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Weighted binomial sums using characters of the symmetric group

Hacène Belbachir[a], **Ihab-Eddine Djellas**[a,b]

[a] *USTHB, Faculty of Mathematics, RECITS Laboratory, P. Box 32 El Alia, Algiers, 16111, Algeria*
[b] *CERIST Scientific and Technical Information Research Center, Algiers, Algeria*

Abstract. Irreducible characters plays a crucial role in the representation theory of finite groups, namely the symmetric group $\mathfrak{S}_n$. In the current paper, we use the irreducible characters of $\mathfrak{S}_n$ to define new weighted binomial coefficient sums. We use the Murnaghan-Nakayama rule to establish recurrence relations for these sums. As application, we employ the recurrences to derive explicit formulas for particular cases, for instance, Euler's formula for the Stirling numbers of second kind is obtained in one of the particular cases. In the Appendix, we compute the initial values of the weighted sums.

1. Introduction

It is well known that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \tag{1}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0. \tag{2}$$

The thought of generalizing sums (1) and (2) was natural; hence, Golombek [7] considered the following sum for positive integers n and s:

$$\sum_{k=1}^{n} k^s \binom{n}{k}, \tag{3}$$

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Email addresses: hacenebelbachir@gmail.com (Hacène Belbachir), hbelbachir@usthb.dz (Hacène Belbachir), ihabeddinedjellas@gmail.com (Ihab-Eddine Djellas), idjellas@usthb.dz (Ihab-Eddine Djellas)

and he established the first values of (3) for $s = 0, 1, 2$ as $2^n, n2^{n-1}$ and $n(n+1)2^{n-2}$ respectively. In general, he established the following formula:

$$\sum_{k=1}^{n} k^s \binom{n}{k} = \frac{\partial^s}{\partial t^s}(1+e^t)^n|_{t=0}. \tag{4}$$

From an arithmetic point of view, Guichard [9] gave an explicit formula of the following sum for fixed m, r $(0 \le r < m)$

$$\sum_{k=0}^{n} A_k \binom{n}{k},$$

where,

$$A_k = \begin{cases} 1 & \text{if } k = r \ (mod\ m), \\ 0 & \text{otherwise.} \end{cases}$$

In a recent work, Simsek [17] gave generating functions for the weighted binomial coefficient powers sum:

$$\sum_{k=0}^{n} k^s \binom{n}{k}^p,$$

for fixed integer p.

Several recurrence relations for generalized binomial coefficient sums were established in the literature; among them, we refer to [6], the author affirmed an 1895 conjecture which states that the sum $\sum_{k=0}^{n} \binom{n}{k}^p$ satisfies a recurrence relation of order $\lfloor (p+3)/2 \rfloor$. Years later, McIntosh [13] gave a recurrence relation for $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^p$ and also gave a recurrence relation of order $\lfloor (p+2)/2 \rfloor$ for $\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^p$ for $p \ge 2$. Further recent recurrences for various binomial coefficient sums can be found in [11].

A combinatorial approach led the authors of [1] to establish recurrences for generalized square binomial coefficient sums; the approach is based on summing over the diagonals of the square binomial coefficients triangle.

One brilliant idea by Chan and Lam [2] to generalize the sums (1) and (2) is to use irreducible characters of the symmetric group $\chi_\mu(\lambda)$ where μ and λ are partitions of the integer n, since $\chi_\mu(n) = 1$ for all partitions μ of n and $\chi_\mu(1^n) = \pm 1$ for all partitions μ of n. The authors [2] studied the weighted binomial sums $[\mu; n] = \sum_{k=0}^{n} \chi_\mu(k)\binom{n}{k}$ and they established a recurrence for $[\mu; n]$, also they gave various results to characterize $[\mu; n]$.

The weighted sums $[\mu; n]$ appear in the computation of the immanant of the Laplacian matrices of trees; see [3] and [4], and also in the computation of the immanant of the q-Laplacians of trees [14].

At the end of the paper [2], the authors asked the following question, *"Is there a recurrence relation for a generalization of* $[\mu; n]$ *defined as follows* $[\mu; n]_s = \sum_{k=0}^{n} \chi_\mu(k) k^s \binom{n}{k}$ *?"*.
In this paper, we settle the author's question by giving a new recurrence relation for $[\mu; n]_s$. Then, we derive some explicit formulas using the recurrence relation.

The present paper is organized as follows: In the next section, we provide some definitions and notation needed to read the paper. In Section 3, we give a new recurrence relation for the generalized weighted binomial coefficients sum, namely $[\mu; n]_s = \sum_{k=0}^{n} \chi_\mu(k) k^s \binom{n}{k}$, and we prove that $[\mu; n]_s$ is a multiple of power two. As an application, in Section 4, we use the recurrence relation to derive some explicit formulas. Finally, we provide a few concerns that were not covered in this study. In the Appendix, we give the initial values of the weighted sums.

2. Preliminaries and notation

In order to render the work self-contained, we recall some background on the representations of symmetric groups along with some notation that will be used later.
Let $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ be a partition of an integer m, where $\mu_i \in \mathbb{N}$, for all $i \in \{1, 2, \dots, k\}$ and the μ_i's are taken in decreasing order. A partition μ of m, denoted $\mu \vdash m$, can be represented by a Ferrers diagram, which is a collection of boxes left justified, with μ_1 boxes in the first line, μ_2 boxes in the second one, and so on. If we neglect the decreasing order constraint, μ is called a *composition* of m.

