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Extended Jacobson's lemma for the generalized inverse

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Abstract. In this paper, we present an extended Jacobson's lemma for g-Drazin inverse in Banach algebras. Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

Then $1 - ac \in \mathcal{A}^d$ if and only if $1 - bd \in \mathcal{A}^d$. Related generalized Jacobson's lemma for Drazin, core and p-core inverses in a Banach algebra are thereby obtained.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. An element $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) provided that there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a - a^2b \in \mathcal{A}^{qnil}.$$

The preceding b is unique if exists, we denote it by a^d. Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1}$ for every $\lambda \in \mathbb{C}\}$. If we replace $\mathcal{A}^{qnil}$ in the above definition with the set of nilpotents $\mathcal{A}^{nil}$, then b is called the Drazin inverse of a. For a complex matrix, the g-Drazin and Drazin inverses coincide with each other. The g-Drazin inverse plays an important role in matrix and operator theory. Many authors have been studying this subject from different views (see [1, 2, 16, 19] and [18]).

Jacobson's lemma states that $1 - ab \in \mathcal{A}^{-1}$ if and only if $1 - ba \in \mathcal{A}^{-1}$. In [19, Theorem 2.3], Zhuang, Chen and Cui gave a Jacobson's lemma for generalized Drazin inverse. They proved that $1-ab \in \mathcal{A}^d$ if and only if $1 - ba \in \mathcal{A}^d$. Recently, many generalized Jacobson's lemmas are established by many authors. Let $a, b, c \in R$ with $(ac)^2a = acaba = abaca = a(ba)^2$. The authors proved that $1 - ac \in R^d$ if and only if $1 - ba \in R^d$ (see [2, Theorem 2.2]). Let $a, b, c, d \in R$ with $aca = dba, acd = dbd$. Yan, Zeng Zhu proved that $1 - ac \in R^d$ if and only if $1 - bd \in R^d$ (see [15, Theorem 3.3]). The motivation of this paper is to provide a new generalized Jacobson's lemma for generalized Drazin inverse in a Banach algebra. This makes the preceding known results as our special cases.

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Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

We prove that $1 - bd \in \mathcal{A}^d$ if and only if $1 - ac \in \mathcal{A}^d$. The generalized Jacobson's lemmas for Drazin and group inverses are also established.

An involution of a Banach algebra $\mathcal{A}$ is an anti-automorphism whose square is the identity map 1. A Banach algebra $\mathcal{A}$ with involution $*$ is called a Banach *-algebra, e.g., C^*-algebra. Let $\mathcal{A}$ be a C^*-algebra. An element $a \in \mathcal{A}$ has p-core inverse (i.e., pseudo core inverse) if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$xa^{k+1} = a^k, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{D}}$. We say that $a \in \mathcal{A}$ has core inverse if there exists some $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$. An element a in a Banach *- algebra $\mathcal{A}$ has core inverse if and only if there exist $x \in \mathcal{A}$ such that

$$a = axa, x\mathcal{A} = a\mathcal{A}, \mathcal{A}x = \mathcal{A}a^*.$$

Recently, many authors have studied core and p-core inverses from many different views, e.g., [4, 5, 10, 13]. An element $a \in \mathcal{A}$ has $\{1, 4\}-$ inverse provided that there exists some $x \in \mathcal{A}$ such that $a = axa$ and $(xa)^* = xa$. Let $\mathcal{A}$ be a Banach *-algebra, and let $a, b, c, d \in \mathcal{A}$ satisfy $1 - bd \in \mathcal{A}^{\textcircled{D}}$. Finally, in the last section, we prove that $1 - ac \in \mathcal{A}^{\textcircled{D}}$ if and only if $acd(1 - bd)^{\pi}bacac \in \mathcal{A}^{\{1,4\}}$ under the preceding same conditions. Moreover, related generalized Jacobson's lemma for the core inverse in a Banach algebra is established.

Throughout the paper, all Banach *-algebras are complex with an identity. An element $p \in \mathcal{A}$ is a projection if $p^2 = p = p^*$. $\mathcal{A}^D, \mathcal{A}^{\textcircled{D}}, \mathcal{A}^{\textcircled{\#}}$ and $\mathcal{A}^{nil}$ denote the sets of all Drazin, p-core invertible, core invertible and nilpotent elements in $\mathcal{A}$ respectively. Let $a \in \mathcal{A}^D$. We use a^{π} to stand for the spectral idempotent of a corresponding to $\{0\}$, i.e., $a^{\pi} = 1 - aa^D$.

