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Non-equal neighboring model for factorials (mod p)

Cristian Cobeli[a], Alexandru Pitrop[b], Alexandru Zaharescu[c,a]

[a]*'Simion Stoilow' Institute of Mathematics of the Romanian Academy, 21 Calea Griviței, P. O. Box 1-764, RO-014700, Bucharest, Romania*
[b]*Brasenose College, Radcliffe Square, Oxford, OX1 4AJ, United Kingdom*
[c]*Department of Mathematics, University of Illinois at Urbana-Champaign, Altgeld Hall, 1409 W. Green Street, Urbana, IL, 61801, USA*

Abstract. We study the non-equal-neighbor model introduced by Broughan and Barnett (2009) to study the distribution of the residue classes of factorials modulo p and obtain an estimate which shows that the average analogue sequence has a Poisson distribution.

1. Introduction

The sequence of factorials $\{x!\}_{x\geq 1}$, generated by the fundamental multiplicative rule, has contrasting properties through its particular elements, and by itself as a whole through the general distribution that its components follow. Thus, on the one hand, the elements $x! := 1 \times 2 \times 3 \times \cdots \times x$ of the sequence store an ample number of arithmetic properties as x increases, and, on the other hand, the numbers $x!$ grow rapidly, becoming more and more distant from one another, and, according to the Stirling formula, their size is

$$x! \sim x^x e^{-x}\sqrt{2\pi x}\sum_{k\geq 0}\frac{a_k}{x^k},$$

where $a_0 = 1, a_1 = 1/12, a_2 = 139/51840, \dots$, the general explicit formula of the coefficients a_k being obtained by Nemes [21, Corollary 2]. This unfolds convincingly by looking at the residue classes $x! \pmod p$, where p is a prime number. If p increases, the original order in the resulting finite sequence $\{x! \pmod p\}_{1\leq x\leq p}$ is significantly altered, its outlook being taken on and showing the attributes of a random sequence. The fact was formally stated in 1963 as an open problem by Stauduhar [22, Problem 7] (see also Guy [15, Problem F11] and [5, 6]).

Conjecture 1 (R. Stauduhar). *Let p be a prime number and let $h(p)$ be the number of distinct residue classes of* $1!, 2!, \dots, (p-1)! \pmod p$. *Then*

$$\lim_{p\to\infty}\frac{h(p)}{p} = 1 - \frac{1}{e}.$$

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Communicated by Paola Bonacini
Email addresses: cristian.cobeli@imar.ro (Cristian Cobeli), alexandru-petre.pitrop@bnc.ox.ac.uk (Alexandru Pitrop), zaharesc@illinois.edu (Alexandru Zaharescu)

Questions related to the main matter of the distribution of factorials modulo p have been often considered (see [7–9, 13, 14, 16, 17, 19]). Some effective results on the problem have been obtained (see [1, 4, 5, 10, 12]), the most recent showing that the number of distinct residue classes modulo p achieved by the sequence $1!, 2!, \dots, (p-1)!$ is at least $\left(\sqrt{2} - o(1)\right)\sqrt{p}$ (Grebennikov et al. [14]). Note the square root of p in the lower bound is still quite far from $(1 - 1/e)p$, or to at least a positive proportion of the residue classes mod p that are expected to be covered by the factorials as p gets large.

As the problem is hereditary, in the sense that the distribution of the factorials being fairly uniform along the residue classes mod p is likely the same over short intervals, the matter also attracted attention. Thus, on short intervals, various nontrivial results have been obtained by adapting techniques ranging from the evaluation of the associated sums of characters, exponential estimates or the moments counting the number of representations of a residue class as factorials or analyzing the distribution of the products of several factorials in residue classes mod p (see [8–11, 14, 16, 18, 20]).

As several numerical experiments show the random character of the sequence of residue classes $x! \pmod p$, one should reckon its distribution to resemble that of a Poisson process, and the following conjecture formally states this expectation.

Conjecture 2 (CZ [6]). *For any integer $k \geq 0$, the proportion of residue classes $y \in \mathbb{Z}_p$ for which there are exactly k positive integers $1 \leq x \leq p-1$ for which $x! \equiv y \pmod p$ tends to the limit $1/(ek!)$, as $p \to \infty$.*

Notice that when $k = 0$ Conjecture 2 reduces to Stauduhar's Conjecture and the general statement says that if p is sufficiently large, then the elements of the sequence of factorials $\{x! \pmod p\}_{1 \leq x \leq p}$ have appearances as the events in a Poisson process with mean $\lambda = 1$.

Numerical evidence and a statistical model adapted to measure with precise estimates the distribution of sequences analogue to that of the factorials modulo p are presented in [6]. In this paper we take a step further. While the order is broken and many arithmetical properties of the factorials are lost with it, a feature that is still preserved is the fact that modulo p the adjacent elements of the sequence, that is, $x! \pmod p$ and $(x+1)! \pmod p$, always remain distinct if $1 \leq x \leq p-2$. The statistical model associated with this '*non-equal-neighbor*' property is investigated by Broughan and Barnett [3], who showed that the average sequence analogue to that of the factorials should satisfy Conjecture 2 if $k = 0$. Our aim is to show that an average sequence of numbers satisfying the 'non-equal-neighbor' property does indeed behave like a Poisson process.

To state things precisely, let us suppose that $\mathcal{M}$ is a set of $M > 0$ positive integers. If n is a positive integer, consider the subset of n-tuples $\mathcal{U} \in \mathcal{M}^n$ that have no equal adjacent components. Also, let $\mathcal{N} \subset \mathcal{M}$ be a subset with n elements.

In the particular case of the problem with the factorials $\pmod p$, $\mathcal{M}$ would consist of all residue classes $\pmod p$, and $\mathcal{N} \subset \mathcal{M}$ would contain the actual values taken by the residue classes $x! \pmod p$ for $1 \leq x \leq p$, classes that can be repeated a number of $k \geq 0$ times.

The following k-counter function labels the tuples $\mathbf{x} \in \mathcal{U}$ by the number of repetitions their components would have as elements of $\mathcal{N}$:

$$m_k(\mathbf{x}) = \left|\{y \in \mathcal{N} : y \text{ is represented exactly } k \text{ times in } \mathbf{x}\}\right|. \tag{1}$$

Our main object is to estimate the proportion of sequences of n elements from $\mathcal{M}$, whose neighboring terms are always distinct, and are reached exactly k times by the elements of $\mathcal{N}$. The next theorem shows that almost all sequences with non-equal adjacent terms can be described as 'random' since, in the limit, they follow a Poisson distribution.

Theorem 1. *Let $\lambda \in (0, 1]$, $\gamma \in [0, 1)$, $\delta \in (0, (1-\gamma)/2)$ and let $k \geq 0$ be integer. Suppose the integers n and M satisfy the inequalities $k \leq n \leq M$ and $M = n/\lambda + O(n^\gamma)$, uniformly on λ and k as n tends to infinity while γ is fixed. Then*

$$\frac{1}{|\mathcal{U}|} \cdot \left|\left\{\mathbf{x} \in \mathcal{U} : \left|\frac{m_k(\mathbf{x})}{n} - \frac{\lambda^k}{k!}e^{-\lambda}\right| < n^{-\delta}\right\}\right| = 1 - O_{k,\lambda}\left(n^{\gamma+2\delta-1}\right).$$

(The notation $O_{k,\lambda}\left(n^{\gamma+2\delta-1}\right)$ as n tends to infinity represents a function that is asymptotically bounded above by a constant multiple of $n^{\gamma+2\delta-1}$, but the constant may change with k and λ. In other words, it means that the function grows no faster than $n^{\gamma+2\delta-1}$ as n becomes very large, while k and λ are kept fixed.)

