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Modified inertial-like subgradient extragradient rule with adaptive step sizes for hierarchical variational inequalities with systems of variational inequalities

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Abstract. In this paper, based on the viscosity approximation method, hybrid steepest-descent method and Korpelevich extragradient method, we suggest and analyze two modified inertial-like subgradient extragradient algorithms with adaptive step sizes for solving a hierarchical variational inequality (HVI) with a variational inequality problem for pseudomonotone and Lipschitz continuous mapping (VIP), a general system of variational inequalities for two inverse-strongly monotone mappings (GSVI) and a common fixed-point problem of finitely many nonexpansive mappings and an asymptotically nonexpansive mapping (CFPP) constraints. Under mild restrictions, we demonstrate the strong convergence of the proposed algorithms to a common solution of the VIP, GSVI and CFPP, which is also the unique solution of this HVI.

1. Introduction

In a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the norm $\|\cdot\|$, we denote by P_C the metric projection from H onto a convex and closed set $C \neq \emptyset$ in H. Let $S : C \to H$ be a nonlinear operator on C. Let the $\mathrm{Fix}(S)$, $\mathbf{R}$, $\to$ and $\rightharpoonup$ indicate the fixed point set of S, the set of all real numbers, the strong convergence and the weak convergence, respectively. An operator $S : C \to C$ is referred to as being asymptotically nonexpansive if $\exists\{\theta_k\} \subset [0, +\infty)$ s.t. $\lim_{k\to\infty} \theta_k = 0$ and $\|S^k u - S^k v\| \leq (1 + \theta_k)\|u - v\|, \forall u, v \in C, k \geq 1$. In particular, if $\theta_k = 0, \forall k \geq 1$, then S is referred to as nonexpansive.

Given an operator $A : H \to H$. The classical variational inequality problem (VIP) is to find $u^* \in C$ s.t. $\langle Au^*, v - u^*\rangle \geq 0, \forall v \in C$. The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. Currently, one of the most effective methods for solving the VIP is the extragradient method invented by Korpelevich [20] in 1976. However, it was proven in [20] that this method enjoys only the weak convergence if $\mathrm{VI}(C, A) \neq \emptyset$. The Korpelevich extragradient approach has received great attention given by many authors, who ameliorated it in various manners; see e.g., [1–9, 11–15, 17–26, 28–41]. Besides, let $B_1, B_2 : H \to H$ be two operators. The general system of variational inequalities (GSVI) is to find $(u^*, v^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 v^* + u^* - v^*, x - u^*\rangle \geq 0, \ \forall x \in C, \\ \langle \mu_2 B_2 u^* + v^* - u^*, y - v^*\rangle \geq 0, \ \forall y \in C, \end{cases} \tag{1}$$

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with constants $\mu_1, \mu_2 \in (0, \infty)$. In particular, if $B_1 = B_2 = A$ and $x^* = y^*$, then the GSVI (1) reduces to the classical VIP. Note that, problem (1) can be transformed into a fixed point problem (FPP) in the following manner.

Lemma 1.1 ([8]). *For given $u^*, v^* \in C$, (u^*, v^*) is a solution of GSVI (1) if and only if $u^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $v^* = P_C(I - \mu_2 B_2)u^*$.*

Assume that $\Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \neq \emptyset$, where $S : C \to C$ is an asymptotically nonexpansive mapping with $\{\theta_k\}$ and $\mathrm{Fix}(G)$ is the same set as in Lemma 1.1. In 2018, using a modified extragradient method, Cai et al. [2] introduced a viscosity implicit rule for finding a solution of a hierarchical variational inequality (HVI) with the GSVI and FPP constraints, i.e., for any initial $x_1 \in C$, the sequence $\{x_k\}$ is constructed below

$$\begin{cases} u_k = s_k x_k + (1 - s_k)p_k, \\ v_k = P_C(u_k - \mu_2 B_2 u_k), \\ p_k = P_C(v_k - \mu_1 B_1 v_k), \\ x_{k+1} = P_C[\alpha_k f(x_k) + (I - \alpha_k \rho F)S^k p_k], \end{cases} \tag{2}$$

where $f : C \to C$ is a δ-contraction with $\delta \in [0, 1)$, and the following hold for $\{\alpha_k\}, \{s_k\} \subset (0, 1]$: (i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$; (ii) $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k} = 0$; (iii) $0 < \varepsilon \le s_k \le 1$, $\sum_{k=1}^{\infty} |s_{k+1} - s_k| < \infty$; and (iv) $\sum_{k=1}^{\infty} \|S^{k+1}p_k - S^k p_k\| < \infty$. It was proved in [2] that $x_k \to x^* \in \Omega$, which solves the HVI: $\langle (\rho F - f)x^*, x - x^* \rangle \ge 0, \forall x \in \Omega$.

