Filomat 38:13 (2024), 4417–4440
https://doi.org/10.2298/FIL2413417G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Functional characterizations of realized homogeneous Besov and Triebel-Lizorkin spaces

Bochra Gheribi[a], Madani Moussai[a,*]

[a]*University of M'sila, University Pole, Road Bordj Bou Arreridj, M'sila 28000, Algeria*

Abstract. As the realizations of homogeneous Besov and Triebel-Lizorkin spaces are "true" distributions and not modulo polynomials, we will show some functional properties of these functions related to *BMO* space.

1. Introduction

For $s \in \mathbb{R}$ and $0 < p, q \leq \infty$, the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$, abbreviated by B and F, respectively, are defined modulo polynomials, since $\|f\|_{\dot{B}^s_{p,q}} = \|f\|_{\dot{F}^s_{p,q}} = 0$ if and only if f is a polynomial on $\mathbb{R}^n$, this fact because they are embedded in $\mathcal{S}'_\infty(\mathbb{R}^n)$ "the set of tempered distributions modulo polynomials in $\mathbb{R}^n$", see, e.g., [26, 27, 30]. Thus, the elements of these spaces are not "true" distributions and can not "sometimes" satisfy some functional properties. For instance, we know that if a function f belongs to $BMO(\mathbb{R}^n)$ "the John-Nirenberg space which is defined as the set of all locally integrable functions f (modulo constants) such that $\|f\|_{BMO} := \sup_Q |Q|^{-1} \int_Q |f(x) - m_Q f| \mathrm{d}x < \infty$, where the supremum is taken over all finite cubes Q in $\mathbb{R}^n$", then

$$A_d(f) := \int_{\mathbb{R}^n} (1 + |x|^{n+d})^{-1} |f(x)| \, \mathrm{d}x < \infty \qquad \text{(for all } d > 0) \tag{1}$$

$$|Q|^{-1} \int_Q |f(x) - m_Q f|^p \, \mathrm{d}x \leq c \|f\|^p_{BMO} \quad \text{(for all cubes } Q \text{ in } \mathbb{R}^n), \tag{2}$$

where $p < \infty$ and the constant c depends on p, see, e.g., [13, 15, 17, 29], but these estimates are false if we replace $BMO(\mathbb{R}^n)$ by $\dot{B}^s_{p,q}(\mathbb{R}^n)$ or $\dot{F}^s_{p,q}(\mathbb{R}^n)$. Indeed, let $f(x) := x_1^{n+2}$ and $g(x) := x_1$, we have $A_1(f) = \infty$ and $\int_{[-1,1]^n} |g(x) - m_{[-1,1]^n} g|^p \mathrm{d}x = \frac{2^n}{p+1}$, while $\|f\|_{\dot{B}^s_{p,q}} = \|g\|_{\dot{B}^s_{p,q}} = 0$, and similarly for $\dot{F}^s_{p,q}(\mathbb{R}^n)$. However, it is possible to obtain such inequalities using the realizations mapping [10], which leads us to work on counterparts of B and F the realized spaces $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ and $\dot{\tilde{F}}^s_{p,q}(\mathbb{R}^n)$, respectively; they are defined by distributions f such that $f^{(\alpha)}$ $(\forall |\alpha| = \nu)$ vanish at the infinity in the weak sense, where the positive integer ν is defined as

$$\nu := \begin{cases} ([s - \frac{n}{p}] + 1)_+ & \text{if } \ s - \frac{n}{p} \notin \mathbb{N}_0 \ \text{ or } \ q > 1 \text{ in } B\text{-case } (p > 1 \text{ in } F\text{-case}), \\ s - \frac{n}{p} & \text{if } \ s - \frac{n}{p} \in \mathbb{N}_0 \ \text{ and } \ q \leq 1 \text{ in } B\text{-case } (p \leq 1 \text{ in } F\text{-case}); \end{cases} \tag{3}$$

2020 *Mathematics Subject Classification*. 30H35, 46E35.
Keywords. Besov spaces, *BMO* spaces, Triebel-Lizorkin spaces, Realizations.
Received: 20 September 2022; Accepted: 25 December 2023
Communicated by Dragan S. Djordjević
* Corresponding author: Madani Moussai
Email addresses: bochra.gheribi@univ-msila.dz (Bochra Gheribi), madani.moussai@univ-msila.dz (Madani Moussai)

this natural number presents a minimal value such that $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is embedded in $\mathcal{S}'_\nu(\mathbb{R}^n)$ "the set of tempered distributions modulo polynomials of degree $< \nu$"; similarly for F-case.

In this paper, we will give some functional characterizations of $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ and $\dot{\tilde{F}}^s_{p,q}(\mathbb{R}^n)$ related to $BMO(\mathbb{R}^n)$ space, where essentially we find that $d > (s - \frac{n}{p})_+$ is *sufficient* to obtain (1) and it is an *optimal value*; see Section 5.1 below. On the other hand, the realizations commuting with translations and/or dilations (the so-called classification) will play an important role to get (1) and (2). For this reason, the following three cases with respect to parameters n, s, p and q are needed (cf. [10, thms. 4.1, 4.2]):

$$\text{either } s < \tfrac{n}{p} \quad \text{or} \quad s = \tfrac{n}{p} \text{ and } q \le 1 \text{ in } B\text{-case } (p \le 1 \text{ in } F\text{-case}), \tag{4}$$

$$\text{either } s - \tfrac{n}{p} \in \mathbb{R}^+ \backslash \mathbb{N}_0 \quad \text{or} \quad s - \tfrac{n}{p} \in \mathbb{N} \text{ and } q \le 1 \text{ in } B\text{-case } (p \le 1 \text{ in } F\text{-case}), \tag{5}$$

$$s - \tfrac{n}{p} \in \mathbb{N}_0 \text{ and } q > 1 \text{ in } B\text{-case } (p > 1 \text{ in } F\text{-case}). \tag{6}$$

Concerning the realized spaces $\dot{\tilde{B}}^s_{p,q}(\mathbb{R}^n)$ and $\dot{\tilde{F}}^s_{p,q}(\mathbb{R}^n)$ we refer to Subsection 3.2.2 below and to [8, 9, 21], the case $\dot{\tilde{F}}^s_{\infty,q}(\mathbb{R}^n)$ can be found in [2]. Then, our treatment will be based on these conditions, where (4) is the so-called "canonical case", however for (5)–(6) we quote the comment given at the end of page 68 in [1], in which it has mentioned, for a specific subject, that working in such spaces is much more difficult to handle. So for this reason, it seems that some functional properties of these spaces have not yet been studied in detail, since realizations, except the canonical case, are defined up to a polynomial whose degree depends on $s - \frac{n}{p}$ and q in B-case (p in F-case), cf. [26, pp. 55-56]. In this context, let us recall some previous works related to topics in functional analysis:
– convolution inequalities, [3],
– inequalities of Gagliardo-Nirenberg type, [4],
– pointwise multiplications in canonical case, [6],
– the boundedness of pseudodifferential operators in B-case, [19],
– some Hardy type estimates between $L_p(\mathbb{R}^n; |x|^{-s}\mathrm{d}x)$ and B-space (F-space), [22].
There are also other papers on application of the realizations on other homogeneous function spaces, e.g., [12].

Notation. All spaces occurring in this paper are defined on Euclidean space $\mathbb{R}^n$, we will omit $\mathbb{R}^n$. As usual, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by $Q_\lambda(x_0)$ the cube with side length λ centered at $x_0 \in \mathbb{R}^n$, i.e., $Q_\lambda(x_0) := x_0 + [-\lambda, \lambda]^n$. If $a \in \mathbb{R}$ we put $a_+ := \max(a, 0)$. If $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t. The symbol $\hookrightarrow$ means a continuous embedding. By $\|\cdot\|_p$ we denote the quasi-norm of the Lebesgue space L_p. We denote by L_p^{loc} the set of all functions f such that $f \in L_p(K)$ for all compact sets $K \subset \mathbb{R}^n$. If $1 \le p \le \infty$, $p' := \frac{p}{p-1}$ is its conjugate exponent. If $f \in L_1$

$$\mathcal{F} f(\xi) = \widehat{f(\xi)} := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x$$

is its Fourier transform on $\mathbb{R}^n$; the operator $\mathcal{F}$ can be extended to the whole of tempered distributions $\mathcal{S}'$ in the usual way. For a function f we defined translation and dilatation operators by $\tau_a f := f(\cdot - a)$, $a \in \mathbb{R}^n$, and $h_\lambda f := f(\lambda^{-1}\cdot)$, $\lambda > 0$, respectively. For a locally integrable function f we denote by $m_B f := |B|^{-1}\int_B f\mathrm{d}x$ its mean value with respect to the set B, where $|B|$ is the Lebesgue measure of B. We put $\mathcal{P}_0 := \{0\}$, $\mathcal{P}_1 := \{c : c \in \mathbb{C}\}$, $\mathcal{P}_m$ the set of all polynomials in $\mathbb{R}^n$ of degree $< m$ ($m = 2, 3, \ldots$) and $\mathcal{P}_\infty$ the set of all polynomials in $\mathbb{R}^n$.

For $m \in \mathbb{N}_0 \cup \{\infty\}$, the symbol $\mathcal{S}_m$ will be used for the set of functions φ in Schwartz space $\mathcal{S}$ such that $\langle u, \varphi\rangle = 0$ for all $u \in \mathcal{P}_m$ (e.g., $\mathcal{S}_0 = \mathcal{S}$), its topological dual is denoted by $\mathcal{S}'_m$. If $f \in \mathcal{S}'$ then $[f]_m$ denotes the equivalence class of f modulo $\mathcal{P}_m$. The quotient space $\mathcal{S}'/\mathcal{P}_m$ can be identified with $\mathcal{S}'_m$.

We will use the notation $\dot{A}^s_{p,q}$ for $\dot{B}^s_{p,q}$ or $\dot{F}^s_{p,q}$, and $\dot{\tilde{A}}^s_{p,q}$ for $\dot{\tilde{B}}^s_{p,q}$ or $\dot{\tilde{F}}^s_{p,q}$, when it is no need to distinguish between them.

The constants $c, c_1, \dots$ are strictly positive, depend only on the fixed parameters as $n, s, p, \dots$ and some fixed functions, their values may change from line to line.

Finally, we will multiple use of the following well-known assertion: If $0 < p \leq q \leq \infty$, then there exists a constant $c > 0$ such that the inequality

$$\|f^{(\alpha)}\|_q \leq cR^{|\alpha|+n(1/p-1/q)} \left\|f\right\|_p \tag{7}$$

holds for all $f \in L_p$ and all $R > 0$, such that $\operatorname{supp} \widehat{f} \subset \{\xi : |\xi| \leq R\}$, see, e.g., [30, rem. 1.3.2/1]. In case $|\alpha| = 0$, the constant c can be given explicitly, $c = p_0^{n(1/p-1/q)}$ where p_0 is the smallest integer not less than $p/2$, cf. [24, thm. 4].

Plan. This work is organized as follows. In Section 2 we state our main results. In Section 3, we recall definitions and some properties of $\dot{A}^s_{p,q}$ and their realizations. Section 4 is devoted to the proofs. In a last section, we first discuss the optimal conditions on "d" such that (1) holds, then we give an extension to inhomogeneous and Morrey spaces.

2. Statement of the main results

We will prove the following statements, where for brevity, if for a given function $f \in \dot{\tilde{A}}^s_{p,q}$ there exists a polynomial $u_f \in \mathcal{P}_\nu$ then we set

$$\tilde{f} := f + u_f. \tag{8}$$

Theorem 2.1. *Let $0 < p, q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Let d be a real number such that $d > (s - \frac{n}{p})_+$.*

(i) *If $f \in \dot{\tilde{A}}^s_{p,q}$, then there exists a polynomial $u_f \in \mathcal{P}_\nu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{|\tilde{f}(x)|}{1 + |x|^{n+d}} \, \mathrm{d}x \leq c\|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{9}$$

holds; the constant c is independent of f.

(ii) *If moreover $p < \infty$ in F-case and that* (4) *is satisfied; here we have realizations commuting with translations, then the inequality*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x - x_0|^{n+d}} \, \mathrm{d}x \leq c\|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{10}$$

holds for all $x_0 \in \mathbb{R}^n$; the constant c is independent of f and x_0.

Remark 2.2. If (4) is satisfied and $s > 0$, it is necessary that $p < \infty$.

Remark 2.3. It is well-known that $BMO = \dot{\tilde{F}}^0_{\infty,2}$ and $\nu = 1$, see [2, rem. 7]. Then, by Theorem 2.1 we obtain $\int_{\mathbb{R}^n} (1 + |x|^{n+d})^{-1} |f(x) + c| \mathrm{d}x < \infty$ for all $f \in \dot{\tilde{F}}^0_{\infty,2}$ and all $c \in \mathbb{C}$, also for all $f \in BMO$ (here $d > 0$), see (1); in [13] it has been chosen $d = 1$.

Theorem 2.4. *Let s, p, q and d be the same as in Theorem 2.1.*

(i) *If $f \in \dot{\tilde{A}}^s_{p,q}$, then there exists a polynomial $u_f \in \mathcal{P}_\nu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{\left|\tilde{f}(x) - m_{Q_1(0)}\tilde{f}\right|}{1 + |x|^{n+d}} \, \mathrm{d}x \leq c\|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{11}$$

holds; the constant c is independent of f.

(ii) *Assume that $p < \infty$ in F-case and that either* (4) *or* (5) *is satisfied; here we have realizations commuting with dilations. If $f \in \mathring{\tilde{A}}^s_{p,q}$, then there exists a polynomial $u_f \in \mathcal{P}_\nu$ such that the inequality*

$$\int_{\mathbb{R}^n} \frac{\left|\tilde{f}(x) - m_{Q_\lambda(0)}\tilde{f}\right|}{\lambda^{n+d} + |x|^{n+d}} \, \mathrm{d}x \le c\lambda^{s-d-n/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{12}$$

holds for all $\lambda > 0$; the constant c is independent of f and λ.

(iii) *In particular, if* (4) *is satisfied ($p < \infty$ in F-case); here we have realizations commuting with translations, then the inequality*

$$\int_{\mathbb{R}^n} \frac{\left|f(x) - m_{Q_\lambda(x_0)}f\right|}{\lambda^{n+d} + |x - x_0|^{n+d}} \, \mathrm{d}x \le c\lambda^{s-d-n/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{13}$$

holds for all $f \in \mathring{\tilde{A}}^s_{p,q}$, all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f, λ and x_0.