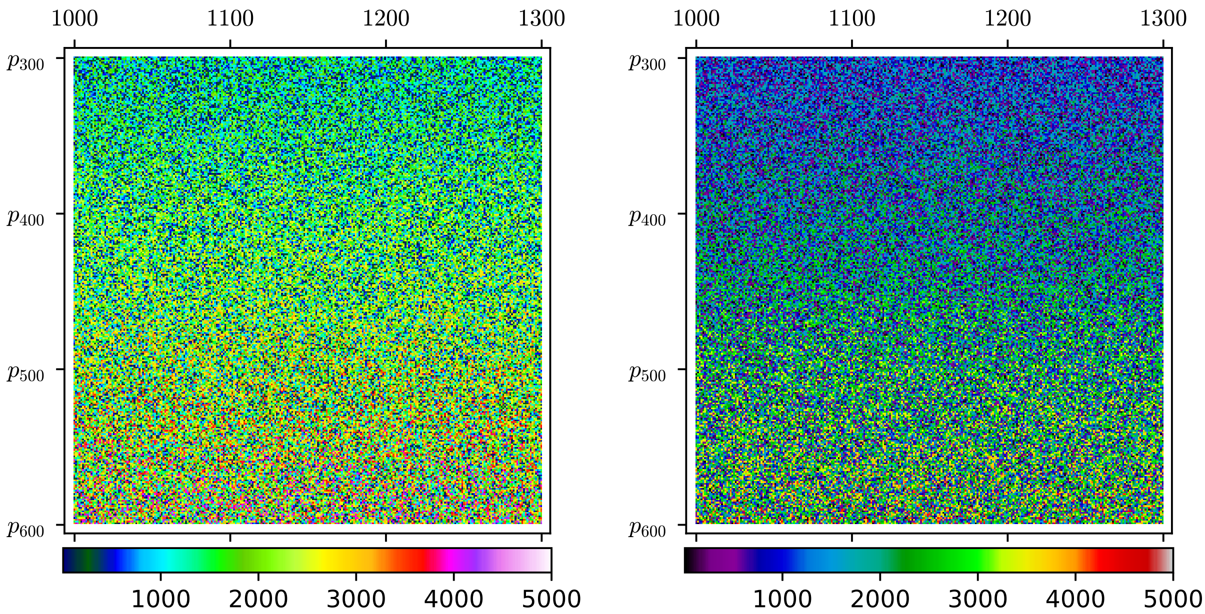

Figure 1. *For each prime* p *in the interval* $[p_{300}, p_{600}) = [1987, 4397)$ *are represented the residue classes* $x! \pmod p$ *with* $x \in [1000, 1300)$ *(left) and* 300 *randomly selected residue classes* $\pmod p$ *with non-equal neighbors (right). Two slightly different color-maps are used for the two representations, as the textures would otherwise be almost indistinguishable to the naked eye.*

While the 'non-equal-neighbor' model proves successful covering a large class of subsequences of n elements chosen from $\mathcal{M}$, one needs to impose supplementary conditions on the subsequences on which the averages are calculated. Further studies are needed in order to obtain results on the distribution of thinner sets of sequences that resemble the factorials and get closer to the precise statements such as that in Conjecture 2.

Various numerical experiments fully support the type of distribution summarized in Conjecture 2. A few of them are discussed in [6]. A different test is shown in Figure 1. There, the modular values of factorials taken from a fixed interval of length 300 are shown for comparison with those of a set of 300 residue classes $\pmod p$ randomly chosen so that they satisfy the non-equal-neighbor condition. The square representations have three hundred lines each, corresponding to the prime numbers starting with the 300th prime $p_{300} = 1987$. Notice how the average brightness and color hues of the two color-maps used gradually change with increasing p due to the larger and larger number of residue classes that have to be represented.

In Figure 2 one can see the values of two other types of 'factorial-alike' sequences $\pmod p$, for 300 consecutive primes from two different intervals. For each $p \in [p_{2000}, p_{2300})$, in the image on the left, on the line corresponding to p are shown in different colors the values of the products $2 \times 3 \times 5 \times \cdots \times p_x \pmod p$ for $1000 \le x < 1300$. In the representation on the right-side of Figure 2, the 'factorials' of the so called '*square-prime*' numbers are shown analogously. The recently introduced square-prime numbers [2] are defined as the elements of the union of sets

$$SP = \bigcup_{k \ge 2} \{k^2 p \,:\, p \text{ prime}\} \tag{2}$$

ordered increasingly. The first 26 square-prime numbers are: 8, 12, 18, 20, 27, 28, 32, 44, 45, 48, 50, 52, 63, 68, 72, 75, 76, 80, 92, 98, 99, 108, 112, 116, 117, 124. Let s_k denote the kth square-prime number. The sequence $\{s_k\}_{k\ge1}$

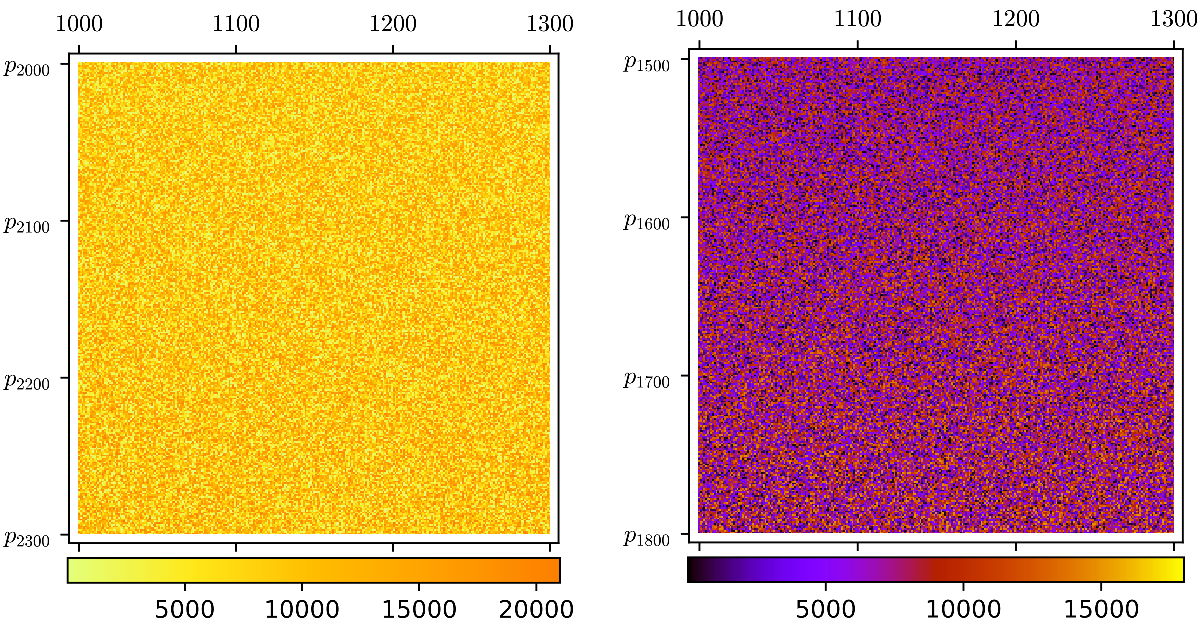

FIGURE 2. *'Factorials' of primes (left) and 'factorials' of SP numbers belonging to the set (2) (right). In the representation on the left, the primes p belong to the interval $[p_{2000}, p_{2300}) = [17389, 20357)$ and, on the representation on the right, the primes are chosen from $[p_{1500}, p_{1800}) = [12553, 15401)$. Then, for each $x \in [1000, 1300)$, the point of coordinates (p, x) represents the residue class modulo p of the product of primes $\prod_{1\le k\le x} p_k$ (left), and the product of square-primes $\prod_{1\le k\le x} s_k$ (right).*

is interesting because it has mixed properties, some of which are similar or closely related to the sequence of prime numbers. In the square on the right of Figure 2 the point of coordinates $(p, x) \in [p_{1500}, p_{1800}) \times [1000, 1300)$ has the color associated to the residue class of $s_1 \times s_2 \times \cdots \times s_x \pmod{p}$. Note that both these 'factorial-alike' sequences satisfy the non-equal-neighbor property and their distributions appear to be no different than those of the sequences shown in Figure 1.

The paper is organized as follows: in Section 2 we introduce a few combinatorial results that are needed in the subsequent arguments of the proof of Theorem 1. In Section 3 we obtain an estimate for the average of proportions $m_k(\mathbf{x})/n$ over all sequences $\mathbf{x}$ of n elements in $\mathcal{M}$ that satisfy the non-equal-neighbor condition. Then, in Section 4, we show that the associated second moment about this average is by comparison much smaller. This allows us to complete the proof of the main Theorem 1 in the final Section 5.

2. Notations and preliminary results

Suppose $n \ge 2$, $M \ge 3$ and let $\mathcal{U} = \mathcal{U}(\mathcal{M}, n)$ be the set of all n-tuples with components in $\mathcal{M}$ and no two neighbor components equal, that is,

$$\mathcal{U}(\mathcal{M}, n) = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{M}^n \ : \ x_j \ne x_{j+1},\ 1 \le j \le n-1\}.$$

Note that the cardinality of $\mathcal{U}$ is equal to $M(M-1)^{n-1}$, because there are M possible ways to choose x_1 as the first component of an n-tuple, and for each choice of x_1 there are $M-1$ ways to choose x_2 different from x_1. And so on, there are $(M-1)$ ways to choose x_n different from x_{n-1}. Similarly, for any fixed $y \in \mathcal{N}$, the number of $\mathbf{x} \in \mathcal{U}$ with no component equal to y is equal to $(M-1)(M-2)^{n-1}$. We state the remark in summary in next lemma.