Assume that $\Omega = \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A) \neq \emptyset$, where $\bigcap_{i=1}^{N} \mathrm{Fix}(S_i)$ is the common fixed-point set of finitely many nonexpansive self-mappings $\{S_i\}_{i=1}^{N}$ on C and $\mathrm{VI}(C, A)$ is the solution set of the VIP for pseudomonotone and Lipschitz continuous mapping A. In 2020, Ceng et al. [4] put forward an inertial-like subgradient extragradient method with adaptive step sizes for solving a HVI with the VIP and CFPP constraints. Given a contraction $f : H \to H$ with constant $\delta \in [0, 1)$, and an η-strongly monotone and κ-Lipschitzian mapping $F : H \to H$ with $\delta < \ell := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, 2\eta/\kappa^2)$. Let $\{\beta_k\}, \{\gamma_k\}, \{\epsilon_k\}$ be the sequences in $(0, 1)$ s.t. $\beta_k + \gamma_k < 1$ $\forall k \ge 1$, $\beta_k \to 0$ and $\epsilon_k/\beta_k \to 0$. In addition, one writes $S_k := S_{k \bmod N}$ for each $k \ge 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, i.e., if $k = jN + q$ for some integers $j \ge 0$ and $0 \le q < N$, then $S_k = S_N$ if $q = 0$ and $S_k = S_q$ if $0 < q < N$.

Algorithm 1.2 ([4]). Initialization: *Give $\tau_1 > 0$, $\alpha > 0$, $\mu \in (0, 1)$. Let $u_0, u_1 \in H$ be arbitrary.*

Step 1. Given the iterates u_{k-1} and u_k $(k \ge 1)$, choose α_k s.t. $0 \le \alpha_k \le \widetilde{\alpha}_k$, where

$$\widetilde{\alpha}_k = \begin{cases} \min\{\alpha, \frac{\epsilon_k}{\|u_k - u_{k-1}\|}\}, & if\ u_k \neq u_{k-1}, \\ \alpha, & otherwise. \end{cases}$$

Step 2. Calculate $w_k = S_k u_k + \alpha_k(S_k u_k - S_k u_{k-1})$ and $v_k = P_C(w_k - \tau_k A w_k)$.

Step 3. Construct the half-space $C_k := \{u \in H : \langle w_k - \tau_k A w_k - v_k, u - v_k \rangle \le 0\}$, and calculate $t_k = P_{C_k}(w_k - \tau_k A v_k)$.

Step 4. Compute $u_{k+1} = \beta_k f(u_k) + \gamma_k u_k + ((1 - \gamma_k)I - \beta_k \rho F)t_k$, and update

$$\tau_{k+1} = \begin{cases} \min\{\mu \frac{\|w_k - v_k\|^2 + \|t_k - v_k\|^2}{2\langle A w_k - A v_k, t_k - v_k \rangle}, \tau_k\}, & if\ \langle A w_k - A v_k, t_k - v_k \rangle > 0, \\ \tau_k, & otherwise. \end{cases}$$

Put $k := k + 1$ and return to Step 1.

Under suitable assumptions, it was proven in [4] that $u_k \to u^* \in \Omega$, which solves the HVI: $\langle (\rho F - f)u^*, v - u^* \rangle \ge 0, \forall v \in \Omega$. Subsequently, Thong et al. [25] suggested a new inertial subgradient extragradient method with adaptive step sizes for solving the VIP for pseudomonotone and Lipschitz continuous mapping A. Let $\mathrm{VI}(C, A) \neq \emptyset$. Suppose that $\{\lambda_k\} \subset (\lambda, 1) \subset (0, 1)$ and $\{\epsilon_k\}, \{\beta_k\} \subset (0, 1)$ are such that $\beta_k \to 0$ and $\epsilon_k/\beta_k \to 0$ as $k \to \infty$.

Algorithm 1.3 ([25]). *Give* $\tau_1 > 0,\ \alpha > 0,\ \mu \in (0,1)$. *Let* $u_0, u_1 \in H$ *be arbitrary. Choose* α_k *s.t.*

$$0 \le \alpha_k \le \widetilde{\alpha}_k := \begin{cases} \min\{\alpha, \frac{\epsilon_k}{\|u_k - u_{k-1}\|}\}, & if\ u_k \ne u_{k-1}, \\ \alpha, & otherwise, \end{cases}$$

Compute

$$\begin{cases} w_k = (1-\beta_k)[u_k + \alpha_k(u_k - u_{k-1})], \\ v_k = P_C(w_k - \tau_k A w_k), \\ C_k := \{u \in H : \langle w_k - \tau_k A w_k - v_k, u - v_k\rangle \le 0\}, \\ u_{k+1} = (1-\lambda_k) w_k + \lambda_k P_{C_k}(w_k - \tau_k A v_k). \end{cases}$$

Update

$$\tau_{k+1} = \begin{cases} \min\{\mu \frac{\|w_k - v_k\|^2 + \|t_k - v_k\|^2}{2\langle A w_k - A v_k, t_k - v_k\rangle}, \tau_k\}, & if\ \langle A w_k - A v_k, t_k - v_k\rangle > 0, \\ \tau_k, & otherwise, \end{cases}$$

where $t_k := P_{C_k}(w_k - \tau_k A v_k)$.