In case of modulo constants ($\nu = 1$, i.e., u_f = constant), we can extend the third assertion in this theorem to the cases (5) and (6), that is the following formulation:

Corollary 2.5. *Let $0 < p, q \le \infty$. Suppose that $\nu = 1$ in either* (5) *or* (6)*. Then, the inequality* (13) *remains true.*

Theorem 2.6. *Let $0 < p, q \le \infty$ and $s > (\frac{n}{p} - n)_+$. Let $0 < p_1 < \infty$. In case $1 < p_1 < \infty$ we assume that*

in F-case: $\frac{n}{p_1} \ge \frac{n}{p} - s$, (14)

in B-case: either $\frac{n}{p_1} > \frac{n}{p} - s$, or $\frac{n}{p_1} = \frac{n}{p} - s$ and $q \le p_1$. (15)

(i) *If $f \in \mathring{\tilde{A}}^s_{p,q}$, then there exists a polynomial $u_f \in \mathcal{P}_\nu$ such that the inequality*

$$\left(\int_{Q_1(0)} \left|\tilde{f}(x) - m_{Q_1(0)}\tilde{f}\right|^{p_1} \mathrm{d}x \right)^{1/p_1} \le c\|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{16}$$

holds; the constant c is independent of f.

(ii) *Assume that $p < \infty$ in F-case and that either* (4) *or* (5) *is satisfied; here we have realizations commuting with dilations. If $f \in \mathring{\tilde{A}}^s_{p,q}$, then there exists a polynomial $u_f \in \mathcal{P}_\nu$ such that the inequality*

$$\left(\frac{1}{|Q_\lambda(0)|} \int_{Q_\lambda(0)} \left|\tilde{f}(x) - m_{Q_\lambda(0)}\tilde{f}\right|^{p_1} \mathrm{d}x \right)^{1/p_1} \le c\lambda^{s-n/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{17}$$

holds for all $\lambda > 0$; the constant c is independent of f and λ.

(iii) *In particular, if* (4) *is satisfied ($p < \infty$ in F-case); here we have realizations commuting with translations, then the inequality*

$$\left(\frac{1}{|Q_\lambda(x_0)|} \int_{Q_\lambda(x_0)} \left|f(x) - m_{Q_\lambda(x_0)}f\right|^{p_1} \mathrm{d}x \right)^{1/p_1} \le c\lambda^{s-n/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}} \tag{18}$$

holds for all $f \in \mathring{\tilde{A}}^s_{p,q}$, all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$; the constant c is independent of f, λ and x_0.

Remark 2.7. In the left-hand sides of inequalities (9), (11), (12), (16) and (17) one can replace $\int$ and $\tilde{f}$ by $\inf_{u \in \mathcal{P}_\infty} \int$ and $f + u$, respectively; this is in the spirit of, e.g., [30, 5.2.4(2)].

Remark 2.8. For Theorem 2.6 the assumption $\frac{n}{p_1} \geq \frac{n}{p} - s$ is necessary at least in (17). Indeed, assume that $s - \frac{n}{p} + \frac{n}{p_1} < 0$, then by dividing each term in (17) by $\lambda^{s-n/p}$ and letting $\lambda \to \infty$ we obtain a contradiction. We note that assumptions (14) and (15) coincide with the necessary conditions so that the inhomogeneous $A^s_{p,q}$ is embedded in L_{p_1}, cf. [27, coro. 2.2.4/2] (see Subsection 5.2 below for the definition of $A^s_{p,q}$).

As in Corollary 2.5, we extend the assertion in Theorem 2.6/(iii) to functions $f \in \dot{\tilde{A}}^s_{p,q}$ such that $\nu = 1$. Where in this formulation we will find interesting examples with cases $s = \frac{n}{p}$ or $1 + \frac{n}{p}$.

Corollary 2.9. *Let s, p, p_1 and q be the same as in Theorem 2.6, in particular the conditions* (14)–(15). *Let either $(\frac{n}{p} - n)_+ < s < 1 + \frac{n}{p}$ or $s = 1 + \frac{n}{p}$ and $q \leq 1$ in B-case ($p \leq 1$ in F-case). Then, the inequality* (18) *remains true.*

Remark 2.10. In Corollary 2.9 we have $\nu = 0$ or 1. If $\nu \geq 2$, there exist functions $f \in \dot{\tilde{A}}^s_{p,q}$ for which the inequality (18) is false. Indeed, the first example is given by the function $h(x) := x_1^2$. Clearly $h \in \dot{\tilde{A}}^{3+n/p}_{p,q}$ since $\nu \geq 3$. By a direct calculation one finds $\int_{Q_1(0)} |h(x) - m_{Q_1(0)}h| \mathrm{d}x = \frac{4}{9\sqrt{3}}2^n$. Thus, if $p_1 \geq 1$ by Hölder inequality we obtain

$$\int_{Q_1(0)} |h(x) - m_{Q_1(0)}h| \, \mathrm{d}x \leq c\left(\int_{Q_1(0)} \left|h(x) - m_{Q_1(0)}h\right|^{p_1} \mathrm{d}x\right)^{1/p_1},$$

if $0 < p_1 < 1$, since $|h(x) - m_{Q_1(0)}h| \leq |h(x)| + \frac{1}{3} \leq \frac{4}{3}$ for all $x \in Q_1(0)$, we have

$$\begin{aligned}\int_{Q_1(0)} |h(x) - m_{Q_1(0)}h| \, \mathrm{d}x &= \int_{Q_1(0)} |h(x) - m_{Q_1(0)}h|^{p_1} |h(x) - m_{Q_1(0)}h|^{1-p_1} \, \mathrm{d}x \\ &\leq \left(\tfrac{4}{3}\right)^{1-p_1} \int_{Q_1(0)} |h(x) - m_{Q_1(0)}h|^{p_1} \, \mathrm{d}x,\end{aligned}$$

this is the left-hand side of (18), while $\|[h]_\infty\|_{\dot{A}^s_{p,q}} = 0$. The second example in $\dot{\tilde{A}}^{1+n/p}_{p,q}$ (with $q > 1$ in B-case and $p > 1$ in F-case, here $\nu = 2$) for the function $g(x) := x_1$ with the same calculations given for h.

Remark 2.11. In case $s \geq \frac{n}{p}$ and $1 \leq p_1 < \infty$, a careful examination of the proof of Theorem 2.6 shows that the constant c appears in the right-hand sides of (16)–(18) has the form $c_1 p_1$, where $c_1 > 1$ independent of p_1.

The aim of Corollary 2.9 is to extend the so-called John-Nirenberg inequality, proved on BMO see, e.g., [15, 18, 28] and [29, p. 144], to realized spaces:

Theorem 2.12. *Let $0 < p < \infty$ and $0 < q \leq \infty$. Let either $\frac{n}{p} \leq s < 1 + \frac{n}{p}$ or $s = 1 + \frac{n}{p}$ and $q \leq 1$ in B-case ($p \leq 1$ in F-case). Then, there exist two positive constants c_1, c_2, such that the inequality*

$$\left|\left\{x \in Q : \ |f(x) - m_Q f| > R\right\}\right| \leq c_1 |Q| \exp\left(-\frac{c_2 R}{|Q|^{s/n-1/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}}}\right) \tag{19}$$

holds for all $f \in \dot{\tilde{A}}^s_{p,q}$, all $R > 0$ and all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes.

We recall that the estimates (9)–(12) are satisfied if we replace $\dot{\tilde{A}}^s_{p,q}$ by BMO, see, e.g., [13]. In that case, the right-hand side of (12) is $\lambda^{-d}\|f\|_{BMO}$ with any $d > 0$. Theorem 2.12 has the well-known consequence (in our situation)

$$\frac{1}{|Q|} \int_Q \exp\left(\frac{b}{|Q|^{s/n-1/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}}} \left|f(x) - m_Q f\right|\right) \mathrm{d}x \leq c, \quad c := c(n, b),$$

where $b < c_2$ (c_2 is the constant given in (19)), see again [15, coro. 3.1.7] and [29, p. 146].

3. Preliminaries

In this section, the numbers s, p and q satisfy: $s \in \mathbb{R}$ and $p, q \in]0, \infty]$, unless otherwise stated.

3.1. Besov and Triebel-Lizorkin spaces

To define Besov and Triebel-Lizorkin spaces, we briefly recall the Littlewood-Paley decomposition. We introduce a C^∞ radial function ρ such that $0 \leq \rho \leq 1$, $\rho(\xi) = 1$ if $|\xi| \leq 1$ and $\rho(\xi) = 0$ if $|\xi| \geq \frac{3}{2}$. We put $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$. Then γ vanishes outside the annulus $\frac{1}{2} \leq |\xi| \leq \frac{3}{2}$ and $\gamma(\xi) = 1$ if $\frac{3}{4} \leq |\xi| \leq 1$. We have the following identity

$$\sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1\,, \quad (\xi \in \mathbb{R}^n \setminus \{0\}).$$

We also introduce the convolution operators $\widehat{Q_j f}(\xi) := \gamma(2^{-j}\xi)\widehat{f}(\xi)$ $(\forall j \in \mathbb{Z})$. The operators Q_j are defined on $\mathcal{S}'$, also on $\mathcal{S}'_\infty$ since $Q_j f = 0$ iff $f \in \mathcal{P}_\infty$, then we make use of the following convention:

If $f \in \mathcal{S}'_\infty$ we define $Q_j f := Q_j g$ for all $g \in \mathcal{S}'$ such that $[g]_\infty = f$.

They take values in the space of analytical functions of exponential type, see Paley-Wiener theorem. They are also uniformly bounded in $\mathcal{L}(L_p)$ for any $1 \leq p \leq \infty$ (the Young inequality). Finally, the desired decompositions are described by the following well-known statement:

Proposition 3.1. (i) *For every $f \in \mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, resp.), it holds that $f = \sum_{j\in\mathbb{Z}} Q_j f$ in $\mathcal{S}_\infty$ ($\mathcal{S}'_\infty$, resp.).*

(ii) *For every $f \in \mathcal{S}$ ($\mathcal{S}'$, resp.) and every $k \in \mathbb{Z}$, it holds that $f = \rho_k * f + \sum_{j>k} Q_j f$ in $\mathcal{S}$ ($\mathcal{S}'$, resp.), where $\widehat{\rho_k}(\xi) := \rho(2^{-k}\xi)$.*

Definition 3.2. (i) *The homogeneous Besov space $\dot{B}^s_{p,q}$ is the set of all $f \in \mathcal{S}'_\infty$ such that*
$\|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{j\in\mathbb{Z}} (2^{js} \|Q_j f\|_p)^q \Big)^{1/q} < \infty.$

(ii) *Let $0 < p < \infty$. The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is the set of all $f \in \mathcal{S}'_\infty$ such that*
$\|f\|_{\dot{F}^s_{p,q}} := \Big\| \Big(\sum_{j\in\mathbb{Z}} (2^{js}|Q_j f|)^q \Big)^{1/q} \Big\|_p < \infty.$

Definition 3.3. (i) *Let $0 < q < \infty$. The homogeneous space $\dot{F}^s_{\infty,q}$ is the set of all $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{F}^s_{\infty,q}} := \sup_{k\in\mathbb{Z},\, \mu\in\mathbb{Z}^n} \left(2^{kn} \int_{P_{k,\mu}} \sum_{j\geq k} 2^{jsq} |Q_j f(x)|^q \mathrm{d}x \right)^{1/q} < \infty,$$

where $P_{k,\mu}$ ($k \in \mathbb{Z}$, $\mu \in \mathbb{Z}^n$) is the set (dyadic cube) of $x \in \mathbb{R}^n$ such that $2^{-k}\mu_\ell \leq x_\ell < 2^{-k}(\mu_\ell + 1)$ ($\ell = 1, \ldots, n$).

(ii) *We put $\dot{F}^s_{\infty,\infty} = \dot{B}^s_{\infty,\infty}$.*

Remark 3.4. The spaces $\dot{A}^s_{p,q}$ are quasi-Banach for the above defined quasi-seminorms. The above definitions are independent of the choice of ρ, see, e.g., [26, 30] and [14, coro. 5.3].

We have $\mathcal{S}_\infty \hookrightarrow \dot{A}^s_{p,q} \hookrightarrow \mathcal{S}'_\infty$. We also have $\dot{B}^s_{p,\min(p,q)} \hookrightarrow \dot{F}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\max(p,q)}$ ($p < \infty$ in F-case) and the following two statements which are proved in [16] and [2, lem. 3], respectively:

Proposition 3.5. *For $0 < r \leq \infty$, $s_1 > s_2$ and $0 < p_1 < p_2 < \infty$ such that $s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$, it holds that $\dot{B}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,q} \hookrightarrow \dot{B}^{s_2-n/p_2}_{\infty,q}$, $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{B}^{s_2}_{p_2,p_1}$ and $\dot{F}^{s_1}_{p_1,q} \hookrightarrow \dot{F}^{s_2}_{p_2,r}$.*

Proposition 3.6. *For all $q > 0$ it holds $\dot{F}^s_{\infty,q} \hookrightarrow \dot{B}^s_{\infty,\infty}$.*

The following statement is a variant of the so-called Nikol'skij representation method for $\dot{A}^s_{p,q}$, see [27, prop. 2.3.2/1] or [32]. It is of self-contained interest, where its part of $\dot{F}^s_{\infty,q}$, is a main result in this section.

Proposition 3.7. *Let $0 < a < b$. Let $(u_j)_{j\in\mathbb{Z}}$ be a sequence in $\mathcal{S}'$ such that*

- *$\widehat{u_j}$ is supported by the compact annulus $a2^j \le |\xi| \le b2^j$,*
- *$A := (\sum_{j\in\mathbb{Z}} 2^{jsq}\|u_j\|_p^q)^{1/q} < \infty$ in B-case,*
- *$A := \|(\sum_{j\in\mathbb{Z}} 2^{jsq}|u_j|^q)^{1/q}\|_p < \infty$ in F-case if $p < \infty$,*
- *$A := \sup_{k\in\mathbb{Z},\,\mu\in\mathbb{Z}^n} \Big(2^{kn}\int_{P_{k,\mu}} \sum_{j\ge k} 2^{jsq}|u_j(x)|^q \mathrm{d}x\Big)^{1/q}$ in F-case if $p = \infty$.*

(i) *Then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'_\infty$ to a limit u satisfying $\|u\|_{\dot{A}^s_{p,q}} \le cA$, where the constant c depends only on n, s, p, q, a and b.*

(ii) *If in addition $s > (\frac{n}{p} - n)_+$ in B-case, $s > (\frac{n}{\min(p,q)} - n)_+$ in F-case, the same conclusion holds for $a = 0$.*

Proof. We only prove the F-case with $p = \infty$, since other cases can be found in, e.g., [11, prop. 4], [20, prop. 3.4] and [21, props. 2.15, 2.17].