2. generalized Jacobson's lemma

In this section, we investigate new extension of Jacobson's lemma for generalized Drazin inverse in a Banach algebra. We begin with

Lemma 2.1. *(see [2, Lemma 2.1]) Let $\mathcal{A}$ be a Banach algebra, let $m \in \mathbb{N}$ and let $a \in \mathcal{A}$. Then a has g-Drazin inverse if and only if there exists $b \in \mathcal{A}$ such that*

$$ab = ba, \left[ab - (ab)^2\right]^m = 0, a - a^2b \in \mathcal{A}^{qnil}.$$

In this case,

$$a^d = (a + 1 - e)^{-1}e, e = \sum_{i=0}^{m-1} \binom{2m}{i} (ab)^{2m-i}(1 - ab)^i.$$

We are ready to prove:

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

Then $\alpha = 1 - bd \in \mathcal{A}^d$ if and only if $\beta = 1 - ac \in \mathcal{A}^d$. In this case,

$$\beta^d = (\beta + 1 - e)^{-1}e, e = 4(y\beta)^3 - 3(y\beta)^4,$$
$$y = \left[1 - acd\alpha^{\pi}(1 - \alpha\alpha^{\pi}(1 + bd + bdbd))^{-1}bac\right](1 + ac + acac) + acd\alpha^d bac.$$

Proof. $\Longrightarrow$ Let $p = \alpha^{\pi}, x = \alpha^{d}$. Then $1 - p\alpha(1 + bd + bdbd) \in \mathcal{A}^{-1}$. Let

$$y = \Big[1 - acdp(1 - p\alpha(1 + bd + bdbd))^{-1}bac\Big](1 + ac + acac) + acdxbac.$$

Step 1. $\Big[y\beta - (y\beta)^2\Big]^2 = 0$. We compute that

$$y\beta a = a - [acacac - acdxbac(1 - ac)]a - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}bac[1 - (ac)^3]a$$
$$= a - ac[dbac - dxbac(1 - ac)]a - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}\Big[bac - bac(ac)^3\Big]a$$
$$= a - ac[dbac - dx(bac - bacac)]a - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}(bac - bacacdbac)a$$
$$= a - ac[dbac - dx(bac - bacac)]a - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}(1 - bacacd)baca$$
$$= a - ac[dbac - dx(1 - bd)bac]a - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}[1 - (bd)^3]baca$$
$$= a - acdpbac - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}p\alpha(1 + bd + bdbd)baca$$
$$= a - acdp[a - p\alpha(1 + bd + bdbd)]^{-1}\Big[(1 - p\alpha(1 + bd + bdbd)) + p\alpha(1 + bd + bdbd)\Big]baca$$
$$= [1 - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}bac]a.$$

Clearly, $(bacacd)(bd) = b[(ac)^2d]bd = b[(db)^2d]bd = bdb[(db)^2d] = bdb[(ac)^2d] = (bd)(bacacd)$; hence, $(bacacd)\alpha = \alpha(bacacd)$. This implies that $(bacacd)x = x(bacacd)$. We verify that

$$acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}bac(acdxbac)$$
$$= acd[1 - p\alpha(1 + bd + bdbd)]^{-1}(px)bacacdbac = 0.$$

Write $1 - y\beta = az$ for some $z \in \mathcal{A}$. Therefore we have

$$y\beta(1 - y\beta)$$
$$= (y\beta a)z$$
$$= [1 - acdp\Big(1 - p\alpha(1 + bd + bdbd)\Big)^{-1}bac](az)$$
$$= [1 - acdp\Big(1 - p\alpha(1 + bd + bdbd)\Big)^{-1}bac](1 - y\beta)$$
$$= 1 - y\beta - acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}bac(1 - y\beta).$$

Let $\delta = acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}$. Then we check that

$$\delta bacacdp[1 - p\alpha(1 + bd + bdbd)]^{-1}bac(1 + ac + acac)$$
$$= acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}b(dbdbd)p[1 - p\alpha(1 + bd + bdbd)]^{-1}bac(1 + ac + acac)$$
$$= acdp[1 - p\alpha(1 + bd + bdbd)]^{-2}bdbdbdbac(1 + ac + acac).$$

Clearly, $acdp = \delta(1 - p\alpha(1 + bd + bdbd))$. We easily see that,

$$bacaca = b(ac)^2a = b(dbaca) = bdbaca,$$

$$bacacaca = (bacaca)ca = (bdbaca)ca =$$
$$bd(bacaca) = bd(bdbaca) = (bd)^2baca.$$

Then we have

$$acdp[1 - p\alpha(1 + bd + bdbd)]^{-1}bacy\beta(1 - y\beta)$$

$$= \delta bacy(1-ac)az$$
$$= \delta bac\Big[1 - acdp(1-p\alpha(1+bd+bdbd))^{-1}bac\Big](1+ac+acac)a(1-ca)z$$
$$= \Big[\delta bac(1+ac+acac) - \delta bacacdp(1-p\alpha(1+bd+bdbd))^{-1}bac(1+ac+acac)\Big]a(1-ca)z$$
$$= \Big[\delta(1+bd+bdbd)bac - acdp(1-p\alpha(1+bd+bdbd))^{-2}(bd)^3(1+bd+bdbd)bac\Big]a(1-ca)z$$
$$= [\delta - \delta(1-p\alpha(1+bd+bdbd))^{-1}(bd)^3](1+bd+bdbd)bac\Big]a(1-ca)z$$
$$= \Big[\delta(1-p\alpha(1+bd+bdbd))^{-1}\Big[p - p\alpha(1+bd+bdbd) - p(bd)^3\Big](1+bd+bdbd)bac\Big]a(1-ca)z = 0.$$

Hence,
$y\beta(1-y\beta)y\beta = (1-y\beta)y\beta - acdp[1-p\alpha(1+bd+bdbd]^{-1}bac(1-y\beta)y\beta = (1-y\beta)y\beta$, and so $(1-y\beta)^2y\beta = 0$. Therefore $\Big[y\beta - (y\beta)^2\Big]^2 = [(1-y\beta)^2y\beta)]y\beta = 0$.

