Filomat 38:13 (2024), 4649–4662
https://doi.org/10.2298/FIL2413649Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# GSEP elements in a ring with involution

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**Abstract.** In this paper, we characterize SEP elements and GSEP elements by various methods. SEP elements are mainly characterized by a element in a given set, and then extended to GSEP elements. GSEP elements are characterized by (b,c)-inverse, inner invertible elements and equations. We obtain a lot of new characterizations of SEP elements and GSEP elements.

## 1. Introduction

Generalized inverses of rings plays an important role in solving equations with one or two variables [1–5]. Therefore, generalized inverses of rings have important applications in many fields, such as mathematical statistics, system theory, optimization theory, modern control theory, and so on. Now more and more people explore the generalized inverses of rings [6–17]. In recent years, with the help of the expression of the solution of the generalized inverse equation in an involution ring, people have adopted some new methods to characterize EP elements, SEP elements and normal elements. Many new characterization of EP elements and SEP elements are obtained [18–27].

In this paper, we introduce a new kind of generalized inverse, so-called GSEP, which is between EP and SEP. The goal of this paper is to give some new characterizations of SEP elements and GSEP elements.

Let $R$ be a ring and $* : R \to R$ be a map satisfying

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^* \text{ for } a,\ b \in R.$$

Then $R$ is called an involution ring or a $*-$ring .

Let $R$ be a ring and $a \in R$. Then $a$ is called the group invertible element if there exists $a^{\#} \in R$ such that

$$a = aa^{\#}a,\ a^{\#} = a^{\#}aa^{\#},\ aa^{\#} = a^{\#}a.$$

We usually write $R^{\#}$ to denote the set of all group invertible elements in $R$.

It is well known that $a \in R^{\#}$ if and only if $a \in a^2R \cap Ra^2$.

Let $R$ be a $*-$ring and $a \in R$. Then $a$ is called the Moore-Penrose invertible element if there exists $a^+ \in R$ such that

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a.$$

2020 *Mathematics Subject Classification*. 15A03, 15A09, 15A24, 15B57, 16W10.
*Keywords*. SEP element, GSEP element, (b,c)-inverse, inner invertible element, equation.
Received: 09 March 2023; Revised: 24 December 2023; Accepted: 29 December 2023
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (11901510, 11761017), Natural Science Foundation of Jiangsu Province (BK20200944, BK20170589).
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We always use $R^+$ to denote the set of all Moore-Penrose invertible elements in $R$.

Noting that if $a^\#$ and $a^+$ exist, then they are all unique.

Let $a \in R^\# \cap R^+$, if $a^\# = a^+$, then $a$ is called an EP element.

We denote the set of all EP elements in $R$ by $R^{EP}$.

If $a \in R$ and $a = aa^*a$, then $a$ is said to be the PI element and we use $R^{PI}$ to denote the set of all PI elements in $R$.

If $a \in R^{EP} \cap R^{PI}$, then $a$ is said to be the strong EP element. We used to write $R^{SEP}$ to represent the set of all strong EP elements in $R$.

Let $R$ be a ring and $a \in R$. Then $a$ is called projection, if $a^2 = a = a^*$. We always use $PE(R)$ to denote the set of all projections in $R$.

Clearly, $aa^+, a^+a \in PE(R)$.

Let $a, b, c \in R$. If there exists $y \in R$, such that

$$y \in bRy \cap yRc, \ \ b = yab, \ \ c = cay,$$

then $a$ is called a $(b, c)-$invertible, $y$ is unique when it exists and $y$ is called the $(b, c)-$inverse of $a$ and write it by $a^{\|(b,c)}$, that is $y = a^{\|(b,c)}$.

Let $a \in R^\# \cap R^+$, we take $\chi_a = \{a, a^\#, a^+, a^*, (a^+)^*, (a^\#)^*\}$, $\tau_a = \{a, a^\#, (a^+)^*\}$ and $\gamma_a = \{a^+, a^*, (a^\#)^*\}$. Cleraly, $\chi_a = \tau_a \cup \gamma_a$.

An element $a \in R$ is regular if there exists some $b \in R$ satisfying $aba = a$. In this case $b$ is called an inner inverse of $a$.

## 2. Characterizing SEP elements by projections

It is well known that $a \in R^{SEP}$ if and only if $a^\# = a^+$ and $a = (a^+)^*$. This implies $a^\#(a^+)^* = a^+a \in PE(R)$. Hence, we have the following theorem.

**Theorem 2.1.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^\#(a^+)^* \in PE(R)$.*

*Proof.* We only need to show the sufficiency: From the hypothesis, we have

$$a^\#(a^+)^* = (a^\#(a^+)^*)^2 = (a^\#(a^+)^*)^*.$$

This induces

$$a^\#(a^+)^* = a^+(a^\#)^* = a^+(a^\#)^*aa^+ = a^\#(a^+)^*aa^+,$$

and

$$(a^+)^* = aa^\#(a^+)^* = aa^\#(a^+)^*aa^+ = (a^+)^*aa^+.$$

Applying the involution on the equality, one gets $a^+ = aa^+a^+$. Hence, $a \in R^{EP}$ by [11, Theorem 1.2.1]. Now we have

$$(a^+)^* = aa^\#(a^+)^* = a(a^\#(a^+)^*)^2 = (a^+)^*a^\#(a^+)^*,$$

and

$$a = (a^+)^*a^*a = (a^+)^*a^\#(a^+)^*a^*a = (a^+)^*a^\#aa^+a = (a^+)^*a^\#a = (a^+)^*,$$

one obtains $a \in R^{PI}$. Thus $a \in R^{SEP}$. □

It is well known that $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$. Hence, Theorem 2.1 implies the following corollary.

**Corollary 2.2.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $(a^\#)^*a^+ \in PE(R)$.*

It is easy to show that $a \in PE(R)$ if and only if $a^* \in PE(R)$. Hence, Theorem 2.1 induces the following corollary.

