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The (anti)-η-Hermitian solution of quaternion linear system

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Abstract. Based on semi-tensor product of quaternion matrices we study the sufficient and necessary conditions for the existence of (anti)-η-Hermitian solutions of the system of equations over the quaternion algebra, and for some quaternion matrices with special structure, we use $\mathcal{GH}$-representation method to extract their independent elements. Then the special solution of quaternion linear system is obtained by using classical matrix theory. The effectiveness of the algorithm is verified, and the comparison with the algorithms under other methods shows that the algorithm in this paper is more efficient. And the application in time-varying linear system of the presented algorithm is represented.

1. Introduction

Notation:

• $\mathbb{R}/\mathbb{R}^m/\mathbb{R}^{m\times n}$ represent the set of all real numbers/real column vectors with m-dimension/$m \times n$ real matrices, respectively.

• $\mathbb{Q}/\mathbb{Q}^m/\mathbb{Q}^{m\times n}$ represent the set of all quaternions/quaternion column vectors with m-dimension/$m \times n$ quaternion matrices, respectively.

• $\mathbf{S}_n/\mathbf{AS}_n$ represent the set of all $n\times n$ real symmetric matrices/real skew-symmetric matrices, respectively.

• I_n represents the unit matrix with n-dimension.

• For $A \in \mathbb{Q}^{m\times n}$, $Row_i(A)(i = 1, 2, \cdots, m)$ represents the ith row of matrix A, $Col_j(A)(j = 1, 2, \cdots, n)$ represents the jth column of matrix A.

• $A^{\mathrm{T}}/A^{\mathrm{H}}/A^{\dagger}$ represent the transpose/the conjugate transpose/Moore-Penrose inverse of matrix A, respectively.

• $\|\cdot\|_F$ represents the Frobenius norm of real matrix or Euclidean norm of real vector.

As a hypercomplex number, quaternion is a linear combination of real number and three imaginary units i, j, k, expressed as

$$\mathbb{Q} = \{q = q_1 + q_2\mathrm{i} + q_3\mathrm{j} + q_4\mathrm{k} | q_1, q_2, q_3, q_4 \in \mathbb{R}\},$$

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where i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. The set of quaternion matrices is expressed as

$$\mathbb{Q}^{m\times n} = \{Q = Q_1 + Q_2\mathrm{i} + Q_3\mathrm{j} + Q_4\mathrm{k} | Q_1, Q_2, Q_3, Q_4 \in \mathbb{R}^{m\times n}\}.$$

The conjugate matrix of Q is defined as $\overline{Q} = Q_1 - Q_2\mathrm{i} - Q_3\mathrm{j} - Q_4\mathrm{k}$, and the Frobenius norm of Q is defined as $\|Q\| = \sqrt{\|Q_1\|_F^2 + \|Q_2\|_F^2 + \|Q_3\|_F^2 + \|Q_4\|_F^2}$.

Quaternion matrices have important applications in some new fields such as color images[1–3], rendering and three-dimensional fractal display. The mathematical models of these applications are mostly transformed into problems for solving the quaternion matrix equations. Therefore, the study of quaternion matrix equations has attracted the attention of many scholars. For example, based on real representation method, Zhang[4–6] investigated the minimal norm least squares (anti)-η-Hermitian solution, (anti)-j-self-conjugate solution on quaternion matrix equations $AXB + CXD = E$, $AXB + CYD = E$ and $X - A\widehat{X}B = C$, respectively; based on complex representation method, Yuan[7, 8] discussed the minimal norm least squares (anti)-η-Hermitian solution and Hermitian solution of quaternion matrix equations $AXB + CYD = E$ and $(AXB, CXD) = (E, F)$, respectively. Furthermore, some scholars also considered using the rank method[9–11] to study quaternion matrix equations.

As a new type of algorithm applicable to any matrix dimension, the proposal of semi-tensor product of matrices[12] greatly facilitates the study of matrix theory. At present, semi-tensor product of matrices is widely used in biological system and life science[13, 14], game theory[15, 16], graph theory and formation control[17, 18], fuzzy control[19, 20], coding theory and algorithm implementation[21–23]. Moreover, some scholars proposed real vector representations of quaternion based on semi-tensor product of matrices, the operation properties of quaternion matrix based on real vector representation were studied. For example, based on real vector representation of quaternion, Ding[24] and Wang[25] proposed the least squares (anti-)**J**-self-conjugate solution of quaternion matrix equation $X - A\widehat{X}B = C$, and derived the expressions of the least squares Hermitian, Persymmetric and Bisymmetric solution for the quaternion matrix equation $AXB = C$, respectively. In this paper, by using semi-tensor product of matrices, we propose a vector representation of quaternion matrix, and the operation relation of vector representation of quaternion matrix is also studied. In addition, with the aid of representation matrix of the multiplication of quaternion, we show different matrix representations of quaternion matrices by using semi-tensor product of matrices, and define matrix representation that satisfy certain conditions as $\mathcal{L}$-representation. As a practice, we study the (anti)-η-Hermitian solutions of the system of equations

$$\begin{cases} A_1XB_1 = C_1 \\ A_2YB_2 = C_2 \end{cases} \tag{1}$$

over the quaternion algebra combined with vector representation and $\mathcal{L}$-representation of quaternion matrix, $\mathcal{GH}$-representation of special quaternion matrix.

Definition 1.1. *[5] For any $A = A_1 + A_2\mathrm{i} + A_3\mathrm{j} + A_4\mathrm{k} \in \mathbb{Q}^{n\times n}$, and $\eta \in \{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$, denote*

$$A^{\eta\mathrm{H}} = -\eta A^{\mathrm{H}}\eta.$$

If $A^{\eta\mathrm{H}} = A$, A is called η-Hermitian matrix, the set of all $n \times n$ η-Hermitian matrices is denoted by $\eta\mathbb{HQ}^{n\times n}$; if $A^{\eta\mathrm{H}} = -A$, A is called anti-η-Hermitian matrix, the set of all $n \times n$ anti-η-Hermitian matrices is denoted by $\eta\mathbb{AHQ}^{n\times n}$.

The following problems will be solved in this paper.

Problem Let $A_i, B_i, C_i \in \mathbb{Q}^{n\times n}, (i = 1, 2)$, and

$$M_r = \{[X, Y] | X \in \eta\mathbb{HQ}^{n\times n}, Y \in \eta\mathbb{AHQ}^{n\times n}, \|A_1XB_1 - C_1\|^2 + \|A_2YB_2 - C_2\|^2 = \min\}.$$

Find out $[X_\eta, Y_{a\eta}] \in M_r$, such that

$$\|[X_\eta, Y_{a\eta}]\| = \min_{[X,Y]\in M_r} \|[X, Y]\|.$$

The solution $[X_\eta, Y_{a\eta}]$ is called the minimal norm least squares (anti)-η-Hermitian solution of the system of equations (1) over the quaternion algebra. If min = 0, $[X_\eta, Y_{a\eta}]$ is called the minimal norm (anti)-η-Hermitian solution of the system of equations (1) over the quaternion algebra.

Special matrix is an important branch of matrix theory, (anti)-η-Hermitian matrix is a special kind of quaternion matrix. It is clearly known that if a quaternion matrix is (anti)-η-Hermitian matrix, then the structure of each part of the quaternion matrix is closely related to the (skew)-symmetric matrix. (Skew)-Symmetric matrix is an important part of special matrix. In the fields of numerical analysis, engineering technology, computer network, image and so on, according to different models and special requirements, there are a variety of special (skew)-symmetric matrices. Statistical signal processing and convergence analysis in linear modeling[26] are some of the main application fields of (anti)-η-Hermitian matrix. This particularity of (anti)-η-Hermitian matrix makes some scholars[27–29] devote themselves to study the (anti)-η-Hermitian solution of quaternion matrix equations.