Step 2. $y \in comm(\beta)$. Let $s = ac$. Then we have

Claim 1. $\beta(acdxbac) = (acdxbac)\beta$. Obviously, we have

$$p = (bd)^3p[1-p\alpha(1+bd+bdbd)]^{-1} = (bd)^6p[1-p\alpha(1+bd+bdbd)]^{-2}.$$

(1) We prove that $acd\alpha xbacs = sacd\alpha xbac$.

$$sacdpbac = acacd(bd)^6p[1-p\alpha(1+bd+bdbd)]^{-2}bac$$
$$= acd(bd)^7p[1-p\alpha(1+bd+bdbd)]^{-2}bac$$
$$= acdp[1-p\alpha(1+bd+bdbd)]^{-2}(bd)^7bac$$
$$= acdp[1-p\alpha(1+bd+bdbd)]^{-2}(bd)^5b(dbdba)c$$
$$= acdp[1-p\alpha(1+bd+bdbd)]^{-2}(bd)^5b(dbaca)c$$
$$= acdp[1-p\alpha(1+bd+bdbd)]^{-2}(bd)^6bacac$$
$$= acdpbacs,$$

as claimed.

Moreover, we have

$$acdbd\alpha xbacs = ac(acd\alpha xbac)s$$
$$= acs(acd\alpha xbac)$$
$$= s(acacd)\alpha xbac$$
$$= sacdbd\alpha xbac.$$

Therefore

$$acd(1+bd)\alpha xbacs = sacd(1+bd)\alpha xbac.$$

That is,

$$acdxbacs - acd(bd)^2xbacs = sacdxbac - sacd(bd)^2xbac.$$

(2) We compute that

$$\begin{aligned} acd(bd)^2xbacs &= acdbdbdxbacac \\ &= acdxbdb(dbaca)c \\ &= acdxbdb(dbdba)c \\ &= acdbdbdbdxbac \\ &= sacd(bd)^2xbac. \end{aligned}$$

Hence $s(acdxbac) = (acdxbac)s$, and so $\beta(acdxbac) = (acdxbac)\beta$.

Claim 2.

$$s\Big[1-acdp(1-p\alpha(1+bd+bdbd)^{-1}bac\Big](1+ac+acac)$$

$$=\Big[1-acdp(1-p\alpha(1+bd+bdbd)^{-1}bac\Big](1+ac+acac)s.$$

Set $t=acdp(1-p\alpha(1+bd+bdbd)^{-1}bac(1+ac+acac)$. Then we compute that

$$\begin{array}{rcl} st &=& acacdp[1-p\alpha(1+bd+bdbd)]^{-1}bac(1+ac+acac)\\ &=& acacd(bd)^6p[1-p\alpha(1+bd+bdbd)]^{-3}bac(1+ac+acac)\\ &=& acdp[1-p\alpha(1+bd+bdbd)]^{-3}(bd)^7bac(1+ac+acac) \end{array}$$

Also we have

$$\begin{array}{rcl} ts &=& acdp[1-p\alpha(1+bd+bdbd)]^{-1}bac(1+ac+acac)ac\\ &=& acdp[1-p\alpha(1+bd+bdbd)]^{-3}(bd)^6bacac(1+ac+acac)\\ &=& acdp[1-p\alpha(1+bd+bdbd)]^{-3}(bd)^5b(dbaca)c(1+ac+acac)\\ &=& acdp[1-p\alpha(1+bd+bdbd)]^{-3}(bd)^5b(dbdba)c(1+ac+acac). \end{array}$$

Then $st=ts$; hence, $\beta t=t\beta$. Accordingly, $y\in comm(\beta)$.

Step 3. $\beta-\beta y\beta\in\mathcal{A}^{qnil}$. As is well known, $\mathcal{A}^{qnil}=\{r\in\mathcal{A}\mid 1+zr\in\mathcal{A}^{-1}$ if $zr=rz\}$. Then we have

$$\begin{array}{rl} & bacacdbdbdbdbdbdbdp\alpha[1-p\alpha(1+bd+bdbd)]^{-3}\\ = & (bd)^9[1-p\alpha(1+bd+bdbd)]^{-3}(\alpha-\alpha^2\alpha^d)\\ \in & \mathcal{A}^{qnil}. \end{array}$$

