6) $A^*A = A^*B$ *and* $A^{\#}B = BA^\#$;
(7) $A^{\#}B = AA^\#$ *and* $BA^{\#}$ *is Hermitian;*
(8) $A^{\#}B = AA^\#$ *and* $B(A^\#)^2 = (A^{\#})^2B$;
(9) $A^{\#}B = AA^\#$ *and* $B(A^{\#})^2 = A^{\#}$;
(10) $A^{\#}B = AA^\# = BA^\#$;
(11) A^*B *is Hermitian and* $B(A^\#)^2 = (A^{\#})^2B$;
(12) A^*B *is Hermitian and* $B(A^{\#})^2 = A^{\#}$;
(13) A^*B *is Hermitian and* $A^{\#}B = AA^\#$.

Proof. (1)⇒(2)-(13) If $A\overset{\#}{\leq}B$, then A, B have decomposition forms as in (11) and

$$A^{\#} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*,\ A^\# = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*.$$

Then,

$$A^*A = A^*B = U\begin{bmatrix} T^*T & T^*S \\ S^*T & S^*S \end{bmatrix}U^*,$$
$$BA^{\#} = U\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}U^*,$$
$$B(A^\#)^2 = (A^{\#})^2B = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*,$$
$$B(A^{\#})^2 = A^{\#} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$
$$BA^\# = AA^\# = A^{\#}B = U\begin{bmatrix} I_p & T^{-1}S \\ 0 & 0 \end{bmatrix}U^*.$$

Obviously, (2)-(13) are true.

Let $A = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$ be the core-EP decomposition of A, where $T\in\mathbb{C}^{p\times p}$ is nonsingular with $p = \mathrm{rk}(A)$ and U is unitary. Then,

$$A^{\#} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*,\ A^\# = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*.$$

Partitioning B in conformation with partition of A, gives $B = U\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}U^*$. Then

$$\begin{aligned} A^*A &= U\begin{bmatrix} T^*T & T^*S \\ S^*T & S^*S \end{bmatrix}U^*,\ A^*B = U\begin{bmatrix} T^*B_{11} & T^*B_{12} \\ S^*B_{11} & S^*B_{12} \end{bmatrix}U^*, \\ BA^{\#\!\!\!\bigcirc} &= U\begin{bmatrix} B_{11}T^{-1} & 0 \\ B_{21}T^{-1} & 0 \end{bmatrix}U^*,\ A^{\#\!\!\!\bigcirc}B = U\begin{bmatrix} T^{-1}B_{11} & T^{-1}B_{12} \\ 0 & 0 \end{bmatrix}U^*, \\ AA^{\#} &= U\begin{bmatrix} I_p & T^{-1}S \\ 0 & 0 \end{bmatrix}U^*,\ BA^{\#} = U\begin{bmatrix} B_{11}T^{-1} & B_{11}T^{-1}S \\ B_{21}T^{-1} & B_{21}T^{-1}S \end{bmatrix}U^*, \\ BA^2 &= U\begin{bmatrix} B_{11}T^2 & B_{11}TS \\ B_{21}T^2 & B_{21}TS \end{bmatrix}U^*,\ ABA = U\begin{bmatrix} TB_{11}T + SB_{21}T & TB_{11}S + SB_{21}S \\ 0 & 0 \end{bmatrix}U^*, \\ (A^{\#\!\!\!\bigcirc})^2B &= U\begin{bmatrix} T^{-2}B_{11} & T^{-2}B_{12} \\ 0 & 0 \end{bmatrix}U^*. \end{aligned}$$

(2)⇒(1) From $A^*A = A^*B$ and T is nonsingular, we can obtain

$$B_{11} = T,\ B_{12} = S. \tag{27}$$

From $BA^{\#\!\!\!\bigcirc}$ is Hermitian and T is nonsingular, we can obtain

$$B_{21} = 0. \tag{28}$$

Applying (27) and (28), we have

$$A = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*,\ B = U\begin{bmatrix} T & S \\ 0 & B_{22} \end{bmatrix}U^*,$$

Then by Theorem 3.1, $A\overset{\#\!\!\!\bigcirc}{\leq}B$.

Similarly, under the above decomposition of A and B, as long as the conditions (27) and (28) are true, then $A\overset{\#\!\!\!\bigcirc}{\leq}B$. It is easy to see that each of conditions (3)-(13) implies (27) and (28). □

4. Some new characterizations of the strong core orthogonality

In this section, we give some new results on strong core orthogonality.