The main contributions of this paper are as follows:

• Firstly, by using the property of quaternion conjugate operation, we find the operation relation of quaternion matrix under vector representation.

• Secondly, under the structure matrix of the multiplication of quaternion, we establish different matrix representation of quaternion matrix by semi-tensor product of quaternion matrices. And according to our requirements, the matrix representation that meets certain conditions is defined as $\mathcal{L}$-representation.

• Thirdly, the $\mathcal{H}$-representation given by Zhang[30] establishes the relationship between special matrices and their independent elements. However, it cannot be directly applied to complex matrices or quaternion matrices(such as Hermitian matrix) whose real and imaginary parts have different structures. So we define the $\mathcal{GH}$-representation on quaternion matrix on the basis of $\mathcal{L}$-representation.

The main structure arrangement of this article is as follows: In Section 2, the basics of quaternion and the definition of semi-tensor product of quaternion matrices are introduced. In Section 3, the main conclusions of this paper about semi-tensor product of quaternion matrices are given, the quaternion matrix operation under vector representation is studied firstly; the conclusion of the relationship between structure matrix and matrix representation of quaternion matrix is investigated, and the $\mathcal{L}$-representation is proposed secondly; $\mathcal{GH}$-representation applied to special quaternion matrices are presented thirdly. Using the description given earlier in this article, in Section 4, we study the system of equations over the quaternion algebra, corresponding numerical examples are given to verify the effectiveness of the proposed method in Section 5 and the time comparison with the algorithms under other methods is also displayed. Finally, the article is summarised in Section 6.

2. Preliminaries

As an extension of classical matrix multiplication, semi-tensor product of matrices proposed by Cheng is limited to the field of real numbers and complex numbers. Some properties of semi-tensor product of matrices cannot be directly extended to quaternion. In this part our main work is to apply semi-tensor product of matrices to quaternion. For more details about semi-tensor product of matrices, please refer to the literature [33].

Definition 2.1. *[33] Suppose $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{p\times q}$, the left semi-tensor product of A and B is defined by*

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}),$$

and the right semi-tensor product of A and B is defined by

$$A \rtimes B = (I_{t/n} \otimes A)(I_{t/p} \otimes B),$$

where $t = lcm(n, p)$ is the least common multiple of n and p, $\otimes$ represents the Kronecker product of matrices. If $n = p$, semi-tensor product of quaternion matrices reduces to the classical matrix product. Moreover, when a proposition holds for both the left semi-tensor product of quaternion matrices and the right semi-tensor product of quaternion matrices, we use the symbol $\bowtie$.

Example 2.2. *Suppose* $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$. *The left semi-tensor product of A and B can be calculated by*

$$A \ltimes B = (A \otimes I_2)B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 5 \\ -1 & -2 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} Row_1(A) \ltimes Col_1(B) & Row_1(A) \ltimes Col_2(B) \\ Row_2(A) \ltimes Col_1(B) & Row_2(A) \ltimes Col_2(B) \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ -\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 1 \end{pmatrix} & -\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 5 \\ -1 & -2 \\ -3 & -1 \end{bmatrix}.$$

Nevertheless, for the right semi-tensor product of matrices

$$A \rtimes B = (I_2 \otimes A)B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ -1 & -2 \\ 4 & 7 \\ -2 & -1 \end{bmatrix}$$

$$\neq \begin{bmatrix} Row_1(A) \rtimes Col_1(B) & Row_1(A) \rtimes Col_2(B) \\ Row_2(A) \rtimes Col_1(B) & Row_2(A) \rtimes Col_2(B) \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \rtimes \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rtimes \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \rtimes \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix} & \begin{pmatrix} -1 \\ 0 \end{pmatrix} \rtimes \begin{pmatrix} 2 \\ -1 \\ 1 \\ 3 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 4 & 7 \\ -1 & -2 \\ -2 & -1 \end{bmatrix}.$$

Remark 2.3. *Classical matrix block multiplication holds for the left semi-tensor product of matrices, but not for the right semi-tensor product of matrices. This is the biggest difference between the left semi-tensor product of matrices and the right semi-tensor product of matrices, and also makes the left semi-tensor product of matrices far superior to the right semi-tensor product of matrices in application.*

Theorem 2.4. *[34] Suppose* $\alpha, \beta \in \mathbb{R}$, A, B, C *be quaternion matrices, then*
(1) (Associative rule)

$$(A \bowtie B) \bowtie C = A \bowtie (B \bowtie C),$$

(2) (Distributive rule)

$$A \bowtie (\alpha B + \beta C) = \alpha A \bowtie B + \beta A \bowtie C, (\alpha B + \beta C) \bowtie A = \alpha B \bowtie A + \beta C \bowtie A.$$

In the operation of semi-tensor product of quaternion matrices, swap matrix plays an extremely important role, which can realize the exchange of factor order.

Definition 2.5. *[33] A swap matrix* $W_{[m,n]}$ *is a* $mn \times mn$ *matrix, which is defined as*

$$W_{[m,n]} = [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \ldots, I_n \otimes \delta_m^m],$$

where $\delta_m^i (i = 1, 2, \ldots, m)$ *represents the ith column of unit matrix* I_m.

Definition 2.6. *For $A = (a_{ij}) \in \mathbb{Q}^{m\times n}$, the column vector representation of quaternion matrix A is defined as*

$$V_c(A) = (a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn})^{\mathrm{T}},$$

the row vector representation of quaternion matrix A is defined as

$$V_r(A) = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn})^{\mathrm{T}}.$$

The swap matrix has the following properties.

Theorem 2.7. *[34] (1)Suppose $A \in \mathbb{Q}^{s\times t}$, then for any integer $m > 0$ have*

$$W_{[s,m]} \ltimes A \ltimes W_{[m,t]} = I_m \otimes A.$$

(2)Suppose $A \in \mathbb{Q}^{m\times n}$, then

$$W_{[m,n]} V_r(A) = V_c(A), W_{[n,m]} V_c(A) = V_r(A).$$

3. Main Results

We obtain some conclusions of semi-tensor product of quaternion matrices on quaternion algebra and apply these conclusions to solve the quaternion matrix equation.

3.1. Vector Representation of Quaternion Matrix

As we all know, quaternion multiplication is not satisfy the commutative law,

$$V_c(AXB) = (B^{\mathrm{T}} \otimes A) V_c(X) \tag{2}$$

is not tenable on quaternion. Therefore, the research of quaternion matrix equation is mostly to transform the quaternion matrix equation into real or complex linear equation by using the real representation[1] or complex representation method[2] of quaternion matrix, and then use equation (2). We use semi-tensor product of quaternion matrices to research the operation of vector representation of quaternion matrix which is based on the conjugate property of quaternion. This implements a direct transformation from a quaternion matrix equation to a vector matrix equation.

Theorem 3.1. *Suppose $A \in \mathbb{Q}^{m\times n}$, $X \in \mathbb{Q}^{n\times q}$, $Y \in \mathbb{Q}^{p\times m}$, then*

(1)$V_r(AX) = A \ltimes V_r(X); V_c(AX) = A \rtimes V_c(X);$

(2)$V_c(\overline{YA}) = A^{\mathrm{H}} \ltimes V_c(\overline{Y}); V_r(\overline{YA}) = A^{\mathrm{H}} \rtimes V_r(\overline{Y}).$

Proof. (1) For $A = (a_{ij}) \in \mathbb{Q}^{m\times n}$, $X = (x_{js}) \in \mathbb{Q}^{n\times q}$, suppose $C = AX$, the left semi-tensor product of the ith row of A and $V_r(X)$ is

$$Row_i(A) \ltimes V_r(X) = Row_i(A) \ltimes \begin{bmatrix} Row_1(X)^{\mathrm{T}} \\ \vdots \\ Row_n(X)^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} a_{ik}x_{k1} \\ \vdots \\ \sum_{k=1}^{n} a_{ik}x_{kq} \end{bmatrix} = Row_i(C)^{\mathrm{T}},$$

then we have $V_r(AX) = A \ltimes V_r(X)$.