By hypothesis, we see that $d\alpha bdbdbdbd\in a\mathcal{A}$. By virtue of [9, Lemma 3.1], we get

$$(1-y\beta)d\alpha bdbdbdbdp(1-p\alpha(1+bd+bdbd))^{-2}bac$$

$$=acdp[1-p\alpha(1+bd+bdbd)]^{-1}bacd\alpha bdbdbdbdp(1-p\alpha(1+bd+bdbd))^{-2}bac$$

$$=acdp[1-p\alpha(1+bd+bdbd)]^{-1}bdbd\alpha bdbdbdbdp(1-p\alpha(1+bd+bdbd))^{-2}bac$$

$$=acdbdbdbdbdbdbdp\alpha[1-p\alpha(1+bd+bdbd)]^{-3}bac\in\mathcal{A}^{qnil}.$$

Since $1-y\beta\in a\mathcal{A}$, by using [9, Lemma 3.1], we have

$$beta(1-y\beta)^3$$

$$=\beta[acdp\Big(1-p\alpha(1+bd+bdbd)\Big)^{-1}bac][acdp\Big(1-p\alpha(1+bd+bdbd)\Big)^{-1}bac](1-y\beta)$$

$$=\beta[acdp\Big(1-p\alpha(1+bd+bdbd)\Big)^{-1}](bdbdbdp)[\Big(1-p\alpha(1+bd+bdbd)\Big)^{-1}bac](1-y\beta)$$

$$=\beta acdbdbdbdp\Big(1-p\alpha(1+bd+bdbd)\Big)^{-2}bac(1-y\beta)$$

$$=(1-ac)a(cdbdbdbd)p(1-p\alpha(1+bd+bdbd))^{-2}bac(1-y\beta)$$

$$=(acdbdbdbd-acacdbdbdbd)p(1-p\alpha(1+bd+bdbd))^{-2}bac(1-y\beta)$$

$$=(dbdbdbdbd-dbdbdbdbdbd)p(1-p\alpha(1+bd+bdbd))^{-2}bac(1-y\beta)$$

$$=d\alpha bdbdbdbdp(1-p\alpha(1+bd+bdbd))^{-2}bac(1-y\beta)\in\mathcal{A}^{qnil}.$$

Then $\beta(1-y\beta)^3\in\mathcal{A}^{qnil}$, and so $(\beta-\beta^2y)^3=\beta(1-y\beta)^3\beta^2\in\mathcal{A}^{qnil}$. Hence $\beta-\beta^2y\in\mathcal{A}^{qnil}$. Therefore we are through by Lemma 2.1.

$\Longleftarrow$ Since $1-ac\in\mathcal{A}^d$, it follows by Jacobson's Lemma that $1-ca\in\mathcal{A}^d$. Applying the preceding discussion, we obtain that $1-bd\in\mathcal{A}^d$, as desired. □

Corollary 2.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$ satisfying satisfying*

$$(ac)^2a = acaba = abaca = a(ba)^2.$$

Then $\alpha = 1 - ba \in \mathcal{A}^d$ if and only if $\beta = 1 - ac \in \mathcal{A}^d$. In this case,

$$\beta^d = (\beta + 1 - e)^{-1}e, e = 4(y\beta)^3 - 3(y\beta)^4,$$
$$y = \Big[1 - aca\alpha^{\pi}(1 - \alpha\alpha^{\pi}(1 + ba + baba))^{-1}bac\Big](1 + ac + acac) + aca\alpha^a bac.$$

Proof. By hypothesis, we check that

$$(ac)^2a = acaba = abaca = (ab)^2a;$$
$$(ac)^2a = acaba = abaca = (ab)^2a.$$

This completes the proof by Theorem 2.2. □

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra, let $\lambda \in \mathbb{C}$, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

Then $\lambda - bd \in \mathcal{A}^d$ if and only if $\lambda - ac \in \mathcal{A}^d$. In this case, $(ac)^d = a[(bd)^d]^2c$. If $\lambda \neq 0$, then

$$\begin{array}{rcl} (\lambda - ac)^d & = & (\beta + \lambda - \lambda e)^{-1}e, \\ e & = & 4\lambda^{-3}(y\beta)^3 - 3\lambda^{-4}(y\beta)^4, \\ y & = & \lambda^{-2}\Big[1 - acd\alpha^{\pi}(\lambda^3 - \alpha\alpha^{\pi}(\lambda^2 + \lambda bd + bdbd))^{-1}bac\Big] \\ & & (\lambda^2 + \lambda ac + acac) + \lambda^{-2}acd\alpha^d bac. \end{array}$$

Proof. Case 1. $\lambda = 0$. By virtue of [3, Theorem 2.2], we prove that $bd \in \mathcal{A}^d$ if and only if $ac \in \mathcal{A}^d$. Additionally, we have $(ac)^d = a[(bd)^d]^2c$.