**Corollary 2.3.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+(a^\#)^* \in PE(R)$.*

**Lemma 2.4.** *Let $a \in R^{\#}$. If $a \in PE(R)$, then $a^{\#} \in PE(R)$.*

*Proof.* Since $a \in PE(R)$, $a^2 = a = a^*$. This gives

$$(a^{\#})^2 = a(a^{\#})^3 = a^2(a^{\#})^3 = a^{\#},$$

and

$$(a^{\#})^* = (a^*)^{\#} = a^{\#}.$$

Hence, $a^{\#} \in PE(R)$. □

**Lemma 2.5.** *Let $a \in R^{\#} \cap R^{+}$. Then*
*(1) $(a^{\#}(a^{+})^*)^{\#} = aa^{\#}a^*a$;*
*(2) $(a^{\#}(a^{+})^*)^{+} = a^*a^2a^{+}$.*

*Proof.* (1) Since $(a^{\#}(a^{+})^*)(aa^{\#}a^*a) = a^{\#}(a^{+})^*a^*a = a^{\#}aa^{+}a = a^{\#}a$,

$$(a^{\#}(a^{+})^*)(aa^{\#}a^*a)(a^{\#}(a^{+})^*) = a^{\#}a(a^{\#}(a^{+})^*) = a^{\#}(a^{+})^*.$$

Noting that

$$(aa^{\#}a^*a)(a^{\#}(a^{+})^*) = aa^{\#}a^*(a^{+})^* = aa^{\#}a^{+}a = aa^{\#}.$$

Then

$$(aa^{\#}a^*a)(a^{\#}(a^{+})^*)(aa^{\#}a^*a) = aa^{\#}(aa^{\#}a^*a) = aa^{\#}a^*a.$$

Thus $a^{\#}(a^{+})^* \in R^{\#}$ and $(a^{\#}(a^{+})^*)^{\#} = aa^{\#}a^*a$.

(2) Since $(a^{\#}(a^{+})^*)(a^*a^2a^{+}) = a^{\#}aa^{+}a^2a^{+} = aa^{+}$,

$$(a^{\#}(a^{+})^*)(a^*a^2a^{+})(a^{\#}(a^{+})^*) = aa^{+}(a^{\#}(a^{+})^*) = a^{\#}(a^{+})^*.$$

Also

$$(a^*a^2a^{+})(a^{\#}(a^{+})^*)(a^*a^2a^{+}) = (a^*a^2a^{+})aa^{+} = a^*a^2a^{+},$$

and

$$(a^*a^2a^{+})(a^{\#}(a^{+})^*) = a^*aa^{\#}(a^{+})^* = a^*(a^{+})^* = a^{+}a.$$

Hence, $a^{\#}(a^{+})^* \in R^{+}$ and $(a^{\#}(a^{+})^*)^{+} = a^*a^2a^{+}$. □

Lemma 2.4, Lemma 2.5 and Theorem 2.1 lead to the following theorem.

**Theorem 2.6.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^{\#}a^*a \in PE(R)$.*

**Lemma 2.7.** *Let $a \in R^{\#} \cap R^{+}$. If $a \in PE(R)$, then $a \in R^{EP}$.*

*Proof.* It is an immediate result of [11, Theorem 1.4.1], because $a$ is Hermitian. □

**Theorem 2.8.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^*a^2a^{+} \in PE(R)$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{SEP}$, then $a^{\#}(a^{+})^* \in PE(R)$ by Theorem 2.1. And we can get $(a^{\#}(a^{+})^*)^{\#} \in PE(R)$ by Lemma 2.4. Noting that $a^{\#}(a^{+})^* \in PE(R)$, then $a^{\#}(a^{+})^* \in R^{EP}$ by Lemma 2.7. Thus $(a^{\#}(a^{+})^*)^{+} = (a^{\#}(a^{+})^*)^{\#} \in PE(R)$. By Lemma 2.5, $a^*a^2a^{+} \in PE(R)$.

$\Longleftarrow$ From the hypothesis, we have $a^*a^2a^{+} \in PE(R)$. By Lemma 2.7, $a^*a^2a^{+} \in R^{EP}$, then we have $(a^*a^2a^{+})^{+} = (a^*a^2a^{+})^{\#}$. Noting that $a^*a^2a^{+} \in PE(R)$, then $(a^*a^2a^{+})^{\#} \in PE(R)$ by Lemma 2.4. By Lemma 2.5, we have $a^{\#}(a^{+})^* = (a^*a^2a^{+})^{+} = (a^*a^2a^{+})^{\#} \in PE(R)$. Hence, $a \in R^{SEP}$ by Theorem 2.1. □

## 3. Generalized SEP elements

**Theorem 3.1.** *Let $a \in R^{\#} \cap R^{+}$. Then $a^{\#}(a^{+})^{*} \in R^{SEP}$ if and only if $(a^{\#})^{*} = aa^{*}a$.*

*Proof.* $\Longrightarrow$ Assume that $a^{\#}(a^{+})^{*} \in R^{SEP}$. Then $(a^{\#}(a^{+})^{*})^{\#} = (a^{\#}(a^{+})^{*})^{*}$. By Lemma 2.5, $aa^{\#}a^{*}a = a^{+}(a^{\#})^{*}$. Multiplying the equality on the left by $aa^{\#}$, one gets

$$a^{+}(a^{\#})^{*} = aa^{\#}a^{+}(a^{\#})^{*}.$$

This gives

$$a^{+} = a^{+}(a^{\#})^{*}a^{*} = aa^{\#}a^{+}(a^{\#})^{*}a^{*} = aa^{\#}a^{+}.$$

Multiplying the equality on the right by $a$, one gets

$$a^{+}a = aa^{\#}a^{+}a = a^{\#}aa^{+}a = a^{\#}a.$$

Hence, $a \in R^{EP}$ by [12, Theorem 1.2].

It follows that $aa^{*}a = a(aa^{\#}a^{*}a) = aa^{+}(a^{\#})^{*} = aa^{+}(a^{+})^{*} = (a^{+})^{*} = (a^{\#})^{*}$.

$\Longleftarrow$ From the hypothesis, we have $(a^{\#})^{*} = aa^{*}a$. Multiplying the equality on the left by $a^{+}$, one gets

$$a^{+}(a^{\#})^{*} = a^{+}aa^{*}a = a^{*}a.$$

Applying the involution on the equality, one gets $a^{\#}(a^{+})^{*} = a^{*}a$. By Lemma 2.5,

$$(a^{\#}(a^{+})^{*})^{+} = a^{*}a^{2}a^{+} = a^{+}(aa^{*}a)aa^{+} = a^{+}(a^{\#})^{*}aa^{+} = a^{+}(a^{\#})^{*} = a^{*}a.$$

$$(a^{\#}(a^{+})^{*})^{*} = a^{+}(a^{\#})^{*} = a^{*}a.$$

And

$$(a^{\#}(a^{+})^{*})(a^{*}a) = a^{*}aa^{*}a = (a^{*}a)(a^{\#}(a^{+})^{*}).$$

One gets $(a^{\#}(a^{+})^{*})^{\#} = a^{*}a$. Hence

$$(a^{\#}(a^{+})^{*})^{+} = (a^{\#}(a^{+})^{*})^{*} = (a^{\#}(a^{+})^{*})^{\#}.$$

Thus, $a^{\#}(a^{+})^{*} \in R^{SEP}$. □

Let $a \in R^{\#} \cap R^{+}$. If $(a^{\#})^{*} = aa^{*}a$, then $a$ is called a generalized SEP element.