The convergence is obtained since $\dot{F}^s_{\infty,q} \hookrightarrow \dot{B}^s_{\infty,\infty}$, then we prove $\|u\|_{\dot{F}^s_{\infty,q}} \le cA$. Owing to the support of $\widehat{u_j}$, there exist $m_1, m_2 \in \mathbb{Z}$ (with $m_2 = \infty$ in case $a = 0$) depending only on a and b such that $Q_k u = \sum_{j=k+m_1}^{k+m_2} Q_k u_j$. We put $\tilde{u}_k := \sum_{j=k+m_1}^{k+m_2} u_j$ (i.e., $Q_k u = Q_k \tilde{u}_k$) and separate the proof into two cases: $a > 0$ and $a = 0$.

Step 1: the case $a > 0$. Let $d := \min(1, q)$ and $\theta > n$. By (7) we have

$$
\begin{aligned}
|Q_k\tilde{u}_k(x)| &\le \int_{\mathbb{R}^n} 2^{kn}|\check{\gamma}(2^k(x-y))\tilde{u}_k(y)|\,\mathrm{d}y \\
&\le c_1 2^{k(n/d-n)}\bigg(\int_{\mathbb{R}^n} 2^{knd}|\check{\gamma}(2^k(x-y))|^d|\tilde{u}_k(y)|^d\,\mathrm{d}y\bigg)^{1/d} \\
&\le c_2 2^{kn/d}\bigg(\sum_{w\in\mathbb{Z}^n}\int_{P_{J,w}} (1+2^k|x-y|)^{-(n+1)-\theta}|\tilde{u}_k(y)|^d\,\mathrm{d}y\bigg)^{1/d},
\end{aligned}
$$

where $\check{\gamma}$ is the inverse Fourier transform of γ; here we used the fact that $\check{\gamma} \in \mathcal{S}$. On the other hand, for $x \in P_{J,\mu}$ and $y \in P_{J,w}$ ($\mu, w \in \mathbb{Z}^n$, $J \in \mathbb{Z}$) we have

$$1 + |w-\mu| \le 2\sqrt{2n}(1+2^J|x-y|) \le 2\sqrt{2n}(1+2^k|x-y|) \quad \text{if} \quad k \ge J,$$

and use [32, lem. 2.6] (see also, e.g., [23, lem. 3.2]), one has

$$
\begin{aligned}
|Q_k\tilde{u}_k(x)| &\le c_1\bigg(\sum_{w\in\mathbb{Z}^n}(1+|\mu-w|)^{-(n+1)}\int_{\mathbb{R}^n} 2^{kn}(1+2^k|x-y|)^{-\theta}|\tilde{u}_k(y)|^d\chi_{P_{J,w}}(y)\,\mathrm{d}y\bigg)^{1/d} \\
&\le c_2\bigg(\sum_{w\in\mathbb{Z}^n}(1+|\mu-w|)^{-(n+1)}M(|\tilde{u}_k|^d\chi_{P_{J,w}})(x)\bigg)^{1/d}, \qquad (\forall x \in P_{J,\mu}),
\end{aligned}
\tag{20}
$$

where $\chi_{P_{J,w}}$ is the characteristic function on the cube $P_{J,w}$ and M is the maximal function. Since $\frac{q}{d} > 1$, we apply twice the Minkowski's inequality, then from (20) it holds

$$
\begin{aligned}
\int_{P_{J,\mu}}\sum_{k\ge J} 2^{ksq}|Q_k\tilde{u}_k(x)|^q\mathrm{d}x &\le c_1\int_{\mathbb{R}^n}\sum_{k\ge J}2^{ksq}\bigg(\sum_{w\in\mathbb{Z}^n}\dots\bigg)^{q/d}\mathrm{d}x \\
&\le c_1\bigg[\sum_{w\in\mathbb{Z}^n}(1+|\mu-w|)^{-(n+1)}\bigg(\int_{\mathbb{R}^n}\sum_{k\ge J}2^{ksq}\Big(M(|\tilde{u}_k|^d\chi_{P_{J,w}})(x)\Big)^{q/d}\mathrm{d}x\bigg)^{d/q}\bigg]^{q/d}.
\end{aligned}
\tag{21}
$$

Applying the Fefferman-Stein vector-valued inequality in $L_{q/d}(\mathbb{R}^n;\ell_{q/d}(\mathbb{Z}^n))$, see, e.g., [14, thm. A.1, p. 147], we obtain the bound

$$c\Big[\sum_{w\in\mathbb{Z}^n}(1+|\mu-w|)^{-(n+1)}\Big(\int_{P_{J,w}}\sum_{k\geq J}2^{ksq}|\tilde{u}_k(x)|^q\mathrm{d}x\Big)^{d/q}\Big]^{q/d}. \tag{22}$$

Now, we easily have $|\tilde{u}_k(x)|^q\leq c\sum_{j=k+m_1}^{k+m_2}|u_j(x)|^q$ where $c:=c(m_1,m_2,q)$, which implies

$$\sum_{k\geq J}2^{ksq}|\tilde{u}_k(x)|^q\leq c_1\sum_{j\geq J+m_1}|u_j(x)|^q\sum_{k=j-m_1}^{j-m_2}2^{ksq}\leq c_2\sum_{j\geq J+m_1}2^{jsq}|u_j(x)|^q.$$

Also, easily there exist $([2^{m_1}]+2)^n$ disjoint dyadic cubes $\{P_{J+m_1,w_r}\}_{r=1}^{([2^{m_1}]+2)^n}$ such that $P_{J,w}\subset\bigcup_{r=1}^{([2^{m_1}]+2)^n}P_{J+m_1,w_r}$. Then

$$\int_{P_{J,w}}\sum_{k\geq J}2^{ksq}|\tilde{u}_k(x)|^q\mathrm{d}x\leq c_1\sum_{r=1}^{([2^{m_1}]+2)^n}\int_{P_{J+m_1,w_r}}\sum_{j\geq J+m_1}2^{jsq}|u_j(x)|^q\mathrm{d}x\leq c_2 2^{-Jn}A^q.$$

Inserting this estimate into (22); since $\sum_{w\in\mathbb{Z}^n}(1+|\mu-w|)^{-(n+1)}=\sum_{w\in\mathbb{Z}^n}(1+|w|)^{-(n+1)}<\infty$, we get from (21)

$$\int_{P_{J,\mu}}\sum_{k\geq J}2^{ksq}|Q_k\tilde{u}_k(x)|^q\mathrm{d}x\leq c2^{-Jn}A^q.$$

Finally, dividing both sides of the last inequality by 2^{-Jn}, and taking the supremum, the result follows.

Step 2: the case $a=0$. We start with the definition of the homogeneous Triebel-Lizorkin-type space $\dot{F}^{s,\tau}_{p,q}$: for $\tau\geq 0$, $s\in\mathbb{R}$, $p\in]0,\infty[$ and $q\in]0,\infty]$, $\dot{F}^{s,\tau}_{p,q}$ is the set of all $f\in\mathcal{S}'_\infty$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}}:=\sup_{k\in\mathbb{Z},\,\eta\in\mathbb{Z}^n}2^{kn\tau}\Big\|\Big(\sum_{j\geq k}(2^{js}|Q_jf|)^q\Big)^{1/q}\Big\|_{L_p(P_{k,\eta})}<\infty;$$

see, e.g., [5] or [34]. For these space, let us recall the following two properties:

(P1) Let $s>(\frac{n}{p}-n)_+$ and $b>0$. If a sequence $(g_j)_{j\in\mathbb{Z}}$ in $\mathcal{S}'$ satisfies that $\widehat{g_j}$ is supported by the ball $|\xi|\leq b2^j$, then it holds

$$\Big\|\sum_{j\in\mathbb{Z}}g_j\Big\|_{\dot{F}^{s,\tau}_{p,q}}\leq c\sup_{k\in\mathbb{Z},\,\mu\in\mathbb{Z}^n}2^{kn\tau}\Big(\int_{P_{k,\mu}}\Big(\sum_{j\geq k}2^{jsq}|g_j(x)|^q\Big)^{p/q}\mathrm{d}x\Big)^{1/p},$$

see [5, thm. 3.2].

(P2) If $s\in\mathbb{R}$, $v\in]0,\infty[$ and $q\in]0,\infty]$, then $\dot{F}^{s,1/v}_{v,q}=\dot{F}^s_{\infty,q}$ (equivalent quasi-seminorms), see [14, coro. 5.7] or [33, prop. 3.1] or [34, 1.4.4].

We now turn to our assertion. Using in the first (P1) with $g_j:=u_j$, $\tau:=\frac{1}{v}$ and $p:=v$, where $v\in]0,\infty[$ will be chosen later on. Applying in the second (P2), we have

$$\Big\|\sum_{j\in\mathbb{Z}}u_j\Big\|_{\dot{F}^s_{\infty,q}}\leq c\sup_{k\in\mathbb{Z},\,\mu\in\mathbb{Z}^n}2^{kn/v}\Big(\int_{P_{k,\mu}}\Big(\sum_{j\geq k}2^{jsq}|u_j(x)|^q\Big)^{v/q}\mathrm{d}x\Big)^{1/v}, \tag{23}$$

where we need the condition $s > (\frac{n}{v} - n)_+$. We then choose $0 < v < q$, and apply Hölder's inequality with exponents $\frac{q}{v}$ and $\frac{q}{q-v}$ it holds

$$\int_{P_{k,\mu}} \Big(\sum_{j\geq k} 2^{jsq} |u_j(x)|^q \Big)^{v/q} \mathrm{d}x \leq 2^{-kn(1-v/q)} \Big(\int_{P_{k,\mu}} \sum_{j\geq k} 2^{jsq} |u_j(x)|^q \mathrm{d}x \Big)^{v/q} \leq 2^{-kn} A^v. \tag{24}$$

Since $s > (\frac{n}{q} - n)_+$, we choose v such that $v := 1$ if $q \geq 1$ and $(1 + \frac{s}{n})^{-1} < v < q$ if $0 < q < 1$. Then inserting (24) into (23), the desired result follows. □

The name of homogeneity for the space $\dot{A}^s_{p,q}$ is due to the following property:

Proposition 3.8. *There exist two constants $c_1, c_2 > 0$ such that the inequality*

$$c_1 \|f\|_{\dot{A}^s_{p,q}} \leq \lambda^{s-n/p} \|h_\lambda f\|_{\dot{A}^s_{p,q}} \leq c_2 \|f\|_{\dot{A}^s_{p,q}}$$

holds for all $f \in \dot{A}^s_{p,q}$ and all $\lambda > 0$. Moreover, the expression $\lambda^{s-n/p}\|h_\lambda f\|_{\dot{A}^s_{p,q}}$ defines an equivalent quasi-seminorm in $\dot{A}^s_{p,q}$.

Proof. See, e.g., [30, rem. 5.1.3/4]. In case $\dot{F}^s_{\infty,q}$ see [2, lem. 1]. □

For further properties of $\dot{A}^s_{p,q}$ as well as inhomogeneous counterpart (cf., see Definition 5.2 below), we quote, e.g., [26, 27, 30, 31].

3.2. Realizations

3.2.1. Generalities

We recall that a quasi-Banach space of distributions E in $\mathcal{S}'_m$ is a vector subspace of $\mathcal{S}'_m$ endowed with a complete quasi-seminorm such that $E \hookrightarrow \mathcal{S}'_m$.

Definition 3.9. *Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \{0, \ldots, m\}$. Let E be a quasi-Banach space of distributions in $\mathcal{S}'_m$. A realization of E in $\mathcal{S}'_k$ is a continuous linear mapping $\sigma : E \to \mathcal{S}'_k$, such that $[\sigma(f)]_m = f$ for all $f \in E$. The image set $\sigma(E)$ is called the realized space of E with respect to σ.*

In realizations theory, a realization σ is entirely determined by its range, and any quasi-Banach space of distributions E in $\mathcal{S}'_m$ has an infinity of realizations in $\mathcal{S}'_k$ if $k < m$, in case $k = m$ only the identity is the unique realization. More explicitly, we have $\sigma(f) - f \in \mathcal{P}_m$, and we note that in case of an infinity of realizations, whose commute with translation or dilation have some chances to be unique, cf. [10, props 2.2, 2.4]. We also have:

Proposition 3.10. *Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \{0, \ldots, m\}$. Let E be a quasi-Banach space of distributions in $\mathcal{S}'_m$. If $\sigma_0 : E \to \mathcal{S}'_k$ is a realization, then there exists a natural number N (with $N = m$ in case $m < \infty$), such that for all finite family $(\mathcal{L}_\alpha)_{k\leq|\alpha|\leq N}$ of continuous linear functionals on E, the following formula defines a realization of E modulo $\mathcal{P}_k$:*

$$\sigma(f)(x) := \sigma_0(f)(x) + \sum_{k\leq|\alpha|\leq N} \mathcal{L}_\alpha(f)\, x^\alpha.$$

Conversely, any realization of E in $\mathcal{S}'_k$ is defined in such a way.

We refer to [7]–[9] for preceding proposition and further properties of realizations.

3.2.2. The realized spaces

We begin by the following notion:

Definition 3.11. *We say that a distribution $f \in \mathcal{S}'$ vanishes at the infinity in the weak sense if* $\lim_{\lambda\to 0}\langle h_\lambda f, \varphi\rangle = 0$ *for all $\varphi \in \mathcal{S}$. The set of all such distributions is denoted by $\widetilde{C}_0$.*

For some examples of such distributions, we have $f \in \widetilde{C}_0$ if $f \in L_p$ $(1 \le p < \infty)$, $\partial_\ell f \in \widetilde{C}_0$ $(\ell = 1, \dots, n)$ if $f \in L_\infty$ or $f \in \widetilde{C}_0$. A "good" example is given by the Littlewood-Paley decomposition, see Proposition 3.12 below, and other can be found in [4].

On the other hand, we are able to define the realized space $\dot{\tilde{A}}^s_{p,q}$, that is

$$\dot{\tilde{A}}^s_{p,q} := \Big\{ f \in \mathcal{S}'_\nu : \ [f]_\infty \in \dot{A}^s_{p,q} \ \text{ and } \ f^{(\alpha)} \in \widetilde{C}_0, \ \forall\, |\alpha| = \nu \Big\},$$

which is endowed with the same quasi-seminorm of $\dot{A}^s_{p,q}$. Recall, it is easy to see that $f \in \mathcal{S}'_\nu$ implies $f^{(\alpha)} \in \mathcal{S}'$ if $|\alpha| = \nu$ since $\partial^\alpha(\mathcal{S}) \subset \mathcal{S}_\nu$.