Let $\mathfrak{S}_m$ be the symmetric group of degree m, K a commutative field, and V a vector space over K. The *representation* of the symmetric group $\mathfrak{S}_m$ is the morphism of groups of $\mathfrak{S}_m$ in the general linear group $GL(V)$. ($GL(V)$ is the group of automorphisms of V)
Furthermore, a subrepresentation of $\mathfrak{S}_m$ is the determination of a subspace W of V which is invariant by the action of $\mathfrak{S}_m$. A representation of $\mathfrak{S}_m$ is said to be *irreducible* if it has only trivial subrepresentations.

If V is a finite dimensional vector space, then $GL(V)$ is the group of invertible matrices, and the *character* of a representation of $\sigma \in \mathfrak{S}_m$ is defined to be the trace of the associated matrix and is denoted χ_σ.
An irreducible character of $\mathfrak{S}_m$ is the character of an irreducible representation.

It is well known that the characters of a finite group are class functions, and hence the irreducible characters χ_μ of the symmetric group $\mathfrak{S}_m$ are indexed by partitions μ of m; see [12, Chapter 3.6].

Several recurrences are used to compute the irreducible characters of $\mathfrak{S}_m$. The branching rule (see, for example [10] and [12]) states that we can express the irreducible characters of $\mathfrak{S}_m$ in terms of irreducible characters of $\mathfrak{S}_{m-1}$. A more general rule by Nakayama [15], is the Murnaghan-Nakayama rule based on combinatorial aspects.
Before stating the Murnaghan-Nakayama rule we need to define some combinatorial objects. Given two partitions μ and λ, a *skew diagram* $\mu\backslash\lambda$ is the set of all cells belonging to μ but not λ where $\mu_i \geq \lambda_i$, for all i. For example, if we let $\mu = (6, 2, 1^2)$ and $\lambda = (2, 1^3)$, the associated skew diagram is shown in Figure 1.

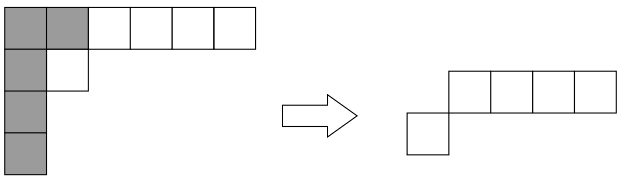

Figure 1: Skew diagram associated to $(6, 2, 1^2)\backslash(2, 1^3)$.

A skew diagram with no 2×2 square is called a *border strip*. A *skew hook* (known also as a "rim hook") ξ is a connected border strip. The height $h(\xi)$ of a skew hook is the number of its lines minus one. We can now give the Murnaghan-Nakayama rule as follows (here we are interested in the recursive version; see for example [16]):

Theorem 2.1. *If μ is a partition of m and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a composition of m, then we have*

$$\chi_\mu(\alpha) = \sum_{\xi} (-1)^{h(\xi)} \chi_{\mu\backslash\xi}(\alpha\backslash\alpha_1), \tag{5}$$

where the sum runs over all skew hooks ξ of μ having α_1 cells, $\mu\backslash\xi$ is a partition of $m - \alpha_1$, and $\alpha\backslash\alpha_1 = (\alpha_2, \alpha_3, \dots, \alpha_k)$.

We use similar notation to [2]. Let μ be a partition of m, and we define the sets $I(\mu)$ (respectively $J(\mu)$ and $K(\mu)$) as the sets containing the subpartitions of size $m - 2$ obtained from μ by removing a skew hook of type $\square\square$ (respectively $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, and two non adjacent rim hooks of size one, as follows: $\square$, $\square$). For example, let $\mu = (6, 2, 1^2)$, then $I(\mu) = \{(4, 2, 1^2)\}$, $J(\mu) = \{(6, 2)\}$ and $K(\mu) = \{(6, 1^2), (5, 2, 1), (5, 1^3)\}$.

Now, we give the proper definition of $[\mu; n]_s$. Let μ be a partition of the integer m and for $k = 0, 1, \dots, \lfloor m/2 \rfloor$; we consider the character $\chi_\mu(k)$ as the character of μ at the conjugacy class σ having exactly k distinct transpositions; i.e. σ is of the cycle type $\sigma = (2^k, 1^{m-2k})$. And define $[\mu; n]_s$ for $1 \le n \le \lfloor m/2 \rfloor$ as follows:

$$[\mu; n]_s = \sum_{k=0}^{n} \chi_\mu(k) k^s \binom{n}{k}. \tag{6}$$

For special partitions of m, classical sums are obtained. When $\mu = (m)$ we have $\chi_{(m)}(\lambda) = 1$ for all conjugacy classes λ; hence, $[(m); n]_s = \sum_{k=0}^{n} k^s \binom{n}{k}$. These sums were considered and studied by Golombek [7]. Similarly, alternating sums could be established by considering the character associated with the partition (1^m) of m; hence, we get $[(1^m); n]_s = \sum_{k=0}^{n} (-1)^k k^s \binom{n}{k}$, this sum is related to the Stirling numbers of second kind, and we establish the explicit formula for $[(1^m); n]_s$ in Section 4. Other partitions are also considered.