Lemma 1. *Let $n \ge 2$ and $M \ge 3$. Then*

$$|\mathcal{U}(\mathcal{M}, n)| = M(M-1)^{n-1}, \tag{3}$$

and, for any $y \in \mathcal{N}$, we have

$$\left|\{\mathbf{x} \in \mathcal{U} : \mathbf{x} = (x_1, \dots, x_n) \text{ and } x_j \neq y \text{ for } 1 \leq j \leq n\}\right| = (M-1)(M-2)^{n-1}\,. \tag{4}$$

The following lemma is elementary, but being applied several times in slightly modified forms, for the clarity of the presentation, we state it here as a reference result.

Lemma 2. *Let $h, z \geq 0$ be integers and let*

$$R(z,h) = \left|\{(s_1, \dots, s_h) \in \mathbb{N}^h \ : \ s_1, \dots, s_h \geq 1;\ s_1 + \cdots + s_h = z\}\right|. \tag{5}$$

If $z = 0$ or $h = 0$ then $R(z,h) = 0$ is set to 0. Then

$$R(z,h) = \binom{z-1}{h-1}.$$

(By convention, if $h = 0$, the set whose cardinality is counted by $R(z,h)$ is empty, and the binomial coefficient is equal to 0 if either of its two parameters is negative.)

Proof. The initial values of $R(z,h)$ are shown in the next table:

	$h=0$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$\cdots$
$z=0$	0	0					
$z=1$	0	1					
$z=2$	0	1	1				
$z=3$	0	1	2	1			
$z=4$	0	1	3	3	1		
$z=5$	0	1	4	6	4	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The values located in the triangle on the upper right side that are missing from the table are all equal to 0.

Any tuple counted in the definition of $R(z,h)$ can be put into a one-to-one correspondence with either a tuple that is counted in $R(z-1,h)$ or with a tuple that is counted in $R(z-1,h-1)$. Precisely, the association is given by

$$\begin{aligned} &(\alpha, \beta, \dots, \omega) \rightsquigarrow (\alpha - 1, \beta, \dots, \omega) && \text{if } \alpha > 1 \text{ or} \\ &(\alpha, \beta, \dots, \omega) \rightsquigarrow (\beta, \dots, \omega) && \text{if } \alpha = 1, \end{aligned}$$

so that, $R(z,h) = R(z-1,h) + R(z-1,h-1)$ and the lemma follows. □

For integers $h, z \geq 0$ and $K \geq 1$ define

$$R_K(z,h) = \left|\{(s_1, \dots, s_h) \in \mathbb{N}^h \ : \ s_1, \dots, s_h \geq K;\ s_1 + \cdots + s_h = z\}\right|. \tag{6}$$

If $z = 0$ or $h = 0$ then, by definition, $R_K(z,h) = 0$. Notice that $R_1(z,h) = R(z,h)$. The next lemma gives a closed formula for $R_K(z,h)$.

Lemma 3. *Let $h, z \geq 0$ and $K \geq 1$ be integers. Then*

$$R_K(z,h) = R(z - h(K-1), h) = \binom{z - h(K-1) - 1}{h-1}. \tag{7}$$

(By convention, the binomial coefficient is equal to 0 if either of its two parameters is negative.)

Proof. Let us first note that according to the definition, the table analogous to the one for $K = 1$ in the proof of Lemma 2 has only zeros on the first column, and also on the first K rows. Then, on the 2nd column, starting with the line $z = K$, all values are equal to 1. These entries are enough to build the entire table with the recursive formula

$$R_K(z,h) = R_K(z-1,h) + R_K(z-K,h-1),$$

which follows from the one-to-one correspondence

$$\begin{aligned}(\alpha,\beta,\dots,\omega) &\rightsquigarrow (\alpha-1,\beta,\dots,\omega) && \text{if } \alpha > K \text{ or}\\ (\alpha,\beta,\dots,\omega) &\rightsquigarrow (\beta,\dots,\omega) && \text{if } \alpha = K,\end{aligned}$$

holding between the sets of tuples that are counted in (6).

As a result, we obtain an infinite matrix whose lower left triangle is similar to the Pascal triangle for $K = 1$, with the exception that the columns are shifted down so that column h has above the first nonzero entry hK zeros for all $h \geq 0$. Consequently it follows that $R_K(z,h) = R(z - h(K-1), h)$ and (7) follows from Lemma 2. This completes the proof of the lemma. □

Lemma 4. *Let $h \geq 1$ and $z \geq 0$ be integers and let*

$$U(z,h) = \Big|\{(s_0,\dots,s_h) \in \mathbb{N}^h \ : \ s_0 \geq 0,\ s_1,\dots,s_{h-1} \geq 1; s_h \geq 0;\ s_0 + \cdots + s_h = z\}\Big|. \tag{8}$$

Then

$$U(z,h) = \binom{z+1}{h} = \frac{z^h}{h!} + O_h\big(z^{h-1}\big).$$

The estimate holds as z tends to infinity while h is kept fixed, and the constant in the big-O term depends on h.

Proof. The set whose cardinality is counted in relation (8) is split in four disjoint subsets in which all s_j's are positive, only one of s_0 or s_h is 0, or both s_0 and s_h are zero, respectively. The cardinalities of these subsets are calculated by using Lemma 2. Thus, we obtain:

$$\begin{aligned}U(z,h) &= R(z,h+1) + R(z,h) + R(z,h) + R(z,h-1)\\ &= \binom{z-1}{h} + \binom{z-1}{h-1} + \binom{z-1}{h-1} + \binom{z-1}{h-2}.\end{aligned}$$

The lemma follows after we group the binomial coefficients by two and use their recurrent generating formula:

$$U(z,h) = \binom{z}{h} + \binom{z}{h-1} = \binom{z+1}{h} = \frac{(z+1)z(z-1)\cdots(z-h+2)}{h!} = \frac{z^h}{h!} + O_h\big(z^{h-1}\big).$$

□

Lemma 5. *Let $h \geq 1$ and $K \geq 1$ be integers. Then*

$$R_K(z,h) = \frac{z^{h-1}}{(h-1)!} + O_{h,K}\big(z^{h-2}\big). \tag{9}$$

The estimate holds as z tends to infinity while h and K are kept fixed, and the constant in the big-O term depends on h and K.

Proof. The lemma follows from formula (7). □

For any integer $k \geq 0$, let $\mathcal{P}(k)$ denote the set of partitions of k in exactly $k+1$ non-negative integer parts, that is,

$$\mathcal{P}(k) := \Big\{ \boldsymbol{L} = (l_0, \dots, l_k) \in \mathbb{Z}^{k+1} : \ l_0, \dots, l_k \geq 0; \ l_0 + \cdots + l_k = k \Big\} . \tag{10}$$

Lemma 6. *The elements of $\mathcal{P}(k)$ are the tuples formed by the exponents of the monomials $z_0^{l_0} \cdots z_k^{l_k}$ in the expansion of $(z_0 + \cdots + z_k)^k$. The coefficient of $z_0^{l_0} \cdots z_k^{l_k}$ in the expansion of $(z_0 + \cdots + z_k)^k$ is the multinomial coefficient $\binom{k}{l_0,\dots,k_k} = \frac{k!}{l_0!\cdots l_k!}$.*

Proof. The first part of the lemma follows by the combinatorial interpretation of the definition (10) and the formula by induction build on the binomial theorem. □

3. An average on tuples with no-equal-neighbor components

In this section we calculate the average of the probabilities that an element in $\mathcal{N}$ is represented exactly k times by tuples $\mathbf{x} \in \mathcal{M}^n$ with no equal neighbor components. For any integers $M \geq 1$ and $0 \leq k \leq n$, this is defined by

$$A(k; M, n) := \frac{1}{|\mathcal{U}(\mathcal{M}, n)|} \sum_{\mathbf{x} \in \mathcal{U}(M,n)} \frac{m_k(\mathbf{x})}{n} . \tag{11}$$

3.1. The limit of $A(k; M, n)$ as n/M tends to λ

In order to evaluate the sum of $m_k(\mathbf{x})$'s in (11), according to the definition (1), in a generic tuple $\mathbf{x} \in \mathcal{U}$, we mark the positions whose components are equal to a certain value $y \in \mathcal{N}$ and then make the counting on each of the demarcated clusters.