Proposition 3.12. *Let $f \in \dot{A}^s_{p,q}$. Then $\sum_{j\in\mathbb{Z}} Q_j f$ converges in $\mathcal{S}'_\nu$. Let us define $\sigma(f)$ its sum. Then the mapping $\sigma : \dot{A}^s_{p,q} \to \mathcal{S}'_\nu$ defined in such a way is a translation and dilatation commuting realization, and $\partial^\alpha \sigma(f) \in \widetilde{C}_0$ for all $|\alpha| = \nu$. Furthermore, $\sigma(\dot{A}^s_{p,q}) = \dot{\tilde{A}}^s_{p,q}$.*

Proof. See, e.g., [10, 21]. In case $\dot{F}^s_{\infty,q}$, the proof is given in [2, thm. 1]. For the last equality we have $\sigma(\dot{A}^s_{p,q}) \subset \dot{\tilde{A}}^s_{p,q}$ by the definition. For the converse embedding, if $f \in \dot{\tilde{A}}^s_{p,q}$, then $\partial^\alpha(\sigma([f]_\infty) - f)$ belongs to $\widetilde{C}_0 \cap \mathcal{P}_\infty$ with $|\alpha| = \nu$; but we easily have $\widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$, hence $\sigma([f]_\infty) - f \in \mathcal{P}_\nu$, and $\sigma([f]_\infty) = f$ in $\mathcal{S}'_\nu$. □

In Definition 3.9, by taking $k = 0$ and $m = \nu$, we obtain realizations of $\dot{A}^s_{p,q}$ in $\mathcal{S}'$ using the Littlewood-Paley setting. Namely, we have the following assertion:

Proposition 3.13. *For all $f \in \dot{A}^s_{p,q}$ we define $\sigma_i(f)$ $(i = 1, 2, 3)$ as the following:*

(i) $\sigma_1(f) := \sum_{j\in\mathbb{Z}} Q_j f$ *in case* (4),

(ii) $\sigma_2(f) := \sum_{j\in\mathbb{Z}} \Big(Q_j f - \sum_{|\alpha|<\nu} \frac{1}{\alpha!}(Q_j f)^{(\alpha)}(0) x^\alpha \Big)$ *in case* (5),

(iii) $\sigma_3(f) := \sum_{j\ge 1} Q_j f + \sum_{j\le 0} \Big(Q_j f - \sum_{|\alpha|<\nu} \frac{1}{\alpha!}(Q_j f)^{(\alpha)}(0) x^\alpha \Big)$ *in case* (6).

Then σ_i is a realization of $\dot{A}^s_{p,q}$ in $\mathcal{S}'$ where all above series converge in $\mathcal{S}'$ such that $\partial^\alpha \sigma_i(f) \in \widetilde{C}_0$ $(|\alpha| = \nu)$ and $[\sigma_i(f)]_\infty = f$ in $\mathcal{S}'_\infty$.

Proof. We refer to [10, 21]. In case $\dot{F}^s_{\infty,q}$ the proof can be found in [2], in particular, Remarks 5–6 and proof of Lemma 9 in this reference. □

We then obtain realizations $\sigma_i : \dot{A}^s_{p,q} \to \mathcal{S}'$ $(i = 1, 2, 3)$, defined by (i)–(iii) of Proposition 3.13, such that

$$\text{if} \quad f \in \dot{\tilde{A}}^s_{p,q} \quad \text{then} \quad \sigma_i([f]_\infty) - f \in \mathcal{P}_\nu. \tag{25}$$

We finish this section by presenting realizations commuting with translations and/or dilations (in the sense $\tau_a \circ \sigma = \sigma \circ \tau_a$, $a \in \mathbb{R}^n$, and $h_\lambda \circ \sigma = \sigma \circ h_\lambda$, $\lambda > 0$) needed for this work:

Proposition 3.14. *Let σ_i $(i = 1, 2, 3)$ be realizations given in (i)–(iii) of Proposition 3.13. Suppose that $p < \infty$ in F-case. Then*

(i) σ_1 *commutes with translations (here also $p = \infty$ in F-case) and dilations,*

(ii) σ_2 *commutes with dilations only,*

(iii) σ_3 *does not commute with translations nor dilations.*

Proof. See [10, thms. 4.1, 4.2]. If $p = \infty$ in F-case, then it has been proved in [2, thm. 1] that σ_1 commutes with the translation. □

4. Proofs

4.1. Proof of Theorem 2.1

Step 1: proof of (9). Let $f \in \dot{\tilde{A}}^s_{p,q}$. Owing to (25), we prove (9) with $\sigma_i([f]_\infty)$ $(i = 1, 2, 3)$ instead of $\tilde{f}$, where σ_i are defined in Proposition 3.13. We are reduced to prove

$$\int_{\mathbb{R}^n} \frac{|\sigma_i([f]_\infty)(x)|}{1 + |x|^{n+d}}\, \mathrm{d}x \leq c\|[f]_\infty\|_{\dot{A}^s_{p,q}} \qquad (i = 1, 2, 3), \tag{26}$$

with the restriction on parameters n, s, p and q given in (4)–(6) and $s > (\frac{n}{p} - n)_+$. For reasons of clarity, we will subdivide the proof into several steps.

Substep 1.1: estimate of $\sigma_1([f]_\infty)$ *with either* $(\frac{n}{p} - n)_+ < s < \frac{n}{p}$ *or* $s = \frac{n}{p}$ *and* $0 < q \leq 1$ *in B-case* ($0 < p \leq 1$ *in F-case).* We write $\sigma_1([f]_\infty) = g_1 + g_2$ where $g_1 := \sum_{j\geq 1} Q_j f$ and $g_2 := \sum_{j\leq 0} Q_j f$.

1.1.1: The case $0 < s < \frac{n}{p}$ *and* $1 < p < \infty$. We have

$$|g_2(x)| \leq \|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}} \sum_{j\leq 0} 2^{-j(s-n/p)} \leq c_1\|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}}, \tag{27}$$

then the embedding $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^{s-n/p}_{\infty,\infty}$ yields

$$\int_{\mathbb{R}^n} \frac{|g_2(x)|}{1 + |x|^{n+d}}\, \mathrm{d}x \leq c_2\|[f]_\infty\|_{\dot{A}^s_{p,q}}$$

since $d > 0$. We also have

$$\|g_1\|_p \leq \|[f]_\infty\|_{\dot{B}^s_{p,\infty}} \sum_{j\geq 1} 2^{-js} \leq c_1\|[f]_\infty\|_{\dot{B}^s_{p,\infty}}, \tag{28}$$

thus, by Hölder inequality, we obtain

$$\int_{\mathbb{R}^n} \frac{|g_1(x)|}{1 + |x|^{n+d}}\, \mathrm{d}x \leq \|g_1\|_p \bigg(\int_{\mathbb{R}^n} \frac{1}{(1 + |x|^{n+d})^{p'}} \mathrm{d}x \bigg)^{1/p'} \leq c_2\|[f]_\infty\|_{\dot{B}^s_{p,\infty}},$$

and finish by the embedding $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\infty}$. Thus, we need the condition $(p' - 1)n + p'd > 0$, i.e., $d > -\frac{n}{p}$. It holds the desired result since $d > 0$.

1.1.2: The case $\frac{n}{p} - n < s < \frac{n}{p}$ *and* $0 < p \leq 1$. We introduce a parameter p_1 such that $1 < p_1 < n(\frac{n}{p} - s)^{-1}$. We have $\dot{A}^s_{p,q} \hookrightarrow \dot{A}^{s-n/p+n/p_1}_{p_1,q}$ implies $\dot{\tilde{A}}^s_{p,q} \hookrightarrow \dot{\tilde{A}}^{s-n/p+n/p_1}_{p_1,q}$; indeed, let ν_1 and ν_2 be the associated numbers with respect to (3), respectively. We have $\nu_1 = \nu_2 = 0$ since

$$s < \frac{n}{p} \quad \text{and} \quad 0 < s - \frac{n}{p} + \frac{n}{p_1} < \frac{n}{p_1}.$$

It suffices now to apply the case 1.1.1 with $\dot{\tilde{A}}^{s-n/p+n/p_1}_{p_1,q}$ instead of $\dot{\tilde{A}}^s_{p,q}$, then we need the condition $(p_1' - 1)n + p_1'd > 0$, i.e., $d > -\frac{n}{p_1}$, which it is satisfied since $d > 0$. The desired result holds.

1.1.3: The case $s = \frac{n}{p}$ *and* $0 < q \leq 1$. We treat here the B-case. Trivially, by (7) we have

$$|\sigma_1([f]_\infty)(x)| \leq \sum_{j\in\mathbb{Z}} \|Q_j f\|_\infty \leq c\|[f]_\infty\|_{\dot{B}^{n/p}_{p,q}}. \tag{29}$$

We conclude that (26) holds since $d > 0$.

1.1.4: The case $s = \frac{n}{p}$ and $0 < p \le 1$. Here we see the F-case. We use the estimate

$$\Big(\sum_j a_j\Big)^b \le \sum_j a_j^b \quad \text{for all } a_j \ge 0 \ \text{ and all } \ 0 < b \le 1,$$

then $\sum_{j\in\mathbb{Z}} \|Q_j f\|_\infty \le (\sum_{j\in\mathbb{Z}} \|Q_j f\|_\infty^p)^{1/p}$, and also use $\dot F^{n/p}_{p,q} \hookrightarrow \dot B^0_{\infty,p}$ we get

$$|\sigma_1([f]_\infty)(x)| \le c\|[f]_\infty\|_{\dot F^{n/p}_{p,q}}. \tag{30}$$

We again conclude that (26) holds since $d > 0$.

Substep 1.2: estimate of $\sigma_2([f]_\infty)$ with either $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$ or $s - \frac{n}{p} \in \mathbb{N}$ and $0 < q \le 1$ in B-case ($0 < p \le 1$ in F-case).

1.2.1: The case $s - \frac{n}{p} \in \mathbb{R}^+ \setminus \mathbb{N}_0$. For every integer N we set $f_N(x) := f(2^{-N}x)$; N is at our disposal. We easily obtain $Q_j f(x) = Q_{j-N} f_N(2^N x)$. Then, we have

$$\begin{aligned}\sigma_2([f]_\infty)(x) &= \sum_{j\in\mathbb{Z}}\left(Q_{j-N} f_N(2^N x) - \sum_{|\alpha|<\nu} (Q_{j-N} f_N)^{(\alpha)}(0)\frac{(2^N x)^\alpha}{\alpha!}\right)\\ &= \sum_{k\in\mathbb{Z}}\left(Q_k f_N(2^N x) - \sum_{|\alpha|<\nu} (Q_k f_N)^{(\alpha)}(0)\frac{(2^N x)^\alpha}{\alpha!}\right).\end{aligned}$$

We split $\sigma_2([f]_\infty)$ as $g_3 + g_4$ where $g_3 := \sum_{k\ge1} \ldots$ and $g_4 := \sum_{k\le0} \ldots$ We have

$$|g_3(x)| \le \sum_{k\ge1}\left(\|Q_k f_N\|_\infty + c_1 \sum_{|\alpha|<\nu} |2^N x|^{|\alpha|} \|(Q_k f_N)^{(\alpha)}\|_\infty\right).$$

By (7) we also have $\|(Q_k f_N)^{(\alpha)}\|_\infty \le c 2^{k|\alpha|} \|Q_k f_N\|_\infty$. Thus,

$$|g_3(x)| \le c_2 \|[f_N]_\infty\|_{\dot B^{s-n/p}_{\infty,\infty}} \sum_{k\ge1}\left(2^{-k(s-n/p)} + 2^{-k(s-n/p-\nu+1)} \sum_{|\alpha|<\nu} |2^N x|^{|\alpha|}\right).$$

As $s - \frac{n}{p} - \nu + 1 > 0$, and the fact that $\|[f_N]_\infty\|_{\dot B^{s-n/p}_{\infty,\infty}} \equiv 2^{-(s-n/p)N} \|[f]_\infty\|_{\dot B^{s-n/p}_{\infty,\infty}}$, it holds

$$|g_3(x)| \le c_3 2^{-(s-n/p)N} \|[f]_\infty\|_{\dot B^{s-n/p}_{\infty,\infty}} \left(1 + \sum_{|\alpha|<\nu} |2^N x|^{|\alpha|}\right), \tag{31}$$

where the constant c_3 is independent of f, N and x, on the one hand. On the other, to see g_4 we introduce a parameter $0 < b < 1$, whose value will be fixed later, and use Taylor's formula, then we obtain

$$\begin{aligned}|g_4(x)| \le \sum_{k\le0}&\left(\|Q_k f_N\|_\infty + \sum_{|\alpha|<\nu} \|(Q_k f_N)^{(\alpha)}\|_\infty |2^N x|^{|\alpha|} (\alpha!)^{-1}\right)^{1-b}\\ &\times\left(\nu \sum_{|\alpha|=\nu} (\alpha!)^{-1} |2^N x|^{|\alpha|} \int_0^1 (1-t)^{\nu-1} \left|(Q_k f_N)^{(\alpha)}(2^N t x)\right| \mathrm{d}t\right)^b.\end{aligned} \tag{32}$$

Using the well-known inequality

$$x \geq 0,\ y \geq 0,\ a \geq 0: \quad (x+y)^a \leq \max(1, 2^{a-1})(x^a + y^a), \tag{33}$$

and the fact that $\|(Q_k f_N)^{(\alpha)}\|_\infty \leq c 2^{k\nu} \|Q_k f_N\|_\infty$ ($|\alpha| = \nu$), we get

$$\begin{aligned} |g_4(x)| &\leq c_1 \|[f_N]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}} \sum_{k\leq 0} 2^{k(n/p-s)(1-b)} \Big(1 + \sum_{|\alpha|<\nu} 2^{k|\alpha|} |2^N x|^{|\alpha|}\Big)^{1-b} \Big(2^{k(\nu+n/p-s)} |2^N x|^\nu\Big)^b \\ &\leq c_2 \|[f_N]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}} |2^N x|^{\nu b} \Big(1 + \sum_{|\alpha|<\nu} |2^N x|^{|\alpha|(1-b)}\Big) \sum_{k\leq 0} 2^{k(\nu b + n/p - s)}. \end{aligned} \tag{34}$$

Then we need the condition $\nu b + \frac{n}{p} - s > 0$. Hence, the number b must satisfy

$$\frac{1}{\nu}\Big(s - \frac{n}{p}\Big) < b < 1. \tag{35}$$

Thus, as in (31), we obtain

$$|g_4(x)| \leq c_3 2^{-(s-n/p)N} \|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}} |2^N x|^{\nu b} \Big(1 + \sum_{|\alpha|<\nu} |2^N x|^{|\alpha|(1-b)}\Big), \tag{36}$$

where the constant c_3 is independent of f, N and x.