3. Main results

In this section, we give a new recurrence for $[\mu; n]_s = \sum_{k=0}^{n} \chi_\mu(k) k^s \binom{n}{k}$ for μ a given partition of m and $1 \le n \le \lfloor m/2 \rfloor$. Our main result is

Theorem 3.1. *Let μ be a partition of m, $m \ge 4$, $1 \le n \le \lfloor m/2 \rfloor$ and $s \ge 0$. Then*

$$[\mu; n]_s = 2 \sum_{\lambda \in I(\mu) \cup K(\mu)} [\lambda; n-1]_s + \sum_{\lambda \in I(\mu)} \sum_{j=0}^{s-1} \binom{s}{j} [\lambda; n-1]_j - \sum_{\lambda \in J(\mu)} \sum_{j=0}^{s-1} \binom{s}{j} [\lambda; n-1]_j. \tag{7}$$

where $[\lambda; 0]_0 = \chi_\lambda(0)$ and $[\lambda; 0]_s = 0$ for all $s \ge 1$.

Proof. We apply some elementary operations to the sum. Let μ be any partition of m, we have:

$$\begin{aligned}
[\mu; n]_s &= \sum_{k=0}^{n} k^s \chi_\mu(k) \binom{n}{k} = \sum_{k=0}^{n} k^s \chi_\mu(k) \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] = \sum_{k=0}^{n-1} \left[k^s \chi_\mu(k) + (k+1)^s \chi_\mu(k+1) \right] \binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} \left[k^s \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) + \sum_{\lambda \in J(\mu)} \chi_\lambda(k) + 2 \sum_{\lambda \in K(\mu)} \chi_\lambda(k) \right) + (k+1)^s \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) - \sum_{\lambda \in J(\mu)} \chi_\lambda(k) \right) \right] \binom{n-1}{k} \\
&\quad \text{(by the Murnaghan-Nakayama rule)} \\
&= \sum_{k=0}^{n-1} \left[k^s \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) + \sum_{\lambda \in J(\mu)} \chi_\lambda(k) + 2 \sum_{\lambda \in K(\mu)} \chi_\lambda(k) \right) + \sum_{j=0}^{s} \binom{s}{j} k^j \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) - \sum_{\lambda \in J(\mu)} \chi_\lambda(k) \right) \right] \binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} \left[k^s \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) + \sum_{\lambda \in J(\mu)} \chi_\lambda(k) + 2 \sum_{\lambda \in K(\mu)} \chi_\lambda(k) \right) + k^s \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) - \sum_{\lambda \in J(\mu)} \chi_\lambda(k) \right) \right. \\
&\quad \left. + \sum_{j=0}^{s-1} \binom{s}{j} k^j \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) - \sum_{\lambda \in J(\mu)} \chi_\lambda(k) \right) \right] \binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} k^s . 2 \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) + \sum_{\lambda \in K(\mu)} \chi_\lambda(k) \right) \binom{n-1}{k} + \sum_{k=0}^{n-1} \sum_{j=0}^{s-1} \binom{s}{j} k^j \left(\sum_{\lambda \in I(\mu)} \chi_\lambda(k) - \sum_{\lambda \in J(\mu)} \chi_\lambda(k) \right) \binom{n-1}{k} \\
&= 2 \sum_{\lambda \in I(\mu) \cup K(\mu)} [\lambda; n-1]_s + \sum_{\lambda \in I(\mu)} \sum_{j=0}^{s-1} \binom{s}{j} [\lambda; n-1]_j - \sum_{\lambda \in J(\mu)} \sum_{j=0}^{s-1} \binom{s}{j} [\lambda; n-1]_j.
\end{aligned}$$

□

To illustrate, we give the following examples:

Example 3.2. *Let* $\mu = (6,2,1^2)$ *and we compute* $[(6,2,1^2);3]_s = \sum_{k=0}^{3} \chi_{(6,2,1^2)}(k)k^s\binom{3}{k}$ *for* $s = 1$.
We recall that we use the notation $\chi_\lambda(k)$ *instead of* $\chi_\lambda(2^k, 1^{m-2k})$ *and* $\chi_\lambda(0)$ *instead of* $\chi_\lambda(1^m)$.
First, we perform the following direct computation:

$$\begin{aligned}[(6,2,1^2);3]_1 &= \sum_{k=0}^{3} k\cdot\chi_{(6,2,1^2)}(k)\cdot\binom{3}{k} = 3\cdot\chi_{(6,2,1^2)}(1) + 3\cdot 2\cdot\chi_{(6,2,1^2)}(2) + 3\cdot\chi_{(6,2,1^2)}(3)\\ &= 3\cdot 70 + 6\cdot(-10) + 3\cdot(-10) = 120.\end{aligned}$$