Suppose $k \geq 1$. Let $y \in \mathcal{N}$ and let $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{U}$ and write $\mathbf{x} = (*, y, *, y, *, \cdots, y, *)$, indicating by $*$'s the clusters of components that are not equal to y. Let $s_0, s_1, \dots, s_k$ be the number of components that separate the y's from the margins or between two consecutive of them. (Occasionally, by s_j and not only by j, we refer to the jth cluster.) The non-equal-neighbors requirement is expressed by the following set of conditions

$$\begin{cases} s_0 \geq 0, s_1 \geq 1, \dots, s_{k-1} \geq 1, s_k \geq 0 \\ s_0 + s_1 + \cdots + s_k = n - k. \end{cases} \tag{12}$$

There are two cases: at the margins, before the first or after the last y and in the interior, between consecutive y's. In the interior, there is always at least one different component between two consecutive y's. Therefore, for any j, $1 \leq j \leq k-1$, there are $(M-1)(M-2)^{s_j - 1}$ ways to fill the s_j components different from y and satisfying the non-equal-neighbor requirement. At the margins, the same formula applies, except if s_0 or $s_k = 0$ and there is nothing to be filled.

Let $\varepsilon_0 = 1$ if $s_0 \geq 1$ and $\varepsilon_0 = 0$ if $s_0 = 0$ and similarly, let $\varepsilon_k = 1$ if $s_k \geq 1$ and $\varepsilon_k = 0$ if $s_k = 0$. Then, the above description summarizes as the following formula

$$\begin{aligned} \sum_{\mathbf{x} \in \mathcal{U}} m_k(\mathbf{x}) = n & \sum_{\substack{s_0, s_k \geq 0 \\ s_1, \dots, s_{k-1} \geq 1 \\ s_0 + \cdots + s_k = n-k}} \Big((M-1)(M-2)^{s_0 - 1} \Big)^{\varepsilon_0} \Big((M-1)(M-2)^{s_k - 1} \Big)^{\varepsilon_k} \\ & \qquad \times \prod_{j=1}^{k-1} (M-1)(M-2)^{s_j - 1} \\ = n & \sum_{\substack{s_0, s_k \geq 0 \\ s_1, \dots, s_{k-1} \geq 1 \\ s_0 + \cdots + s_k = n-k}} (M-1)^{k-1+\varepsilon_0+\varepsilon_k} (M-2)^{n-k-(k-1)-\varepsilon_0-\varepsilon_k} . \end{aligned}$$

Regrouping the factors, this can be rewritten as

$$\begin{aligned}\sum_{\mathbf{x}\in\mathcal{U}} m_k(\mathbf{x}) &= n(M-1)^{k-1}(M-2)^{n-2k+1} \sum_{\substack{s_0,s_k\geq 0\\ s_1,\dots,s_{k-1}\geq 1\\ s_0+\dots+s_k=n-k}} \left(1+\frac{1}{M-2}\right)^{\varepsilon_0+\varepsilon_k}\\ &= n(M-1)^{k-1}(M-2)^{n-2k+1}\Big(\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4\Big),\end{aligned} \tag{13}$$

where, the summation in Σ_1 is taken only over tuples with all components positive; the summations in Σ_2 are taken over tuples with $s_0 = 0$ and $s_k > 0$; the summations in Σ_3 are taken over tuples with $s_0 > 0$ and $s_k = 0$; and in Σ_4 the summation is taken over all tuples with both $s_0 = s_k = 0$. These sums can be evaluated using Lemma 2. We have:

$$\begin{aligned}\Sigma_1 &= R(n-k,k+1)\left(1+\frac{1}{M-2}\right)^2 = \binom{n-k-1}{k}\cdot\left(1+\frac{1}{M-2}\right)^2;\\ \Sigma_2 = \Sigma_3 &= R(n-k,k)\left(1+\frac{1}{M-2}\right) = \binom{n-k-1}{k-1}\cdot\left(1+\frac{1}{M-2}\right);\\ \Sigma_4 &= R(n-k,k-1) = \binom{n-k-1}{k-2}.\end{aligned}$$

On inserting these sums in (13), the average in (11) becomes

$$\begin{aligned}A(k;M,n) =& \frac{(M-1)^{k-1}(M-2)^{n-2k+1}}{M(M-1)^{n-1}}\cdot\binom{n-k-1}{k-2}\times\\ &\times\left(\frac{(n-2k+1)(n-2k)}{k(k-1)}\cdot\left(1+\frac{1}{M-2}\right)^2 + 2\frac{n-2k+1}{k}\cdot\left(1+\frac{1}{M-2}\right)+1\right).\end{aligned} \tag{14}$$

We evaluate precisely this product in the following subsection, but first let us see the size of the main term, assuming that k is fixed and M, n tend to infinity such that n/M tends to a constant $\lambda \in (0, 1]$. The first is the largest of the three terms on the second line of (14) and together with the factors from the outside it can be grouped conveniently, so that we find that $A(k;M,n)$ converges asymptotically like

$$\left(1-\frac{1}{M-1}\right)^{n-k}\cdot\frac{M-2}{M}\cdot\frac{1}{k!}\cdot\frac{(n-k-1)\cdots(n-k-k)}{(M-2)^k}\cdot\left(1+\frac{1}{M-2}\right)^2 \to e^{-\lambda}\frac{1}{k!}\lambda^k. \tag{15}$$

3.2. *The error term in the limit*

To make things precise and control the rate of convergence as n, M get large while the ratio n/M is ascertained around $\lambda \in (0, 1]$, let $\gamma \in [0, 1)$ be fixed. Also, we presume that $k \geq 0$ is fixed, $k \leq n \leq M$ and $M = n/\lambda + O(n^\gamma)$ as n tends to infinity and the constant in the big-O does not depend on λ, γ or k. From these assumptions it follows that $n/M = 1/(\lambda^{-1} + O(n^{\gamma-1}))$, so that $1/M = \lambda(n^{-1} + O(\lambda n^{\gamma-2}))$ and

$$n/M = \lambda(1 + O(\lambda n^{\gamma-1}))\,. \tag{16}$$

Then the asymptotic estimates of the exp-log functions involved are:

$$\left(\frac{n}{M}\right)^k = \lambda^k\Big(1+O\big(\lambda n^{\gamma-1}\big)\Big)^k = \lambda^k\Big(1+O\big(\lambda k n^{\gamma-1}\big)\Big) = \lambda^k\Big(1+O_{k,\lambda}\big(n^{\gamma-1}\big)\Big) \tag{17}$$

and, if $a = 1$ or 2, we have

$$\left(1-\frac{a}{M}\right)^{n-k} = e^{(n-k)\log(1-a/M)} = e^{\frac{-an}{M}+O\left(\frac{k}{M}\right)} = e^{-a\lambda}\left(1+O_{k,\lambda}\big(n^{\gamma-1}\big)\right). \tag{18}$$

Now, on using (17) and (18) in(15), which is the appropriate rewriting of the main term of (14), and noting that the other two terms are not greater than the resulting error terms, we find that

$$A(k;M,n) = \frac{1}{k!}\cdot\lambda^k\Big(1+O_{k,\lambda}\big(n^{\gamma-1}\big)\Big)\cdot e^{-\lambda}\Big(1+O_{k,\lambda}\big(n^{\gamma-1}\big)\Big)$$
$$= \frac{\lambda^k}{k!}e^{-\lambda}\Big(1+O_{k,\lambda}\big(n^{\gamma-1}\big)\Big)\,.$$

Consequently, we arrived at the following conclusion.

Theorem 2. *Let $\lambda\in(0,1]$, $\gamma\in[0,1)$ and let $k\geq 0$ be integer. Suppose the integers M and n satisfy the inequalities $k\leq n\leq M$ and $M = n/\lambda + O(n^\gamma)$, uniformly on λ and k as n tends to infinity while γ is fixed. Then*

$$A(k;M,n) = \frac{\lambda^k}{k!}e^{-\lambda}\Big(1+O_{k,\lambda}\big(n^{\gamma-1}\big)\Big). \tag{19}$$

4. The second moment

In this section we evaluate the variance of the k-counter probabilities $m_k(\mathbf{x})/n$ about the main term of their limit mean $A(k;M,n)$.