Under appropriate assumptions, it was proven in [25] that $u_k \to u^* \in \mathrm{VI}(C, A)$, where $u^* = P_{\mathrm{VI}(C,A)} 0$.

Inspired by the above research works, we put forth two modified inertial-like subgradient extragradient algorithms with adaptive step sizes for solving a HVI with the VIP, GSVI and CFPP constraints by using the viscosity approximation method, hybrid steepest-descent method and Korpelevich extragradient method. Here the VIP, GSVI and CFPP represent a variational inequality problem for pseudomonotone and Lipschitz continuous mapping, a general system of variational inequalities for two inverse-strongly monotone mappings and a common fixed-point problem of finitely many nonexpansive mappings and an asymptotically nonexpansive mapping, respectively. Under suitable restrictions, we demonstrate the strong convergence of the suggested algorithms to a common solution of the VIP, GSVI and CFPP, which is also the unique solution of this HVI. In the end, our main results are applied to solve the VIP, GSVI and CFPP in an illustrated example.

This article is arranged below: In Section 2, we present some concepts and basic tools for further use. Section 3 treats the convergence analysis of the suggested algorithms. In the end, Section 4 applies our main results to solve the VIP, GSVI and CFPP in an illustrated example. Our algorithms are more advantageous and more flexible than the above Algorithms 1.2 and 1.3 because they involve solving the VIP, GSVI and CFPP in H. Our results improve and extend the corresponding results announced by some others, e.g., Cai et al. [2], Ceng et al. [4], Thong et al. [25], and Ceng and Shang [7].

2. Preliminaries

Suppose that C is a nonempty closed convex subset of a real Hilbert space H. Given $u \in H$ and $\{u_k\} \subset H$, we use the $u_k \to u$ (resp., $u_k \rightharpoonup u$) to indicate the strong (resp., weak) convergence of $\{u_k\}$ to u. An operator $\Gamma : C \to H$ is refereed to as being

(i) L-Lipschitz continuous or L-Lipschitzian iff $\exists L > 0$ s.t. $\|\Gamma u - \Gamma v\| \le L\|u - v\|, \forall u, v \in C$;

(ii) monotone iff $\langle \Gamma u - \Gamma v, u - v\rangle \ge 0, \forall u, v \in C$;

(iii) pseudomonotone iff $\langle \Gamma u, v - u\rangle \ge 0 \Rightarrow \langle \Gamma v, v - u\rangle \ge 0, \forall u, v \in C$;

(iv) η-strongly monotone iff $\exists \eta > 0$ s.t. $\langle \Gamma u - \Gamma v, u - v\rangle \ge \eta\|u - v\|^2, \forall u, v \in C$;

(v) α-inverse-strongly monotone iff $\exists \alpha > 0$ s.t. $\langle \Gamma u - \Gamma v, u - v\rangle \ge \alpha\|\Gamma u - \Gamma v\|^2, \forall u, v \in C$;

(vi) sequentially weakly continuous iff $\forall \{u_k\} \subset C$, the relation holds: $u_k \rightharpoonup u \Rightarrow \Gamma u_k \rightharpoonup \Gamma u$.

Clearly, every inverse-strongly monotone operator is monotone and Lipschitz continuous but the converse is not true, and every monotone operator is pseudomonotone but the converse is not true. For every $u \in H$, one knows that there is only a nearest point in C, denoted by $P_C u$, s.t. $\|u - P_C u\| \le \|u - v\|, \forall v \in C$. P_C is refereed to as a metric projection of H onto C.

Lemma 2.1 ([16]). *The following hold:*
(i) $\langle u-v, P_C u - P_C v\rangle \geq \|P_C u - P_C v\|^2,\ \forall u, v \in H;$
(ii) $\langle u - P_C u, v - P_C u\rangle \leq 0,\ \forall u \in H, v \in C;$
(iii) $\|u-v\|^2 \geq \|u - P_C v\|^2 + \|v - P_C u\|^2,\ \forall u \in H, v \in C;$
(iv) $\|u-v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u-v, v\rangle,\ \forall u, v \in H;$
(v) $\|\lambda u + (1-\lambda)v\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2 - \lambda(1-\lambda)\|u-v\|^2,\ \forall u, v \in H, \lambda \in [0,1].$

Lemma 2.2 ([12]). *For any* $u \in H$ *and* $\lambda \geq \eta > 0$ *the inequalities hold:* $\frac{\|u - P_C(u-\lambda Au)\|}{\lambda} \leq \frac{\|u - P_C(u-\eta Au)\|}{\eta}$ *and* $\|u - P_C(u - \eta Au)\| \leq \|u - P_C(u - \lambda Au)\|$.

Lemma 2.3 ([11]). *Suppose that* $A : C \to H$ *is pseudomonotone and continuous. Then* $u^* \in C$ *is a solution to the VIP* $\langle Au^*, v - u^*\rangle \geq 0, \forall v \in C$, *if and only if* $\langle Av, v - u^*\rangle \geq 0, \forall v \in C$.