Now we compute $[(6,2,1^2);3]_1$ *using Theorem 3.1. The sets* $I(\mu)$, $J(\mu)$ *and* $K(\mu)$ *corresponds to* $\{(4,2,1^2)\}$, $\{(6,2)\}$ *and* $\{(6,1^2),(5,2,1),(5,1^3)\}$ *respectively.*

$$\begin{aligned}[(6,2,1^2);3]_1 &= 2\sum_{\lambda\in I(\mu)\cup K(\mu)}[\lambda;2]_1 + \sum_{\lambda\in I(\mu)}[\lambda;2]_0 - \sum_{\lambda\in J(\mu)}[\lambda;2]_0\\ &= 2\Big([(4,2,1^2);2]_1 + [(6,1^2);2]_1 + [(5,2,1);2]_1 + [(5,1^3);2]_1\Big) + [(4,2,1^2);2]_0 - [(6,2);2]_0\\ &= 2\Big[2\cdot\chi_{(4,2,1^2)}(1) + 2\cdot\chi_{(4,2,1^2)}(2) + 2\cdot\chi_{(6,1^2)}(1) + 2\cdot\chi_{(6,1^2)}(2) + 2\cdot\chi_{(5,2,1^2)}(1) + 2\cdot\chi_{(5,2,1^2)}(2)\\ &+2\cdot\chi_{(5,1^3)}(1) + 2\cdot\chi_{(5,1^3)}(2)\Big] + \chi_{(4,2,1^2)}(0) + 2\cdot\chi_{(4,2,1^2)}(1) + \chi_{(4,2,1^2)}(2) - \chi_{(6,2)}(0)\\ &- 2\cdot\chi_{(6,2)}(1) - \chi_{(6,2)}(2) \text{ (By definition of } [\lambda;n]_s)\\ &= 120.\end{aligned}$$

Example 3.3. *As in Example 3.2, we set* $\mu = (6,2,1^2)$ *and we compute* $[(6,2,1^2);3]_s$ *this time for* $s = 2$.
Direct computation yields:

$$\begin{aligned}[(6,2,1^2);3]_2 &= \sum_{k=0}^{3} k^2\cdot\chi_{(6,2,1^2)}(k)\cdot\binom{3}{k} = 3\cdot\chi_{(6,2,1^2)}(1) + 3\cdot 4\cdot\chi_{(6,2,1^2)}(2) + 9\cdot\chi_{(6,2,1^2)}(3)\\ &= 3\cdot 70 + 12\cdot(-10) + 9\cdot(-10) = 0.\end{aligned}$$

Next, we compute $[(6,2,1^2);3]_2$ *using the recurrence relation in Theorem 3.1.*

$$\begin{aligned}[(6,2,1^2);3]_2 &= 2\sum_{\lambda\in I(\mu)\cup K(\mu)}[\lambda;2]_2 + \sum_{\lambda\in I(\mu)}\sum_{k=0}^{1}\binom{2}{k}[\lambda;2]_k - \sum_{\lambda\in J(\mu)}\sum_{k=0}^{1}\binom{2}{k}[\lambda;2]_k\\ &= 2\Big([(4,2,1^2);2]_2 + [(6,1^2);2]_2 + [(5,2,1);2]_2 + [(5,1^3);2]_2\Big) + [(4,2,1^2);2]_0 + 2[(4,2,1^2);2]_1\\ &- [(6,2);2]_0 - 2[(6,2);2]_1\\ &= 2\Big[2\cdot\chi_{(4,2,1^2)}(1) + 4\cdot\chi_{(4,2,1^2)}(2) + 2\cdot\chi_{(6,1^2)}(1) + 4\cdot\chi_{(6,1^2)}(2) + 2\cdot\chi_{(5,2,1^2)}(1) + 4\cdot\chi_{(5,2,1^2)}(2)\\ &+2\cdot\chi_{(5,1^3)}(1) + 4\cdot\chi_{(5,1^3)}(2)\Big] + \chi_{(4,2,1^2)}(0) + 2\cdot\chi_{(4,2,1^2)}(1) + \chi_{(4,2,1^2)}(2)\\ &+ 2\Big[2\cdot\chi_{(4,2,1^2)}(1) + 2\cdot\chi_{(4,2,1^2)}(2)\Big] - \chi_{(6,2)}(0) - 2\cdot\chi_{(6,2)}(1) - \chi_{(6,2)}(2)\\ &- 2\Big[2\cdot\chi_{(6,2)}(1) + 2\cdot\chi_{(6,2)}(2)\Big] \text{ (By definition of } [\lambda;n]_s)\\ &= 0.\end{aligned}$$

From Theorem 3.1, if we set $s = 0$ then $[\mu;n]_0 = \sum_{k=0}^{n}\chi_\mu(k)\binom{n}{k}$ and we get the following corollary:

Corollary 3.4 ([2]). *For* $s = 0$,

$$[\mu;n]_0 = 2\sum_{\lambda\in I(\mu)\cup K(\mu)}[\lambda;n-1]_0.$$

Next, we give the following result for $[\mu; n]_s$,

Theorem 3.5. *For a partition μ of m and an integer n such that $2 \leq n \leq \lfloor m/2 \rfloor$ we have $[\mu; n]_s$ is an integral multiple of 2^{n-s}.*

Proof. We induct on n. For $n = 2$ and all partitions μ of m we have,

$$[\mu; 2]_s = \sum_{k=0}^{2} \chi_\mu(k) k^s \binom{2}{k} = 2\chi_\mu(1) + 2^s \chi_\mu(2) = 2^{2-s} \left(\chi_\mu(1) 2^{s-1} + \chi_\mu(2) 2^{2s-2} \right).$$