By the properties of the swap matrix and $V_r(AX) = A \ltimes V_r(X)$, there have

$$V_c(AX) = W_{[m,q]} \ltimes V_r(AX) = W_{[m,q]} \ltimes A \ltimes V_r(X) = W_{[m,q]} \ltimes A \ltimes W_{[q,n]} \ltimes V_c(X) = (I_q \otimes A) \ltimes V_c(X) = A \rtimes V_c(X).$$

(2)For $A = (a_{ij}) \in \mathbb{Q}^{m\times n}$, $Y = (y_{ti}) \in \mathbb{Q}^{p\times m}$, then

$$V_c(\overline{YA}) = V_c(\overline{YCol_1(A)}, \ldots, \overline{YCol_n(A)}) = \begin{bmatrix} \overline{YCol_1(A)} \\ \vdots \\ \overline{YCol_n(A)} \end{bmatrix},$$

by the properties of quaternion we can derive,

$$\overline{YCol_j(A)} = \overline{Col_1(Y)a_{1j}} + \overline{Col_2(Y)a_{2j}} + \ldots + \overline{Col_m(Y)a_{mj}} = \overline{a_{1j}}\,\overline{Col_1(Y)} + \overline{a_{2j}}\,\overline{Col_2(Y)} + \ldots + \overline{a_{mj}}\,\overline{Col_m(Y)} = \begin{bmatrix} \overline{a_{1j}}I_p & \ldots & \overline{a_{mj}}I_p \end{bmatrix} V_c(\overline{Y}).$$

So

$$V_c(\overline{YA}) = \begin{bmatrix} \overline{a_{11}}I_p & \overline{a_{21}}I_p & \cdots & \overline{a_{m1}}I_p \\ \overline{a_{12}}I_p & \overline{a_{22}}I_p & \cdots & \overline{a_{m2}}I_p \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}}I_p & \overline{a_{2n}}I_p & \cdots & \overline{a_{mn}}I_p \end{bmatrix} V_c(\overline{Y}) = (A^{\mathrm{H}} \otimes I_p) V_c(\overline{Y}) = A^{\mathrm{H}} \ltimes V_c(\overline{Y}).$$

By the properties of the swap matrix, we have

$$V_r(\overline{YA}) = W_{[n,p]} \ltimes V_c(\overline{YA}) = W_{[n,p]} \ltimes A^{\mathrm{H}} \ltimes V_c(\overline{Y}) = W_{[n,p]} \ltimes A^{\mathrm{H}} \ltimes W_{[p,m]} \ltimes V_r(\overline{Y}) = (I_p \otimes A^{\mathrm{H}}) V_r(\overline{Y}) = A^{\mathrm{H}} \rtimes V_r(\overline{Y}).$$

□

3.2. *$\mathcal{L}$-representation of Quaternion Matrix*

As mentioned in the previous section, the study of quaternion matrix equation in reference [1, 2] is based on the real representation and complex representation of quaternion matrix, which is also a common method for the study of quaternion matrix equation. But relatively speaking, there are many kinds of real representations of quaternion, for example, several different real representations of quaternion matrices are given in reference [35]. At present, there is no specific definition of real representation. Until the emergence of semi-tensor product of quaternion matrices, it provides a powerful tool for us to systematically study the matrix representation of quaternion matrix.

Definition 3.2. *[33] Let $W_i (i = 0, 1, \ldots, n)$ be vector spaces. The mapping $F : \prod_{i=1}^{n} W_i \to W_0$ is a multilinear mapping. If $dim(W_i) = k_i$ and $(\delta_{k_i}^1, \delta_{k_i}^2, \ldots, \delta_{k_i}^{k_i})$ is the basis of W_i. Denote*

$$F(\delta_{k_1}^{j_1}, \delta_{k_2}^{j_2}, \ldots, \delta_{k_n}^{j_n}) = \sum_{s=1}^{k_0} c_s^{j_1, j_2, \ldots, j_n} \delta_{k_0}^s, \tag{3}$$

in which $j_t = 1, 2, \ldots, k_t, t = 1, 2, \ldots, n$. Then

$$\left\{ c_s^{j_1, j_2, \ldots, j_n} | j_t = 1, 2, \ldots, k_t, t = 1, 2, \ldots, n; s = 1, 2, \ldots, k_0 \right\}$$

are called structure constants of F. According to the dictionary sorting, the matrix

$$M_F^1 = \begin{bmatrix} c_1^{11\cdots 1} & \cdots & c_1^{11\cdots k_n} & \cdots & c_1^{k_1 k_2 \cdots k_n} \\ c_2^{11\cdots 1} & \cdots & c_2^{11\cdots k_n} & \cdots & c_2^{k_1 k_2 \cdots k_n} \\ \vdots & & \vdots & & \vdots \\ c_{k_0}^{11\cdots 1} & \cdots & c_{k_0}^{11\cdots k_n} & \cdots & c_{k_0}^{k_1 k_2 \cdots k_n} \end{bmatrix}$$

is defined as the right structure matrix of F. The matrix

$$M_F^2 = \begin{bmatrix} c_1^{11\cdots 1} & \cdots & c_1^{k_1 1\cdots 1} & \cdots & c_1^{k_1 k_2 \cdots k_n} \\ c_2^{11\cdots 1} & \cdots & c_2^{k_1 1\cdots 1} & \cdots & c_2^{k_1 k_2 \cdots k_n} \\ \vdots & & \vdots & & \vdots \\ c_{k_0}^{11\cdots 1} & \cdots & c_{k_0}^{k_1 1\cdots 1} & \cdots & c_{k_0}^{k_1 k_2 \cdots k_n} \end{bmatrix}$$

is defined as the left structure matrix of F. And we refer to the left structure matrix and the right structure matrix as the structure matrix.

Example 3.3. *For $x = a_1 + b_1\mathrm{i} + c_1\mathrm{j} + d_1\mathrm{k}$, $y = a_2 + b_2\mathrm{i} + c_2\mathrm{j} + d_2\mathrm{k} \in \mathbb{Q}$, the products of x and y are arranged in the order of each part of x, then*

$$xy = a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathrm{i} + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathrm{j} + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathrm{k}.$$

Fix an ordered basis $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}\}$ in a 4-dimensional quaternion space $\mathbb{Q}$. The basis is normalized to

$$1 \sim \delta_4^1, \mathrm{i} \sim \delta_4^2, \mathrm{j} \sim \delta_4^3, \mathrm{k} \sim \delta_4^4.$$

Under the standardization of this basis, according to the definition of right structure matrix, the right structure matrix of the multiplication of quaternion can be obtained as

$$M_q^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

it is easy to obtain the left structure matrix of the multiplication of quaternion such as

$$M_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Using semi-tensor product of quaternion matrices and the structure matrix of the multiplication of quaternion, we explore the different matrix representations of quaternion matrix. For $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \mathbb{Q}^{n\times n}$, we denote

$$\widehat{X} = \begin{bmatrix} \pm X_1 \\ \pm X_2 \\ \pm X_3 \\ \pm X_4 \end{bmatrix}.$$

Definition 3.4. *Suppose $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \mathbb{Q}^{n\times n}$, where $X_t \in \mathbb{R}^{n\times n} (t = 1, 2, 3, 4)$, then the matrix representation of quaternion matrix X can be represented as*

$$\varphi(X) = M_q \ltimes (I_4 \otimes \widehat{X}).$$

And we define the first column of the matrix representation of the quaternion matrix as

$$\varphi^c(X) = \varphi(X) \ltimes \delta_4^1.$$

$\varphi(X)$ and $\varphi^c(X)$ are determined by M_q and $\widehat{X}$.