First, from (2) and Theorem 3.2, it is easy to check that

$$A\perp_{\#\!\!\!\bigcirc}(B-A) \Leftrightarrow A\overset{\#\!\!\!\bigcirc}{\leq}B \Leftrightarrow A\overset{-}{\leq}B \text{ and } B^{\#\!\!\!\bigcirc} - A^{\#\!\!\!\bigcirc} \in (B-A)\{1,3\}.$$

Considering the matrices A and B in Example 3.3, then

$$\begin{aligned} A^{\#\!\!\!\bigcirc}(B-A) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0, \\ (B-A)A^{\#\!\!\!\bigcirc} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0, \\ A(B-A)^{\#\!\!\!\bigcirc} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which mean that $A\perp_{\#\!\!\!\bigcirc}(B-A)$, but $A\perp_{\#\!\!\!\bigcirc,S}(B-A)$ does not hold.

Next, we add some conditions to make it be equivalent to strong core orthogonality.

Theorem 4.1. *Let $A, B \in \mathbb{C}_n^{\mathrm{GM}}$, then the following statements are equivalent:*
(1) $A \perp_{\textcircled{\#},S} (B-A)$;
(2) $A \overset{-}{\leq} B$, $B^{\textcircled{\#}} - A^{\textcircled{\#}} \in (B-A)\{1,3\}$ and $AB^{\textcircled{\#}} = B^{\textcircled{\#}}ABB^{\textcircled{\#}}$;
(3) $A \perp_{\textcircled{\#}} (B-A)$ and $AB^{\textcircled{\#}} = B^{\textcircled{\#}}ABB^{\textcircled{\#}}$;
(4) $A \overset{\textcircled{\#}}{\leq} B$ and $AB^{\textcircled{\#}} = B^{\textcircled{\#}}ABB^{\textcircled{\#}}$.

Proof. (1)⇒(2) It can be easily proved by decomposition forms of A, B in Lemma 2.6.

(2)⇒(1) Let $B = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*$ be the core-EP decomposition of B, where $T \in \mathbb{C}^{p\times p}$ is nonsingular with $p = \mathrm{rk}(B)$ and U is unitary. From Lemma 2.2, we have

$$A = U\begin{bmatrix} A_{11} & A_{11}T^{-1}S \\ 0 & 0 \end{bmatrix}U^*,\ B-A = U\begin{bmatrix} T-A_{11} & (T-A_{11})T^{-1}S \\ 0 & 0 \end{bmatrix}U^*,$$

where $A_{11} = A_{11}T^{-1}A_{11}$.

From $A \overset{-}{\leq} B$, $B^{\textcircled{\#}} - A^{\textcircled{\#}} \in (B-A)\{1,3\}$ and Theorem 3.2, we can obtain

$$F_{11} = T_1^{-1},\ F_{21} = 0 \text{ and } T_1F_{12} + S_1F_{22} = 0. \tag{29}$$

Because $AB^{\textcircled{\#}} = B^{\textcircled{\#}}ABB^{\textcircled{\#}}$ and

$$AB^{\textcircled{\#}} = U\begin{bmatrix} A_{11} & A_{11}T^{-1}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} A_{11}T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$
$$B^{\textcircled{\#}}ABB^{\textcircled{\#}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_{11} & A_{11}T^{-1}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^{-1}A_{11} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

we get $A_{11}T^{-1} = T^{-1}A_{11}$. Substitute (29) into it, and because T is nonsingular, we can obtain

$$\begin{aligned} A_{11}T^{-1} = T^{-1}A_{11} \quad &\Rightarrow \quad \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^{-1} & F_{12} \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} T_1^{-1} & F_{12} \\ 0 & F_{22} \end{bmatrix}\begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow \quad \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_t & T_1^{-1}S_1 \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow \quad S_1 = 0. \end{aligned}$$

Furthermore,

$$A_{11} = V\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}V^*,\ T^{-1} = V\begin{bmatrix} T_1^{-1} & 0 \\ 0 & F_{22} \end{bmatrix}V^*,\ T = V\begin{bmatrix} T_1 & 0 \\ 0 & F_{22}^{-1} \end{bmatrix}V^*.$$