Lemma 2.4 ([27]). *Suppose that* $\{a_k\}$ *is a sequence of nonnegative reals s.t.* $a_{k+1} \leq (1-\lambda_k)a_k + \lambda_k\gamma_k, \forall k \geq 1$, *where* $\{\lambda_k\}$ *and* $\{\gamma_k\}$ *are sequences of reals s.t. (a)* $\{\lambda_k\} \subset [0,1]$ *and* $\sum_{k=1}^{\infty}\lambda_k = \infty$, *and (b)* $\limsup_{k\to\infty}\gamma_k \leq 0$ *or* $\sum_{k=1}^{\infty}|\lambda_k\gamma_k| < \infty$. *Then* $\lim_{k\to\infty} a_k = 0$.

Lemma 2.5 ([10]). *Assume that* X *is a Banach space which admits a weakly continuous duality mapping,* C *is a nonempty, convex and closed set in* X, *and* $T : C \to C$ *is an asymptotically nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Then* $I - T$ *is demiclosed at zero, i.e., for any* $\{u_k\} \subset C$ *with* $u_k \rightharpoonup u \in C$, *the relation holds:* $(I-T)u_k \to 0 \Rightarrow u \in \mathrm{Fix}(T)$, *where* I *is the identity mapping of* X.

Lemma 2.6 ([31]). *Let the mapping* $B : H \to H$ *be* α*-inverse-strongly monotone. Then, for a given* $\mu \geq 0$, $\|(I-\mu B)x - (I-\mu B)y\|^2 \leq \|x-y\|^2 - \mu(2\alpha-\mu)\|Bx - By\|^2$. *In particular, if* $0 \leq \mu \leq 2\alpha$, *then* $I - \mu B$ *is nonexpansive.*

Using Lemma 2.6, we immediately obtain the following lemma.

Lemma 2.7 ([31]). *Let the mappings* $B_1, B_2 : H \to H$ *be* α*-inverse-strongly monotone and* β*-inverse-strongly monotone, respectively. Let the mapping* $G : H \to C$ *be defined as* $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. *If* $0 \leq \mu_1 \leq 2\alpha$ *and* $0 \leq \mu_2 \leq 2\beta$, *then* $G : H \to C$ *is nonexpansive.*

The following lemma is very useful to analyze the convergence of the suggested algorithms in this paper.

Lemma 2.8 ([25]). *Let* $\{\Gamma_k\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{k_j}\}$ *of* $\{\Gamma_k\}$ *which satisfies* $\Gamma_{k_j} < \Gamma_{k_j+1}$ *for each integer* $j \geq 1$. *Define the sequence* $\{\hbar(k)\}_{k\geq k_0}$ *of integers as follows:*

$$\hbar(k) = \max\{j \leq k : \Gamma_j < \Gamma_{j+1}\},$$

where integer $k_0 \geq 1$ *such that* $\{j \leq k_0 : \Gamma_j < \Gamma_{j+1}\} \neq \emptyset$. *Then, the following hold:*
(i) $\hbar(k_0) \leq \hbar(k_0+1) \leq \cdots$ *and* $\hbar(k) \to \infty$;
(ii) $\Gamma_{\hbar(k)} \leq \Gamma_{\hbar(k)+1}$ *and* $\Gamma_k \leq \Gamma_{\hbar(k)+1}, \forall k \geq k_0$.

Lemma 2.9 ([27]). *Given* $\lambda \in (0,1]$. *Suppose that the mapping* $S : C \to H$ *is nonexpansive, and the operator* $S^\lambda : C \to H$ *is formulated by* $S^\lambda u := Su - \lambda\rho F(Su), \forall u \in C$, *where* $F : H \to H$ *is* κ*-Lipschitzian and* η*-strongly monotone. Then* S^λ *is a contraction provided* $0 < \rho < \frac{2\eta}{\kappa^2}$, *i.e.,* $\|S^\lambda u - S^\lambda v\| \leq (1 - \lambda\ell)\|u - v\|, \forall u, v \in C$, *with* $\ell = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \in (0,1]$.

3. Main Results

Let the feasible set C be nonempty, convex and closed in a real Hilbert space H. In what follows, we assume always that the following hold:

(o1) $S : H \to H$ is an asymptotically nonexpansive operator with $\{\theta_n\} \subset [0, +\infty)$ and $S_l : H \to H$ is nonexpansive for $l = 1, ..., N$ s.t. $\{S_n\}_{n=1}^{\infty}$ is formulated as in Algorithm 1.2.

(o2) $B_1, B_2 : H \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $G : H \to C$ is defined as $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$.

(o3) $A : H \to H$ is L-Lipschitz continuous, pseudomonotone on H, s.t. $\|Au\| \le \liminf_{n\to\infty} \|Au_k\|\ \forall \{u_k\} \subset C$ with $u_k \rightharpoonup u$, and $\Omega = \bigcap_{l=0}^{N} \mathrm{Fix}(S_l) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$ with $S_0 := S$.

(o4) $f : H \to H$ is a contraction with coefficient $\delta \in [0, 1)$, and $F : H \to H$ is η-strongly monotone and κ-Lipschitzian s.t. $\delta < \ell := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$.