Case 2. $\lambda \neq 0$. Set $a' = \lambda^{-1}a$ and $b' = \lambda^{-1}b$. Then

$$(a'c)^2a' = a'cdb'a' = db'a'ca' = (db')^2a';$$
$$(a'c)^2d = a'cdb'd = db'a'cd = (db')^2d.$$

In view of Theorem 2.2, $1 - b'd \in \mathcal{A}^d$ if and only if $1 - a'c \in \mathcal{A}^d$. Obviously, we have

$$\begin{array}{rcl} \lambda - bd & = & \lambda[1 - (\lambda^{-1}b)d], \\ \lambda - ac & = & \lambda[1 - (\lambda^{-1}a)c]. \end{array}$$

Therefore $\lambda - bd \in \mathcal{A}^d$ if and only if $\lambda - ac \in \mathcal{A}^d$.

Further, we prove that

$$\begin{array}{rcl} (\lambda - ac)^d & = & \lambda^{-1}(1 - a'c)^d \\ & = & \lambda^{-1}(\beta' + 1 - e)^{-1}e, e = (y\beta')^4 + 4(y\beta')^3(1 - y\beta'), \\ y & = & \Big[1 - a'cd(\alpha')^{\pi}(1 - \alpha'(\alpha')^{\pi}(1 + b'd + b'db'd))^{-1}b'a'c\Big] \\ & & (1 + a'c + a'ca'c) + a'cd(\alpha')^d b'a'c, \end{array}$$

where $\alpha' = 1 - b'd$ and $\beta' = 1 - a'c$.

Set $\alpha = \lambda - bd$ and $\beta = \lambda - ac$. Then $\alpha' = \lambda^{-1}\alpha$ and $\beta' = \lambda^{-1}\beta$. Then we compute that

$$\begin{array}{rcl} (\lambda - ac)^d & = & (\beta + \lambda - \lambda e)^{-1}e, \\ e & = & 4(y\beta')^3 - 3(y\beta')^4 \\ & = & 4\lambda^{-3}(y\beta)^3 - 3\lambda^{-4}(y\beta)^4, \\ y & = & \Big[1 - a'cd\alpha^{\pi}(1 - \alpha'\alpha^{\pi}(1 + b'd + b'db'd))^{-1}b'a'c\Big] \\ & & (1 + a'c + a'ca'c) + a'cd(\alpha')^d b'a'c \\ & = & \lambda^{-2}\Big[1 - acd\alpha^{\pi}(\lambda^3 - \alpha\alpha^{\pi}(\lambda^2 + \lambda bd + bdbd))^{-1}bac\Big] \\ & & (\lambda^2 + \lambda ac + acac) + \lambda^{-2}acd\alpha^d bac. \end{array}$$

This completes the proof. □

The g-Drazin spectrum is defined by

$$\sigma_d(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \mathcal{A}^d\}.$$

Corollary 2.5. *Let $A, B, C, D \in \mathcal{A}$ such that*

$$(AC)^2A = ACDBA = DBACA = (DB)^2A;$$
$$(AC)^2D = ACDBD = DBACD = (DB)^2D.$$

then $\sigma_d(BD) = \sigma_d(AC)$.

Proof. This corollary is a direct sequence of Theorem 2.4. □

3. extended Jacobson's lemma for Drazin inverse

The aim of this section is to investigate the generalzied Jacobson's lemma for Drazin inverse. We now derive

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

Then $\alpha = 1 - bd \in \mathcal{A}^D$ if and only if $\beta = 1 - ac \in \mathcal{A}^D$. In this case,

$$\beta^D = [1 - acd\alpha^\pi rbacac][1 + ac + (ac)^2 + (ac)^3] + acd\alpha^D bacac,$$

where $r = \sum\limits_{j=0}^{n-1}[1-(bd)^4]^j$ and $n = i(\alpha)$.

Proof. $\Longrightarrow$ Let $y = 1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac$. Then we check that

$$\begin{aligned} y\beta &= 1 - acacacac + acd\alpha^D bacac(1 - ac) \\ &= 1 - (acdbac)ac + acd\alpha^D(1 - bd)bacac \\ &= 1 - acd(1 - \alpha\alpha^D)bacac \\ &= 1 - acd\alpha^\pi bacac, \\ \beta y &= 1 - acacacac + (1 - ac)acd\alpha^D bacac \\ &= 1 - acdbacac + acd\alpha^D(1 - bd)bacac \\ &= 1 - acd(1 - \alpha\alpha^D)bacac \\ &= 1 - acd\alpha^\pi bacac, \end{aligned}$$

Therefore $y\beta = \beta y$. Moreover, $1 - y\beta = acd\alpha^\pi bacac$.