- Observe that (36) is also valid when we take $s = \frac{n}{p}$ and $1 < q \leq \infty$ in B-case ($1 < p \leq \infty$ in F-case) i.e., $\nu = 1$, see the estimate of g_6 in Substep 1.3 below.

We turn to the estimate of $\sigma_2([f]_\infty)$. We choose $N := N(x) \in \mathbb{Z}$ such that $2^{-N} \leq |x| < 2^{-N+1}$ (i.e., $|x| \sim 2^{-N}$), then from (31) and (36), it holds

$$|\sigma_2([f]_\infty)(x)| \leq c|x|^{s-n/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}},$$

where the constant c is independent of f and x. Hence, (26) is satisfied since $d > s - \frac{n}{p}$.

1.2.2: The case $s - \frac{n}{p} \in \mathbb{N}$ and $0 < q \leq 1$ in B-case ($0 < p \leq 1$ in F-case). Applying again Taylor's formula, we get

$$|\sigma_2([f]_\infty)(x)| \leq \nu \sum_{k\in\mathbb{Z}} \sum_{|\alpha|=\nu} (\alpha!)^{-1} |x|^{|\alpha|} \int_0^1 (1-t)^{\nu-1} \Big|(Q_k f)^{(\alpha)}(tx)\Big| \, \mathrm{d}t.$$

As

$$\sum_{k\in\mathbb{Z}} |(Q_k f)^{(\alpha)}(tx)| \leq \sum_{k\in\mathbb{Z}} \|(Q_k f)^{(\alpha)}\|_\infty \leq c_1 \|[f]_\infty\|_{\dot{B}^\nu_{\infty,1}}$$

(see (7)), and by the embeddings

$$\dot{B}^s_{p,q} \hookrightarrow \dot{B}^\nu_{\infty,q} \hookrightarrow \dot{B}^\nu_{\infty,1} \ \text{ in } B\text{-case}, \quad \dot{F}^s_{p,q} \hookrightarrow \dot{B}^\nu_{\infty,p} \hookrightarrow \dot{B}^\nu_{\infty,1} \ \text{ in } F\text{-case}, \tag{37}$$

i.e., $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^\nu_{\infty,1}$, (recall that $\nu = s - \frac{n}{p}$), we have

$$|\sigma_2([f]_\infty)(x)| \leq c_2 |x|^\nu \|[f]_\infty\|_{\dot{A}^s_{p,q}}, \tag{38}$$

where the constant c_2 is independent of f and x. Now, (26) is obtained since $d > s - \frac{n}{p}$.

Substep 1.3: estimate of $\sigma_3([f]_\infty)$ with $s-\frac{n}{p}\in\mathbb{N}_0$ and $1<q\le\infty$ in B-case ($1<p\le\infty$ in F-case). Recall that here $\nu = s-\frac{n}{p}+1\ge 1$. As above we split $\sigma_3([f]_\infty)$ as g_5+g_6 where $g_5 := \sum_{k>0} Q_k f$ and $g_6 := \sum_{k\le 0}\left(Q_k f - \sum_{|\alpha|<\nu}\frac{1}{\alpha!}(Q_k f)^{(\alpha)}(0)x^\alpha\right)$.

1.3.1: The case $s=\frac{n}{p}$. Here $p<\infty$, since by assumption $s>0$. We introduce a parameter p_1 satisfying $\max(p,1)<p_1<\infty$, we then first have

$$\Big\|\sum_{k>0} Q_k f\Big\|_{p_1} \le \sum_{k>0} 2^{-kn/p_1}(2^{kn/p_1}\|Q_k f\|_{p_1}) \le c\|[f]_\infty\|_{\dot{B}^{n/p_1}_{p_1,\infty}}. \tag{39}$$

For g_5, combining (39) with Hölder inequality, the embedding $\dot{A}^{n/p}_{p,q}\hookrightarrow \dot{B}^{n/p_1}_{p_1,\infty}$ (which is true in case $q=\infty$, see Proposition 3.5) and the fact that $d>0$, it follows

$$\int_{\mathbb{R}^n}\frac{|g_5(x)|}{1+|x|^{n+d}}\,\mathrm{d}x \le c_1\Big\|\sum_{k>0}Q_k f\Big\|_{p_1}\left(\int_{\mathbb{R}^n}\left(\frac{1}{1+|x|^{n+d}}\right)^{p_1'}\mathrm{d}x\right)^{1/p_1'} \le c_2\|[f]_\infty\|_{\dot{A}^{n/p}_{p,q}}. \tag{40}$$

To estimate g_6, we proceed as in the case 1.2.1 for g_4, thus by (35) and (36), we get

$$|g_6(x)| \le c|x|^b\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}}, \tag{41}$$

where $0<b<1$. Choosing now $0<b<\min(1,d)$ in (41) and using the embedding $\dot{A}^{n/p}_{p,q}\hookrightarrow\dot{B}^0_{\infty,\infty}$, thus

$$\int_{\mathbb{R}^n}\frac{|g_6(x)|}{1+|x|^{n+d}}\,\mathrm{d}x \le c_1\|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}}\int_{\mathbb{R}^n}\frac{|x|^b}{1+|x|^{n+d}}\,\mathrm{d}x \le c_2\|[f]_\infty\|_{\dot{A}^{n/p}_{p,q}}. \tag{42}$$

Hence, (40) and (42) yield the desired estimate (26).

1.3.2: The case $s-\frac{n}{p}=: m\in\mathbb{N}$. We observe that the estimate

$$\sum_{k>0}\|Q_k f\|_\infty = \sum_{k>0}2^{-km}2^{km}\|Q_k f\|_\infty \le c\|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}}$$

gives

$$|g_5(x)| \le c\|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}}. \tag{43}$$

The estimate for g_6, which is the more difficult one, relies on the case 1.2.1. We consider the following two cases:

• Assume that $|x|<1$, then we introduce an integer $N\ge 1$ such that $|x|\sim 2^{-N}$. Thus, by (35) with $\frac{m}{m+1}<b<1$ and by (36), we get

$$|g_6(x)| \le c|x|^m\|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}} \le c\|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}}. \tag{44}$$

• Assume now $|x|\ge 1$. Let us define an integer $N\ge 0$ such that $|x|\sim 2^N$. We set $f_N(x) := f(2^N x)$, and use the equality (for $k\le 0$)

$$Q_k f(x) = Q_{k+N} f_N(2^{-N}x) = Q_j f_N(2^{-N}x) \quad \text{with } j := k+N\le N,$$

then, we can write $g_6 = g_7+g_8$ where

$$g_7(x) := \sum_{j<0}\left(Q_j f_N(2^{-N}x) - \sum_{|\alpha|<\nu}(Q_j f_N)^{(\alpha)}(0)\frac{(2^{-N}x)^\alpha}{\alpha!}\right),$$

$$g_8(x) := \sum_{0\le j\le N}\left(Q_j f_N(2^{-N}x) - \sum_{|\alpha|<\nu}(Q_j f_N)^{(\alpha)}(0)\frac{(2^{-N}x)^\alpha}{\alpha!}\right).$$

For g_7, we have as in g_4 (see (35) with $\frac{m}{m+1} < b < 1$ and (36)),

$$|g_7(x)| \leq c|x|^m \|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}}. \tag{45}$$

To estimate g_8, we use an inequality similar to (32) with $0 < b_1 < 1$ instead of b, that is

$$|g_8(x)| \leq \sum_{0\leq j\leq N} \Big(\|Q_j f_N\|_\infty + \sum_{|\alpha|<\nu} \|(Q_j f_N)^{(\alpha)}\|_\infty |2^{-N}x|^{|\alpha|}(\alpha!)^{-1} \Big)^{1-b_1}$$
$$\times \Big(\nu \sum_{|\alpha|=\nu} (\alpha!)^{-1} |2^{-N}x|^{|\alpha|} \int_0^1 (1-t)^{\nu-1} \big|(Q_j f_N)^{(\alpha)}(2^{-N}tx)\big| \,\mathrm{d}t \Big)^{b_1}.$$

We get

$$\begin{aligned} |g_8(x)| &\leq c_1 \|[f_N]_\infty\|_{\dot{B}^m_{\infty,\infty}} \sum_{0\leq j\leq N} 2^{j(\nu b_1 - m)} \Big(1 + \sum_{|\alpha|<\nu} 2^{j|\alpha|(1-b_1)} \Big) \\ &\leq c_2 \|[f_N]_\infty\|_{\dot{B}^m_{\infty,\infty}} \Big(\sum_{0\leq j\leq N} 2^{j(\nu b_1 - m)} + \sum_{0\leq j\leq N} 2^{jb_1} \Big). \end{aligned} \tag{46}$$

Now, we choose b_1 such that

$$\nu b_1 - m < 0 \qquad \Big(\text{i.e.,} \sum_{0\leq j\leq N} 2^{j(\nu b_1 - m)} \leq \sum_{j\geq 0} 2^{j(\nu b_1 - m)} = c < \infty\Big) \tag{47}$$

where the constant c is independent of N. Using the following elementary inequality

$$\sum_{0\leq j\leq N} 2^{jb_1} \leq c_1 2^{Nb_1} \leq c_2 |x|^{b_1}$$

(the constants c_1, c_2 depend only on b_1 and n), and $\|[f_N]_\infty\|_{\dot{B}^m_{\infty,\infty}} \leq c|x|^m \|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}}$, then (46) becomes

$$|g_8(x)| \leq c_3 |x|^m (1 + |x|^{b_1}) \|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}}. \tag{48}$$

Summarizing, from (43)–(45) and (48), the resulting estimate reduces to

$$\int_{\mathbb{R}^n} \frac{|\sigma_3([f]_\infty)(x)|}{1+|x|^{n+d}} \,\mathrm{d}x \leq c \|[f]_\infty\|_{\dot{B}^m_{\infty,\infty}} \Big(\int_{\mathbb{R}^n} \frac{1}{1+|x|^{n+d}} \,\mathrm{d}x + \int_{|x|\geq 1} \frac{|x|^m(1+|x|^{b_1})}{1+|x|^{n+d}} \mathrm{d}x \Big).$$

Thus, we need the condition $m + b_1 < d$. Then, from this condition and (47) (recall that $\nu = m+1$), we choose b_1 satisfying

$$0 < b_1 < \min\Big(\frac{m}{m+1}, d - m\Big),$$

and obtain the desired estimate (26) for σ_3.

Step 2: proof of (10). By Proposition 3.14, we have $\sigma_1([\tau_{-x_0} f]_\infty)(x) = \sigma_1([f]_\infty)(x + x_0)$, $x_0 \in \mathbb{R}^n$. In (26), we change f by $\tau_{-x_0} f$ and take into account that $\|\cdot\|_{\dot{A}^s_{p,q}}$ is translation invariant, then

$$\int_{\mathbb{R}^n} \frac{|\sigma_1([f]_\infty)(x)|}{1+|x-x_0|^{n+d}} \,\mathrm{d}x \leq c \|[f]_\infty\|_{\dot{A}^s_{p,q}},$$

with the restriction on parameters n, s, p and q given in (i) of Proposition 3.13 (i.e., (4)). We deduce the desired estimate by (25) since $\sigma_1([f]_\infty) - f \in \mathcal{P}_0 = \{0\}$. $\square$

4.2. Proof of Theorem 2.4

Let $f \in \dot{\tilde{A}}^s_{p,q}$. We subdivide the proof into several steps, where, as in the preceding proof, by (25) we use the realizations σ_i ($i = 1, 2, 3$) defined in (i)–(iii) of Proposition 3.13.

Step 1: proof of (i). By (26), it is clear that

$$\int_{Q_1(0)} |\sigma_i([f]_\infty)(x)|\,\mathrm{d}x \le c_1 \int_{\mathbb{R}^n} \frac{|\sigma_i([f]_\infty)(x)|}{1+|x|^{n+d}}\,\mathrm{d}x \le c_2 \|[f]_\infty\|_{\dot{A}^s_{p,q}}.$$

Also, clearly we have $|m_{Q_1(0)}\,(\sigma_i([f]_\infty))\,| \le c\|[f]_\infty\|_{\dot{A}^s_{p,q}}$. Now, these two inequalities give

$$\int_{\mathbb{R}^n} \frac{|\sigma_i([f]_\infty)(x) - m_{Q_1(0)}\,(\sigma_i([f]_\infty))\,|}{1+|x|^{n+d}}\,\mathrm{d}x \le c_1 \|[f]_\infty\|_{\dot{A}^s_{p,q}} \left(1 + \int_{\mathbb{R}^n} \frac{1}{1+|x|^{n+d}}\,\mathrm{d}x\right) \le c_2 \|[f]_\infty\|_{\dot{A}^s_{p,q}}, \tag{49}$$

and then the desired estimate (11) holds.

Step 2: proof of (ii). Owing to Proposition 3.14, we only proceed with realizations commuting with dilations σ_i ($i = 1, 2$). By replacing f by $f(\lambda\,\cdot)$, $\lambda > 0$, in (49), and using the equality $\sigma_i([h_{1/\lambda} f]_\infty)(x) = \sigma_i([f]_\infty)(\lambda x)$, and by Proposition 3.8, we then get

$$\int_{\mathbb{R}^n} \frac{|\sigma_i([f]_\infty)(x) - m_{Q_\lambda(0)}\,(\sigma_i([f]_\infty))\,|}{\lambda^{n+d}+|x|^{n+d}}\,\mathrm{d}x \le c\lambda^{s-d-n/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}} \qquad (i = 1, 2). \tag{50}$$

The desired inequality (12).