Noting that if $a \in R^{SEP}$, then $(a^{\#})^{*} = a = aa^{*}a$. So SEP elements are generalized SEP.

By Theorem 3.1, we have $a$ is a generalized SEP element if and only if $a^{\#}(a^{+})^{*} \in R^{SEP}$.

We write $R^{GSEP}$ to denote the set of all generalized SEP elements of $R$. Hence, $R^{SEP} \subseteq R^{GSEP}$.

Let $R = M_2(\mathbb{Z}_5)$ with the transposition involution $*$. Choose $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $a^{\#} = a^{+} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ and $a^{*} = a \neq a^{\#}$. It follows that $aa^{*}a = a^{3} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = a^{\#} = (a^{\#})^{*}$. Hence, $a \in R^{GSEP}$, but $a \notin R^{SEP}$. Thus the generalized SEP elements are proper generalization of SEP.

Following from the following Theorem 3.2, we have $R^{GSEP} \subseteq R^{EP}$. We claim that $R^{GSEP} \subsetneqq R^{EP}$.

In fact, choose $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$. Then $a^{\#} = a^{+} = a^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, so $a \in R^{EP}$. However, $aa^{*}a = \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = (a^{\#})^{*}$. Hence, $a \notin R^{GSEP}$.

Therefore we have $R^{SEP} \subsetneqq R^{GSEP} \subsetneqq R^{EP}$.

**Theorem 3.2.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if $a \in R^{EP}$ and $a^{*} = a^{+}(a^{\#})^{*}a^{+}$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $(a^{\#})^* = aa^*a = (aa^*a)a^+a = (a^{\#})^*a^+a$, this implies $a^{\#} = a^+aa^{\#}$. Multiplying the equality on the right by $a$, one gets

$$a^{\#}a = a^+aa^{\#}a = a^+a.$$

Hence, $a \in R^{EP}$ by [12, Theorem 1.2]. Now $a^+(a^{\#})^*a^+ = a^+(aa^*a)a^+ = a^*$.

$\Longleftarrow$ Since $a \in R^{EP}$ and $a^* = a^+(a^{\#})^*a^+$, $aa^+ = a^+a$ and $aa^*a = aa^+(a^{\#})^*a^+a = a^+a(a^{\#})^*aa^+ = (a^{\#})^*$. Hence, $a \in R^{GSEP}$. □

**Lemma 3.3.** *Let $a \in R^{\#} \cap R^+$. Then the followings are equivalent:*
*(1) $a \in R^{GSEP}$;*
*(2) $a^* = a^{\#}(a^{\#})^*a^+$;*
*(3) $a^* = a^+(a^{\#})^*a^{\#}$.*

*Proof.* (1) $\Longrightarrow$ (2) It is an immediate result of Theorem 3.2 because $a^+ = a^{\#}$.

(2) $\Longrightarrow$ (1) Suppose that $a^* = a^{\#}(a^{\#})^*a^+$. Then

$$aa^+a^* = aa^+a^{\#}(a^{\#})^*a^+ = a^{\#}(a^{\#})^*a^+ = a^*.$$

Hence, $a \in R^{EP}$ by [11, Theorem 1.2.1], this gives $a^{\#} = a^+$. So $a^* = a^+(a^{\#})^*a^+$. By Theorem 3.2, $a \in R^{GSEP}$.

Similarly, we can show (1) $\Leftrightarrow$ (3). □

**Theorem 3.4.** *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{GSEP}$ if and only if $a^{\#}(a^{\#})^*a^+(a^+)^* \in PE(R)$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $a^* = a^{\#}(a^{\#})^*a^+$ by Lemma 3.3. This leads to $a^{\#}(a^{\#})^*a^+(a^+)^* = a^*(a^+)^* = a^+a \in PE(R)$.

$\Longleftarrow$ Form the assumption, one has

$$a^{\#}(a^{\#})^*a^+(a^+)^* = (a^{\#}(a^{\#})^*a^+(a^+)^*)^2 = (a^{\#}(a^{\#})^*a^+(a^+)^*)^*.$$

This gives

$$a^{\#}(a^{\#})^*a^+(a^+)^* = a^+(a^+)^*a^{\#}(a^{\#})^* = a^+a(a^+(a^+)^*a^{\#}(a^{\#})^*) = a^+a(a^{\#}(a^{\#})^*a^+(a^+)^*),$$

and

$$a^{\#}(a^{\#})^*a^+ = a^{\#}(a^{\#})^*a^+(a^+)^*a^* = a^+aa^{\#}(a^{\#})^*a^+(a^+)^*a^* = a^+aa^{\#}(a^{\#})^*a^+.$$

Multiplying the equality on the right by $aa^*a^+a$, one gives $a^{\#} = a^+aa^{\#}$. Hence, $a \in R^{EP}$. Now we have

$$a^{\#}(a^{\#})^*a^+ = a^{\#}(a^{\#})^*a^+(a^+)^*a^* = (a^{\#}(a^{\#})^*a^+(a^+)^*)^2a^* = a^{\#}(a^{\#})^*a^+(a^+)^*a^{\#}(a^{\#})^*a^+,$$

and

$$a^{\#}(a^{\#})^* = a^{\#}(a^{\#})^*aa^+ = a^{\#}(a^{\#})^*a^+a = a^{\#}(a^{\#})^*a^+(a^+)^*a^{\#}(a^{\#})^*a^+a = a^{\#}(a^{\#})^*a^+(a^+)^*a^{\#}(a^{\#})^*.$$