Hence, $(1 - y\beta)\beta = acd\alpha^\pi bacac(1 - ac) = acd\alpha^\pi \alpha bacac$. Set $n = i(\alpha)$. Then $\alpha^\pi\alpha^n = \alpha^n - \alpha^D\alpha^{n+1} = 0$. By induction, we have $(1 - y\beta)\beta^n = acd\alpha^\pi\alpha^n bacac = 0$; hence, $\beta^n = \beta^{n+1}y$. This implies that β has Drazin inverse. Moreover, $\beta^D = \beta^n y^{n+1}$. We check that

$$[1 - acd\alpha^\pi bacac]acd\alpha^D bacac = acd\alpha^D bacac.$$

Thus, $[1 - acd\alpha^\pi bacac]^n acd\alpha^D bacac = acd\alpha^D bacac$. Accordingly, we compute that

$$\begin{array}{rcl} \beta^D & = & (\beta y)^n y \\ & = & [1 - acd\alpha^\pi bacac]^n[1 + ac + (ac)^2 + (ac)^3] \\ & + & [1 - acd\alpha^\pi bacac]^n acd\alpha^D bacac \\ & = & [1 - acd\alpha^\pi bacac]^n[1 + ac + (ac)^2 + (ac)^3] + acd\alpha^D bacac. \end{array}$$

We compute that

$$
\begin{aligned}
&[1 - acd\alpha^{\pi} bacac]^2\\
&= 1 - acd\alpha^{\pi} bacac - acd\alpha^{\pi} bacac + acd\alpha^{\pi} b(acaca)cd\alpha^{\pi} bacac\\
&= 1 - acd\alpha^{\pi} bacac - acd\alpha^{\pi} bacac + acd\alpha^{\pi} bdb(dbacd)bacac\\
&= 1 - acd\alpha^{\pi} bacac - acd\alpha^{\pi} bacac + acd\alpha^{\pi} bdb(dbdbd)bacac\\
&= 1 - acd\alpha^{\pi}[1 + (1 - (bd)^4)]bacac,\\
&[1 - acd\alpha^{\pi} bacac]^3\\
&= [1 - acd\alpha^{\pi}[1 + (1 - (bd)^4)]bacac][1 - acd\alpha^{\pi} bacac]\\
&= 1 - acd\alpha^{\pi}[1 + (1 - (bd)^4)]bacac\\
&\quad - acd\alpha^{\pi}[1 - [1 + (1 - (bd)^4)]bacacacd\alpha^{\pi}]bacac\\
&= 1 - acd\alpha^{\pi}[1 + (1 - (bd)^4)]bacac\\
&\quad - acd\alpha^{\pi}[1 - [1 + (1 - (bd)^4)](bd)^4\alpha^{\pi}]bacac\\
&= 1 - acd\alpha^{\pi}[1 + (1 - (bd)^4)]bacac\\
&\quad - acd\alpha^{\pi}[1 - (bd)^4 - (1 - (bd)^4)(bd)^4]bacac\\
&= 1 - acd\alpha^{\pi}[1 + (1 - (bd)^4) + (1 - (bd)^4)^2]bacac.
\end{aligned}
$$

By induction, we have

$$[1 - acd\alpha^{\pi} bacac]^n = 1 - acd\alpha^{\pi} rbacac,$$

where $r = \sum_{j=0}^{n-1}[1 - (bd)^4]^j$. Therefore we get

$$
\begin{aligned}
\beta^D &= (\beta y)^n y\\
&= [1 - acd\alpha^{\pi} rbacac][1 + ac + (ac)^2 + (ac)^3] + acd\alpha^D bacac,
\end{aligned}
$$

as desired.

$\Longleftarrow$ This is symmetric. □

As a consequence of Theorem 3.1, we have

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$
\begin{aligned}
(ac)^2 a = acdba = dbaca = (db)^2 a;\\
(ac)^2 d = acdbd = dbacd = (db)^2 d.
\end{aligned}
$$

Then $\alpha = 1 - bd \in \mathcal{A}^{\#}$ if and only if $\beta = 1 - ac \in \mathcal{A}^{\#}$. In this case,

$$\beta^{\#} = [1 - acd\alpha^{\pi} bacac][1 + ac + (ac)^2 + (ac)^3] + acd\alpha^{\#} bacac.$$

Proof. Suppose that $\alpha = 1 - bd \in \mathcal{A}^{\#}$. Then $\alpha \in \mathcal{A}^D$. In view of Theorem 3.1, $\beta^D = [1 - acd\alpha^{\pi} bacac][1 + ac + (ac)^2 + (ac)^3] + acd\alpha^{\#} bacac$. Then $\beta\beta^D = \beta^D\beta$ and $\beta^D = \beta^D\beta\beta^D$. Set $y = 1 + ac + (ac)^2 + (ac)^3 + acd\alpha^{\#} bacac$. Moreover, we have $\beta^D = \beta y^2$ and $\beta = \beta^2 y$. Then

$$\beta\beta^D\beta = \beta^2(\beta y^2) = (\beta^2 y)(\beta y) = \beta^2 y = \beta.$$

Therefore $\beta^{\#} = \beta^D$, as desired.

The proof of the opposite implication is similar to the above. □

Example 3.3.

Let $\mathcal{A} = \mathbb{C}^{3\times 3}$. Choose

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}.$$

Then we check that

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

But $(ac)^2a \neq a(ba)^2$ and $aca \neq dba$.

4. generalized Jacobson's lemma in Banach *-algebras

The purpose of this section is to establish generalized Jacobson's lemma for core and p-core inverses in a Banach *-algebra. For future use, we now record the following.