Now we suppose, for all $2 \leq j \leq n-1$, that the result holds hence, $[\mu; j]_s = \alpha \cdot 2^{j-s}$ where $\alpha \in \mathbb{Z}$. And we prove the result for $j = n$.
We use recurrence given from Theorem 3.1,

$$[\mu; n]_s = 2 \sum_{\lambda \in I(\mu) \cup K(\mu)} [\lambda; n-1]_s + \sum_{\lambda \in I(\mu)} \sum_{k=0}^{s-1} \binom{s}{k} [\lambda; n-1]_k - \sum_{\lambda \in J(\mu)} \sum_{k=0}^{s-1} \binom{s}{k} [\lambda; n-1]_k,$$

and by recurrence hypothesis, we get

$$[\mu; n]_s = 2 \sum_{\lambda \in I(\mu) \cup K(\mu)} \alpha_i \cdot 2^{n-s-1} + \sum_{\lambda \in I(\mu)} \sum_{k=0}^{s-1} \binom{s}{k} \beta_j \cdot 2^{n-k-1} - \sum_{\lambda \in J(\mu)} \sum_{k=0}^{s-1} \binom{s}{k} \gamma_k \cdot 2^{n-k-1},$$

where $1 \leq i \leq |I(\mu) \cup K(\mu)|$, $1 \leq j \leq |I(\mu)|$ and $1 \leq k \leq |J(\mu)|$.
The result follows by regrouping the sums. □

4. Explicit formulas

In the present section, we use the recurrence relation given in Theorem 3.1 to establish some explicit formulas.

We consider particular partitions μ of m and evaluate the sum $[\mu; n]_s$ explicitly in what follows. The partitions considered are in the set: $\{(m), (m-1, 1), (2, 1^{m-2}), (1^m)\}$.

Let $\mu = (m)$ be the trivial partition of the integer m, then we have (see [7])

$$[(m); n]_s = \sum_{k=0}^{n} k^s \binom{n}{k} = \frac{\partial^s}{\partial t^s} (1 + e^t)^n |_{t=0}. \tag{8}$$

Now we consider the partition (1^m) of m; hence, $[(1^m); n]_s = \sum_{k=0}^{n} (-1)^k k^s \binom{n}{k}$ and we get the following corollary:

Corollary 4.1. *Let $\mu = (1^m)$ be a partition of m with $1 \leq n \leq \lfloor m/2 \rfloor$ and $s \geq 0$. Then*

$$[(1^m); n]_s = -\sum_{j=0}^{s-1} \binom{s}{j} [(1^{m-2}); n-1]_j, \tag{9}$$

where for all partitions λ of m, $[\lambda; 0]_0 = \chi_\lambda(0)$ and $[\lambda; 0]_s = 0$ for all $s \geq 1$.

Proof. Let $\mu = (1^m)$ be a partition of m. The sets $I(\mu)$ and $K(\mu)$ are empty in this case, and $J(\mu)$ contains only one partition, that is $\lambda = (1^{m-2})$, and the result is deduced directly from Theorem 3.1. □

Next, we establish the following result,

Theorem 4.2. *Let $\mu = (1^m)$ be a partition of m with $1 \le n \le \lfloor m/2 \rfloor$ and $s \ge 0$. Then*

$$[(1^m);n]_s = \sum_{k=0}^{n}(-1)^k k^s \binom{n}{k} = (-1)^n n! \left\{ {s \atop n} \right\}, \tag{10}$$

where $\left\{ {s \atop n} \right\}$ is the Stirling number of the second kind.

Proof. We use mathematical induction on n. For $s \ge 0$ and $n = 1$, $[(1^m);1]_s = -1 = (-1)^1 1! \left\{ {s \atop 1} \right\}$.
Suppose now Formula (10) is true, for $2 \le k \le n-1$, an let us prove it for n. By Corollary 4.1, [5, Equation 3*c* p. 209], and the induction hypothesis we have:

$$\begin{aligned}
[(1^m);n]_s &= -\sum_{j=0}^{s-1}[(1^{m-2});n-1]_j = -\sum_{j=0}^{s-1}\binom{s}{j}(-1)^{n-1}(n-1)!\left\{ {j \atop n-1} \right\} \\
&= (-1)^n (n-1)! \sum_{j=1}^{s-1}\binom{s}{j}\left\{ {j \atop n-1} \right\} = (-1)^n (n-1)! n \left\{ {s \atop n} \right\} \\
&= (-1)^n n! \left\{ {s \atop n} \right\}.
\end{aligned}$$

□

Remark 4.3. *The Equation* (10) *is Euler's closed form of the Stirling numbers of the second kind, see for example [5, Equation* 1*b p.* 204*]. We provided an alternative proof using the recurrence in Corollary 4.1.*

We deduce the following result from Theorem 4.2,

Corollary 4.4. *Consider $\mu = (1^m)$ the partition of the integer m with $1 \le n \le \lfloor m/2 \rfloor$ and $s \ge 0$. Then $[(1^m);n]_s = 0$ for all $n > s$.*