Example 3.5. *Let $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \mathbb{Q}^{n\times n}$, by the M_q^1 given in Example 3.3, we can get different matrix representations of quaternion matrix X,*

$$\varphi^1(X) = M_q^1 \ltimes \left(I_4 \otimes \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \right) = \begin{bmatrix} X_1 & -X_2 & -X_3 & -X_4 \\ X_2 & X_1 & X_4 & -X_3 \\ X_3 & -X_4 & X_1 & X_2 \\ X_4 & X_3 & -X_2 & X_1 \end{bmatrix}; \ \varphi^2(X) = M_q^1 \ltimes \left(I_4 \otimes \begin{bmatrix} X_1 \\ -X_2 \\ -X_3 \\ -X_4 \end{bmatrix} \right) = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ -X_2 & X_1 & -X_4 & X_3 \\ -X_3 & X_4 & X_1 & -X_2 \\ -X_4 & -X_3 & X_2 & X_1 \end{bmatrix}.$$

Note that when $\widehat{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$ or $\widehat{X} = \begin{bmatrix} X_1 \\ -X_2 \\ -X_3 \\ -X_4 \end{bmatrix}$, the matrix representation used in references [36, 37] can be obtained by using the M_q^2 in Example 3.3. For all the obtained matrix representations, we make certain restrictions on the properties they satisfy, the matrix representations that satisfy our needs are defined as $\mathcal{L}$-representation.

Definition 3.6. *Suppose $X \in \mathbb{Q}^{m\times n}$, $Y \in \mathbb{Q}^{n\times p}$, $\varphi(X)$ is called the $\mathcal{L}$-representation of quaternion matrix if and only if $\varphi(X)$ satisfies the following equations,*

$$(1)\varphi(XY) = \varphi(X)\varphi(Y),$$
$$(2)\varphi^c(XY) = \varphi(X)\varphi^c(Y).$$

It is easy to verify that the matrix representation $\varphi^1(X)$ Example 3.5 does not satisfy the two conditions for the $\mathcal{L}$-representation, but the matrix representation given by $\varphi^2(X)$ does. It is clear that Definition 3.6 has the following equivalent form.

Definition 3.7. *Suppose $X \in \mathbb{Q}^{m\times n}$, $Y \in \mathbb{Q}^{n\times p}$, $\varphi(X)$ is called the $\mathcal{L}$-representation of quaternion matrix if and only if $\varphi(X)$ satisfies the following equations,*

$$(1)(M_q \otimes I_m)(I_4 \otimes \widehat{XY}) = (M_q \otimes I_m)(M_q \otimes \widehat{X})(I_4 \otimes \widehat{Y}),$$
$$(2)(M_q \otimes I_m)(\delta_4^1 \otimes \widehat{XY}) = (M_q \otimes I_m)(M_q \otimes \widehat{X})(\delta_4^1 \otimes \widehat{Y}).$$

Proof. By Definition 3.4, the proof is straightforward. For the second one, we know that $\varphi^c(XY)$ can be expressed as

$$\varphi^c(XY) = \varphi(XY) \ltimes \delta_4^1 = M_q \ltimes (I_4 \otimes \widehat{XY}) \ltimes \delta_4^1 = (M_q \otimes I_m)(\delta_4^1 \otimes \widehat{XY}).$$

And $\varphi(X)\varphi^c(Y)$ can be expressed as

$$\varphi(X)\varphi^c(Y) = M_q \ltimes (I_4 \otimes \widehat{X}) \ltimes M_q \ltimes (I_4 \otimes \widehat{Y}) \ltimes \delta_4^1 = (M_q \otimes I_m)(M_q \otimes \widehat{X})(\delta_4^1 \otimes \widehat{Y}).$$

Then the equivalent condition holds.
□

3.3. $\mathcal{GH}$-Representation of Special Matrices

The $\mathcal{H}$-representation method helps us find the relationship between some elements of special structure matrices and their independent elements. $\mathcal{H}$-representation is defined as follows.

Definition 3.8. *[30] Consider a p-dimensional real matrix subspace $\mathbb{X} \subset \mathbb{R}^{n\times n}$. Assume that $e_1, e_2, \ldots, e_p$ form the basis of $\mathbb{X}$, with a $p \times 1$ vector $\widetilde{X} = \begin{bmatrix} x_1 & x_2 & \ldots & x_p \end{bmatrix}^{\mathrm{T}}$ and $X = \sum_{i=1}^{p} x_i e_i$, then define $H = \begin{bmatrix} V_c(e_1) & V_c(e_2) & \ldots & V_c(e_p) \end{bmatrix}$. For each $X \in \mathbb{X}$, if we express $\psi(X) = V_c(X)$ in the form of*

$$\psi(X) = H\widetilde{X},$$

then $H\widetilde{X}$ is called an $\mathcal{H}$-representation of $\psi(X)$, and H is called an $\mathcal{H}$-representation matrix of $\psi(X)$.

Remark 3.9. *The $\mathcal{H}$-representation method can well help us extract the independent elements of special real matrices. But for complex or quaternion Hermitian matrix, the matrices corresponding to its real and imaginary parts are symmetric matrix and anti-symmetric matrix, respectively, so we cannot consider complex or quaternion Hermitian matrix as a whole for $\mathcal{H}$-representation. Instead, we need to discuss the real and imaginary part matrices separately.*

For $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \mathbb{Q}^{n\times n}$, let $S = \{X_1, X_2, X_3, X_4\}$. A permutation σ on S is a one-to-one mapping from S to S. The definition of $\mathcal{H}$-representation motivates us to define a $\mathcal{GH}$-representation of quaternion matrix.

Definition 3.10. *Consider a quaternion matrix subspace $\mathbb{X} \subset \mathbb{Q}^{n\times n}$, for each $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \mathbb{X}$, denote $\mathcal{X} = \begin{bmatrix} \pm\sigma(X_1) & \pm\sigma(X_2) & \pm\sigma(X_3) & \pm\sigma(X_4) \end{bmatrix}$. If we express $\Phi(X) = V_c(\mathcal{X})$ in the form of*

$$\Phi(X) = \mathrm{H_G}\widetilde{X},$$

where $\mathrm{H_G} = \begin{bmatrix} H_{\sigma(\mathrm{X}_1)} & & & \\ & H_{\sigma(\mathrm{X}_2)} & & \\ & & H_{\sigma(\mathrm{X}_3)} & \\ & & & H_{\sigma(\mathrm{X}_4)} \end{bmatrix}$, $\widetilde{X} = \widetilde{V_c(\mathcal{X})}$ *represents a permutation of independent elements for each part of* $V_c(\mathcal{X})$. *Then* $\mathrm{H_G}\widetilde{X}$ *is called a* $\mathcal{GH}$*-representation of* $\Phi(\mathrm{X})$, *and* $\mathrm{H_G}$ *is called a* $\mathcal{GH}$*-representation matrix of* $\Phi(\mathrm{X})$.

The interest of this paper lies in the structure of quaternion (anti)-η-Hermitian matrix. For quaternion η-Hermitian matrices, its real and imaginary parts have the following characteristics. By Definition 1.1, if A is a quaternion η-Hermitian matrix, we can obtain

$$A_1^{\mathrm{T}} = A_1; A_2^{\mathrm{T}} = \begin{cases} -A_2, \eta = \mathrm{i} \\ A_2, \eta \neq \mathrm{i} \end{cases}; A_3^{\mathrm{T}} = \begin{cases} -A_3, \eta = \mathrm{j} \\ A_3, \eta \neq \mathrm{j} \end{cases}; A_4^{\mathrm{T}} = \begin{cases} -A_4, \eta = \mathrm{k} \\ A_4, \eta \neq \mathrm{k} \end{cases}. \tag{4}$$

If A is a quaternion anti-η-Hermitian matrix, we can obtain

$$A_1^{\mathrm{T}} = -A_1; A_2^{\mathrm{T}} = \begin{cases} A_2, \eta = \mathrm{i} \\ -A_2, \eta \neq \mathrm{i} \end{cases}; A_3^{\mathrm{T}} = \begin{cases} A_3, \eta = \mathrm{j} \\ -A_3, \eta \neq \mathrm{j} \end{cases}; A_4^{\mathrm{T}} = \begin{cases} A_4, \eta = \mathrm{k} \\ -A_4, \eta \neq \mathrm{k} \end{cases}. \tag{5}$$

According to the definition of $\mathcal{GH}$-representation, if you want to apply $\mathcal{GH}$-representation to quaternion matrix with special structure, the first thing to do should be to analyze real and imaginary parts of quaternion matrix. Since the quaternion (anti)-η-Hermitian matrix closely related to the (skew)-symmetric matrix, we choose the standard basis for the (skew)-symmetric matrix as follows.