4.1. Definition and first transformation

For any integers $0\leq n\leq M$, let

$$M_2(k;M,n) := \frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}\in\mathcal{U}}\left(\frac{m_k(\mathbf{x})}{n}-\frac{\lambda^k}{k!}e^{-\lambda}\right)^2 \tag{20}$$

be the second moment about the mean. Note that this is not exactly the formula for the second moment about the mean, since in the binomials of the summation the average is replaced by the main term in its estimate in Theorem 2, but this form suffices for our needs.

Expanding the binomial, we find that

$$M_2(k;M,n) = \frac{\lambda^{2k}}{k!^2}e^{-2\lambda} - \frac{2\lambda^k}{k!}e^{-\lambda}A(k;M,n) + S_2(k;M,n)\,, \tag{21}$$

where, using (3),

$$S_2(k;M,n) = \frac{1}{|\mathcal{U}|}\sum_{\mathbf{x}\in\mathcal{U}}\left(\frac{m_k(\mathbf{x})}{n}\right)^2 = \frac{1}{M(M-1)^{n-1}n^2}\sum_{\mathbf{x}\in\mathcal{U}}m_k^2(\mathbf{x}). \tag{22}$$

4.2. Ground state notations

Let $\mathbf{x}=(x_1,\dots,x_n)\in\mathcal{U}$ and write $\mathbf{x}=(*,y,*,y,*,\dots,y,*)$, where the components equal to y are separated by other components different from y. These components are grouped in clusters of lengths $s_0,s_1,\dots,s_k$, which satisfy conditions (12). In the following, let $\boldsymbol{S}$ be the shorthand notation for such a generic cluster.

Let $l(0),\dots,l(k)$ be the number of components equal to y' that are in these clusters of lengths $s_0,\dots,s_k$, respectively. These numbers satisfy conditions:

$$0\leq l(0),\dots,l(k)\leq k \quad\text{and}\quad l(0)+\cdots+l(k)=k, \tag{23}$$

so that, $\boldsymbol{L}=(l_0,\dots,k_k)$ belongs to the set $\mathcal{P}(k)$ defined by (10).

Given an integer k and a cluster of integers $\boldsymbol{S}=(s_0,\dots,s_k)$ that satisfy conditions (12), let $\mathcal{P}(k,\boldsymbol{S})$ denote by the subset of $\mathcal{P}(k)$ formed by those partitions of k whose components are bounded above by s_j's, that is

$$\mathcal{P}(k,\boldsymbol{S}) := \Big\{(l(0),\dots,l(k))\in\mathcal{P}(k) : l(0)\leq s_0,\dots,l(k)\leq s_k\Big\}. \tag{24}$$

Further, let $t_{i,j}$ be the number of consecutive components in the groups of components that separate the y_2's in the jth cluster, for $0 \le i \le l(j)$ and $0 \le j \le k$. So that, they satisfy conditions:

$$\begin{cases} t_{0,j} \ge 0, t_{1,j} \ge 1, \dots, t_{l(j)-1,j} \ge 1, t_{l(j),j} \ge 0 \\ t_{0,j} + \cdots + t_{l(j),j} = s_j - l(j). \end{cases} \tag{25}$$

In order to indicate the presence or the absence at the margins of some components different from the selected ones, say y and y', we denote

$$\varepsilon_0(j) = \begin{cases} 0, & \text{if } t_{0,j} = 0 \\ 1, & \text{if } t_{0,j} > 0 \end{cases} \quad \text{and} \quad \varepsilon_{l(j)}(j) = \begin{cases} 0, & \text{if } t_{l(j),j} = 0 \\ 1, & \text{if } t_{l(j),j} > 0 \end{cases}.$$

The cluster s_j, which may be any of the $*$'s from $\mathbf{x} = (*, y, \dots)$ or $\mathbf{x} = (\dots, y, *, y, \dots)$ or $\mathbf{x} = (\dots, y, *)$, has in turn the form $*, y', *, y', *, \dots, y', *$, where the sequences of components indicated also by $*$'s have lengths $t_{0,j}$, $t_{1,j}, \dots, t_{l(j),j}$ and contain no y's and no y''s. Then, because of the non-equal-neighbor condition, there are $M-2$ ways to choose the first component on the left (2 less than M because y and y' are not allowed) and $M-3$ ways to choose the following components (since y, y' and also, the value chosen for its left neighbor are not allowed).

4.3. Disjunction, amalgamation and partial estimation

With the above notations, the sum on the right side of relation (22) is expressed as

$$\begin{aligned} \sum_{\mathbf{x}\in\mathcal{U}} m_k^2(\mathbf{x}) = & n^2 \sum_{\substack{s_0,s_k\ge 0\\ s_1,\dots,s_{k-1}\ge 1\\ s_0+\cdots+s_k=n-k}} \sum_{L\in\mathcal{P}(k,S)} \prod_{j=0}^{k} \sum_{\substack{t_{0,j},t_{l(j),j}\ge 0\\ t_{1,j},\dots,t_{l(j)-1,j}\ge 1\\ t_{0,j}+\cdots+t_{l(j),j}=s_j-l(j)}} \Big((M-2)(M-3)^{t_{0,j}-1}\Big)^{\varepsilon_0(j)} \\ & \times \Big((M-2)(M-3)^{t_{l(j),j}-1}\Big)^{\varepsilon_{l(j)}(j)} \prod_{i=1}^{l(j)-1} (M-2)(M-3)^{t_{i,j}-1}. \end{aligned} \tag{26}$$

Using the fact that $t_{k,j}\epsilon_k(j) = t_{k,j}$, for $k = 0, \dots, l(j)$, we have

$$\begin{aligned} \sum_{\mathbf{x}\in\mathcal{U}} m_k^2(\mathbf{x}) = & n^2 \sum_{\substack{s_0,s_k\ge 0\\ s_1,\dots,s_{k-1}\ge 1\\ s_0+\cdots+s_k=n-k}} \sum_{L\in\mathcal{P}(k,S)} \prod_{j=0}^{k} \sum_{\substack{t_{0,j},t_{l(j),j}\ge 0\\ t_{1,j},\dots,t_{l(j)-1,j}\ge 1\\ t_{0,j}+\cdots+t_{l(j),j}=s_j-l(j)}} (M-2)^{l(j)-1+\epsilon_0(j)+\epsilon_{l(j)}(j)} \\ & \qquad \times (M-3)^{s_j-2l(j)+1-\epsilon_0(j)-\epsilon_{l(j)}(j)} \\ = & n^2 \sum_{\substack{s_0,s_k\ge 0\\ s_1,\dots,s_{k-1}\ge 1\\ s_0+\cdots+s_k=n-k}} \sum_{L\in\mathcal{P}(k,S)} \prod_{j=0}^{k} (T_1+T_2+T_3+T_4), \end{aligned} \tag{27}$$

where the four terms, which depend on the cluster j, $T_h = T_h(j)$, for $h = 1, 2, 3, 4$, represent the sums that are split according to the values taken by the pairs $(\epsilon_0(j), \epsilon_{l(j)}(j))$ in the set $\{(0,0); (0,1); (1,0); (1,1)\}$, respectively. The sums T_1, T_2, T_3, T_4 can be expressed in closed form using the notation (5) as follows:

$$\begin{aligned} T_1 &= R\Big(s_j - l(j), l(j) - 1\Big)(M-2)^{l(j)-1}(M-3)^{s_j-2l(j)+1} \\ T_2 = T_3 &= R\Big(s_j - l(j), l(j)\Big)(M-2)^{l(j)}(M-3)^{s_j-2l(j)} \\ T_4 &= R\Big(s_j - l(j), l(j) + 1\Big)(M-2)^{l(j)+1}(M-3)^{s_j-2l(j)-1}. \end{aligned} \tag{28}$$

Then, factoring out the main terms and employing Lemma 2, the expressions from relation (28) become

$$\begin{aligned} T_1 &= M^{s_j-l(j)}\binom{s_j-l(j)-1}{l(j)-2}\Big(1-\frac{2}{M}\Big)^{l(j)-1}\Big(1-\frac{3}{M}\Big)^{s_j-2l(j)+1},\\ T_2 = T_3 &= M^{s_j-l(j)}\binom{s_j-l(j)-1}{l(j)-1}\Big(1-\frac{2}{M}\Big)^{l(j)}\Big(1-\frac{3}{M}\Big)^{s_j-2l(j)},\\ T_4 &= M^{s_j-l(j)}\binom{s_j-l(j)-1}{l(j)}\Big(1-\frac{2}{M}\Big)^{l(j)+1}\Big(1-\frac{3}{M}\Big)^{s_j-2l(j)-1}. \end{aligned} \tag{29}$$

(Recall that, by the convention in Lemma 2, the binomial coefficients are equal to 0 if either of their two parameters is negative.)