Suppose that the sequences $\{\lambda_n\}, \{\epsilon_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$ satisfy
(i) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$ and $\lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$;
(iv) $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}] \subset (0, 1)$.

Algorithm 3.1. Initialization: *Given* $\tau_1 > 0,\ \alpha > 0,\ \mu \in (0, 1)$. *Let* $x_0, x_1 \in H$ *be arbitrary and choose* α_n *s.t.*

$$0 \le \alpha_n \le \widetilde{\alpha}_n := \begin{cases} \min\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & if\ x_n \neq x_{n-1}, \\ \alpha, & otherwise. \end{cases} \tag{3}$$

Step 1. Set $u_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1})$, *and compute*

$$\begin{cases} v_n = P_C(u_n - \mu_2 B_2 u_n), \\ q_n = P_C(v_n - \mu_1 B_1 v_n). \end{cases}$$

Step 2. Compute

$$\begin{cases} w_n = \beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \beta_n \rho F)q_n, \\ y_n = P_C(w_n - \tau_n A w_n). \end{cases}$$

Step 3. Compute

$$x_{n+1} = (1 - \lambda_n)w_n + \lambda_n S^n P_{C_n}(w_n - \tau_n A y_n)$$

where

$$C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \le 0\}.$$

Step 4. Update

$$\tau_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A w_n - A y_n, z_n - y_n \rangle}, \tau_n\}, & if\ \langle A w_n - A y_n, z_n - y_n \rangle > 0, \\ \tau_n, & otherwise, \end{cases} \tag{4}$$

where $z_n = P_{C_n}(w_n - \tau_n A y_n)$. *Again set* $n := n + 1$ *and go to Step 1.*

It is remarkable that, putting $\gamma_n = 0\ \forall n \ge 1$, $f = 0$ and $G = S_l = S = \rho F = I$ for $l = 1, ..., N$, we transform Algorithm 3.1 into Algorithm 1.3, where $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. It is clear that, from (3) one has $\frac{\alpha_n}{\beta_n}\|x_{n-1} - x_n\| \to 0$ as $n \to \infty$. In fact, from $\alpha_n\|x_{n-1} - x_n\| \le \epsilon_n, \forall n \ge 1$, it follows that $\frac{\alpha_n}{\beta_n}\|x_{n-1} - x_n\| \le \frac{\epsilon_n}{\beta_n} \to 0$ (due to condition (ii)).

Lemma 3.2. *Suppose* $\{\tau_n\}$ *is constructed by* (4). *Then* $\{\tau_n\}$ *is nonincreasing s.t.* $\tau_n \ge \tau := \min\{\tau_1, \frac{\mu}{L}\}, \forall n \ge 1$, *and* $\lim_{n\to\infty} \tau_n \ge \tau := \min\{\tau_1, \frac{\mu}{L}\}$.

Proof. From (4), it can be easily seen that $\{\tau_n\}$ is nonincreasing. Note that $L\|w_n - y_n\|\|z_n - y_n\| \ge \langle A w_n - A y_n, z_n - y_n \rangle$ and $\frac{1}{2}(\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \ge \|w_n - y_n\|\|z_n - y_n\|$. So it follows that $\tau_{n+1} \ge \min\{\tau_n, \frac{\mu}{L}\}$. This arrives at $\lim_{n\to\infty} \tau_n \ge \tau := \min\{\tau_1, \frac{\mu}{L}\}$. □

It is remarkable that, by Lemmas 2.2 and 3.2, we know that if $w_n = y_n$ or $Ay_n = 0$, then y_n lies in VI(C, A). In fact, if $w_n = y_n$ or $Ay_n = 0$, one has $0 = \|y_n - P_C(y_n - \tau_n Ay_n)\| \geq \|y_n - P_C(y_n - \tau Ay_n)\|$. Hence we get $y_n \in$ VI(C, A).

he following lemmas are very useful for the convergence analysis of our algorithm.

Lemma 3.3. *Let $\{z_n\}, \{x_n\}, \{w_n\}$ be the sequences constructed in Algorithm 3.1. Then there exists an integer $n_1 \geq 1$ such that for each $n \geq n_1$, $0 < 1 - \mu\frac{\tau_n}{\tau_{n+1}}$ and*

$$\theta_n(2+\theta_n)\|z_n - u\|^2 - \frac{1}{2}(1 - \mu\frac{\tau_n}{\tau_{n+1}})\lambda_n(\|w_n - y_n\| + \|z_n - y_n\|)^2 + \|w_n - u\|^2 \geq \|x_{n+1} - u\|^2, \ \forall u \in \Omega. \quad (5)$$

Proof. We first claim that

$$\frac{\mu}{\tau_{n+1}}\|z_n - y_n\|^2 + \frac{\mu}{\tau_{n+1}}\|w_n - y_n\|^2 \geq 2\langle Aw_n - Ay_n, z_n - y_n\rangle, \ \forall n \geq 1. \quad (6)$$