Step 3: proof of (iii). By (50) with σ_1 and $\tau_{-x_0} f$ ($x_0 \in \mathbb{R}^n$) instead of f, and by the fact that σ_1 commutes with translations, then this case can be done as in the proof of Theorem 2.1/Step 2. We omit details. □

4.3. Proof of Corollary 2.5

To prove (13) with f instead of $\tilde{f}$ under conditions (5)–(6), we deal with realizations σ_i ($i = 2, 3$). We change f by $\tau_{-x_0} h_{1/\lambda} f = f(\lambda \cdot + x_0)$, $\lambda > 0$, $x_0 \in \mathbb{R}^n$, and we take into account that $\sigma_i([\tau_{-x_0} h_{1/\lambda} f]_\infty) = \tau_{-x_0} h_{1/\lambda} f + c_{i,\lambda,x_0}$ where $c_{i,\lambda,x_0} \in \mathbb{C}$, cf. (25). Thus, as $m_{Q_1(0)}\,(\tau_{-x_0} h_{1/\lambda} f) = m_{Q_\lambda(x_0)} f$ and $m_{Q_1(0)}\,(c_{i,\lambda,x_0}) = c_{i,\lambda,x_0}$, we have

$$\sigma_i([\tau_{-x_0} h_{1/\lambda} f]_\infty) - m_{Q_1(0)}\,(\sigma_i([\tau_{-x_0} h_{1/\lambda} f]_\infty)) = \tau_{-x_0} h_{1/\lambda} f - m_{Q_\lambda(x_0)} f. \tag{51}$$

Then, using Theorem 2.4/(i) with a suitable change of variables, and the equivalence quasi-seminorm

$$\|[\tau_{-x_0} h_{1/\lambda} f]_\infty\|_{\dot{A}^s_{p,q}} = \|[h_{1/\lambda} f]_\infty\|_{\dot{A}^s_{p,q}} \equiv \lambda^{s-n/p} \|[f]_\infty\|_{\dot{A}^s_{p,q}}, \tag{52}$$

we obtain the desired estimate. □

4.4. Proof of Theorem 2.6

Let $f \in \dot{\tilde{A}}^s_{p,q}$. Here also, by (25) we use the realizations σ_i ($i = 1, 2, 3$) given in Proposition 3.13.

Step 1: proof of (i). We will prove (16) in several steps, where we will present explicitly the constant which appears in its right side.

Substep 1.1: $0 < p_1 \le 1$. We first see the case $p_1 = 1$. As above choosing a real number d such that $d > (s - \frac{n}{p})_+$, then by (11) it holds

$$\int_{Q_1(0)} \left|\tilde{f}(x) - m_{Q_1(0)} \tilde{f}\right|\mathrm{d}x \le c_1 \int_{Q_1(0)} \frac{\left|\tilde{f}(x) - m_{Q_1(0)} \tilde{f}\right|}{1+|x|^{n+d}}\,\mathrm{d}x \le c_2 \|[f]_\infty\|_{\dot{A}^s_{p,q}}. \tag{53}$$

In the second, assume that $0 < p_1 < 1$. We set $v := \frac{1}{p_1}$. Then by Hölder inequality with exponents v and v', we get

$$\left(\int_{Q_1(0)} \left|\tilde{f}(x) - m_{Q_1(0)}\tilde{f}\right|^{p_1} \mathrm{d}x\right)^{1/p_1} \leq 2^{n(1-p_1)/p_1} \int_{Q_1(0)} \left|\tilde{f}(x) - m_{Q_1(0)}\tilde{f}\right| \mathrm{d}x,$$

which leads to apply the first case (i.e., when $p_1 = 1$) and obtain the desired estimate with the constant $c_2 2^{n(1-p_1)/p_1}$.

Substep 1.2: $1 < p_1 < \infty$. We separate the argument according to σ_i ($i = 1, 2, 3$). We will use systematically the inequality (33).

1.2.1: Estimate of $\sigma_1([f]_\infty)$. Recall that $\sigma_1([f]_\infty) = g_1 + g_2$ where $g_1 := \sum_{j\geq 1} Q_j f$ and $g_2 := \sum_{j\leq 0} Q_j f$.
• If $s = \frac{n}{p}$ and $0 < q \leq 1$ in B-case ($0 < p \leq 1$ in F-case), we use (29) and (30), in B-case and in F-case, respectively. Then we have

$$\begin{aligned}\left|\sigma_1([f]_\infty)(x) - m_{Q_1(0)}\left(\sigma_1([f]_\infty)\right)\right|^{p_1} &\leq (2c_1)^{p_1-1} \|[f]_\infty\|^{p_1-1}_{\dot{A}^{n/p}_{p,q}} \left|\sigma_1([f]_\infty)(x) - m_{Q_1(0)}\left(\sigma_1([f]_\infty)\right)\right| \\ &\leq (2c_1)^{p_1-1} \|[f]_\infty\|^{p_1-1}_{\dot{A}^{n/p}_{p,q}} |\tilde{f}(x) - m_{Q_1(0)}\tilde{f}|,\end{aligned}$$

hence, it suffices to apply (53) and obtain the result with the constant $(2c_1)^{1-1/p_1} c_2 \leq c_3$.
• If $s < \frac{n}{p}$ and $p_1 \leq p < \infty$, we apply Hölder inequality (with exponents $v := \frac{p}{p_1}$ and v'), also as $s > 0$ and $2^{n/p_1 - n/p} < 2^n$, we have

$$\|g_1\|_{L_{p_1}(Q_1(0))} \leq 2^{n/p_1 - n/p}\left(\int_{Q_1(0)} \left|g_1(x)\right|^p \mathrm{d}x\right)^{1/p} \leq 2^n c \|[f]_\infty\|_{\dot{B}^s_{p,\infty}}, \tag{54}$$

cf. (28), then the embedding $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^s_{p,\infty}$ yields the result.
• If $s < \frac{n}{p}$, $p < p_1 < \infty$ and $\frac{n}{p} - s < \frac{n}{p_1} < \frac{n}{p}$, we have (cf. (7))

$$\|Q_j f\|_{p_1} \leq \|Q_j f\|_p^{p/p_1} \|Q_j f\|_\infty^{1-p/p_1} \leq 2^{-j(n/p_1 - n/p + s)} (c_1)^{1-p/p_1} \|[f]_\infty\|_{\dot{B}^s_{p,\infty}}.$$

Using the elementary inequality $\sum_{j\geq 1} 2^{-j\beta} \leq \frac{1}{\beta \log 2}$ ($\forall \beta > 0$), it holds

$$\|g_1\|_{L_{p_1}(Q_1(0))} \leq c_2 (c_1)^{1-p/p_1} \left(\frac{n}{p_1} - \frac{n}{p} + s\right)^{-1} \|[f]_\infty\|_{\dot{B}^s_{p,\infty}}.$$

• If $s < \frac{n}{p}$, $p < p_1 < \infty$ and $\frac{n}{p} - s = \frac{n}{p_1} < \frac{n}{p}$ (here $q \leq p_1$ in B-case), we have

$$\|g_1\|_{L_{p_1}(Q_1(0))} \leq \left\| \sum_{j\geq 1} |Q_j f| \right\|_{p_1} \leq \|[f]_\infty\|_{\dot{F}^0_{p_1,1}},$$

thus, since $\dot{F}^s_{p,q} \hookrightarrow \dot{F}^0_{p_1,1}$ the last estimate is bounded by $c\|[f]_\infty\|_{\dot{F}^s_{p,q}}$, and we obtain the result in F-case. However, in B-case, we apply the embedding $B^{n/p-n/p_1}_{p,q} \hookrightarrow L_{p_1}$ (with $p_1 \geq 1$, $p < p_1$ and $q \leq p_1$) cf. Remark 2.8 and [27, p. 36], where $B^{n/p-n/p_1}_{p,q}$ is the inhomogeneous Besov space, thus by using an assertion similar to Proposition 3.7 for $B^{n/p-n/p_1}_{p,q}$ cf. [32, thm. 3.6], we get that $\|g_1\|_{L_{p_1}(Q_1(0))}$ is bounded by

$$c_1 \left\| \sum_{j\geq 1} Q_j f \right\|_{B^{n/p-n/p_1}_{p,q}} \leq c_2 \left(\sum_{j\geq 1} (2^{js} \|Q_j f\|_p)^q\right)^{1/q} \leq c_2 \|[f]_\infty\|_{\dot{B}^s_{p,q}}.$$

The different constants here depend only on n, s, p, q since $p_1 = n(\frac{n}{p} - s)^{-1}$.

• We now see g_2 if $s < \frac{n}{p}$. By (27) we obtain

$$\|g_2\|_{L_{p_1}(Q_1(0))} \le c_1 \|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}} \left(\int_{Q_1(0)} \mathrm{d}x \right)^{1/p_1} \le c_2 2^{n/p_1} \|[f]_\infty\|_{\dot{A}^s_{p,q}}.$$

On the other hand, since

$$\int_{Q_1(0)} \left| \sigma_1([f]_\infty)(x) \right| \mathrm{d}x \le c \int_{Q_1(0)} \frac{\left| \sigma_1([f]_\infty)(x) \right|}{1 + |x|^{n+d}} \mathrm{d}x,$$

then by (26) and the fact that $2^{n/p_1} < 2^n$ we have

$$\left(\int_{Q_1(0)} \left| m_{Q_1(0)}\left(\sigma_1([f]_\infty)\right) \right|^{p_1} \mathrm{d}x \right)^{1/p_1} = 2^{n/p_1} \left| m_{Q_1(0)}\left(\sigma_1([f]_\infty)\right) \right| \le c 2^n \|[f]_\infty\|_{\dot{A}^s_{p,q}}. \tag{55}$$

Hence (16) is proved for σ_1.

1.2.2: Estimate of $\sigma_2([f]_\infty)$. As above, we have $\sigma_2([f]_\infty) = g_3 + g_4$, where

$$g_3 := \sum_{j \ge 1} \left(Q_j f - \sum_{|\alpha| < \nu} \frac{1}{\alpha!} (Q_j f)^{(\alpha)}(0) x^\alpha \right)$$

and g_4 is defined in the same way but replacing $j \ge 1$ by $j \le 0$.

Assume first $s - \frac{n}{p} \in \mathbb{R}^+ \backslash \mathbb{N}_0$. To estimate $\|g_3\|_{L_{p_1}(Q_1(0))}$, thus $\|\sum_{j \ge 1} Q_j f\|_{L_{p_1}(Q_1(0))}$ can be treated as in (54) if $p_1 \le p$; however, if $p_1 > p$, we have

$$\begin{aligned}
\Big\| \sum_{j \ge 1} Q_j f \Big\|_{L_{p_1}(Q_1(0))} &\le \Big\| \sum_{j \ge 1} Q_j f \Big\|_p^{p/p_1} \left(\sum_{j \ge 1} \|Q_j f\|_\infty \right)^{1 - p/p_1} \qquad r := \min(1, p) \\
&\le \Big((3/2)([p/2] + 1) \Big)^{n/p - n/p_1} \left(\sum_{j \ge 1} \|Q_j f\|_p^r \right)^{p/(r p_1)} \left(\sum_{j \ge 1} 2^{jn/p} \|Q_j f\|_p \right)^{1 - p/p_1} \quad \text{(cf. (7))} \\
&\le c_1 \|[f]_\infty\|_{\dot{B}^s_{p,\infty}} \left(\sum_{j \ge 1} 2^{-jsr} \right)^{p/(r p_1)} \left(\sum_{j \ge 1} 2^{-j(s - n/p)} \right)^{1 - p/p_1} \\
&\le c_2 \|[f]_\infty\|_{\dot{B}^s_{p,\infty}}.
\end{aligned}$$

To continue with the second term in g_3, we have

$$\begin{aligned}
\left(\int_{Q_1(0)} \Big| \sum_{j \ge 1} \sum_{|\alpha| < \nu} (Q_j f)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!} \Big|^{p_1} \mathrm{d}x \right)^{1/p_1} &\le c_1 \sum_{|\alpha| < \nu} \left(\int_{Q_1(0)} |x|^{p_1 |\alpha|} \mathrm{d}x \right)^{1/p_1} \sum_{j \ge 1} \|Q_j f^{(\alpha)}\|_\infty \\
&\le c_2 2^{n/p_1} \|[f]_\infty\|_{\dot{B}^s_{p,\infty}} \sum_{j \ge 1} 2^{j(n/p - s + \nu - 1)} \le c_3 2^n \|[f]_\infty\|_{\dot{A}^s_{p,q}},
\end{aligned}$$

since $\frac{n}{p} - s + \nu - 1 < 0$ and $\int_{Q_1(0)} |x|^{p_1 |\alpha|} \mathrm{d}x \le \int_{Q_1(0)} \mathrm{d}x \le 2^n$. We now see $\|g_4\|_{L_{p_1}(Q_1(0))}$; by (34) with $N = 0$, we obtain

$$\begin{aligned}
\|g_4\|_{L_{p_1}(Q_1(0))} &\le c_1 \|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}} \left(\int_{Q_1(0)} |x|^{\nu b p_1} \Big(1 + \sum_{|\alpha| < \nu} |x|^{|\alpha|(1-b)} \Big)^{p_1} \mathrm{d}x \right)^{1/p_1} \\
&\le c_2 2^{n/p_1} \|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}} \le c_2 2^n \|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,\infty}},
\end{aligned}$$

where the number b satisfies the condition (35), and we finish by the embedding $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^{s-n/p}_{\infty,\infty}$.

Second, if $s - \frac{n}{p} \in \mathbb{N}$ and $0 < q \leq 1$ in B-case ($0 < p \leq 1$ in F-case), we apply (38) and obtain $\|\sigma_2([f]_\infty)\|_{L_{p_1}(Q_1(0))} \leq 2^{n/p_1} c \|[f]_\infty\|_{\dot{A}^s_{p,q}}$ (see also (37)).

Finally, the estimate of $\Big(\int_{Q_1(0)} |m_{Q_1(0)}(\sigma_2([f]_\infty))|^{p_1} \mathrm{d}x \Big)^{1/p_1}$ can be done similar to (55). The desired estimate is obtained with a constant $c > 1$ independent of p_1.