Next, we express the small factors on the right side of the products of relation (29) using the estimate

$$\Big(1-\tfrac{a}{M}\Big)^s = e^{s\log(1-a/M)} = \exp\Big\{-s\Big(a/M+\frac{(a/M)^2}{2}+\frac{(a/M)^3}{3}+\cdots\Big)\Big\} = e^{-s\frac{a}{M}(1+O(a/M))}, \tag{30}$$

where $M\to\infty$ and a is constant. Then

$$\begin{aligned} T_1+T_2 &= M^{s_j-l(j)}\Big[\binom{s_j-l(j)-1}{l(j)-2}e^{\frac{-3s_j+4l(j)-1}{M}}+\binom{s_j-l(j)-1}{l(j)-1}e^{\frac{-3s_j+4l(j)}{M}}\Big]e^{O(1/M)}\\ &= M^{s_j-l(j)}e^{\frac{-3s_j+4l(j)}{M}}\Big[\binom{s_j-l(j)-1}{l(j)-2}+\binom{s_j-l(j)-1}{l(j)-1}\Big]\Big(1+O\big(\tfrac{1}{M}\big)\Big), \end{aligned} \tag{31}$$

and

$$\begin{aligned} T_3+T_4 &= M^{s_j-l(j)}\Big[\binom{s_j-l(j)-1}{l(j)-1}e^{\frac{-3s_j+4l(j)}{M}}+\binom{s_j-l(j)-1}{l(j)}e^{\frac{-3s_j+4l(j)+1}{M}}\Big]e^{O(1/M)}\\ &= M^{s_j-l(j)}e^{\frac{-3s_j+4l(j)}{M}}\Big[\binom{s_j-l(j)-1}{l(j)-1}+\binom{s_j-l(j)-1}{l(j)}\Big]\Big(1+O\big(\tfrac{1}{M}\big)\Big). \end{aligned} \tag{32}$$

The binomial coefficients in (31) and (32) add together into a binomial coefficient on the next line of Pascal's triangle. Therefore

$$\begin{aligned} T_1+T_2 &= M^{s_j-l(j)}e^{\frac{-3s_j+4l(j)}{M}}\binom{s_j-l(j)}{l(j)-1}\Big(1+O\big(\tfrac{1}{M}\big)\Big)\\ T_3+T_4 &= M^{s_j-l(j)}e^{\frac{-3s_j+4l(j)}{M}}\binom{s_j-l(j)}{l(j)}\Big(1+O\big(\tfrac{1}{M}\big)\Big). \end{aligned} \tag{33}$$

And once again, the binomial coefficients in(33) add together into a binomial coefficient on the next line of Pascal's triangle, so that we get the following simple concise form of the total sums

$$T_1+T_2+T_3+T_4 = M^{s_j-l(j)}e^{\frac{-3s_j+4l(j)}{M}}\binom{s_j-l(j)+1}{l(j)}\Big(1+O\big(\tfrac{1}{M}\big)\Big). \tag{34}$$

Inserting the expression (34) into (27), we find that

$$\sum_{\mathbf{x}\in\mathcal{U}} m_k^2(\mathbf{x}) = n^2 \sum_{\substack{s_0,s_k\ge 0\\ s_1,\dots,s_{k-1}\ge 1\\ s_0+\dots+s_k=n-k}} \sum_{L\in\mathcal{P}(k,S)} \prod_{j=0}^{k}\Big[M^{s_j-l(j)}e^{\frac{-3s_j+4l(j)}{M}}\binom{s_j-l(j)}{l(j)}\Big(1+O\big(\tfrac{1}{M}\big)\Big)\Big],$$

whence, by employing condition $l(0)+\cdots+l(k)=k$ (of the presence of the components y' in the clusters separated by y's, formula (23)) yields

$$\sum_{\mathbf{x}\in\mathcal{U}} m_k^2(\mathbf{x}) = n^2 M^{n-2k} e^{\frac{-3n+7k}{M}} \sum_{\substack{s_0,s_k\geq 0\\ s_1,\dots,s_{k-1}\geq 1\\ s_0+\cdots+s_k=n-k}} \sum_{\boldsymbol{L}\in\mathcal{P}(k,\boldsymbol{S})} \left[\prod_{j=0}^{k}\binom{s_j-l(j)}{l(j)}\right]\left(1+O\Big(\frac{k}{M}\Big)\right). \tag{35}$$

Notice that in all the above factorization of the factors $\big(1+O(1/M)\big)$ the estimates are correct because only positive numbers are added, otherwise cancellations might occur and the main term might be smaller, making the equalities false.

4.4. *The combinatorial sum*

We check the size of the sum over $\boldsymbol{L}\in\mathcal{P}(k,\boldsymbol{S})$ of the product on the right side of relation (35) starting with a few cases with small k.

For any integers k,n, and cluster $\boldsymbol{S}=(s_0,\dots,s_k)$ satisfying the set of condition (12) imposed under the multiple summation in (35), denote

$$SSP(k,n,\boldsymbol{S}) := \sum_{\boldsymbol{L}\in\mathcal{P}(k,\boldsymbol{S})}\left[\prod_{j=0}^{k}\binom{s_j-l(j)}{l(j)}\right]. \tag{36}$$

According to the condition in Lemma 2 by which the binomial coefficients with any negative parameter is equal to 0, the summation can be thought over all partitions of k, but only those with s_j's larger than the corresponding l_j's being the ones that count.

We call *admissible* a cluster $\boldsymbol{S}=(s_0,\dots,s_k)$ for which $s_j\geq l(j)$ for all $0\leq j\leq k$, so that all the binomial coefficients in (36) are non-zero.

Next we evaluate the size of $SSP(k,n,\boldsymbol{S})$ with $\boldsymbol{S}$ admissible.

1. $k=0$. Then $s_0=n$ and the only $\boldsymbol{L}$ in $\mathcal{P}(0)$ is $\boldsymbol{L}=(0)$. Therefore

$$SSP(n,0,\boldsymbol{S}) = \binom{n}{0} = 1.$$

2. $k=1$. The only possible components of a partition of 1 are $l(0),l(1)\in\{0,1\}$, so that $\mathcal{P}(1)=\{(0,1);(1,0)\}$, which implies

$$SSP(n,1,\boldsymbol{S}) = \binom{s_0}{0}\binom{s_1-1}{1}+\binom{s_0-1}{1}\binom{s_1}{0} = (s_1-1)+(s_0-1) = n+O(1).$$

3. $k=2$. The set of partitions is $\mathcal{P}(2)=\{(0,0,2);(0,2,0);(2,0,0);(0,1,1);(1,0,1);(1,1,0)\}$. Then

$$\begin{aligned}
SSP(n,2,\boldsymbol{S}) =& \binom{s_0}{0}\binom{s_1}{0}\binom{s_2-2}{2}+\binom{s_0}{0}\binom{s_1-2}{2}\binom{s_2}{0}+\binom{s_0-2}{2}\binom{s_1}{0}\binom{s_2}{0}\\
&+\binom{s_0}{0}\binom{s_1-1}{1}\binom{s_2-1}{1}+\binom{s_0-1}{1}\binom{s_1}{0}\binom{s_2-1}{1}+\binom{s_0-1}{1}\binom{s_1-1}{1}\binom{s_2}{0}\\
=&\frac{(s_2-2)(s_2-3)}{2}+\frac{(s_1-2)(s_1-3)}{2}+\frac{(s_0-2)(s_0-3)}{2}\\
&+(s_1-1)(s_2-1)+(s_0-1)(s_2-1)+(s_0-1)(s_1-1)\\
=&\frac{(s_0+s_1+s_2)^2}{2}+O(s_0+s_1+s_2)\\
=&n^2/2+O(n).
\end{aligned}$$

Notice that here, the constant in the big-O term depends on k and it is not larger than $T_{k+1} \cdot |\mathcal{P}(k, \boldsymbol{S})|$, where T_k is the kth triangular number. (Here T_{k+1} counts the coefficients of the terms of degree two in the first sum, and $|\mathcal{P}(k, \boldsymbol{S})|$ is the number of all the terms in all sums. The reason for this bound is effective is that the terms that appear after multiplying the factors of the sums are regrouped in expressions of the form $(s_0 + s_1 + s_2 + s_3)^r$ with $r = 0, 1, 2$, and the total number of these expressions is $\le |\mathcal{P}(k, \boldsymbol{S})|$.)