Noting that $a^+ = a^{\#}$ and $a^+(a^{\#})^*a^* = a^+$. Then

$$a^{\#} = a^+ = a^+(a^{\#})^*a^* = a^{\#}(a^{\#})^*a^* = a^{\#}(a^{\#})^*a^+(a^+)^*a^{\#}(a^{\#})^*a^* = a^{\#}(a^{\#})^*a^+(a^+)^*a^{\#},$$

$$a = aa^{\#}a = a(a^{\#}(a^{\#})^*a^+(a^+)^*a^{\#})a = aa^{\#}(a^+)^*a^+(a^+)^*a^{\#}a = (a^+)^*a^+(a^+)^*,$$

$$a^* = a^+(a^+)^*a^+ = a^+(a^{\#})^*a^+.$$

By Theorem 3.2, $a \in R^{GSEP}$. □

**Theorem 3.5.** *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{GSEP}$ if and only if $aa^* = (a^{\#})^*a^+$.*

*Proof.* $\Longrightarrow$ Since $a \in R^{GSEP}$, $a \in R^{EP}$ and $a^* = a^+(a^\#)^*a^+$ by Theorem 3.2. This gives

$$aa^* = aa^+(a^\#)^*a^+ = a^+a(a^\#)^*a^+ = (a^\#)^*a^+.$$

$\Longleftarrow$ Suppose that $aa^* = (a^\#)^*a^+$. Then

$$a^+a^2a^* = a^+a((a^\#)^*a^+) = (a^\#)^*a^+ = aa^*,$$

and

$$a^\# = a(a^\#)^2 = aa^*(a^+)^*(a^\#)^2 = a^+a^2a^*(a^+)^*(a^\#)^2 = a^+a^2(a^\#)^2 = a^+a^\#a.$$

Hence, $a \in R^{EP}$ by [13, Theorem 2.1] and $a^* = a^+aa^* = a^+(a^\#)^*a^+$. Thus $a \in R^{GSEP}$ by Theorem 3.2. □

Similarly, we have the following theorem.

**Theorem 3.6.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $a^*a = a^+(a^\#)^*$.*

## 4. Using (*b*, *c*)-inverse to characterize GSEP elements

**Theorem 4.1.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $(a^\#)^*a^+(a^+)^*$ is $(a^*, a^*)$-invertible with $((a^\#)^*a^+(a^+)^*)^{\|(a^*,a^*)} = a^\#$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $a^* = a^\#(a^\#)^*a^+ = a^+(a^\#)^*a^\#$ by Lemma 3.3 and $a \in R^{EP}$ by Theorem 3.2. Hence

$$a^\# = a^+aa^\# = a^*(a^+)^*a^\# \in a^*Ra^\#,$$

and

$$a^\# = a^\#aa^+ = a^\#(a^+)^*a^* \in a^\#Ra^*.$$

Noting that

$$a^\#((a^\#)^*a^+(a^+)^*)a^* = a^\#(a^\#)^*a^+ = a^*,$$

$$a^*((a^\#)^*a^+(a^+)^*)a^\# = (a^*(a^\#)^*a^+)((a^\#)^*a^\#) = a^+(a^\#)^*a^\# = a^*.$$

Hence, $((a^\#)^*a^+(a^+)^*)^{\|(a^*,a^*)} = a^\#$.

$\Longleftarrow$ From the assumption, we have

$$a^* = a^\#((a^\#)^*a^+(a^+)^*)a^* = a^\#(a^\#)^*a^+.$$

By Lemma 3.3, $a \in R^{GSEP}$. □

**Theorem 4.2.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $(a^\#)^*a^+(a^+)^*$ is $(a^*, x)$-invertible with $((a^\#)^*a^+(a^+)^*)^{\|(a^*,x)} = a^\#$ for some $x \in \chi_a$.*

*Proof.* $\Longrightarrow$ It is an immediate result in instead of by choosing $x = a^*$ by Theorem 4.1.

$\Longleftarrow$ From the proof of the sufficiency of Theorem 4.1, we easy know that $a \in R^{GSEP}$. □

**Theorem 4.3.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $(a^+(a^+)^*)^{\|(a^*,a^*a)}$ exists and $(a^+(a^+)^*)^{\|(a^*,a^*a)} = a^\#(a^\#)^*$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $a^* = a^\#(a^\#)^*a^+ = (a^\#(a^\#)^*)(a^+(a^+)^*)a^*$ by Lemma 3.3. Noting that

$$(a^*a)(a^+(a^+)^*)(a^\#(a^\#)^*) = a^*(a^+)^*a^\#(a^\#)^* = a^+aa^\#(a^\#)^*.$$

By Theorem 3.2, $a \in R^{EP}$, this gives

$$(a^*a)(a^+(a^+)^*)(a^\#(a^\#)^*) = a^+(a^\#)^*.$$

By Theorem 3.6, $a^*a = a^+(a^\#)^*$. Hence, $(a^*a)(a^+(a^+)^*)(a^\#(a^\#)^*) = a^*a$. Since $a \in R^{EP}$,

$$a^\#(a^\#)^* = a^*(a^+)^*a^\#(a^\#)^* \in a^*Ra^\#(a^\#)^*,$$

$$a^\#(a^\#)^* = a^\#(a^\#)^*a^+(a^+)^*a^*a \in a^\#(a^\#)^*Ra^*a.$$

Hence, $(a^+(a^+)^*)^{\|(a^*,a^*a)} = a^\#(a^\#)^*$.

$\Longleftarrow$ From the assumption, we have

$$a^* = (a^\#(a^\#)^*)(a^+(a^+)^*)a^* = a^\#(a^\#)^*a^+.$$

By Lemma 3.3, $a \in R^{GSEP}$. □

## 5. Characterizing GSEP elements by inner invertible elements

**Theorem 5.1.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $\begin{pmatrix} a^\# \\ a^+ \end{pmatrix}$ is a regular vector with an inner inverse $(a - aa^*a, (a^\#)^*)$.*

*Proof.* Clearly, $\begin{pmatrix} a^\# \\ a^+ \end{pmatrix}(a - aa^*a, (a^\#)^*)\begin{pmatrix} a^\# \\ a^+ \end{pmatrix} = \begin{pmatrix} a^\# - a^\#aa^*aa^\# + a^\#(a^\#)^*a^+ \\ a^+aa^\# - a^*aa^\# + a^+(a^\#)^*a^+ \end{pmatrix}$.

$\Longrightarrow$ Since $a \in R^{GSEP}$, $a \in R^{EP}$ by Theorem 3.2 and $a^\#(a^\#)^*a^+ = a^*$ by Lemma 3.3. This gives

$$a^\# - a^\#aa^*aa^\# + a^\#(a^\#)^*a^+ = a^\# - a^+aa^*aa^+ + a^* = a^\#,$$

$$a^+aa^\# - a^*aa^\# + a^+(a^\#)^*a^+ = a^+aa^+ - a^*aa^+ + a^\#(a^\#)^*a^+ = a^+ - a^* + a^* = a^+.$$

Hence, $\begin{pmatrix} a^\# \\ a^+ \end{pmatrix}(a - aa^*a, (a^\#)^*)\begin{pmatrix} a^\# \\ a^+ \end{pmatrix} = \begin{pmatrix} a^\# \\ a^+ \end{pmatrix}$.