Substitute the results into the following equations:

$$A^{\textcircled{\#}}(B-A) = U\begin{bmatrix} A_{11}^{\textcircled{\#}}(T-A_{11}) & A_{11}^{\textcircled{\#}}(T-A_{11}T^{-1}S) \\ 0 & 0 \end{bmatrix}U^*,$$
$$(B-A)A^{\textcircled{\#}} = U\begin{bmatrix} (T-A_{11})A_{11}^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$
$$A(B-A)^{\textcircled{\#}} = U\begin{bmatrix} A_{11}(T-A_{11})^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Then it is easy to check that $A^{\textcircled{\#}}(B-A) = 0, (B-A)A^{\textcircled{\#}} = 0, A(B-A)^{\textcircled{\#}} = 0$, i.e., $A \perp_{\textcircled{\#},S} (B-A)$.

It is obvious that (2)⇔(3)⇔(4). □

Theorem 4.2. *Let $A, B \in \mathbb{C}_n^{\mathrm{GM}}$, then the following statements are equivalent:*
(1) $A \perp_{\textcircled{\#},S} (B - A)$;
(2) $A \overset{-}{\leq} B$, $B^{\textcircled{\#}} - A^{\textcircled{\#}} \in (B - A)\{1, 3\}$ *and* $BA^{\textcircled{\#}} = A^{\textcircled{\#}} B^2 B^{\textcircled{\#}}$;
(3) $A \perp_{\textcircled{\#}} (B - A)$ *and* $BA^{\textcircled{\#}} = A^{\textcircled{\#}} B^2 B^{\textcircled{\#}}$;
(4) $A \overset{\textcircled{\#}}{\leq} B$ *and* $BA^{\textcircled{\#}} = A^{\textcircled{\#}} B^2 B^{\textcircled{\#}}$.

Proof. (1)⇒(2) It can be easily proved by decomposition forms of A, B in Lemma 2.6.

(2)⇒(1) We continue to use decomposition forms of A, B, A_{11} and T^{-1} as in Theorem 4.1. From $A \overset{-}{\leq} B$, $B^{\textcircled{\#}} - A^{\textcircled{\#}} \in (B - A)\{1, 3\}$ and Theorem 3.2, we can obtain

$$F_{11} = T_1^{-1}, \ F_{21} = 0 \text{ and } \ T_1 F_{12} + S_1 F_{22} = 0. \tag{30}$$

Because

$$BA^{\textcircled{\#}} = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} TA_{11}^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$
$$A^{\textcircled{\#}} B^2 B^{\textcircled{\#}} = U \begin{bmatrix} A_{11}^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^2 \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} A_{11}^{\textcircled{\#}} T & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

and $BA^{\textcircled{\#}} = A^{\textcircled{\#}} B^2 B^{\textcircled{\#}}$, we get $TA_{11}^{\textcircled{\#}} = A_{11}^{\textcircled{\#}} T$. Because T and T_1 is nonsingular and substitute result (30) into it, we can obtain

$$\begin{aligned} TA_{11}^{\textcircled{\#}} = A_{11}^{\textcircled{\#}} T \quad & \Rightarrow \quad A_{11}^{\textcircled{\#}} T^{-1} = T^{-1} A_{11}^{\textcircled{\#}} \\ & \Rightarrow \quad \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{-1} & F_{12} \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} T_1^{-1} & F_{12} \\ 0 & F_{22} \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ & \Rightarrow \quad \begin{bmatrix} T_1^{-2} & T_1^{-1} F_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^{-2} & 0 \\ 0 & 0 \end{bmatrix} \\ & \Rightarrow \quad F_{12} = 0. \end{aligned}$$

Because T is nonsingular, we can obtain F_{22} is nonsingular. By using (30), we have $S_1 = 0$. It follows that

$$A_{11} = U \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} U^*, \ T^{-1} = U \begin{bmatrix} T_1^{-1} & 0 \\ 0 & F_{22} \end{bmatrix} U^*.$$

Therefore, $A \perp_{\textcircled{\#},S} (B - A)$.
It is obvious that (2)⇔(3)⇔(4). □

Remark 4.3. *According to Theorem 4.1, Theorem 4.2 and the characterizations of core partial order (e.g. Theorem 3.6), one can derive a number of equivalent conditions of strong core orthogonality.*

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