Lemma 3.11. *(1)For* $\mathbb{X} = \mathbf{S}_n$, *let*

$$\{E_{ij} : 1 \leq j \leq i \leq n\},$$

where $E_{ij} = (e_{st})_{n\times n}, e_{ij} = e_{ji} = 1$,

$$\widetilde{X_s} = (x_{11}, \ldots, x_{n1}, x_{22}, \ldots, x_{n2}, \ldots, x_{nn})^{\mathrm{T}}. \tag{6}$$

(2)For $\mathbb{X} = \mathbf{AS}_n$, *let*

$$\{F_{ij} : 1 \leq j < i \leq n-1\},$$

where $F_{ij} = (f_{st})_{n\times n}, f_{ij} = -f_{ji} = 1$,

$$\widetilde{X_a} = (x_{21}, \ldots, x_{n1}, x_{32}, \ldots, x_{n2}, \ldots, x_{n,n-1})^{\mathrm{T}}. \tag{7}$$

Example 3.12. *Let* $\mathbb{X} = \mathbf{S}_2$, $X = (x_{ij}) \in \mathbb{X}$, *and then* $dim(\mathbb{X}) = 3$. *If we select a basis of* $\mathbb{X}$ *as*

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to compute

$$\Psi(\mathrm{X}) = V_c(\mathrm{X}) = (x_{11}, \quad x_{21}, \quad x_{21}, \quad x_{22})^{\mathrm{T}},$$

and

$$\widetilde{X} = (x_{11}, \quad x_{21}, \quad x_{22})^{\mathrm{T}},\ H_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 3.13. *Let* $\mathbb{X} = \mathbf{AS}_3$, $X = (x_{ij}) \in \mathbb{X}$, *and then* $dim(\mathbb{X}) = 3$. *If we select a basis of* $\mathbb{X}$ *as*

$$F_{21} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, F_{31} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, F_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to compute

$$\Psi(X) = V_c(X) = (0,\quad x_{21},\quad x_{31},\quad -x_{21},\quad 0,\quad x_{32},\quad -x_{31},\quad -x_{32},\quad 0)^{\mathrm{T}},$$

and

$$\widetilde{X} = (x_{21},\quad x_{31},\quad x_{32})^{\mathrm{T}},\ H_s = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}^{\mathrm{T}}.$$

Definition 3.14. *Define* $\varphi : X_s = (x_{ij}) \in \mathbf{S}_n \longmapsto \widetilde{X_s}$*, where* $\widetilde{X_s}$ *is defined in* (6)*; define* $\chi : X_a = (x_{ij}) \in \mathbf{AS}_n \longmapsto \widetilde{X_a}$*, where* $\widetilde{X_a}$ *is defined in* (7)*.*

For the sake of clarity, we denote the $\mathcal{H}$-representation matrix corresponding to $\mathbb{X} = \mathbf{S}_n$ by H_s, the $\mathcal{H}$-representation matrix corresponding to $\mathbb{X} = \mathbf{AS}_n$ by H_a.

Theorem 3.15. *(1)If* $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \mathbb{Q}^{n\times n}$*, then*

$$X \in \eta\mathbb{HQ}^{n\times n} \Longleftrightarrow \Phi(X) = \mathrm{H}_{\eta q} V_{c\eta q}(X).$$

(2)If $Y = Y_1 + Y_2\mathrm{i} + Y_3\mathrm{j} + Y_4\mathrm{k} \in \mathbb{Q}^{n\times n}$*, then*

$$Y \in \eta\mathbb{AHQ}^{n\times n} \Longleftrightarrow \Phi(Y) = \mathrm{H}_{\eta aq} V_{c\eta aq}(Y).$$

By analyzing the structure of (anti)-η-Hermitian matrix in (4) *and* (5)*, we know that* $\mathrm{H}_{\eta q}$ *and* $\mathrm{H}_{\eta aq}$ *is a diagonal matrix composed of* H_s *and* H_a*, and the positions of* H_s *and* H_a *change with the value of* $\eta \in \{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$ *and* $\Phi(X)$*. And* $V_{c\eta q}(X)$ *and* $V_{c\eta aq}(Y)$ *represents the combinations of column vectors consisting of independent elements corresponding to each part of* $\Phi(X)$ *and* $\Phi(Y)$*.*

Proof. (1) When $\eta = \mathrm{i}$, we have $X_k^{\mathrm{T}} = X_k (k = 1, 3, 4)$ and $X_2^{\mathrm{T}} = -X_2$ for quaternion matrix $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \eta\mathbb{HQ}^{n\times n}$. Then by Definition 3.8 and Definition 3.10, if we select $\mathcal{X} = \begin{bmatrix}\sigma(X_1) & \sigma(X_2) & \sigma(X_3) & \sigma(X_4)\end{bmatrix}$, and $\sigma(X_1) = X_1, \sigma(X_2) = X_2, \sigma(X_3) = X_3, \sigma(X_4) = X_4$, then we have

$$\Phi(X) = \begin{bmatrix} V_c(X_1) \\ V_c(X_2) \\ V_c(X_3) \\ V_c(X_4) \end{bmatrix} = \begin{bmatrix} H_s & & & \\ & H_a & & \\ & & H_s & \\ & & & H_s \end{bmatrix} \begin{bmatrix} \widetilde{X_1} \\ \widetilde{X_2} \\ \widetilde{X_3} \\ \widetilde{X_4} \end{bmatrix} = \mathrm{H}_{\eta q} V_{c\eta q}(X),$$

where the final form of $\mathrm{H}_{\eta q}$ and $V_{c\eta q}(X)$ is determined by the permutation σ. Other cases can be proved similarly. □

4. Numerical Solution of Problem 1

Lemma 4.1. *[38] The least squares solutions of the linear system of equations* $Ax = b$*, with* $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$*, can be represented as*

$$x = A^{\dagger}b + (I - A^{\dagger}A)y,$$

where $y \in \mathbb{R}^n$ *is an arbitrary vector. The minimal norm least squares solution of the linear system of equations* $Ax = b$ *is* $A^{\dagger}b$*.*

Lemma 4.2. *[38] The linear system of equations* $Ax = b$*, with* $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$*, has a solution* $x \in \mathbb{R}^n$ *if and only if*

$$AA^{\dagger}b = b.$$

In this case, it has the general solution

$$x = A^{\dagger}b + (I - A^{\dagger}A)y,$$

where $y \in \mathbb{R}^n$ *is an arbitrary vector. And the minimal norm solution of the linear system of equations* $Ax = b$ *is* $A^{\dagger}b$*.*

According to the above description, through the discussion of Problem 1, the following conclusion is obtained. First we define the symbols to be used as

$$U=\varphi(I_n\otimes A_1)K_{n^2}\varphi(B_1^{\mathrm{H}}\otimes I_n)K_{n^2},\ V=\varphi(I_n\otimes A_2)K_{n^2}\varphi(B_2^{\mathrm{H}}\otimes I_n)K_{n^2},$$

$$K_{n^2}=\begin{bmatrix} I_{n^2} & 0 & 0 & 0\\ 0 & -I_{n^2} & 0 & 0\\ 0 & 0 & -I_{n^2} & 0\\ 0 & 0 & 0 & -I_{n^2}\end{bmatrix},\ H_\eta=\begin{bmatrix}\mathrm{H}_{\eta q} & 0\\ 0 & \mathrm{H}_{\eta aq}\end{bmatrix},\ E=\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}H_\eta.$$