$\Longleftarrow$ From the assumption, one gets $\begin{cases} a^\# = a^\# - a^\#aa^*aa^\# + a^\#(a^\#)^*a^+ & (1) \\ a^+ = a^+aa^\# - a^*aa^\# + a^+(a^\#)^*a^+ & (2) \end{cases}$.

From (1), we have

$$a^\#aa^*aa^\# = a^\#(a^\#)^*a^+ = a^\#(a^\#)^*a^+aa^+ = a^\#aa^*aa^\#aa^+ = a^\#aa^*,$$

and

$$a^* = a^+aa^* = a^+aa^\#aa^* = a^+aa^\#aa^*aa^\# = a^*aa^\#.$$

Hence, $a \in R^{EP}$ by [11, Theorem 1.1.3]. Now we obtain

$$a^* = a^\#aa^*aa^\# = a^\#(a^\#)^*a^+.$$

By Lemma 3.3, $a \in R^{GSEP}$. □

Similarly, we have the following theorem.

**Theorem 5.2.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $\begin{pmatrix} a^\# \\ a^+ \end{pmatrix}$ is a regular vector with an inner inverse $(a - (a^\#)^*, a^+a^2a^*a)$.*

**Theorem 5.3.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $a \in R^{EP}$ and $a^\#(a^\#)^*$ is a regular with an inner inverse $a^+(a^\#)^*$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $a^* = a^\#(a^\#)^*a^+$ by Lemma 3.3, and $a \in R^{EP}$ by Theorem 3.2. Now we have

$$(a^\#(a^\#)^*)(a^+(a^\#)^*)(a^\#(a^\#)^*) = (a^\#(a^\#)^*a^+)(a^\#)^*a^+(a^\#)^* = a^*(a^\#)^*a^+(a^\#)^* = a^+(a^\#)^* = a^\#(a^\#)^*.$$

Hence, $a^+(a^\#)^*$ is an inner inverse of $a^\#(a^\#)^*$.

$\Longleftarrow$ From the assumption, we have

$$a^\#(a^\#)^* = (a^\#(a^\#)^*)(a^+(a^\#)^*)(a^\#(a^\#)^*).$$

Multiplying the equality on the left by $a^+a^2$, one gets

$$(a^\#)^* = (a^\#)^*a^+(a^\#)^*a^\#(a^\#)^*.$$

Multiplying the equality on the right by $a^*$ and remind in heart that $a \in R^{EP}$, one obtains

$$aa^\# = (aa^\#)^* = (a^\#)^*a^+(a^\#)^*a^\#(a^\#)^*a^* = (a^\#)^*a^+(a^\#)^*a^+(aa^\#)^* = (a^\#)^*a^+(a^\#)^*a^+.$$

This gives

$$a^* = a^*aa^+ = a^*aa^\# = a^*(a^\#)^*a^+(a^\#)^*a^+ = a^+(a^\#)^*a^+,$$

and

$$a^*a = a^+(a^\#)^*a^+a = a^+(a^+)^*a^+a = a^+(a^+)^* = a^+(a^\#)^*.$$

By Theorem 3.6, $a \in R^{GSEP}$. □

Let $a \in R$, if there exists $b \in R$ such that $a = ba^2$, then $a$ is called a left strongly regular and $b$ is called a left strongly regular inverse of $a$.

**Theorem 5.4.** *Let $a \in R^\# \cap R^+$. Then $a \in R^{GSEP}$ if and only if $a^\#(a^\#)^*$ is a left strongly regular with a left strongly regular inverse $a^+(a^\#)^*$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $a^* = a^\#(a^\#)^*a^+$ and $a^\# = a^+$. This gives

$$(a^+(a^\#)^*)(a^\#(a^\#)^*)^2 = (a^\#(a^\#)^*a^+)(a^\#)^*a^+(a^\#)^* = a^*(a^\#)^*a^+(a^\#)^* = a^+(a^\#)^* = a^\#(a^\#)^*.$$

$\Longleftarrow$ From the hypothesis, one gets

$$a^\#(a^\#)^* = a^+(a^\#)^*(a^\#(a^\#)^*)^2.$$

Multiplying the equality on the right by $a^*a^+a$, one gives

$$a^\# = a^+(a^\#)^*a^\#(a^\#)^*a^\#.$$

This leads to

$$aa^+ = a^\#a^2a^+ = a^+(a^\#)^*a^\#(a^\#)^*a^\#a^2a^+ = a^+(a^\#)^*a^\#(a^\#)^* = a^+a(a^+(a^\#)^*a^\#(a^\#)^*) = a^+a^2a^+,$$

and

$$a = aa^+a = a^+a^2a^+a = a^+a^2.$$

Hence, $a \in R^{EP}$, it follows that

$$a^* = aa^+a^* = a^+(a^\#)^*a^\#(a^\#)^*a^* = a^+(a^\#)^*a^+(a^\#)^*a^* = a^+(a^\#)^*a^+,$$

and

$$a^*a = a^+(a^\#)^*a^+a = a^+(a^\#)^*aa^+ = a^+(a^\#)^*.$$

By Theorem 3.6, $a \in R^{GSEP}$. □

## 6. Constructing invertible elements to characterize GSEP elements

**Theorem 6.1.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if there exists invertible element $u$ such that $a^* = a^{\#}(a^{\#})^* u$ and $ua^{+}a = a^{+}$.*

*Proof.* $\Longrightarrow$ Suppose that $a \in R^{GSEP}$. Then $a^* = a^{\#}(a^{\#})^* a^{+}$ and $a^{+} = a^{\#}$. Choose $u = a^{+} + 1 - aa^{\#}$. Then $u \in U(R)$ with $u^{-1} = a + 1 - aa^{\#}$ and $ua^{+}a = ua^{\#}a = (a^{+} + 1 - aa^{\#})a^{\#}a = a^{+}a^{\#}a = a^{+}$.

Also, $a^{\#}(a^{\#})^* u = a^{\#}(a^{\#})^*(a^{+} + 1 - aa^{\#}) = a^{\#}(a^{\#})^* a^{+} + a^{\#}(a^{\#})^* - a^{\#}(a^{\#})^* aa^{+} = a^*$.