Proof. From (10) and [5, Equation 1*a* p. 204], the result is easily seen. □

After that, we consider the partition $\mu = (m-1,1)$ of the integer m and the weighted sum that corresponds to $\mu = (m-1,1)$ in this case is: $[(m-1,1);n]_s = \sum_{k=0}^{n} \chi_{(m-1,1)}(k) k^s \binom{n}{k}$. We give the following corollary:

Corollary 4.5. *Let $\mu = (m-1,1)$ be a partition of m with $1 \le n \le \lfloor m/2 \rfloor$ and $s \ge 0$. Then*

$$[(m-1,1);n]_s = 2[(m-3,1);n-1]_s + 2[(m-2);n-1]_s + \sum_{j=0}^{s-1}\binom{s}{j}[(m-3,1);n-1]_j, \tag{11}$$

where for all partitions λ of m, $[\lambda;0]_0 = \chi_\lambda(0)$ and $[\lambda;0]_s = 0$ for all $s \ge 1$.

Proof. Let $\mu = (m-1,1)$ be a partition of m. The result is deduced from Theorem 3.1 since the sets $I(\mu)$, $J(\mu)$, and $K(\mu)$ corresponds to $\{(m-3,1)\}$, $\{\emptyset\}$, and $\{(m-2)\}$ respectively. □

Next, in Theorem 4.6, we establish the explicit formula for the sum $[(m-1,1);n]_1 = \sum_{k=0}^{n} \chi_{(m-1,1)}(k) k \binom{n}{k}$.

Theorem 4.6. *Let $\mu = (m-1,1)$ be a partition of m with $1 \le n \le \lfloor m/2 \rfloor$. Then*

$$[(m-1,1);n]_1 = \left(-n^2 + (m-2)n\right)2^{n-1}. \tag{12}$$

Proof. We use mathematical induction on n. For $n = 1$, and by Corollary 4.5:
$[(m-1,1);1]_1 = [(m-3,1);0]_0 = \chi_{(m-3,1)}(0) = \binom{m-3}{m-4} = m-3$. (By the Hook length formula; see [16, p.92])
Suppose now Formula (12) is true, for $2 \le k \le n-1$, an let us prove it for n. By Corollary 4.5, the induction hypothesis, Equation (4), and [2, Theorem 3.1] we have:

$$\begin{aligned}[(m-1,1);n]_1 &= 2[(m-3,1);n-1]_1 + 2[(m-2);n-1]_1 + [(m-3,1);n-1]_0 \\ &= 2\Big((-(n-1)^2 + ((m-2)-2)(n-1))2^{n-2}\Big) + 2\Big((n-1)2^{n-2}\Big) + 2^{n-1}\binom{(m-2)-(n-1)-1}{(m-3)-(n-1)-1} \\ &= \Big(-n^2 + (m-2)n\Big)2^{n-1}.\end{aligned}$$

□

Another interesting partition μ of the integer m is the partition $\mu = (2, 1^{m-2})$. The associated character is used to define the generalized matrix function known as the second immanant [8]. We will consider the case of our sum where the partition is $\mu = (2, 1^{m-2})$, first we give the recurrence relation and then the explicit formula.
The following corollary is a direct application of Theorem 3.1 by identifying the sets $I(\mu)$, $J(\mu)$ and $K(\mu)$,

Corollary 4.7. *Let $\mu = (2, 1^{m-2})$ be a partition of m with $1 \le n \le \lfloor m/2 \rfloor$ and $s \ge 0$. Then*

$$[(2,1^{m-2});n]_s = 2[(1^{m-2});n-1]_s - \sum_{j=0}^{s-1}\binom{s}{j}[(2,1^{m-4});n-1]_j, \tag{13}$$

where for all partitions λ of m, $[\lambda;0]_0 = \chi_\lambda(0)$ and $[\lambda;0]_s = 0$ for all $s \ge 1$.

To establish the explicit formula for: $[(2,1^{m-2});n]_s$ we need the following result which gives the exact value of $\chi_{(2,1^{m-2})}(k)$, that is the irreducible character of $\mathfrak{S}_m$ associated to the partition $\mu = (2, 1^{m-2})$ evaluated at the conjugacy class having exactly k distinct transpositions,

Corollary 4.8. *Let m be a positive integer, then*

$$\chi_{(2,1^{m-2})}(k) = (-1)^k(m-2k-1). \tag{14}$$

Proof. In [8], it is stated that: $\chi_{(2,1^{m-2})}(\sigma) = \varepsilon(\sigma)(F(\sigma)-1)$, where σ is any conjugacy class of order m, $\varepsilon(\sigma)$ is the alternating character and $F(\sigma)$ is the number of fixed points of σ. Since we need the value of character at the conjugacy class σ having exactly k distinct transpositions, then: $\varepsilon(\sigma) = (-1)^k$ and $F(\sigma) = m-2k$ and the result follows. □

We establish now the explicit formula of $[(2,1^{m-2});n]_s = \sum_{k=0}^{n}\chi_{(2,1^{m-2})}(k)k^s\binom{n}{k}$.