4. $k = 3$. Now the set of partitions contains $4 + 12 + 4 = 20$ quadruples. Of these, there are four with a non-zero component equal to 3 and four with only one component equal to 0 and the other 1's. The other twelve quadruples have two component equal to 0 and the others are a 1 and a 2. Then the terms of the sum $SSP(n, 3, \boldsymbol{S})$ are

$$SSP(n,3,\boldsymbol{S}) = \sum_h \frac{(s_h-3)(s_h-4)(s_h-5)}{3!} + \sum_{h,i} \frac{(s_h-2)(s_h-3)}{2!} \cdot \frac{s_i-1}{1!} + \sum_{h,i,j} \frac{s_h-1}{1!} \cdot \frac{s_i-1}{1!} \cdot \frac{s_j-1}{1!},$$

where h, i, j are distinct and run to cover all quadruples of $\mathcal{P}(3)$, as described above the relation. Here, the terms of degree 3 (the highest, and equal to the value of k) are

$$\frac{1}{6}\sum_h s_h^3 + \frac{1}{2}\sum_{h,i} s_h^2 s_i + \sum_{h,i,j} s_h s_i s_j = \frac{1}{6}(s_0+s_1+s_2+s_3)^3 = \frac{(n-3)^3}{3!}.$$

The remaining terms are of degree 2, 1 and 0, and they combine, as seen in the cases with smaller k into closed formulas $(s_0 + s_1 + s_2 + s_3)^r = (n - 3)^r$ with $r = 2, 1, 0$, so that their total sum is $O(n^2)$. The constant in the big-O term depends on k and it is not larger than $T_{k+1} \cdot |\mathcal{P}(k, \boldsymbol{S})|$.

Similarly, for larger k, we have

$$\begin{aligned} SSP(n,k,\boldsymbol{S}) = &\sum_h \frac{(s_h-k)\big(s_h-(k+1)\big)\cdots\big(s_h-(2k-1)\big)}{k!} \\ &+ \sum_{h,i} \frac{\big(s_h-(k-1)\big)(s_h-k)\cdots\big(s_h-(2k-3)\big)}{(k-1)!} \cdot \frac{s_i-1}{1!} + \cdots \\ &\cdots + \sum_{h,i,j,\ldots} \frac{s_h-1}{1!} \cdot \frac{s_i-1}{1!} \cdot \frac{s_j-1}{1!} \cdots . \end{aligned}$$

Multiplying all the factors in the numerators of the fractions above and then sorting the resulted monomials by their degrees, we see that the generic one of the highest degree has the form $s_0^{l_0} \cdots s_k^{l_k}$, where $l_0 + \cdots + l_k = k$, and its coefficient in the total sum is $\frac{1}{l_0! \cdots l_k!}$. Then, factoring out $1/k!$ from the sum of all these highest degree monomials, we obtain an expression which, according to Lemma 6, equals $(s_0 + \cdots + s_k)^k/k!$.

The sum of the lower degree terms also add up in similar expressions of degrees strictly lower than k, and even if some necessary terms were missing, one could add them to complete the power of k-nomials $(s_0 + \cdots + s_k)^r$, with $r < k$. Therefore, the total number of these lower degree terms depends only on k, and hence, since $s_0 + \cdots + s_k = n - k$, we conclude that

$$SSP(n,k,\boldsymbol{S}) = \frac{n^k}{k!} + O_k(n^{k-1}), \tag{37}$$

for any admissible cluster $\boldsymbol{S}$.

We remark that the constant in the big-O term depends on k and, by the same reasoning above from the cases with small k and Lemma 6, it is not larger than k^{k+2}.

If the cluster $\boldsymbol{S}$ is not admissible, then some of the terms that could complete the expression $(s_0 + \cdots + s_k)$ of the largest degree might be missing, so the resulting main term might be smaller. In any case, we have

$$SSP(n,k,\boldsymbol{S}) = O_k(n^k), \tag{38}$$

for any non-admissible cluster $\boldsymbol{S}$.

4.5. The estimate of $\sum_{\mathbf{x}\in\mathcal{U}} m_k^2(\mathbf{x})$

We insert the results on the combinatorial sums from the previous subsection into the relation (35) by separating the outer sum from the right side into two parts according to whether the cluster $\boldsymbol{S}$ is admissible or not. On combining (37) and (38) we have

$$\sum_{\boldsymbol{S}} SSP(n,k,\boldsymbol{S}) = \sum_{\boldsymbol{S}\text{ admissible}} \left(\frac{n^k}{k!} + O_k(n^{k-1})\right) + \sum_{\boldsymbol{S}\text{ non-admissible}} O_k(n^k)\,. \tag{39}$$

Here the summands no longer depend on $\boldsymbol{S}$, so that the result depends only on the cardinality of the set of clusters $\boldsymbol{S}$ that are admissible or non-admissible. Also, without changing the final outcome, on the right-hand side of (39) we can move some terms from the first sum to the second. Indeed, on the one hand, from Lemma 4 we know that the number of all $\boldsymbol{S}$'s that satisfy conditions (12) is $\frac{(n-k)^k}{k!} + O_k\big(n^{k-1}\big) = \frac{n^k}{k!} + O_k\big(n^{k-1}\big)$. On the other hand, all clusters $\boldsymbol{S}$ whose components are all sufficiently large, say larger than $2k$, are admissible and, by Lemma 5 with $h = k+1$, $z = n-k$ and $K = 2k$, we know that their number is $\frac{n^k}{k!} + O_k\big(n^{k-1}\big)$, also. Therefore the number of clusters $\boldsymbol{S}$ that have some components smaller than $2k$ is of the order $O_k\big(n^{k-1}\big)$. It then follows that

$$\sum_{\boldsymbol{S}} SSP(n,k,\boldsymbol{S}) = \sum_{\substack{s_0,\dots,s_k\geq 2k\\ s_0+\cdots+s_k=n-k}} \left(\frac{n^k}{k!} + O_k(n^{k-1})\right) + O_k(n^{2k-1}) = \frac{n^{2k}}{(k!)^2} + O_k(n^{2k-1})\,.$$

On inserting this estimate into the right side of (35), we find that

$$\begin{aligned}\sum_{\mathbf{x}\in\mathcal{U}} m_k^2(\mathbf{x}) &= n^2M^{n-2k}e^{\frac{-3n+7k}{M}}\left(\frac{n^{2k}}{(k!)^2} + O_k(n^{2k-1})\right)\left(1 + O\Big(\frac{k}{M}\Big)\right)\\ &= n^2M^{n-2k}e^{\frac{-3n}{M}}\left(\frac{n^{2k}}{(k!)^2} + O_k(n^{2k-1})\right).\end{aligned} \tag{40}$$

4.6. Completion of the estimate of the second moment

By inserting (40) into (22), we obtain

$$\begin{aligned}S_2(k;M,n) &= \frac{M^{n-2k-1}}{(M-1)^{n-1}}e^{\frac{-3n}{M}}\left(\frac{n^{2k}}{(k!)^2} + O_k(n^{2k-1})\right)\\ &= \frac{1}{(k!)^2}\left(\frac{n}{M}\right)^{2k}\left(1 + \frac{1}{M-1}\right)^{n-1} e^{-3\frac{n}{M}}\left(1 + O_k(n^{-1})\right).\end{aligned} \tag{41}$$

Now, in the hypothesis that the integer $k \geq 0$ and $\lambda \in (0,1]$, $\gamma \in [0,1)$ are fixed, while $n \leq M$ both tend to infinity keeping their rate of growth controlled by $M = n/\lambda + O(n^\gamma)$, uniformly on λ and k, using the analogue of the estimates (17) and (18) in (41), yields

$$S_2(k;M,n) = \left(\frac{\lambda^k}{k!}\right)^2 e^{\lambda}e^{-3\lambda}\left(1 + O_{k,\lambda}(n^{\gamma-1})\right) = \left(\frac{\lambda^k e^{-\lambda}}{k!}\right)^2\left(1 + O_{k,\lambda}(n^{\gamma-1})\right). \tag{42}$$

Then, on inserting this estimate into (21) together with the one obtained for the average $A(k;M,n)$ in Theorem 2, we obtain

$$M_2(k;M,n) = \frac{\lambda^{2k}}{k!^2}e^{-2\lambda} - \frac{2\lambda^k}{k!}e^{-\lambda}\cdot\frac{\lambda^k}{k!}e^{-\lambda}\left(1 + O_{k,\lambda}\big(n^{\gamma-1}\big)\right) + \left(\frac{\lambda^k e^{-\lambda}}{k!}\right)^2\left(1 + O_{k,\lambda}(n^{\gamma-1})\right).$$

Notice that here the main term cancels out, which makes the second moment to become small as n, M tend to infinity under the assumed condition. We state the result as the following theorem.