$\Longleftarrow$ From the assumption, we have $a^* = a^{\#}(a^{\#})^* u$ and $ua^{+}a = a^{+}$ for some $u \in U(R)$. This gives

$$a^{+} = (ua^{+}a)a^{+}a = a^{+}a^{+}a.$$

Hence, $a \in R^{EP}$. It follows that

$$a^* = a^* a^{+} a = a^{\#}(a^{\#})^* ua^{+}a = a^{\#}(a^{\#})^* a^{+}.$$

Hence, $a \in R^{GSEP}$ by Lemma 3.3. □

**Theorem 6.2.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if $a^* = a^{\#}ua^{+}$ and $uaa^{\#} = (a^{\#})^*$ for some $u \in U(R)$.*

*Proof.* $\Longrightarrow$ Since $a \in R^{GSEP}$, $a \in R^{EP}$ and $a^* = a^{\#}(a^{\#})^* a^{+}$, one has $aa^{\#} = (aa^{\#})^*$. Choose $u = (a^{\#})^* + 1 - aa^{\#}$. Then $u^{-1} = a^* + 1 - aa^{\#}$, this implies $u \in U(R)$ and $uaa^{\#} = u(aa^{\#})^* = (a^{\#})^*$ and $a^{\#}ua^{+} = a^{\#}((a^{\#})^* + 1 - aa^{\#})a^{+} = a^{\#}(a^{\#})^* a^{+} = a^*$.

$\Longleftarrow$ From the hypothesis, one yields

$$a^* = a^{\#}ua^{+} = aa^{+}(a^{\#}ua^{+}) = aa^{+}a^*.$$

Hence, $a \in R^{EP}$ by [11, Theorem 1.2.1]. It follows that

$$a^* = a^{\#}ua^{+} = a^{\#}u(aa^{\#})^* a^{+} = a^{\#}uaa^{\#}a^{+} = a^{\#}(a^{\#})^* a^{+}.$$

Thus $a \in R^{GSEP}$ by Lemma 3.3. □

**Theorem 6.3.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if $a^* = u(a^{\#})^* a^{+}$ and $aa^{+}u = a^{+}$ for some $u \in U(R)$.*

*Proof.* $\Longrightarrow$ Choose $u = a^{\#} + 1 - aa^{\#}$. Then we are done.

$\Longleftarrow$ Since $a^{+} = aa^{+}u = aa^{\#}(aa^{+}u) = aa^{\#}a^{+}$. Multiplying the equality on the right by $a$, one gets $a^{+}a = aa^{\#}$. Hence, $a \in R^{EP}$ by [12, Theorem 1.2]. This gives

$$a^* = aa^{+}a^* = aa^{+}u(a^{\#})^* a^{+} = a^{+}(a^{\#})^* a^{+}.$$

Hence, $a \in R^{GSEP}$ by Theorem 3.2. □

**Theorem 6.4.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^* = u(a^{\#})^* a^{+}$ and $aa^{+}u = a^*$ for some $u \in U(R)$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{SEP}$. Then $a \in R^{GSEP}$ and $a^{+} = a^*$. By Theorem 6.3, we are done.

$\Longleftarrow$ Since $a^* = aa^{+}u = aa^{\#}(aa^{+}u) = aa^{\#}a^*$, then $a \in R^{EP}$ by [11, Theorem 1.1.3]. This gives

$$a^* = aa^{+}a^* = aa^{+}u(a^{\#})^* a^{+} = a^*(a^{\#})^* a^{+} = a^{+}.$$

Hence, $a \in R^{SEP}$. □

## 7. Using the consistency of certain equation to characterize GSEP elements

Now we construct the following equation

$$a^{\#}xa^{+} = a^{*}. \tag{7.1}$$

**Lemma 7.1.** *Let $a \in R^{\#} \cap R^{+}$. Then Eq.*(7.1) *is consistent if and only if $a \in R^{EP}$. In this case, the general solution to Eq.*(7.1) *is given by*

$$x = aa^{*}a + u - a^{+}aua^{+}a, u \in R. \tag{7.2}$$

*Proof.* $\Longrightarrow$ If Eq.(7.1) is consistent, then there exists $x_0 \in R$ such that $a^{\#}x_0a^{+} = a^{*}$, this gives $a^{*} = aa^{\#}a^{*}$. Hence, $a \in R^{EP}$ by [11, Theorem 1.1.3].

$\Longleftarrow$ If $a \in R^{EP}$, then $a^{*} = a^{\#}aa^{*}aa^{+}$, it follows that $x = aa^{*}a$ is a solution. Hence, Eq.(7.1) is consistent.

Now, if $a \in R^{EP}$, then $x = aa^{*}a$ is a solution. Hence

$$a^{\#}(aa^{*}a + u - a^{+}aua^{+}a)a^{+} = a^{\#}aa^{*}aa^{+} = a^{*},$$

it follows that the formula (7.2) is the solution to Eq.(7.1).

Let $x = x_0$ be any solution to Eq.(7.1). Then $a^{\#}x_0a^{+} = a^{*}$.

Choose $u = x_0$. Then

$$a^{+}aua^{+}a = a^{+}a^{2}(a^{\#}x_0a^{+})a = a^{+}a^{2}a^{*}a = aa^{*}a.$$

It follows that $x_0 = aa^{*}a + u - a^{+}aua^{+}a$ has the form of the formula (7.2).

Thus the general solution to Eq.(7.1) is given by (7.2). □

**Theorem 7.2.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if Eq.*(7.1) *is consistent and the general solution to Eq.*(7.1) *is given by*

$$x = (a^{\#})^{*} + u - a^{+}aua^{+}a, u \in R. \tag{7.3}$$

*Proof.* It is an immediate result of Theorem 3.2 and Lemma 7.1. □

**Lemma 7.3.** *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to the following equation is given by* (7.3).

$$a^{+}axa^{*}a^{*} = a^{*}. \tag{7.4}$$

*Proof.* First, we have

$$a^{+}a((a^{\#})^{*} + u - a^{+}aua^{+}a)a^{*}a^{*} = a^{+}a(a^{\#})^{*}a^{*}a^{*} = a^{*}.$$

Hence, the formula (7.3) is the solution to Eq.(7.4).