In fact, in the case when $0 \geq \langle Aw_n - Ay_n, z_n - y_n\rangle$, it is clear that (6) holds. Otherwise, from (4) one gets (6). Note that for all $u \in \Omega \subset C \subset C_n$,

$$\begin{aligned}\|z_n - u\|^2 &= \|P_{C_n}(w_n - \tau_n Ay_n) - u\|^2 \leq \langle z_n - u, w_n - \tau_n Ay_n - u\rangle \\ &= \frac{1}{2}\|z_n - u\|^2 + \frac{1}{2}\|w_n - u\|^2 - \frac{1}{2}\|z_n - w_n\|^2 - \langle z_n - u, \tau_n Ay_n\rangle,\end{aligned}$$

which hence yields

$$-2\langle z_n - u, \tau_n Ay_n\rangle - \|z_n - w_n\|^2 + \|w_n - u\|^2 \geq \|z_n - u\|^2. \quad (7)$$

Thanks to $u \in$ VI(C, A), one gets $\langle Au, p - u\rangle \geq 0, \forall p \in C$. Using the pseudomonotonicity of A on C one gets $\langle Ap, p - u\rangle \geq 0, \forall p \in C$. Setting $p := y_n \in C$ one gets $\langle Ay_n, u - y_n\rangle \leq 0$. Thus,

$$\langle Ay_n, y_n - z_n\rangle \geq \langle Ay_n, y_n - z_n\rangle + \langle Ay_n, u - y_n\rangle = \langle Ay_n, u - z_n\rangle. \quad (8)$$

From (7) and (8), one has

$$2\langle w_n - \tau_n Ay_n - y_n, z_n - y_n\rangle - \|y_n - w_n\|^2 - \|z_n - y_n\|^2 + \|w_n - u\|^2 \geq \|z_n - u\|^2. \quad (9)$$

Since $z_n = P_{C_n}(w_n - \tau_n Ay_n)$ and $C_n := \{z \in H : \langle w_n - \tau_n Aw_n - y_n, z - y_n\rangle \leq 0\}$, one gets

$$\begin{aligned}2\tau_n\langle Aw_n - Ay_n, z_n - y_n\rangle &\geq 2\tau_n\langle Aw_n - Ay_n, z_n - y_n\rangle + 2\langle w_n - \tau_n Aw_n - y_n, z_n - y_n\rangle \\ &= 2\langle w_n - \tau_n Ay_n - y_n, z_n - y_n\rangle,\end{aligned}$$

which together with (6), leads to

$$\mu\frac{\tau_n}{\tau_{n+1}}\|z_n - y_n\|^2 + \mu\frac{\tau_n}{\tau_{n+1}}\|w_n - y_n\|^2 \geq 2\langle w_n - \tau_n Ay_n - y_n, z_n - y_n\rangle.$$

This together with (9), yields

$$-(1 - \mu\frac{\tau_n}{\tau_{n+1}})(\|z_n - y_n\|^2 + \|y_n - w_n\|^2) + \|w_n - u\|^2 \geq \|z_n - u\|^2. \quad (10)$$

Besides, from Algorithm 3.1 one obtains

$$\begin{aligned}\|x_{n+1} - u\|^2 &\leq (1 - \lambda_n)\|w_n - u\|^2 + \lambda_n(1+\theta_n)^2\|z_n - u\|^2 \\ &\leq (1 - \lambda_n)\|w_n - u\|^2 + \lambda_n\|z_n - u\|^2 + \theta_n(2+\theta_n)\|z_n - u\|^2,\end{aligned}$$

which together with (10), attains

$$\|x_{n+1} - u\|^2 \leq \|w_n - u\|^2 - (1 - \mu\frac{\tau_n}{\tau_{n+1}})\lambda_n(\|w_n - y_n\|^2 + \|z_n - y_n\|^2) + \theta_n(2+\theta_n)\|z_n - u\|^2.$$

Since $\lim_{n\to\infty}(1 - \mu\frac{\tau_n}{\tau_{n+1}}) = 1 - \mu > 0$, we know that $\exists n_1 \geq 1$ s.t. $\forall n \geq n_1$, $0 < 1 - \mu\frac{\tau_n}{\tau_{n+1}}$ and (5) holds. □

Lemma 3.4. *Suppose that* $\{u_n\}, \{x_n\}, \{z_n\}$ *are bounded sequences constructed in Algorithm 3.1 such that* $x_n - x_{n+1} \to 0$, $x_n - u_n \to 0$, $x_n - z_n \to 0$, $u_n - Gu_n \to 0$ *and* $x_n - S^n x_n \to 0$. *If* $S^n x_n - S^{n+1} x_n \to 0$ *and* $\exists \{x_{n_k}\} \subset \{x_n\}$ *s.t.* $x_{n_k} \rightharpoonup z \in H$, *then* $z \in \Omega$.