Theorem 4.9. *Let $\mu = (2, 1^{m-2})$ be a partition of m with $1 \le n \le \lfloor m/2 \rfloor$. Then*

$$[(2,1^{m-2});n]_s = (-1)^n n!\left[(m-1)\left\{{s \atop n}\right\} - 2\left\{{s+1 \atop n}\right\}\right], \tag{15}$$

where $\left\{{s \atop n}\right\}$ is the Stirling number of the second kind.

Proof. We use Theorem 4.2 and Corollary 4.8, we get

$$\begin{aligned}[(2,1^{m-2});n]_s &= \sum_{k=0}^{n}\chi_{(2,1^{m-2})}(k)k^s\binom{n}{k} = \sum_{k=0}^{n}(-1)^k(m-2k-1)k^s\binom{n}{k} \\ &= (m-1)\sum_{k=0}^{n}(-1)^k k^s\binom{n}{k} - 2\sum_{k=0}^{n}(-1)^k k^{s+1}\binom{n}{k} \\ &= (m-1)[(1^m);n]_s - 2[(1^m);n]_{s+1} \\ &= (-1)^n n!\left[(m-1)\left\{{s \atop n}\right\} - 2\left\{{s+1 \atop n}\right\}\right].\end{aligned}$$

□

Remark 4.10. *The last result could be established also by induction on n using Corollary 4.7, since the partitions involving in the recurrence are* $\mu = (1^{m-2})$ *and* $\mu = (2, 1^{m-4})$.

We derive also the following characterisation of $[(2, 1^{m-2}); n]_s$,

Corollary 4.11. *Let* $\mu = (2, 1^{m-2})$ *be the partition of the integer m with* $1 \leq n \leq \lfloor m/2 \rfloor$ *and* $s \geq 0$. *Then* $[(2, 1^{m-2}); n]_s = 0$ *for all* $n > s + 1$.

Proof. The result is obtained from Theorem 4.9 and [5, Equation 1*a* p. 204]. □

We conclude the present section with the case of Hook partitions. A Hook partition of m is a partition of the form $\mu = (k, 1^{m-k})$, $k \geq 1$.

If we restrict $\mu = (k, 1^{m-k})$ to be a Hook partition of m, the sets $I(\mu)$, $J(\mu)$, and $K(\mu)$ corresponds to $\{(k-2, 1^{m-k})\}$, $\{(k, 1^{m-k-2})\}$, and $\{(k-1, 1^{m-k-1})\}$ respectively, and we get the following corollary from Theorem 3.1:

Corollary 4.12. *Let* $\mu = (k, 1^{m-k})$ *be a Hook partition of m with* $k \geq 3$, $m \geq 5$, $1 \leq n \leq \lfloor m/2 \rfloor$ *and* $s \geq 0$. *Then*

$$[(k, 1^{m-k}); n]_s = 2[(k-2, 1^{m-k}); n-1]_s + 2[(k-1, 1^{m-k-1}); n-1]_s + \sum_{j=0}^{s-1} \binom{s}{j} [(k-2, 1^{m-k}); n-1]_j - \sum_{j=0}^{s-1} \binom{s}{j} [(k, 1^{m-k-2}); n-1]_j, \quad (16)$$

where for all partitions λ *of m,* $[\lambda; 0]_0 = \chi_\lambda(0)$ *and* $[\lambda; 0]_s = 0$ *for all* $s \geq 1$.

5. Conclusion

In this work, we presented a new recurrence relation for the sum of binomial coefficients with weights taken to be irreducible characters of the symmetric group, namely $\sum_{k=0}^{n} \chi_\mu(k) k^s \binom{n}{k}$. Then, we used the recurrence relation to establish closed formulas when the irreducible character is related to partitions in the set: $\{(m), (m-1, 1), (2, 1^{m-2}), (1^m)\}$. Also, we derived some particular cases when the sum $\sum_{k=0}^{n} \chi_\mu(k) k^s \binom{n}{k}$ is zero.

Further explorations are still possible, indeed, we ask if it is possible to use the recurrence in Theorem 3.1 to determine explicit formulas for $[\mu; n]_s$ for other particular partitions μ of the integer m, namely the the Hook partitions.

Additionally, looking at the first values of $[\mu; n]_s$; as shown in the Appendix, we can remark the presence of negative values for these sums. Therefore, under what circumstances is $[\mu; n]_s$ a positive quantity?

Appendix

We provide first values of $[\mu; n]_s$ for the integers $2 \leq m \leq 7$, $0 \leq s \leq 3$ and all possible partitions μ of m. For the case $s = 0$, the reader is referred to the appendix of [2]. For $s = 1$, $[\mu; n]_1$ are as follows:

$m = 2$

μ	$n = 1$
(1^2)	−1
(2)	1

$m = 3$

μ	$n = 1$
(1^3)	−1
$(2, 1)$	0
(3)	1

$m = 4$

μ	$n = 1$	$n = 2$
(1^4)	−1	0
$(2, 1^2)$	−1	−4
(2^2)	0	4
$(3, 1)$	1	0
(4)	1	4

$m = 5$

μ	$n = 1$	$n = 2$
(1^5)	−1	0
$(2, 1^3)$	−2	−4
$(2^2, 1)$	−1	0
$(3, 1^2)$	0	−4
$(3, 2)$	1	4
$(4, 1)$	2	4
(5)	1	4