Now, let $x = x_0$ be any solution to Eq.(7.4). Then

$$a^{+}ax_0a^{*}a^{*} = a^{*}.$$

This gives

$$a^{+}ax_0a^{+}a = a^{+}ax_0a^{*}(a^{+})^{*} = (a^{+}ax_0a^{*}a^{*})(a^{\#})^{*}(a^{+})^{*} = a^{*}(a^{\#})^{*}(a^{+})^{*} = (a^{\#})^{*}a^{+}a.$$

Choose $u = x_0 - (a^{\#})^{*}$. Then

$$a^{+}aua^{+}a = a^{+}ax_0a^{+}a - a^{+}a(a^{\#})^{*}a^{+}a = (a^{\#})^{*}a^{+}a - (a^{\#})^{*}a^{+}a = 0,$$

and

$$x_0 = (a^{\#})^{*} + x_0 - (a^{\#})^{*} = (a^{\#})^{*} + u = (a^{\#})^{*} + u - a^{+}aua^{+}a.$$

Thus the general solution to Eq.(7.4) is given by (7.3). □

Theorem 7.2 and Lemma 7.3 induce the following theorem.

**Theorem 7.4.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if Eq.*(7.1) *has the same solution as Eq.*(7.4).

Now we establish the following euqation

$$a^{+}axa^{*}a^{+}(a^{\#})^{*} = a^{*}a. \tag{7.5}$$

**Theorem 7.5.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if Eq.*(7.5) *is consistent and the general solution to Eq.*(7.5) *is given by* (7.3).

*Proof.* $\Longrightarrow$ Since $a \in R^{GSEP}$, $a^{*}a = a^{+}(a^{\#})^{*}$ by Theorem 3.6. This gives $x = (a^{\#})^{*} + u - a^{+}aua^{+}a$ is the solution to Eq.(7.5).

Now let $x = x_0$ be any solution to Eq.(7.5), one has

$$a^{+}ax_0a^{*}a^{+}(a^{\#})^{*} = a^{*}a.$$

This induces

$$\begin{aligned} a^{+}ax_0a^{+}a &= a^{+}ax_0a^{*}a^{+}a(a^{\#})^{*}a^{+}a = (a^{+}ax_0a^{*})(a^{+}(a^{+})^{*}a^{*})(a(a^{\#})^{*}a^{+}a) \\ &= (a^{+}ax_0a^{*}a^{+}(a^{\#})^{*})a^{*}a(a^{\#})^{*}a^{+}a = a^{*}aa^{*}a(a^{\#})^{*}a^{+}a = a^{*}aa^{*}a(a^{\#})^{*} \\ &= a^{+}(a^{\#})^{*}a^{*}a(a^{\#})^{*} = a^{+}a(a^{\#})^{*} = (a^{\#})^{*}. \end{aligned}$$

Choose $u = x_0$. Then

$$x_0 = (a^{\#})^{*} + x_0 - (a^{\#})^{*} = (a^{\#})^{*} + u - a^{+}ax_0a^{+}a = (a^{\#})^{*} + u - a^{+}aua^{+}a.$$

Hence, the general solution to Eq.(7.5) is given by (7.3).

$\Longleftarrow$ It follows from the assumption that

$$a^{+}a((a^{\#})^{*} + u - a^{+}aua^{+}a)a^{*}a^{+}(a^{\#})^{*} = a^{*}a.$$

One gets $a^{+}(a^{\#})^{*} = a^{*}a$. Hence, $a \in R^{GSEP}$. □

The following corollary follows from Theorem 7.2 and Theorem 7.5.

**Corollary 7.6.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if Eq.*(7.5) *has the same solution as Eq.*(7.1).

## 8. Characterizing GSEP elements by the form of solution to related equations

Now, we know that if $a \in R^{GSEP}$, then $(a^{\#})^{*} = aa^{*}a$ and $a \in R^{EP}$ by Theorem 3.2, so we get

$$a^{*}a = a^{\#}(a^{\#})^{*} = a^{\#}(a^{\#})^{*}a^{+}a.$$

This inspires us to give the following equation

$$a^{*}x = a^{\#}(a^{\#})^{*}a^{+}x. \tag{8.1}$$

**Theorem 8.1.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if Eq.*(8.1) *has at least one solution in $\chi_a = \{a, a^{\#}, (a^{+})^{*}, a^{+}, a^{*}, (a^{\#})^{*}\}$.*

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $a^{*} = a^{\#}(a^{\#})^{*}a^{+}$. Hence, every element in $\chi_a$ is a solution to Eq.(8.1).

$\Longleftarrow$ From the hypothesis, there exists $x_0 \in \chi_a$ such that

$$a^{*}x_0 = a^{\#}(a^{\#})^{*}a^{+}x_0.$$

If $x_0 \in \tau_a = \{a, a^\#, (a^+)^*\}$, then $x_0 x_0^+ = aa^+$ by [26, Theorem 4.2], this induces

$$a^* = a^*aa^+ = a^*x_0x_0^+ = a^\#(a^\#)^*a^+x_0x_0^+ = a^\#(a^\#)^*a^+aa^+ = a^\#(a^\#)^*a^+.$$

If $x \in \gamma_a = \{a^+, a^*, (a^\#)^*\}$, then $x_0x_0^+ = a^+a$, this leads to

$$a^*a^+a = a^*x_0x_0^+ = a^\#(a^\#)^*a^+x_0x_0^+ = a^\#(a^\#)^*a^+a^+a.$$

Multiplying the equality on the right by $(a^\#)^*a^*$, one gets

$$a^* = a^\#(a^\#)^*a^+.$$

In any case, we have $a \in R^{GSEP}$. □

Now we revise Eq.(8.1) as follows

$$a^*x = a^\#(a^\#)^*ya^+. \tag{8.2}$$

**Theorem 8.2.** *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(8.2) is given by*

$$\begin{cases} x = (a^\#)^*a^+a^2pa^+ + u - aa^+u \\ y = a^*a^+a^3p + v - aa^+va^+a \end{cases}, \quad p,\ u,\ v \in R \text{ with } ap = a^+a^2p. \tag{8.3}$$

*Proof.* First, we claim that the formula (8.3) is the solution to Eq.(8.2). In fact

$$a^*x = a^*((a^\#)^*a^+a^2pa^+ + u - aa^+u) = a^*(a^\#)^*a^+a^2pa^+ = a^+a^2pa^+ = apa^+$$
$$= a^\#(a^\#)^*a^*a^+a^3pa^+ = a^\#(a^\#)^*(a^*a^+a^3p + v - aa^+va^+a)a^+ = a^\#(a^\#)^*ya^+.$$