Proof. From Algorithm 3.1 and the conditions $x_n - x_{n+1} \to 0$ and $x_n - u_n \to 0$, we obtain that

$$\begin{aligned} \|S_n x_n - x_n\| &= \|u_n - x_n - \alpha_n (S_n x_n - S_n x_{n-1})\| \\ &\leq \|u_n - x_n\| + \alpha_n \|x_n - x_{n-1}\| \to 0 \quad (n \to \infty). \end{aligned}$$

Note that

$$\begin{aligned} \|w_n - x_n\| &= \|(1-\gamma_n)(Gu_n - x_n) + \beta_n (f(x_n) - \rho F G u_n)\| \\ &= \|(1-\gamma_n)[Gu_n - u_n + u_n - x_n] + \beta_n (f(x_n) - \rho F G u_n)\| \\ &\leq (1-\gamma_n)\|Gu_n - u_n + u_n - x_n\| + \beta_n \|f(x_n) - \rho F G u_n\| \\ &\leq \|Gu_n - u_n\| + \|u_n - x_n\| + \beta_n (\|f(x_n)\| + \|\rho F G u_n\|). \end{aligned}$$

Since $\beta_n \to 0$, $\|u_n - x_n\| \to 0$ and $\|u_n - Gu_n\| \to 0$, by the boundedness of $\{x_n\}$ and $\{Gu_n\}$ we deduce that

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{11}$$

It is clear that $\{w_n\}$ is bounded. From (10) it follows that

$$\begin{aligned} (1 - \mu \frac{\tau_n}{\tau_{n+1}})(\|z_n - y_n\|^2 + \|y_n - w_n\|^2) &\leq \|w_n - u\|^2 - \|z_n - u\|^2 \\ &\leq \|w_n - z_n\|(\|w_n - u\| + \|z_n - u\|) \\ &\leq (\|w_n - x_n\| + \|x_n - z_n\|)(\|w_n - u\| + \|z_n - u\|). \end{aligned}$$

Since $1 - \mu \frac{\tau_n}{\tau_{n+1}} \to 1 - \mu > 0$ and $\|x_n - z_n\| \to 0$ (due to the condition), from (11) one gets

$$\lim_{n\to\infty} \|y_n - z_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|w_n - y_n\| = 0.$$

It is clear that $\{y_n\}$ is bounded. Also, owing to the conditions $x_n - z_n \to 0$, $x_n - S^n x_n \to 0$ and $S^n x_n - S^{n+1} x_n \to 0$, one has that

$$\begin{aligned} \|z_n - S^n z_n\| &\leq \|z_n - x_n\| + \|x_n - S^n x_n\| + \|S^n x_n - S^n z_n\| \\ &\leq (2 + \theta_n)\|z_n - x_n\| + \|x_n - S^n x_n\| \to 0 \quad (n \to \infty), \end{aligned}$$

and hence

$$\begin{aligned} \|z_n - S z_n\| &\leq \|z_n - S^n x_n\| + \|S^n x_n - S^{n+1} x_n\| + \|S^{n+1} x_n - S z_n\| \\ &\leq (2 + \theta_1)\|z_n - S^n x_n\| + \|S^n x_n - S^{n+1} x_n\| \\ &\leq (2 + \theta_1)(\|z_n - S^n z_n\| + \|S^n z_n - S^n x_n\|) + \|S^n x_n - S^{n+1} x_n\| \\ &\leq (2 + \theta_1)\{\|z_n - S^n z_n\| + (1 + \theta_n)\|z_n - x_n\|\} + \|S^n x_n - S^{n+1} x_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{12}$$

We claim that $\lim_{n\to\infty} \|x_n - S_l x_n\| = 0$ for $l = 1, ..., N$. In fact, observe that for $l = 1, ..., N$,

$$\begin{aligned} \|x_n - S_{n+l} x_n\| &\leq \|x_n - S_{n+l} x_{n+l}\| + \|S_{n+l} x_{n+l} - S_{n+l} x_n\| \\ &\leq \|x_n - S_{n+l} x_{n+l}\| + \|x_{n+l} - x_n\| \\ &\leq \|S_{n+l} x_{n+l} - x_{n+l}\| + 2\|x_{n+l} - x_n\|. \end{aligned}$$

So, from $x_n - S_n x_n \to 0$ and $x_n - x_{n+1} \to 0$, one gets $\lim_{n\to\infty} \|x_n - S_{n+l} x_n\| = 0$ for $l = 1, ..., N$. This hence ensures that

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0, \ \forall l \in \{1, 2, ..., N\}. \tag{13}$$

Noticing $y_n = P_C(w_n - \tau_n A w_n)$, one has $\langle w_{n_k} - \tau_{n_k} A w_{n_k} - y_{n_k}, y - y_{n_k}\rangle \le 0, \forall y \in C$, and hence

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, y - y_{n_k}\rangle + \langle A w_{n_k}, y_{n_k} - w_{n_k}\rangle \le \langle A w_{n_k}, y - w_{n_k}\rangle,\ \forall y \in C. \tag{14}$$

Using the boundedness of $\{w_{n_k}\}$ and Lipschitz continuity of A, one gets the boundedness of $\{Aw_{n_k}\}$. Thanks to $\tau_n \ge \tau := \min\{\tau_1, \frac{\mu}{L}\}$ and $w_n - y_n \to 0$, from (14) one gets

$$\liminf_{k\to\infty}\langle A w_{n_k}, y - w_{n_k}\rangle \ge 0,\ \forall y \in C.$$