Theorem 4.3. *Suppose $A_i, B_i, C_i \in \mathbb{Q}^{n\times n}, (i=1,2)$, and $X=X_1+X_2\mathrm{i}+X_3\mathrm{j}+X_4\mathrm{k}\in\eta\mathbb{H}\mathbb{Q}^{n\times n}$, $Y=Y_1+Y_2\mathrm{i}+Y_3\mathrm{j}+Y_4\mathrm{k}\in\eta\mathbb{A}\mathbb{H}\mathbb{Q}^{n\times n}$. The (anti)-η-Hermitian solution set of the system of equations* (1) *over the quaternion algebra can be represented as*

$$M_r=\left\{X\in\eta\mathbb{H}\mathbb{Q}^{n\times n}, Y\in\eta\mathbb{A}\mathbb{H}\mathbb{Q}^{n\times n}\,\middle|\,\begin{bmatrix}\varphi^c(V_c(X))\\ \varphi^c(V_c(Y))\end{bmatrix}=H_\eta E^\dagger\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}+H_\eta(I_{4n^2}-E^\dagger E)y\right\},$$

where $\forall y\in\mathbb{R}^{4n^2}$. And the minimal norm least squares (anti)-η-Hermitian solution of the system of equations (1) *over the quaternion algebra is*

$$\begin{bmatrix}\varphi^c(V_c(X))\\ \varphi^c(V_c(Y))\end{bmatrix}=H_\eta E^\dagger\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}. \tag{8}$$

Proof. By Theorem 3.1, Theorem 3.15 and $\mathcal{L}$-representation, we can deduce

$$\begin{aligned}
&\|A_1XB_1-C_1\|^2+\|A_2YB_2-C_2\|^2\\
=&\|V_c(A_1XB_1)-V_c(C_1)\|^2+\|V_c(A_2YB_2)-V_c(C_2)\|^2\\
=&\|(I_n\otimes A_1)\overline{(B_1^{\mathrm{H}}\otimes I_n)V_c(\overline{X})}-V_c(C_1)\|^2+\|(I_n\otimes A_2)\overline{(B_2^{\mathrm{H}}\otimes I_n)V_c(\overline{Y})}-V_c(C_2)\|^2\\
=&\|\varphi^c((I_n\otimes A_1)\overline{(B_1^{\mathrm{H}}\otimes I_n)V_c(\overline{X})})-\varphi^c(V_c(C_1))\|^2+\|\varphi^c((I_n\otimes A_2)\overline{(B_2^{\mathrm{H}}\otimes I_n)V_c(\overline{Y})})-\varphi^c(V_c(C_2))\|^2\\
=&\|\varphi(I_n\otimes A_1)\varphi^c(\overline{(B_1^{\mathrm{H}}\otimes I_n)V_c(\overline{X})})-\varphi^c(V_c(C_1))\|^2+\|\varphi(I_n\otimes A_2)\varphi^c(\overline{(B_2^{\mathrm{H}}\otimes I_n)V_c(\overline{Y})})-\varphi^c(V_c(C_2))\|^2\\
=&\|\varphi(I_n\otimes A_1)K_{n^2}\varphi(B_1^{\mathrm{H}}\otimes I_n)K_{n^2}\varphi^c(V_c(X))-\varphi^c(V_c(C_1))\|^2+\|\varphi(I_n\otimes A_2)K_{n^2}\varphi(B_2^{\mathrm{H}}\otimes I_n)K_{n^2}\varphi^c(V_c(Y))-\varphi^c(V_c(C_2))\|^2\\
=&\left\|\begin{bmatrix}U\varphi^c(V_c(X))\\ V\varphi^c(V_c(Y))\end{bmatrix}-\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}\right\|^2=\left\|\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\begin{bmatrix}\varphi^c(V_c(X))\\ \varphi^c(V_c(Y))\end{bmatrix}-\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}\right\|^2\\
=&\left\|\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\begin{bmatrix}\mathrm{H}_{\eta q} & 0\\ 0 & \mathrm{H}_{\eta aq}\end{bmatrix}\begin{bmatrix}V_{c\eta q}(X)\\ V_{c\eta aq}(Y)\end{bmatrix}-\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}\right\|^2=\left\|E\begin{bmatrix}V_{c\eta q}(X)\\ V_{c\eta aq}(Y)\end{bmatrix}-\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}\right\|^2.
\end{aligned}$$

Thus

$$\|A_1XB_1-C_1\|^2+\|A_2YB_2-C_2\|^2=\min,$$

if and only if

$$\left\|E\begin{bmatrix}V_{c\eta q}(X)\\ V_{c\eta aq}(Y)\end{bmatrix}-\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}\right\|^2=\min.$$

For the real matrix equation

$$E\begin{bmatrix}V_{c\eta q}(X)\\ V_{c\eta aq}(Y)\end{bmatrix}=\begin{bmatrix}\varphi^c(V_c(C_1))\\ \varphi^c(V_c(C_2))\end{bmatrix}.$$

The least squares (anti)-η-Hermitian solution can be represented as

$$\begin{bmatrix} V_{c\eta q}(X) \\ V_{c\eta aq}(Y) \end{bmatrix} = E^{\dagger} \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} + (I_{4n^2} - E^{\dagger}E)y, \forall y \in \mathbb{R}^{4n^2}.$$

By $\mathcal{GH}$-representation, we have

$$\begin{bmatrix} \varphi^c(V_c(X)) \\ \varphi^c(V_c(Y)) \end{bmatrix} = \begin{bmatrix} \mathrm{H}_{\eta q} & 0 \\ 0 & \mathrm{H}_{\eta aq} \end{bmatrix} \begin{bmatrix} V_{c\eta q}(X) \\ V_{c\eta aq}(Y) \end{bmatrix} = H_{\eta}E^{\dagger} \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} + H_{\eta}(I_{4n^2} - E^{\dagger}E)y, \forall y \in \mathbb{R}^{4n^2}.$$

And the minimal norm least squares (anti)-η-Hermitian solution of the system of equations (1) over the quaternion algebra is

$$\begin{bmatrix} \varphi^c(V_c(X)) \\ \varphi^c(V_c(Y)) \end{bmatrix} = H_{\eta}E^{\dagger} \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix}.$$

□

Theorem 4.4. *Suppose $A_i, B_i, C_i \in \mathbb{Q}^{n\times n}, (i = 1, 2)$. Then the system of equations* (1) *over the quaternion algebra has solution $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \eta\mathbb{H}\mathbb{Q}^{n\times n}$, $Y = Y_1 + Y_2\mathrm{i} + Y_3\mathrm{j} + Y_4\mathrm{k} \in \eta\mathbb{A}\mathbb{H}\mathbb{Q}^{n\times n}$ if and only if*

$$(EE^{\dagger} - I_{8n^2}) \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} = 0. \tag{9}$$

Proof. By the properties of Moore-Penrose inverse, we can get

$$\begin{aligned} &\|A_1XB_1 - C_1\|^2 + \|A_2YB_2 - C_2\|^2 \\ =& \left\| E \begin{bmatrix} V_{c\eta q}(X) \\ V_{c\eta aq}(Y) \end{bmatrix} - \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} \right\|^2 = \left\| EE^{\dagger}E \begin{bmatrix} V_{c\eta q}(X) \\ V_{c\eta aq}(Y) \end{bmatrix} - \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} \right\|^2 \\ =& \left\| EE^{\dagger} \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} - \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} \right\|^2 = \left\| (EE^{\dagger} - I_{8n^2}) \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} \right\|^2. \end{aligned}$$