1.2.3: Estimate of $\sigma_3([f]_\infty)$. Here $q > 1$ in B-case and $p > 1$ in F-case. Assume that $s - \frac{n}{p} \in \mathbb{N}$; this case can be done similar to the preceding substep when $s - \frac{n}{p} \in \mathbb{R}^+ \backslash \mathbb{N}_0$ and will be omitted. We now see the case $s = \frac{n}{p}$. Recall that the assumption on s implies that $p < \infty$. Here $\sigma_3([f]_\infty) = g_5 + g_6$ where g_5, g_6 are given in the proof of Theorem 2.1/Substep 1.3. By (41) we get

$$\|g_6\|_{L_{p_1}(Q_1(0))} \leq c_1 \left(\int_{Q_1(0)} |x|^{bp_1} \mathrm{d}x \right)^{1/p_1} \|[f]_\infty\|_{\dot{B}^0_{\infty,\infty}} \leq c_2 2^{n/p_1} \|[f]_\infty\|_{\dot{A}^{n/p}_{p,q}}$$

where $0 < b < 1$ and $2^{n/p_1} < 2^n$. Now, if $p_1 > p$ (recall that $p_1 > 1$), by (7) it holds

$$\begin{aligned} \|g_5\|_{L_{p_1}(Q_1(0))} &\leq \big((3/2)([p/2]+1)\big)^{n/p - n/p_1} \sum_{k \geq 1} 2^{k(n/p - n/p_1)} \|Q_k f\|_p \\ &\leq c_1([p/2]+1)^{n/p} \|[f]_\infty\|_{\dot{B}^{n/p}_{p,\infty}} \sum_{k \geq 1} 2^{-kn/p_1} \leq c_2 p_1 ([p/2]+1)^{n/p} \|[f]_\infty\|_{\dot{A}^{n/p}_{p,q}}; \end{aligned}$$

if $p_1 \leq p$, using Hölder inequality (with exponents $v := \frac{p}{p_1}$ and v'), we have

$$\begin{aligned} \|g_5\|_{L_{p_1}(Q_1(0))} &\leq 2^{n/p_1 - n/p} \|g_5\|_{L_p(Q_1(0))} \\ &\leq 2^{n/p_1} \sum_{k \geq 1} 2^{-kn/p} (2^{kn/p} \|Q_k f\|_p) \leq c_1 2^{n/p_1} \|[f]_\infty\|_{\dot{B}^{n/p}_{p,\infty}} \leq c_2 2^n \|[f]_\infty\|_{\dot{A}^{n/p}_{p,q}}. \end{aligned}$$

Also, the estimate of $\Big(\int_{Q_1(0)} |m_{Q_1(0)}(\sigma_3([f]_\infty))|^{p_1} \mathrm{d}x \Big)^{1/p_1}$ can be done similar to (55). By these estimates we obtain the desired result for $\sigma_3([f]_\infty)$ with a constant of type

$$c_1 p_1 ([p/2]+1)^{n/p} + c_2 2^n = c_3 p_1 \qquad \text{with} \quad c_3 > 1\,.$$

Step 2: proof of (ii). The realizations σ_i ($i = 1, 2$) commute with dilations, we then proceed as in the proof of Theorem 2.4/Step 2.

Step 3: proof of (iii). Here as the proof of Theorem 2.1/Step 2 since the realization σ_1 commutes with translations. □

4.5. Proof of Corollary 2.9

By Proposition 3.13 and (25) we have the following three cases:

– If either $(\frac{n}{p} - n)_+ < s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q \leq 1$ in B-case ($p \leq 1$ in F-case), then $\nu = 0$ and $\sigma_1 f = f$. This case has been given in Theorem 2.6/(iii).

– If $s = \frac{n}{p}$ and $q > 1$ in B-case ($p > 1$ in F-case), then $\nu = 1$ and $\sigma_3 f - f = c \in \mathbb{C}$.

– If either $\frac{n}{p} < s < 1 + \frac{n}{p}$ or $s = 1 + \frac{n}{p}$ and $q \leq 1$ in B-case ($p \leq 1$ in F-case), then $\nu = 1$ and $\sigma_2 f - f = c \in \mathbb{C}$.

It suffices to apply Theorem 2.6/(i) with a suitable change of variables and the equivalence quasi-seminorm, cf. (51)–(52). □

4.6. Proof of Theorem 2.12

By Remark 2.11 there exists a constant $c > 1$ such that

$$\frac{1}{|Q_\lambda(x_0)|}\int_{Q_\lambda(x_0)} \left|f(x) - m_{Q_\lambda(x_0)}f\right|^{p_1}\mathrm{d}x \le \left(cp_1\lambda^{s-n/p}\|[f]_\infty\|_{\dot{A}^s_{p,q}}\right)^{p_1} = \left(c_1p_1\right)^{p_1}, \tag{56}$$

where $c_1 := c\lambda^{s-n/p}\|[f]_\infty\|_{\dot{A}^s_{p,q}}$, for all $f \in \mathring{\dot{A}}^s_{p,q}$, all $\lambda > 0$, all $x_0 \in \mathbb{R}^n$ and all $1 \le p_1 < \infty$. Now, the argument is similar to the "famous" proof given in [29], since by (56) we have the correspondent inequality to formula (13)/page 145 in this reference. $\square$

5. A general remarks

5.1. Optimality of the condition

As mentioned before, we study the optimality of the given condition $d > (s - \frac{n}{p})_+$ in Theorem 2.1. We first see the case $d = 0$.

(I). Let us define $f(x) := \mathrm{e}^{ix_1}$, $x \in \mathbb{R}^n$, which satisfies $Q_jf(x) = \gamma(2^{-j}, 0, \ldots, 0)\mathrm{e}^{ix_1}$, then $[f]_\infty \in \dot{B}^0_{\infty,1}$. We apply the following assertion proved in, e.g., [10]:

Lemma 5.1. *Let K be a compact subset of $\mathbb{R}^n\backslash\{0\}$. Every bounded function f such that* supp $\widehat{f} \subset K$ *belongs to* $\widetilde{C}_0$.

As $\widehat{f} = c\delta_{(1,0,\ldots,0)}$ (Dirac distribution), then $f \in \mathring{\dot{B}}^0_{\infty,1}$. Now clearly

$$\int_{\mathbb{R}^n}(1 + |x|^n)^{-1}|f(x)|\mathrm{d}x = \int_{\mathbb{R}^n}(1 + |x|^n)^{-1}\mathrm{d}x = \infty.$$

(II). Assume that $0 < q \le \infty$, $0 < p \le \infty$ ($p < \infty$ in F-case) and $s > (\frac{n}{p} - n)_+$. Let us introduce γ_1 a real valued and C^∞ radial function supported by the ball $|\xi| \le \frac{1}{5}$. We set $\widehat{\psi} := \gamma_1 * \gamma_1$. We have $\psi \ge 0$ and supp $\widehat{\psi} \subset \{\xi : |\xi| \le \frac{2}{5}\}$. Let us define a function f by

$$f(x) := \sum_{j\ge1} a_j\psi(2^{-j}x) \qquad (x \in \mathbb{R}^n),$$

where $(a_j)_{j\ge1}$ is a positive sequence satisfying

$$\left(\sum_{j\ge1}\left(a_j2^{-j(s-n/p)}\right)^t\right)^{1/t} < \infty, \text{ where } t := q \text{ in } B\text{-case}, \ 0 < t < p \text{ in } F\text{-case}. \tag{57}$$

By Proposition 3.7 and (57) we get $[f]_\infty \in \dot{A}^s_{p,q}$; indeed, it is clear in B-case, however in F-case we introduce a parameter p_1 such that $t < p_1 < p$ and $\frac{1}{p_1} > \frac{1}{n}(\frac{n}{p} - s)$, we put $s_1 := s - \frac{n}{p} + \frac{n}{p_1}$ (the assumption on s implies $s_1 > (\frac{n}{p_1} - n)_+$), then we use the embeddings $\dot{B}^{s_1}_{p_1,t} \hookrightarrow \dot{F}^{s_1}_{p_1,t} \hookrightarrow \dot{F}^s_{p,q}$.

To prove that $f^{(\alpha)} \in \widetilde{C}_0$ for $|\alpha| = \nu$, we first consider the case $s - \frac{n}{p} \notin \mathbb{N}_0$ or $s - \frac{n}{p} \in \mathbb{N}_0$ and $q > 1$ in B-case ($p > 1$ in F-case). Then, it suffices to recall that in [10, p. 483] it was proved that for all $\varphi \in \mathcal{S}$, there exists a constant $c := c(\varphi) > 0$ such that

$$|\langle h_\lambda f^{(\alpha)}, \varphi\rangle| \le c\lambda^{\nu-s+n/p}\|[f]_\infty\|_{\dot{A}^s_{p,q}}, \quad \forall\lambda > 0,$$

which tends to 0 with $\lambda \to 0$; recall that $\nu - s + \frac{n}{p} > 0$. We note that the last estimate has been also proved in [21, p. 173] under the condition $s - \frac{n}{p} \notin \mathbb{N}_0$, in which, with the condition $s - \frac{n}{p} \in \mathbb{N}_0$ and $q > 1$ in B-case

($p > 1$ in F-case) can be done complete similarly. Second, we consider the case $s - \frac{n}{p} \in \mathbb{N}_0$ and $q \le 1$ in B-case ($p \le 1$ in F-case); here $\nu - s + \frac{n}{p} = 0$. We observe that $Q_j f = 0$ if $j \ge 0$ since $\widehat{f}$ is supported by the ball $|\xi| \le \frac{1}{5}$, then for a fixed $|\alpha| = \nu$ we set

$$g_k := \sum_{-k \le j \le -1} Q_j f^{(\alpha)} \qquad (k = 1, 2, \ldots).$$

By (7) we have $\|Q_j f^{(\alpha)}\|_\infty \le c 2^{j\nu} \|Q_j f\|_\infty$ for all $j \in \mathbb{Z}$ and all $|\alpha| = \nu$. Then

$$\|g_k\|_\infty \le c_1 \sum_{-k \le j \le -1} 2^{j\nu} \|Q_j f\|_\infty \le c_1 \|[f]_\infty\|_{\dot{B}^{s-n/p}_{\infty,1}} \le c_2 \|[f]_\infty\|_{\dot{A}^s_{p,q}}; \tag{58}$$

we used the embedding $\dot{A}^s_{p,q} \hookrightarrow \dot{B}^{s-n/p}_{\infty,1}$ which is observed before (see (37)). Hence g_k is a *bounded* function, on the one hand. On the other, clearly $\widehat{g_k}$ is supported by $2^{-k-1} \le |\xi| \le \frac{1}{5}$, then Lemma 5.1 yields $g_k \in \widetilde{C}_0$. Since $f^{(\alpha)} = \sum_{j \le -1} Q_j f^{(\alpha)}$ in $\mathcal{S}'_\infty$, and as $\sum_{j \in \mathbb{Z}} \|Q_j f^{(\alpha)}\|_\infty \le c \|[f]_\infty\|_{\dot{A}^s_{p,q}}$ (can be obtained as in (58)), then it holds

$$\lim_{k \to +\infty} \|f^{(\alpha)} - g_k\|_\infty \le \lim_{k \to +\infty} \sum_{j \le -k-1} \|Q_j f^{(\alpha)}\|_\infty = 0.$$

Now, for an arbitrary fixed $\varepsilon > 0$ there exists a positive integer k_ε such that

$$|\langle h_\lambda f^{(\alpha)}, \varphi \rangle| \le \|f^{(\alpha)} - g_k\|_\infty \|\varphi\|_1 + |\langle h_\lambda g_k, \varphi \rangle| \le \varepsilon \|\varphi\|_1 + |\langle h_\lambda g_k, \varphi \rangle|$$

holds for all $k \ge k_\varepsilon$, all $\varphi \in \mathcal{S}$ and all $\lambda > 0$. We then obtain $\lim_{\lambda \to 0} \langle h_\lambda f^{(\alpha)}, \varphi \rangle = 0$. Hence we obtain $f^{(\alpha)} \in \widetilde{C}_0$, and we conclude that $f \in \mathring{\dot{A}}^s_{p,q}$.

We turn to choose $(a_j)_{j \ge 1}$ and d as the following:

(i) in case either $s - \frac{n}{p} \notin \mathbb{N}_0$ or $s - \frac{n}{p} \in \mathbb{N}_0$ and $q \le 1$ in B-case ($p \le 1$ in F-case), we take $-n < d < s - \frac{n}{p}$ and $a_j := 2^{jr}$ where $d < r < s - \frac{n}{p}$,

(ii) in case $s - \frac{n}{p} \in \mathbb{N}_0$ and $q > 1$ in B-case ($p > 1$ in F-case), we take $-n < d = s - \frac{n}{p}$ and $a_j := j^{-r} 2^{jd}$ where $\frac{1}{t} < r \le 1$ (t is defined in (57)).

Then, the condition (1) is false for f with the chosen d. Indeed, let $x_0 \ne 0$ be a fixed number such that $\psi(x_0) > 0$, then by continuity there exist $\beta_i := \beta_i(x_0) > 0$ ($i = 1, 2$) such that $\psi(x) > 0$ for $\beta_1 \le |x| \le \beta_2$. Also, the assumption $n + d > 0$ gives:

$$\text{for all } j \ge 1, \text{ if } \beta_1 \le 2^{-j}|x| \Rightarrow 1 + |x|^{n+d} \le \left(1 + \beta_1^{-(n+d)}\right)|x|^{n+d}.$$

On the other hand, for all fixed $N \in \mathbb{N}$ it holds $f(x) \ge \sum_{j=1}^N a_j \psi(2^{-j}x)$, then

$$\begin{aligned} \mathcal{G} := \int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+d}} \, \mathrm{d}x &\ge \sum_{j=1}^N a_j \int_{\beta_1 \le 2^{-j}|x| \le \beta_2} \frac{\psi(2^{-j}x)}{1 + |x|^{n+d}} \, \mathrm{d}x \\ &\ge \beta_3^{-1} \sum_{j=1}^N a_j \int_{\beta_1 \le 2^{-j}|x| \le \beta_2} \frac{\psi(2^{-j}x)}{|x|^{n+d}} \, \mathrm{d}x \qquad (\beta_3 := 1 + \beta_1^{-(n+d)}) \\ &\ge \beta_3^{-1} \left(\int_{\beta_1 \le |y| \le \beta_2} \frac{\psi(y)}{|y|^{n+d}} \, \mathrm{d}y \right) \sum_{j=1}^N a_j 2^{-jd} = c_1 \sum_{j=1}^N a_j 2^{-jd}, \end{aligned} \tag{59}$$

where c_1 is independent of N. Letting $N \to \infty$ we conclude $\mathcal{G} = \infty$.

In case (i) (with exception of $s - \frac{n}{p} \leq -1$) we can take d such that $\nu - 1 < d < s - \frac{n}{p}$, then the condition (1) becomes also false for $\tilde{f} := f + u_f$ ($\forall u_f \in \mathcal{P}_\nu$) cf. (8), indeed, owing to (59) it suffices to observe that $\int_{\mathbb{R}^n} (1 + |x|^{n+d})^{-1} |u_f(x)| \mathrm{d}x < \infty$.