$m = 6$

μ	$n = 1$	$n = 2$	$n = 3$
(1^6)	−1	0	0
$(2, 1^4)$	−3	−4	0
$(2^2, 1^2)$	−3	−4	−12
(2^3)	−1	0	12
$(3, 1^3)$	−2	−8	−12
$(3, 2, 1)$	0	0	0
(3^2)	1	4	0
$(4, 1^2)$	2	0	−12
$(4, 2)$	3	8	24
$(5, 1)$	3	8	12
(6)	1	4	12

$m = 7$

μ	$n = 1$	$n = 2$	$n = 3$
(1^7)	−1	0	0
$(2, 1^5)$	−4	−4	0
$(2^2, 1^3)$	−6	−8	−12
$(2^3, 1)$	−4	−4	0
$(3, 1^4)$	−5	−12	−12
$(3, 2, 1^2)$	−5	−12	−24
$(3, 2^2)$	−1	0	12
$(3^2, 1)$	1	−4	0
$(4, 1^3)$	0	−8	−24
$(4, 2, 1)$	5	8	12
$(4, 3)$	4	12	24
$(5, 1^2)$	5	8	0
$(5, 2)$	6	16	36
$(6, 1)$	4	12	24
(7)	1	4	12

For $s = 2$, $[\mu; n]_2$ are as follows:

$m = 2$

μ	$n = 1$
(1^2)	−1
(2)	1

$m = 3$

μ	$n = 1$
(1^3)	−1
(2, 1)	0
(3)	1

$m = 4$

μ	$n = 1$	$n = 2$
(1^4)	−1	2
$(2, 1^2)$	−1	−6
(2^2)	0	8
(3, 1)	1	−2
(4)	1	6

$m = 5$

μ	$n = 1$	$n = 2$
(1^5)	−1	2
$(2, 1^3)$	−2	−4
$(2^2, 1)$	−1	20
$(3, 1^2)$	0	−8
(3, 2)	1	6
(4, 1)	2	4
(5)	1	6

$m = 6$

μ	$n = 1$	$n = 2$	$n = 3$
(1^6)	−1	2	0
$(2, 1^4)$	−3	−2	12
$(2^2, 1^2)$	−3	−2	−24
(2^3)	−1	2	36
$(3, 1^3)$	−2	−12	−12
(3, 2, 1)	0	0	0
(3^2)	1	6	−12
$(4, 1^2)$	2	−4	−36
(4, 2)	3	10	48
(5, 1)	3	10	6
(6)	1	6	24

$m = 7$

μ	$n = 1$	$n = 2$	$n = 3$
(1^7)	−1	2	0
$(2, 1^5)$	−4	0	12
$(2^2, 1^3)$	−6	−4	−12
$(2^3, 1)$	−4	0	12
$(3, 1^4)$	−5	−14	0
$(3, 2, 1^2)$	−5	−14	−36
$(3, 2^2)$	−1	2	36
$(3^2, 1)$	1	6	−12
$(4, 1^3)$	0	−16	−48
(4, 2, 1)	5	−14	12
(4, 3)	4	16	36
$(5, 1^2)$	5	6	−24
(5, 2)	6	20	60
(6, 1)	4	16	36
(7)	1	6	24

For $s = 3$, $[\mu; n]_3$ are as follows:

$m = 2$

μ	$n = 1$
(1^2)	-1
(2)	1

$m = 3$

μ	$n = 1$
(1^3)	-1
$(2, 1)$	0
(3)	1

$m = 4$

μ	$n = 1$	$n = 2$
(1^4)	-1	6
$(2, 1^2)$	-1	-10
(2^2)	0	16
$(3, 1)$	1	-6
(4)	1	10

$m = 5$

μ	$n = 1$	$n = 2$
(1^5)	-1	6
$(2, 1^3)$	-2	-4
$(2^2, 1)$	-1	6
$(3, 1^2)$	0	-16
$(3, 2)$	1	10
$(4, 1)$	2	4
(5)	1	10

$m = 6$

μ	$n = 1$	$n = 2$	$n = 3$
(1^6)	-1	6	-6
$(2, 1^4)$	-3	-6	60
$(2^2, 1^2)$	-3	2	-66
(2^3)	-1	6	102
$(3, 1^3)$	-2	-20	0
$(3, 2, 1)$	0	0	0
(3^2)	1	10	-54
$(4, 1^2)$	2	-12	-96
$(4, 2)$	3	14	114
$(5, 1)$	3	14	6
(6)	1	10	54

$m = 7$

μ	$n = 1$	$n = 2$	$n = 3$
(1^7)	-1	6	-6
$(2, 1^5)$	-4	8	36
$(2^2, 1^3)$	-6	4	-24
$(2^3, 1)$	-4	8	36
$(3, 1^4)$	-5	-18	42
$(3, 2, 1^2)$	-5	-18	-66
$(3, 2^2)$	-1	6	102
$(3^2, 1)$	1	10	-54
$(4, 1^3)$	0	-32	-96
$(4, 2, 1)$	5	2	18
$(4, 3)$	4	24	60
$(5, 1^2)$	5	2	-90
$(5, 2)$	6	28	120
$(6, 1)$	4	24	60
(7)	1	10	54

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