The system of equations (1) over the quaternion algebra has (anti)-η-Hermitian solution if and only if

$$\|A_1XB_1 - C_1\|^2 + \|A_2YB_2 - C_2\|^2 = 0 \Longleftrightarrow \left\| (EE^{\dagger} - I_{8n^2}) \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} \right\|^2 = 0 \Longleftrightarrow (EE^{\dagger} - I_{8n^2}) \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} = 0.$$

In case that the system of equations (1) over the quaternion algebra is compatible, its (anti)-η-Hermitian solution satisfies

$$E \begin{bmatrix} V_{c\eta q}(X) \\ V_{c\eta aq}(Y) \end{bmatrix} = \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix}.$$

Moreover, by Lemma 4.2, the minimal norm (anti)-η-Hermitian solution of the system of equations (1) over the quaternion algebra satisfies

$$\begin{bmatrix} V_{c\eta q}(X) \\ V_{c\eta aq}(Y) \end{bmatrix} = E^{\dagger} \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} + (I_{4n^2} - E^{\dagger}E)y, \forall y \in \mathbb{R}^{4n^2}.$$

By $\mathcal{GH}$-representation, we have

$$\begin{bmatrix} \varphi^c(V_c(X)) \\ \varphi^c(V_c(Y)) \end{bmatrix} = \begin{bmatrix} \mathrm{H}_{\eta q} & 0 \\ 0 & \mathrm{H}_{\eta aq} \end{bmatrix} \begin{bmatrix} V_{c\eta q}(X) \\ V_{c\eta aq}(Y) \end{bmatrix} = H_{\eta}E^{\dagger} \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix} + H_{\eta}(I_{4n^2} - E^{\dagger}E)y, \forall y \in \mathbb{R}^{4n^2}.$$

And the minimal norm (anti)-η-Hermitian solution of the system of equations (1) over the quaternion algebra is

$$\begin{bmatrix} \varphi^c(V_c(X)) \\ \varphi^c(V_c(Y)) \end{bmatrix} = H_{\eta}E^{\dagger} \begin{bmatrix} \varphi^c(V_c(C_1)) \\ \varphi^c(V_c(C_2)) \end{bmatrix}.$$

□

5. Numerical algorithm and examples

Algorithm 1 Vector Representation Method of Quaternion Matrix.

Input: Quaternion matrix $A_i, B_i, C_i \in \mathbb{Q}^{n\times n}, (i = 1, 2)$;
Output: Output the minimal norm least squares (anti)-η-Hermitian solution of the system of equations (1) over the quaternion algebra according to (8);
1: Compute $\varphi^c(V_c(C_1)), \varphi^c(V_c(C_2))$;
2: Compute $U = \varphi(I_n \otimes A_1)K_{n^2}\varphi(B_1^{\mathrm{H}} \otimes I_n)K_{n^2}$, $V = \varphi(I_n \otimes A_2)K_{n^2}\varphi(B_2^{\mathrm{H}} \otimes I_n)K_{n^2}$;
3: Input H_s, H_a, compute $H_\eta = \begin{bmatrix} \mathrm{H}_{\eta q} & 0 \\ 0 & \mathrm{H}_{\eta aq} \end{bmatrix}$, $E = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} H_\eta$;
4: Calculate the minimal norm least squares solution of the system of equations over the quaternion algebra according to (8).

We select $\widehat{X} = \begin{bmatrix} X_1 \\ -X_2 \\ -X_3 \\ -X_4 \end{bmatrix}$ for $X = X_1 + X_2\mathrm{i} + X_3\mathrm{j} + X_4\mathrm{k} \in \mathbb{Q}^{n\times n}$ as an example in the following numerical example, in this case, the matrix representation we get is the form of $\varphi^2(X)$ in Example 3.5.

Example 5.1. *Let $A_i, B_i \in \mathbb{Q}^{n\times n}, (i = 1, 2)$ be generated randomly for $n = 2K(K = 1 : 20)$. Randomly generate η-Hermitian matrix X_η and anti-η-Hermitian Y_a, respectively. Then for the left side of the system of equations* (1) *over the quaternion algebra, replace X with X_η, Y with Y_a, calculate* $\begin{cases} C_1 = A_1X_\eta B_1 \\ C_2 = A_2Y_aB_2 \end{cases}$. *For the system of equations* (1) *over the quaternion algebra with $A_i, B_i, C_i, (i = 1, 2)$ above, its numerical solutions can be obtained by using Algorithm 1. As the dimension changes, the order of error between the numerical solution and the real solution is shown in Figure 1.*

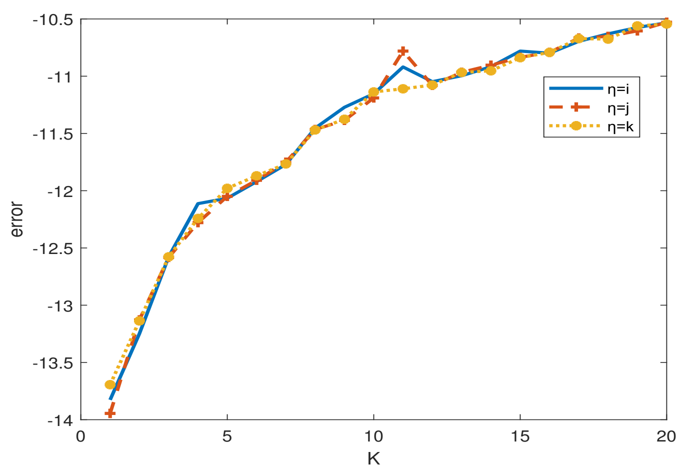

Figure 1: The Order of Error

Figure 1 shows that the error between the numerical solution obtained by Algorithm 1 and the real solution is very small, which fully demonstrates the effectiveness of the algorithm in this paper.

Next, taking the η-Hermitian solution of quaternion matrix equation $AXB = C$ as an example, we compare the method of solving the special solution of quaternion matrix equation in this paper with the method in reference [25], [31] and [32]. For quaternion matrix equation $AXB = C$, we give the numerical calculations of the algorithms in this paper with η-Hermitian solution as follows.

Algorithm 2 Calculate the minimal norm least squares η-Hermitian solution of quaternion matrix equation $AXB = C$ according to vector representation method in this paper.

Input: Quaternion matrix $A, B, C \in \mathbb{Q}^{n\times n}$;

Output: Output the minimal norm least squares η-Hermitian solution $\widehat{X}_\eta$ of quaternion matrix equation $AXB = C$;

1: Compute $\varphi^c(V_c(C))$, $U = \varphi(I_n \otimes A)K_{n^2}\varphi(B^{\mathrm{H}} \otimes I_n)K_{n^2}$,

2: Input H_s, H_a, compute $\mathrm{H}_{\eta q}$, $E = U\mathrm{H}_{\eta q}$,

3: Calculate the minimal norm least squares η-Hermitian solution of quaternion matrix equation $AXB = C$ according to $\widehat{X}_\eta = \mathrm{H}_{\eta q}E^\dagger\overrightarrow{C}$.

The algorithm in reference [25] is based on semi-tensor product of matrices and the real vector representation, and quaternion matrix equation is converted into real vector matrix equation. In reference [31] and [32], direct algorithms for solving quaternion matrix equations are given by using the real and complex representations of quaternion matrices, respectively. Next, we give the comparison results between the algorithm in this paper and the algorithms in reference [25], [31] and [32].