(III). We now consider the counterpart for $\dot{F}^s_{\infty,q}$ with $s > 0$ and $0 < q < \infty$, (recall that in Theorem 2.1 the condition $s > (\frac{n}{\infty} - n)_+$ is reduced to $s > 0$). We only prove the embedding $\dot{\tilde{B}}^s_{\infty,q} \hookrightarrow \dot{\tilde{F}}^s_{\infty,q}$, one then can adapt the reasoning in (II) given for $\dot{B}^s_{\infty,q}$. Indeed, we easily have

$$\int_{P_{k,\mu}} \sum_{j \geq k} 2^{jsq} |Q_j f(x)|^q \mathrm{d}x \leq 2^{-kn} \sum_{j \geq k} 2^{jsq} \|Q_j f\|^q_\infty \leq 2^{-kn} \|[f]_\infty\|^q_{\dot{B}^s_{\infty,q}},$$

then, we get $\dot{B}^s_{\infty,q} \hookrightarrow \dot{F}^s_{\infty,q}$. Let now ν_1 and ν_2 be the associated numbers to $\dot{F}^s_{\infty,q}$ and $\dot{B}^s_{\infty,q}$ with respect to (3), respectively. We have $\nu_1 = [s] + 1$. However, $\nu_2 = [s] + 1$ if ($s \notin \mathbb{N}$) or ($s \in \mathbb{N}$ and $q > 1$), and $\nu_2 = s$ if ($s \in \mathbb{N}$ and $0 < q \leq 1$). In all cases we have $\nu_2 \leq \nu_1$. Then, for all $f \in \dot{\tilde{B}}^s_{\infty,q}$, we have $f^{(\alpha)} \in \widetilde{C}_0$ ($\forall |\alpha| = \nu_2$) implies that $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \nu_1$, (cf. see the example just after Definition 3.11). Then $f \in \dot{\tilde{F}}^s_{\infty,q}$. Hence the desired embedding follows.

5.2. Results for inhomogeneous spaces

We start by recalling the definition of inhomogeneous Besov and Triebel-Lizorkin spaces.

Definition 5.2. *Let $s \in \mathbb{R}$.*

(i) *Let $0 < p, q \leq \infty$. The Besov space $B^s_{p,q}$ is the set of all $f \in \mathcal{S}'$ such that*

$\|f\|_{B^s_{p,q}} := \|\rho_0 * f\|_p + \left(\sum_{j \geq 1} (2^{js} \|Q_j f\|_p)^q \right)^{1/q} < \infty$, *where* $\widehat{\rho_0} := \rho$.

(ii) *Let $0 < p < \infty$ and $0 < q \leq \infty$. The Triebel-Lizorkin space $F^s_{p,q}$ is the set of all $f \in \mathcal{S}'$ such that*

$\|f\|_{F^s_{p,q}} := \|\rho_0 * f\|_p + \left\| \left(\sum_{j \geq 1} (2^{js} |Q_j f|)^q \right)^{1/q} \right\|_p < \infty.$

(iii) *Let $0 < q < \infty$. The space $F^s_{\infty,q}$ is the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{F^s_{\infty,q}} := \|\rho_0 * f\|_\infty + \sup_{k \in \mathbb{N},\, \mu \in \mathbb{Z}^n} \left(2^{kn} \int_{P_{k,\mu}} \sum_{j \geq k} 2^{jsq} |Q_j f(x)|^q \mathrm{d}x \right)^{1/q} < \infty,$$

(see [14, (12.8)]).

We denote by $A^s_{p,q}$ for $B^s_{p,q}$ or $F^s_{p,q}$. The relation between $A^s_{p,q}$ and $\dot{A}^s_{p,q}$ is given by the following well-known statement.

Proposition 5.3. *Let $0 < p, q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Then $A^s_{p,q}$ is the set of all $f \in L_p$ such that $[f]_\infty \in \dot{A}^s_{p,q}$. Moreover the expression $\|f\|_p + \|[f]_\infty\|_{\dot{A}^s_{p,q}}$ defines an equivalent quasi-norm in $A^s_{p,q}$. Here, A and $\dot{A}$ are B-spaces or F-spaces simultaneously.*

Proof. See, e.g., [31, thm. 2.3.3, p. 98]. In case $F^s_{\infty,q}$ see [2, lem. 4]. □

Remark 5.4. With assumptions of Theorem 2.1, we have $A^s_{p,q} \subset \dot{\tilde{A}}^s_{p,q}$. Indeed; if $1 \leq p < \infty$, the assertion follows by $L_p \hookrightarrow \widetilde{C}_0$; if $0 < p < 1$, we have $A^s_{p,q} \hookrightarrow L_1$ since $s > \frac{n}{p} - n$, and again it follows by $L_1 \hookrightarrow \widetilde{C}_0$; if $p = \infty$ in F-case, we have $\nu = [s] + 1 \geq 1$ the associated number to $\dot{F}^s_{\infty,q}$ with respect to (3), let us now take $f \in F^s_{\infty,q}$, as $F^s_{\infty,q} \hookrightarrow L_\infty$ (see [2, lem. 4]) we get $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \nu$ (cf. see the example just after Definition 3.11), i.e., $f \in \dot{\tilde{F}}^s_{\infty,q}$. Now using Proposition 5.3, inequalities (9)–(13) and (16)–(19) are valid if we replace $\dot{\tilde{A}}^s_{p,q}$ by $A^s_{p,q}$ (with $p < \infty$ in F-case), also similarly for (9), (11) and (16) by replacing $\dot{\tilde{F}}^s_{\infty,q}$ by $F^s_{\infty,q}$.

5.3. An application to Morrey spaces

The Morrey space $\mathcal{M}_r^u$ $(0 < r \leq u \leq \infty)$ is the set of all functions $f \in L_r^{\rm loc}$ such that

$$\|f\|_{\mathcal{M}_r^u} := \sup_Q |Q|^{1/u-1/r}\left(\int_Q |f(x)|^r \,\mathrm{d}x\right)^{1/r} < \infty,$$

with the supremum is taken over all cubes Q in $\mathbb{R}^n$, cf. [34, sect. 1.3.2]. As mentioned before, the presence of the polynomials causes that $\dot{A}^s_{p,q}$ can not be embedded in $\mathcal{M}_r^u$. However, in realized spaces one can obtain the following result:

Theorem 5.5. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Suppose that* (4) *is satisfied. Let $0 < p_1 < \infty$ be a real number such that* (14)–(15) *are satisfied if $1 < p_1 < \infty$. We put $\frac{1}{u} := \frac{1}{p} - \frac{s}{n}$. Then it holds $\dot{\tilde{A}}^s_{p,q} \hookrightarrow \mathcal{M}^u_{p_1}$.*

Proof. In Theorem 2.6/(iii) we have proved

$$\left(\frac{1}{\lambda^n}\int_{Q_\lambda(x_0)} \left|\sigma_1([f]_\infty)(x) - m_{Q_\lambda(x_0)}\left(\sigma_1([f]_\infty)\right)\right|^{p_1} \mathrm{d}x\right)^{1/p_1} \leq c\lambda^{s-n/p}\|[f]_\infty\|_{\dot{A}^s_{p,q}}$$

(σ_1 is defined in Proposition 3.13) for all $f \in \dot{\tilde{A}}^s_{p,q}$, all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$. Now, applying (33) and (55) since $m_{Q_\lambda(x_0)}\left(\sigma_1([f]_\infty)\right) = m_{Q_1(0)}\left(\sigma_1([\tau_{-x_0}h_{1/\lambda}f]_\infty)\right)$ and using the equivalence quasi-seminorm, cf. (52), we obtain

$$\left(\frac{1}{\lambda^n}\int_{Q_\lambda(x_0)} \left|\sigma_1([f]_\infty)(x)\right|^{p_1} \mathrm{d}x\right)^{1/p_1} \leq c\lambda^{s-n/p}\|[f]_\infty\|_{\dot{A}^s_{p,q}}.$$

This gives $\|f\|_{\mathcal{M}^u_{p_1}} \leq c\|[f]_\infty\|_{\dot{A}^s_{p,q}}$ since (25), and the desired result. □

If we take $u = p_1$ in the preceding theorem, and use the equality $\mathcal{M}^{p_1}_{p_1} = L_{p_1}$ (see [34, 1.4.3]), we obtain the following interesting embedding between realized spaces and Lebesgue spaces:

Corollary 5.6. *Let $1 \leq p < \infty$ and $1 < p_1 < \infty$ be such that $p < p_1$. Let $0 < q \leq \infty$ (with $0 < q \leq p_1$ in B-case). Then it holds $\dot{\tilde{A}}^{n/p-n/p_1}_{p,q} \hookrightarrow L_{p_1}$.*

In the same spirit, we present an embedding between $\dot{\tilde{A}}^s_{p,q}$ and the Campanato space $\mathcal{L}^{p,\theta}$, where $\mathcal{L}^{p,\theta}$ $(1 \leq p < \infty, 0 \leq \theta < \infty)$ is the set of all functions $f \in L_p^{\rm loc}$ (modulo constants) such that

$$\|f\|_{\mathcal{L}^{p,\theta}} := \sup_Q \left(|Q|^{-\theta/n}\int_Q |f(x) - m_Q f|^p \,\mathrm{d}x\right)^{1/p} < \infty,$$

with the supremum is taken over all cubes Q in $\mathbb{R}^n$, cf. [25] and [34].

Theorem 5.7. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s > (\frac{n}{p} - n)_+$. Suppose that* (4) *is satisfied. Assume that* (5)–(6) *hold with $\nu = 1$, i.e., either $(\frac{n}{p} - n)_+ < s < 1 + \frac{n}{p}$ or $s = 1 + \frac{n}{p}$ and $q \leq 1$ in B-case ($p \leq 1$ in F-case). Let $1 \leq p_1 < \infty$ be a real number such that* (14)–(15) *are satisfied if $1 < p_1 < \infty$. We put $\theta := n + p_1(s - \frac{n}{p})$. Then it holds $\dot{\tilde{A}}^s_{p,q} \hookrightarrow \mathcal{L}^{p_1,\theta}$.*

Proof. Using Corollary 2.9, it is the same preceding proof. □

In this sense, we can use the different properties of $\mathcal{L}^{p,\theta}$ as $\mathcal{L}^{p,n} = BMO$ $(1 \leq p < \infty)$, ..etc, see [25].

Acknowledgments. The authors thank, both, the General Direction of Education and Training (Sub-Directorate of Research Training) MESRS, PRFU project no. C00L03UN280120220005, and the General Direction of Scientific Research and Technological Development DGRSDT, Algeria. We also wish to thank the referee for his valuable comments and suggestions.

References

[1] H. Bahouri, J.Y. Chemin, R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren Math. Wiss. **343**, Springer, 2011.

[2] M. Benallia, M. Moussai, *Realization of homogeneous Triebel-Lizorkin spaces with $p = \infty$ and characterizations via differences*, Ufa Math. J. **11** (2019), 115–130.

[3] M. Benallia, M. Moussai, *Some convolution inequality in realized homogeneous Besov and Triebel-Lizorkin spaces*, Publ. Inst. Math. (Beograd) (N.S.) **106**(120) (2019), 69–84.

[4] M. Benallia, M. Moussai, *Inequalities of Gagliardo-Nirenberg type in realized homogeneous Besov and Triebel-Lizorkin spaces*, Math. Rep. (Bucur.) **22**(72) (2020), 19–39.

[5] F. Bensaid, M. Moussai, *Function algebras of Besov and Triebel-Lizorkin-type*, Czechoslovak Math. J. **73** (2023), 1281–1300.

[6] S. Bissar, M. Moussai, *Pointwise multiplication in the realized homogeneous Besov and Triebel-Lizorkin spaces*, Probl. Anal. Issues Anal. **7**(25) (2018), 3–22.

[7] G. Bourdaud, *Réalisations des espaces de Besov homogènes*, Ark. Mat. **26** (1988), 41–54.

[8] G. Bourdaud, *Ce qu'il faut savoir sur les espaces de Besov*, Preprint, Paris, 2009.

[9] G. Bourdaud, *Realizations of homogeneous Sobolev spaces*, Complex Var. Elliptic Equ. **56** (2011), 857–874.

[10] G. Bourdaud, *Realizations of homogeneous Besov and Lizorkin-Triebel spaces*, Math. Nachr. **286** (2013), 476–491.

[11] G. Bourdaud, M. Moussai, W. Sickel, *Composition operators in Lizorkin-Triebel spaces*, J. Funct. Anal. **259** (2010), 1098–1128.

[12] L. Brandolese, *Application of the realization of homogeneous Sobolev spaces to Navier-Stokes*, SIAM J. Math. Anal. **37** (2005), 673–683.

[13] C. Fefferman, E.M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[14] M. Frazier, B. Jawerth, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. **93** (1990), 34–170.

[15] L. Grafakos, *Modern Fourier analysis*, Springer, New York, 2009.

[16] B. Jawerth, *Some observations on Besov and Lizorkin-Triebel spaces*, Math. Scand. **40** (1977), 94–104.

[17] F. John, L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

[18] J-L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*, Lecture Notes in Mathematics, 994, Springer-Verlag, Berlin, 1983.

[19] S. Meliani, M. Moussai, *Boundedness of pseudodifferential operators on realized homogeneous Besov spaces*, Taiwanese J. Math. **21** (2017), 441–465.

[20] M. Moussai, *Composition operators on Besov algebras*, Rev. Mat. Iberoam. **28** (2012), 239–272.

[21] M. Moussai, *Realizations of homogeneous Besov and Triebel-Lizorkin spaces and an application to pointwise multipliers*, Anal. Appl. (Singap.) **13** (2015), 149–183.

[22] M. Moussai, *Some Hardy-type estimates in realized homogeneous Besov and Triebel–Lizorkin spaces*, Ann. Fac. Sci. Toulouse Math. (6) **29** (2020), 39–55.

[23] M. Moussai, *Characterization by approximation of homogeneous Besov and Triebel-Lizorkin type spaces*, Arch. Math. **122** (2024), 59–69.

[24] R.J. Nessel, G. Wilmes, *Nikolskii-type inequalities for trigonometric polynomials and entire functions of exponential type*, J. Aust. Math. Soc. **25** (1978), 7–18.

[25] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1969), 71–87.

[26] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series I, Durham, N.C., 1976.

[27] T. Runst, W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter, Berlin, 1996.

[28] Y. Sawano, *Theory of Besov spaces*, Springer Nature Singapore Pte Ltd., 2018.

[29] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton U.P., 1993.

[30] H. Triebel, *Theory of function saces*, Birkhäuser, Basel, 1983.

[31] H. Triebel, *Theory of function spaces II*, Birkhäuser, Basel, 1992.

[32] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I: Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA. Math. **33** (1986), 131–174.

[33] D. Yang, W. Yuan, *New Besov-type spaces and Triebel-Lizorkin-type spaces including Q spaces*. Math. Z. **265** (2010), 451–480.

[34] W. Yuan, W. Sickel, D. Yang, *Morrey and Campanato meet Besov, Lizorkin and Triebel*, Lecture Notes in Mathematics, vol. **2005**, Springer-Verlag, 2010.