Lemma 4.1. *(see [13, Theorem 3.3]) Let $\mathcal{A}$ be a C^*-algebra, and let $\alpha \in \mathcal{A}$. Then $\alpha \in \mathcal{A}^{\textcircled{D}}$ if and only if*

(1) $\alpha \in \mathcal{A}^D$;
(2) $\alpha^\pi \in \mathcal{A}^{\{1,4\}}$.

In this case,

$$\alpha^{\textcircled{D}} = \alpha^D[1 - (\alpha^\pi)^{\{1,4\}}\alpha^\pi].$$

We have accumulated all the information necessary to prove the following.

Theorem 4.2. *Let $\mathcal{A}$ be a C^*-algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

If $\alpha = 1 - bd \in \mathcal{A}^{\textcircled{D}}$, then the following are equivalent:

(1) $\beta = 1 - ac \in \mathcal{A}^{\textcircled{D}}$.
(2) $acd\alpha^\pi bacac \in \mathcal{A}^{\{1,4\}}$.

In this case,

$$\begin{aligned} \beta^{\textcircled{D}} &= [1 - acd\alpha^\pi rbacac][1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac] \\ &\quad [1 - (acd\alpha^\pi bacac)^{\{1,4\}} acd\alpha^\pi bacac], \end{aligned}$$

where $r = \sum\limits_{j=0}^{n-1}[1 - (bd)^4]^j$ and $n = i(\alpha)$.

Proof. Since $\alpha = 1 - bd \in \mathcal{A}^{\textcircled{D}}$, we have $\alpha \in \mathcal{A}^D$. In view of Theorem 3.1, $\beta \in \mathcal{A}^D$ and

$$\beta^D = [1 - acd\alpha^\pi rbacac][1 + ac + (ac)^2 + (ac)^3] + acd\alpha^D bacac,$$

where $r = \sum\limits_{j=0}^{n-1}[1 - (bd)^4]^j$. Then

$$\begin{aligned} \beta^\pi &= 1 - \beta\beta^D \\ &= 1 - \beta(\beta^n y^{n+1}) \\ &= 1 - (\beta y)^{n+1} \\ &= 1 - \beta y \\ &= acd\alpha^\pi bacac. \end{aligned}$$

According to Lemma 4.1, $\beta \in \mathcal{A}^{\text{Ⓓ}}$ if and only if $acd\alpha^{\pi}bacac \in \mathcal{A}^{\{1,4\}}$. We compute that

$$\begin{aligned}\beta^{D} &= [1 - acd\alpha^{\pi}rbacac][1 + ac + (ac)^2 + (ac)^3] + acd\alpha^{D}bacac \\ &= [1 - acd\alpha^{\pi}rbacac][1 + ac + (ac)^2 + (ac)^3 + acd\alpha^{D}bacac].\end{aligned}$$

In this case, we have

$$\begin{aligned}\beta^{\text{Ⓓ}} &= \beta^{D}[1 - (\beta^{\pi})^{\{1,4\}}\beta^{\pi}] \\ &= \beta^{D}[1 - (acd\alpha^{\pi}bacac)^{\{1,4\}}acd\alpha^{\pi}bacac] \\ &= [1 - acd\alpha^{\pi}rbacac][1 + ac + (ac)^2 + (ac)^3 + acd\alpha^{D}bacac] \\ &\quad [1 - (acd\alpha^{\pi}bacac)^{\{1,4\}}acd\alpha^{\pi}bacac].\end{aligned}$$

This completes the proof. □

As consequences of Theorem 4.2, we derive

Corollary 4.3. *Let $\mathcal{A}$ be a C^*-algebra, and let $a, b, c \in \mathcal{A}$ satisfying*

$$(ac)^2a = acaba = abaca = a(ba)^2.$$

Then $\alpha = 1 - ba \in \mathcal{A}^{\text{Ⓓ}}$ if and only if

(1) $\beta = 1 - ac \in \mathcal{A}^{\text{Ⓓ}}$;
(2) $aca\alpha^{\pi}bacac \in \mathcal{A}^{\{1,4\}}$.

In this case,

$$\begin{aligned}\beta^{\text{Ⓓ}} &= [1 - aca\alpha^{\pi}rbacac][1 + ac + (ac)^2 + (ac)^3 + aca\alpha^{D}bacac] \\ &\quad [1 - (aca\alpha^{\pi}bacac)^{\{1,4\}}aca\alpha^{\pi}bacac].\end{aligned}$$

where $r = \sum_{j=0}^{n-1}[1 - (ba)^4]^j$ and $n = i(\alpha)$.

Corollary 4.4. *Let $\mathcal{A}$ be a C^*-algebra, and let $a, b, c \in \mathcal{A}$ satisfying*

$$aca = dba, acd = dbd.$$

Then $\alpha = 1 - bd \in \mathcal{A}^{\text{Ⓓ}}$ if and only if

(1) $\beta = 1 - ac \in \mathcal{A}^{\text{Ⓓ}}$;
(2) $acd\alpha^{\pi}bacac \in \mathcal{A}^{\{1,4\}}$.