Example 5.2. *Let $A, B \in \mathbb{Q}^{n\times n}$ be generated randomly for $n = 4K(K = 1 : 10)$. Randomly generate η-Hermitian matrix X_η, let $\eta = \mathrm{i}$. Then calculate $C = AX_\eta B$. For the quaternion matrix equation $AXB = C$ with A, B, C above, its computational solutions can be obtained by using Algorithm 2 and the methods in reference [25], [31] and [32]. As the dimension changes, the time consumed by these algorithms is shown in Table 1 and the error comparison is shown in Figure 2. For real vector representation method in reference [25], because the matrix dimension expands rapidly during calculation, we only choose $K = 1 : 5$. If the form of the solution obtained by the algorithm in reference [31] wants to be consistent with the form of the solution obtained by the algorithm in this paper, it needs to be transformed with the help of a large matrix, while the method of finding the generalized inverse by blocks in reference [32] increases the complexity of the algorithm. So from the time comparison results, it can be seen that the method presented in this paper can be used to calculate quaternion linear systems more efficiently.*

Table 1: CPU Time Comparison Results

n	Algorithm 2	Method in [25]	Method in [31]	Method in [32]
4	0.0032	0.0144	0.0045	0.0070
8	0.0073	0.7949	0.0097	0.0237
12	0.0462	8.9890	0.0506	0.1243
16	0.1509	64.8089	0.2353	0.4440
20	0.5913	706.2046	0.7439	1.5042
24	1.2788	*	1.8783	4.0585
28	3.3768	*	4.4529	12.6217
32	7.1360	*	9.8549	37.2051
36	14.5462	*	18.9817	73.0678
40	27.1371	*	36.5951	142.6037

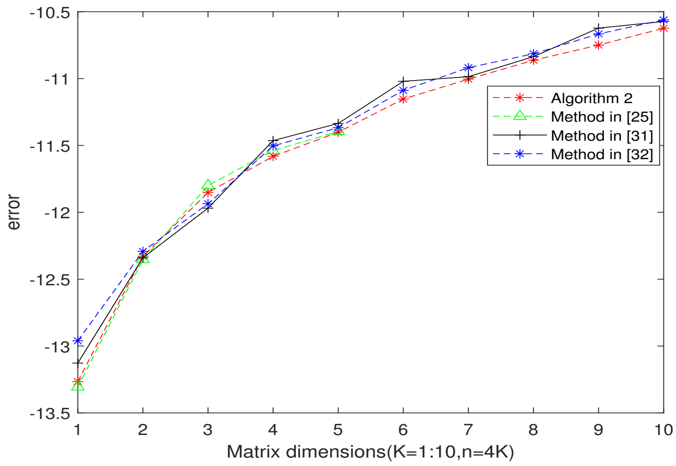

Figure 2: The Order of Error in Different Methods

Next, we apply the algorithm in this paper to time-varying linear systems.

Example 5.3. *Considering the design of the continuous return to zero dynamics of linear time-varying systems, the matrix formalism can define the design formula as*

$$\dot{L}(t) = -\lambda J(\mathbf{L}(t))$$

and the $J(\cdot) : \mathbb{Q}^{n\times n} \rightarrow \mathbb{Q}^{n\times n}$ *is denoted as an array of j functions, then*

$$J(\mathbf{L}(t)) = \begin{bmatrix} j(l_{11}(t)) & j(l_{12}(t)) & \cdots & j(l_{1n}(t)) \\ j(l_{21}(t)) & j(l_{22}(t)) & \cdots & j(l_{2n}(t)) \\ \vdots & \vdots & \ddots & \vdots \\ j(l_{n1}(t)) & j(l_{n2}(t)) & \cdots & j(l_{nn}(t)) \end{bmatrix},$$

where $j(l_{ij}(t)) = \begin{cases} 1 \; l_{ij}(t) > 0 \\ 0 \; l_{ij}(t) \leq 0 \end{cases}$. *And the time derivative is*

$$\mathbf{L}(t) = \dot{A}(t)X(t)B(t) + A(t)\dot{X}(t)B(t) + A(t)X(t)\dot{B}(t).$$

After conversion, the above equation can be equivalently transformed into a quaternion matrix equation $AXB = C$, *defining the parameters as:*

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 \\ 3 & 2 & 2 & 2 \\ 4 & 3 & 4 & 2 \\ 2 & 4 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 2 & 4 \\ 2 & 3 & 0 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 2 & 5 & 4 \end{bmatrix}\mathrm{i} + \begin{bmatrix} 4 & 2 & 5 & 4 \\ 4 & 2 & 4 & 2 \\ 5 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}\mathrm{j} + \begin{bmatrix} 1 & 3 & 1 & 4 \\ 4 & 3 & 0 & 4 \\ 5 & 1 & 0 & 3 \\ 0 & 4 & 4 & 4 \end{bmatrix}\mathrm{k};$$

$$B = \begin{bmatrix} 0 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 3 & 1 & 0 \\ 2 & 0 & 4 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 4 & 4 \\ 5 & 5 & 3 & 1 \\ 1 & 2 & 2 & 3 \\ 2 & 0 & 4 & 2 \end{bmatrix}\mathrm{i} + \begin{bmatrix} 1 & 4 & 1 & 4 \\ 3 & 3 & 2 & 4 \\ 5 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{bmatrix}\mathrm{j} + \begin{bmatrix} 3 & 2 & 0 & 1 \\ 3 & 3 & 5 & 4 \\ 1 & 5 & 5 & 3 \\ 2 & 2 & 2 & 3 \end{bmatrix}\mathrm{k};$$

$$C = \begin{bmatrix} -2937 & -3514 & -4276 & -4152 \\ -2541 & -3272 & -3632 & -3611 \\ -2286 & -2865 & -2966 & -2969 \\ -2366 & -2889 & -3238 & -3301 \end{bmatrix} + \begin{bmatrix} -1543 & -547 & -995 & -1383 \\ -1608 & -952 & -1192 & -1721 \\ -1210 & -905 & -737 & -1590 \\ -1755 & -1186 & -1373 & -2404 \end{bmatrix}\mathrm{i} + \begin{bmatrix} 150 & -199 & 804 & -108 \\ 797 & 381 & 1672 & 841 \\ 1436 & 817 & 2246 & 1633 \\ 1959 & 1251 & 2598 & 1803 \end{bmatrix}\mathrm{j} +$$

$\begin{bmatrix} 54 & 581 & 230 & 758 \\ -37 & 651 & 350 & 713 \\ 700 & 1568 & 1439 & 1741 \\ 910 & 1907 & 2065 & 2103 \end{bmatrix}$k. *Using Algorithm 2, we can calculate that the quaternion matrix equation $AXB = C$ has a unique η-Hermitian($\eta = \mathrm{i}$) solution, which is*

$$X = \begin{bmatrix} 6 & 0 & 7 & 9 \\ 0 & 8 & 8 & 3 \\ 7 & 8 & 4 & 6 \\ 9 & 3 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & -1 \\ -5 & 0 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}\mathrm{i} + \begin{bmatrix} -8 & -3 & -3 & -4 \\ -3 & -4 & -7 & -7 \\ -3 & -7 & 0 & -5 \\ -4 & -7 & -5 & -10 \end{bmatrix}\mathrm{j} + \begin{bmatrix} -4 & -8 & -5 & -6 \\ -8 & -4 & -4 & -4 \\ -5 & -4 & 0 & -7 \\ -6 & -4 & -7 & -4 \end{bmatrix}\mathrm{k}.$$

6. Conclusion

In this article, we mainly study the properties of semi-tensor product of quaternion matrices. And the conclusions of semi-tensor product of quaternion matrices are used to solve the special solution of the system of equations over the quaternion algebra, for the quaternion matrix with special structure, the $\mathcal{GH}$-representation method is used to establish the relationship between the actual column arrangement and the actual column arrangement of independent elements based on the $\mathcal{L}$-representation of quaternion matrix, so as to improve the operation efficiency. Finally the effectiveness of the proposed method is also verified by compared with the other method, the feasibility of the algorithm in this paper is found.

Conflict of interest: The authors declare that there are no conflict of interests to this work.

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