Theorem 3. *Let $\lambda \in (0,1]$, $\gamma \in [0,1)$ and let $k \geq 0$ be integer. Suppose the integers M and n satisfy the inequalities $k \leq n \leq M$ and $M = n/\lambda + O(n^\gamma)$, uniformly on λ and k as n tends to infinity while γ is fixed. Then*

$$M_2(k;M,n) = O_{k,\lambda}\left(n^{\gamma-1}\right). \tag{43}$$

5. Proof of Theorem 1

Now we can prove the uniform result in Theorem 1. Let $\eta > 0$ be fixed and define the sets

$$\mathcal{U}_L(\eta) = \left\{ \mathbf{x} \in \mathcal{U} : \left| \frac{m_k(\mathbf{x})}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right| < \eta \right\} \quad \text{and} \quad \mathcal{U}_R(\eta) = \left\{ \mathbf{x} \in \mathcal{U} : \left| \frac{m_k(\mathbf{x})}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right| \geq \eta \right\}.$$

Note that these two sets are disjoint and any $\mathbf{x} \in \mathcal{U}$ belongs to either $\mathcal{U}_L(\eta)$ or $\mathcal{U}_R(\eta)$, that is, they form a partition of $\mathcal{U}$. Then, by the definition of the second moment (20), it follows that

$$M_2(k; M, n) = \frac{1}{|\mathcal{U}|} \sum_{\mathbf{x} \in \mathcal{U}_L(\eta)} \left(\frac{m_k(\mathbf{x})}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right)^2 + \frac{1}{|\mathcal{U}|} \sum_{\mathbf{x} \in \mathcal{U}_R(\eta)} \left(\frac{m_k(\mathbf{x})}{n} - \frac{\lambda^k}{k!} e^{-\lambda} \right)^2.$$

From here, since the first sum is positive, using the definition of $\mathcal{U}_L(\eta)$ and $\mathcal{U}_R(\eta)$, we deduce that

$$M_2(k; M, n) \geq \frac{1}{|\mathcal{U}|} \sum_{\mathbf{x} \in \mathcal{U}_R(\eta)} \eta^2 = \frac{\eta^2}{|\mathcal{U}|} \Big(|\mathcal{U}| - |\mathcal{U}_L(\eta)| \Big),$$

which implies

$$\frac{|\mathcal{U}_L(\eta)|}{|\mathcal{U}|} \geq 1 - \eta^{-2} M_2(k; M, n). \tag{44}$$

Next, on choosing $\eta = n^{-\delta}$ and using on the right side of (44) the estimate (43), which implies the existance of a constant $C > 0$ such that $Cn^{\gamma-1} \geq -M_2(k; M, n) \geq -Cn^{\gamma-1}$ for n sufficiently large, we find that

$$\frac{|\mathcal{U}_L(n^{-\delta})|}{|\mathcal{U}|} \geq 1 - O_{k,\lambda}\left(n^{\gamma+2\delta-1} \right),$$

where the implied constant in the big-O estimate may change if the a priori fixed k and λ change. This concludes the proof of Theorem 1.

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References

[1] Banks William, Luca Florian, Shparlinski Igor E., Stichtenoth Henning, *On the value set of $n!$ modulo a prime*, Turkish J. Math. **29** (2005), no. 2, 169–174. https://journals.tubitak.gov.tr/cgi/viewcontent.cgi?article=2714&context=math 4712

[2] Bhat Raghavendra N., *Distribution of square-prime numbers*, Missouri J. Math. Sci. 34 (2022), no. 1, 121–126. https://doi.org/10.35834/2022/3401121 4713

[3] Broughan Kevin A., Barnett A. Ross, *On the missing values of $n! \mod p$*, J. Ramanujan Math. Soc. **24** (2009), no. 3, 277–284. 4712

[4] Chen Yong-Gao, Dai Li-Xia, *Congruences with factorials modulo p*, Integers **6** (2006), A21, 3 pp. http://math.colgate.edu/~integers/g21/g21.pdf 4712

[5] Cobeli C., Vâjâitu M., Zaharescu A., *The sequence $n! \pmod p$*, J. Ramanujan Math. Soc. **15** (2000), no. 2, 135–154. 4711, 4712

[6] Cobeli Cristian, Zaharescu Alexandru, *Factorials $\pmod p$ and the average of modular mappings*, preprint arXiv:2011.07582v1, 2020. https://arxiv.org/pdf/2011.07582.pdf 4711, 4712, 4713

[7] Dai Lixia, *Congruences with factorials modulo p. II*, J. Nanjing Norm. Univ. Nat. Sci. Ed. **31** (2008), no. 4, 33–36. http://njsfdxzrb.paperonce.org/OA/pdfdow.aspx?Sid=200804007 4712

[8] Garaev Moubariz Z., García Victor C., *On the number of representations by $n!$ modulo a prime and applications*, Monatsh. Math. **198**, no. 3, 535–545 (2022). https://doi.org/10.1007/s00605-022-01689-y 4712

[9] Garaev M. Z., Hernández J., *A note on $n!$ modulo p*, Monatsh. Math. **182** (2017), no. 1, 23–31. https://doi.org/10.1007/s00605-015-0867-8 4712

[10] Garaev Moubariz Z., Luca Florian, Shparlinski Igor E., *Character sums and congruences with n!*, Trans. Amer. Math. Soc. **356** (2004), no. 12, 5089–5102. https://doi.org/10.1090/S0002-9947-04-03612-8 4712

[11] Garaev Moubariz Z., Luca Florian, Shparlinski Igor E., *Exponential sums and congruences with factorials*, J. Reine Angew. Math. **584** (2005), 29–44. https://doi.org/10.1515/crll.2005.2005.584.29 4712

[12] García Víctor C., *On the value set of $n!m!$ modulo a large prime*, Bol. Soc. Mat. Mexicana (3) **13** (2007), no. 1, 1–6. 4712

[13] García Víctor C., *Representations of residue classes by product of factorials, binomial coefficients and sum of harmonic sums modulo a prime*, Bol. Soc. Mat. Mexicana (3) **14** (2008), no. 2, 165–175. 4712

[14] Grebennikov Alexandr, Sagdeev Arsenii, Semchankau Aliaksei, Vasilevskii Aliaksei, *On the sequence $n!$ mod p*, Rev. Mat. Iberoam. **40** (2024), no. 2, 637–648. https://doi.org/10.4171/rmi/1422 4712

[15] Guy Richard K., *Unsolved problems in number theory*. Third edition. Problem Books in Mathematics. Springer-Verlag, New York, 2004 (first edition 1981). xviii+437 pp. 4711

[16] Klurman Oleksiy, Munsch Marc, *Distribution of factorials modulo p*, J. Théor. Nombres Bordeaux **29** (2017), no. 1, 169–177. https://doi.org/10.5802/jtnb.974 4712

[17] Klurman Oleksiy, Munsch Marc, *Polynomial products modulo primes and applications*, Monatsh. Math. 191 (2020), No. 3, 577–593. https://doi.org/10.1007/s00605-019-01359-6 4712

[18] Lev Vsevolod F., *Permutations in abelian groups and the sequence $n! \pmod p$*, European J. Combin. **27** (2006), no. 5, 635–643. https://doi.org/10.1016/j.ejc.2005.03.003 4712

[19] Luca Florian, Shparlinski Igor E., *On the largest prime factor of $n! + 2^n - 1$*, J. Théor. Nombres Bordeaux **17** (2005), no. 3, 859–870. https://doi.org/10.5802/jtnb.524 4712

[20] Luca Florian, Stănică Pantelimon, *Products of factorials modulo p*, Colloq. Math. **96** (2003), no. 2, 191–205. https://doi.org/10.4064/cm96-2-4 4712

[21] Nemes Gergő, *On the coefficients of the asymptotic expansion of $n!$*, J. Integer Seq. **13**, no. 6, Article ID 10.6.6, 5 p. (2010). https://cs.uwaterloo.ca/journals/JIS/VOL13/Nemes/nemes2.pdf 4711

[22] Stauduhar R., *Problem 7 in 'Proposed problems of the 1963 Number Theory Conference', p. 90*, University of Colorado, Boulder, Col., Aug. 5-24, 1963. Proc. Number Theory Conf., Boulder, 1963, 89–116. 4711