In this case,

$$\begin{aligned}\beta^{\text{Ⓓ}} &= [1 - acd\alpha^{\pi}rbacac][1 + ac + (ac)^2 + (ac)^3 + acd\alpha^{D}bacac] \\ &\quad [1 - (acd\alpha^{\pi}bacac)^{\{1,4\}}acd\alpha^{\pi}bacac].\end{aligned}$$

For the core invertibility, we are ready to prove:

Theorem 4.5. *Let $\mathcal{A}$ be a C^*-algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$\begin{aligned}(ac)^2a = acdba = dbaca = (db)^2a; \\ (ac)^2d = acdbd = dbacd = (db)^2d.\end{aligned}$$

If $\alpha = 1 - bd \in \mathcal{A}^{\text{#}}$, then the following are equivalent:

(1) $\beta = 1 - ac \in \mathcal{A}^{\text{#}}$;
(2) $acd\alpha^{\pi}bacac \in \mathcal{A}^{\{1,4\}}$.

In this case,

$$\begin{aligned}\beta^{\#} &= [1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac]\\ &\quad [1 - (acd\alpha^\pi bacac)^{\{1,4\}} acd\alpha^\pi bacac].\end{aligned}$$

Proof. Since $\alpha = 1 - bd \in \mathcal{A}^{\#}$, it follows by [4, Theorem 2.5] that $\alpha \in \mathcal{A}^{\mathbb{D}}$. In view of Theorem 4.2, $\beta = 1 - ac \in \mathcal{A}^{\mathbb{D}}$. Moreover, we have

$$\begin{aligned}\beta^{\mathbb{D}} &= [1 - acd\alpha^\pi bacac][1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac]\\ &\quad [1 - (acd\alpha^\pi bacac)^{\{1,4\}} acd\alpha^\pi bacac].\end{aligned}$$

In view of Corollary 3.2, $\beta \in \mathcal{A}^{\#}$ and so $\beta \in \mathcal{A}^{\#}$.

We easily check that

$$\begin{aligned}acd\alpha^\pi bacac(ac) &= acd\alpha^\pi bdbdbac\\ &= acd\alpha^\pi b(dbdba)c\\ &= acd\alpha^\pi b(dbaca)c\\ &= (ac)acd\alpha^\pi bacac,\\ acd\alpha^\pi bacac(acd\alpha^D bacac) &= acd\alpha^\pi b(acaca)cd\alpha^D bacac\\ &= acd\alpha^\pi b(dbdba)cd\alpha^D bacac\\ &= acd\alpha^\pi bdbdbdbd\alpha^D bacac\\ &= 0.\end{aligned}$$

Likewise, we have

$$\begin{aligned}(acd\alpha^D bacac)acd\alpha^\pi bacac &= acd\alpha^D bdbdbacd\alpha^\pi bacac\\ &= acd\alpha^D bdbdbdbd\alpha^\pi bacac\\ &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}&[1 - acd\alpha^\pi bacac][1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac]\\ =\ &[1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac][1 - acd\alpha^\pi bacac].\end{aligned}$$

Accordingly,

$$\begin{aligned}\beta^{\#} &= \beta^{\mathbb{D}}\\ &= [1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac]\\ &\quad [1 - acd\alpha^\pi bacac][1 - (acd\alpha^\pi bacac)^{\{1,4\}} acd\alpha^\pi bacac]\\ &= [1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac]\\ &\quad [1 - (acd\alpha^\pi bacac)^{\{1,4\}} acd\alpha^\pi bacac].\end{aligned}$$

This completes the proof. □

Corollary 4.6. *Let $\mathcal{A}$ be a C^*-algebra, and let $a, b, c \in \mathcal{A}$ satisfying*

$$(ac)^2 a = acaba = abaca = a(ba)^2.$$

Then $\alpha = 1 - ab \in \mathcal{A}^{\#}$ if and only if

(1) $\beta = 1 - ac \in \mathcal{A}^{\#}$;
(2) $acd\alpha^\pi bacac \in \mathcal{A}^{\{1,4\}}$.

In this case,

$$\begin{aligned}\beta^{\#} &= [1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac]\\ &\quad [1 - (acd\alpha^\pi bacac)^{\{1,4\}} acd\alpha^\pi bacac].\end{aligned}$$

Corollary 4.7. *Let $\mathcal{A}$ be a C^*-algebra, and let $a, b, c \in \mathcal{A}$ satisfying*

$$aca = dba, acd = dbd.$$

Then $\alpha = 1 - ba \in \mathcal{A}^{\#}$ if and only if

(1) $\beta = 1 - ac \in \mathcal{A}^{\textcircled{\#}}$;
(2) $acd\alpha^{\pi}bacac \in \mathcal{A}^{\{1,4\}}$.

In this case,

$$\beta^{\textcircled{\#}} = [1 + ac + (ac)^2 + (ac)^3 + acd\alpha^D bacac][1 - (acd\alpha^{\pi}bacac)^{\{1,4\}}acd\alpha^{\pi}bacac].$$

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