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution to Eq.(8.2). Then

$$a^*x_0 = a^\#(a^\#)^*y_0a^+.$$

Choose $p = a^+a^\#(a^\#)^*y_0a^+a$, $u = x_0 - (a^\#)^*a^+a^2pa^+$, $v = y_0 - a^*a^+a^3p$. Then

$$ap = aa^+a^\#(a^\#)^*y_0a^+a = a^\#(a^\#)^*y_0a^+a = a^*x_0a = a^+a(a^*x_0a) = a^+a(ap) = a^+a^2p,$$

$$aa^+u = aa^+(x_0 - (a^\#)^*a^+a^2pa^+) = aa^+x_0 - aa^+(a^\#)^*a^+a^2pa^+ = aa^+x_0 - (a^+)^*apa^+$$
$$= aa^+x_0 - (a^+)^*(a^*x_0a)a^+ = aa^+x_0 - aa^+x_0aa^+ = (a^+)^*a^*x_0 - aa^+x_0aa^+$$
$$= (a^+)^*(a^\#(a^\#)^*y_0a^+) - aa^+x_0aa^+ = (a^+)^*(a^\#(a^\#)^*y_0a^+)aa^+ - aa^+x_0aa^+$$
$$= (a^+)^*(a^*x_0)aa^+ - aa^+x_0aa^+ = 0.$$

Thus

$$x_0 = (a^\#)^*a^+a^2pa^+ + u - aa^+u,$$

$$aa^+va^+a = aa^+(y_0 - a^*a^+a^3p)a^+a = aa^+y_0a^+a - aa^+a^*a^+a^3pa^+a$$
$$= aa^+y_0a^+a - aa^+a^*a^+a^3(a^+a^\#(a^\#)^*y_0a^+a)a^+a$$
$$= aa^+y_0a^+a - aa^+a^*a^+a(a^\#)^*y_0a^+a$$
$$= aa^+y_0a^+a - aa^+y_0a^+a = 0.$$

Hence, $y_0 = a^*a^+a^3p + v - aa^+va^+a$. Hence, every solution to Eq.(8.2) has the form of the formula (8.3). □

**Theorem 8.3.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if the general solution to Eq.*(8.2) *is given by*

$$\begin{cases} x = aa^*a^2pa^+ + u - aa^+u \\ y = a^*a^+a^3p + v - aa^+va^+a \end{cases}, \quad p,\ u,\ v \in R. \tag{8.4}$$

*Proof.* $\Longrightarrow$ Since $a \in R^{GSEP}$, $aa^* = (a^{\#})^*a^+$ by Theorem 3.5. Hence, by Theorem 8.2, we are done.

$\Longleftarrow$ The assumption implies

$$a^*(aa^*a^2pa^+ + u - aa^+u) = a^{\#}(a^{\#})^*(a^*a^+a^3p + v - aa^+va^+a)a^+,$$

$$a^*aa^*a^2pa^+ = a^{\#}(a^{\#})^*a^*a^+a^3pa^+ = apa^+ \text{ for all } p \in R.$$

Especially, choose $p = a^{\#}$, one has

$$a^*aa^* = aa^{\#}a^+.$$

Multiplying the equality on the left by $a^+a$, one obtains

$$aa^{\#}a^+ = a^+.$$

Hence, $a \in R^{EP}$. This gives $a^{\#} = aa^{\#}a^+ = a^*aa^*$, that is, $(a^{\#})^* = aa^*a$. Hence, $a \in R^{GSEP}$. $\square$

**Theorem 8.4.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{GSEP}$ if and only if the general solution to Eq.*(8.2) *is given by*

$$\begin{cases} x = (a^{\#})^*a^+a^2pa^+ + u - aa^+u \\ y = a^+(a^{\#})^*a^+a^+a^3p + v - aa^+va^+a \end{cases}, \quad p,\ u,\ v \in R. \tag{8.5}$$

*Proof.* $\Longrightarrow$ Assume that $a \in R^{GSEP}$. Then $a \in R^{EP}$ and $a^*a = a^+(a^{\#})^*$. It follows that

$$a^*a^+a^3p = a^*aa^+a^+a^3p = a^+(a^{\#})^*a^+a^+a^3p.$$

By Theorem 8.2, we are done.

$\Longleftarrow$ According to the assumption, we have

$$a^*((a^{\#})^*a^+a^2pa^+ + u - aa^+u) = a^{\#}(a^{\#})^*(a^+(a^{\#})^*a^+a^+a^3p + v - aa^+va^+a)a^+,$$

this leads to

$$a^+a^2pa^+ = a^{\#}(a^{\#})^*a^+(a^{\#})^*a^+a^+a^3pa^+ \text{ for all } p \in R.$$

Choose $p = a^{\#}$, we yield

$$a^+ = a^{\#}(a^{\#})^*a^+(a^{\#})^*a^+a^+a^2a^+.$$

Multiplying the equality on the left by $aa^{\#}$, we get $a^+ = aa^{\#}a^+$. Then multiplying the equality on the right by $a$, one gives $a^+a = aa^{\#}$. Hence, $a \in R^{EP}$ by [12, Theorem 1.2]. It follows that

$$a^+ = a^{\#}(a^{\#})^*a^+(a^{\#})^*a^+,$$

$$a = aa^+a = aa^{\#}(a^{\#})^*a^+(a^{\#})^*a^+a = aa^{\#}(a^+)^*a^+(a^+)^*a^+a = (a^+)^*a^+(a^+)^*,$$

$$a^*a = a^*(a^+)^*a^+(a^+)^* = a^+(a^+)^* = a^+(a^{\#})^*.$$

Hence, $a \in R^{GSEP}$ by Theorem 3.6. $\square$

**Theorem 8.5.** *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.*(8.2) *is given by*

$$\begin{cases} x = (aa^{\#})^*a^2pa^+ + u - aa^+u \\ y = a^*a^+a^3p + v - aa^+va^+a \end{cases}, \quad p,\ u,\ v \in R. \tag{8.6}$$

*Proof.* $\Longrightarrow$ Assume that $a \in R^{SEP}$. Then $a \in R^{EP}$ and $a^+ = a^*$. Hence, by Theorem 8.2, we are done.

$\Longleftarrow$ From the assumption, we have

$$a^*((aa^\#)^*a^2pa^+ + u - aa^+u) = a^\#(a^\#)^*(a^*a^+a^3p + v - aa^+va^+a)a^+,$$

this leads to

$$a^*a^2pa^+ = a^\#(a^\#)^*a^*a^+a^3pa^+ = apa^+ \text{ for all } p \in R.$$

Choose $p = a^\#$, one gets

$$a^* = aa^\#a^+.$$

Hence, $a \in R^{SEP}$ by [11, Theorem 1.5.3]. □

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