Next, we claim that $z \in \mathrm{VI}(C, A)$. In fact, since $w_n - x_n \to 0$, $w_n - y_n \to 0$ and $x_{n_k} \rightharpoonup z$, we obtain that $x_n - y_n \to 0$ and $y_{n_k} \rightharpoonup z$. Since C is convex and closed, from $\{y_n\} \subset C$ one has $z \in C$. In what follows, we consider two cases. If $Az = 0$, then it is clear that $z \in \mathrm{VI}(C, A)$ (due to $\langle Az, y - z\rangle \ge 0, \forall y \in C$). Assume that $Az \ne 0$. Then, by the hypothesis on A, instead of the sequentially weak continuity of A, we get $0 < \|Az\| \le \liminf_{k\to\infty} \|Ay_{n_k}\|$. Because combining $w_n - y_n \to 0$ and L-Lipschitz continuity of A guarantees $Aw_n - Ay_n \to 0$, it follows that

$$\liminf_{k\to\infty} \|Aw_{n_k}\| \ge \liminf_{k\to\infty}(\|Ay_{n_k}\| - \|Ay_{n_k} - Aw_{n_k}\|) = \liminf_{k\to\infty} \|Ay_{n_k}\| \ge \|Az\| > 0.$$

So, we might assume that $\|Aw_{n_k}\| \ne 0\ \forall k \ge 1$.

We now choose a sequence $\{\eta_k\} \subset (0, 1)$ s.t. $\eta_k \downarrow 0$ as $k \to \infty$. For each $k \ge 1$, let the m_k indicate the smallest positive integer s.t.

$$\langle Aw_{n_l}, y - w_{n_l}\rangle + \eta_k \ge 0, \forall l \ge m_k. \tag{15}$$

Since $\{\eta_k\}$ is decreasing, one knows that $\{m_k\}$ is increasing. Noticing $\{w_{m_k}\} \subset \{w_{n_k}\}$ and $\|Aw_{n_k}\| \ne 0, \forall k \ge 1$, we set $p_{m_k} = \frac{Aw_{m_k}}{\|Aw_{m_k}\|^2}$, and get $\langle Aw_{m_k}, p_{m_k}\rangle = 1, \forall k \ge 1$. So, from (15) one has $\langle Aw_{m_k}, y + \eta_k p_{m_k} - w_{m_k}\rangle \ge 0, \forall k \ge 1$. Also, using the pseudomonotonicity of A one has $\langle A(y+\eta_k p_{m_k}), y+\eta_k p_{m_k} - w_{m_k}\rangle \ge 0, \forall k \ge 1$. This immediately leads to

$$\langle Ay, y - w_{m_k}\rangle \ge \langle Ay - A(y + \eta_k p_{m_k}), y + \eta_k p_{m_k} - w_{m_k}\rangle - \eta_k\langle Ay, p_{m_k}\rangle, \forall k \ge 1. \tag{16}$$

We claim that $\lim_{k\to\infty} \eta_k p_{m_k} = 0$. In fact, noticing $0 < \|Az\| \le \liminf_{k\to\infty} \|Aw_{n_k}\|$, $\{w_{m_k}\} \subset \{w_{n_k}\}$ and $\eta_k \downarrow 0$, we obtain that $0 \le \limsup_{k\to\infty} \|\eta_k p_{m_k}\| = \limsup_{k\to\infty} \frac{\eta_k}{\|Aw_{m_k}\|} \le \frac{\limsup_{k\to\infty}\eta_k}{\liminf_{k\to\infty}\|Aw_{n_k}\|} = 0$. So, one has $\eta_k p_{m_k} \to 0$ as $k \to \infty$.

In the end, we claim that $z \in \Omega$. In fact, using (13) one has $x_{n_k} - S_l x_{n_k} \to 0$ for $l = 1, ..., N$. Since Lemma 2.5 ensures the demiclosedness of $I - S_l$ at zero, from $x_{n_k} \rightharpoonup z$ one has $z \in \mathrm{Fix}(S_l), \forall l \in \{1, ..., N\}$. So, $z \in \bigcap_{l=1}^N \mathrm{Fix}(S_l)$. Also, from $x_n - z_n \to 0$ and $x_{n_k} \rightharpoonup z$, one gets $z_{n_k} \rightharpoonup z$. Using (12) one has $z_{n_k} - Sz_{n_k} \to 0$. From Lemma 2.5 one knows the demiclosedness of $I - S$ at zero, and hence gets $z \in \mathrm{Fix}(S)$. Moreover, from $x_n - u_n \to 0$ and $x_{n_k} \rightharpoonup z$, one gets $u_{n_k} \rightharpoonup z$. According to Lemma 2.7, one knows that the mapping $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ is nonexpansive for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Using the hypothesis $u_n - Gu_n \to 0$ one has $u_{n_k} - Gu_{n_k} \to 0$. From Lemma 2.5 one knows the demiclosedness of $I - G$ at zero, and hence gets $z \in \mathrm{Fix}(G)$. In addition, letting $k \to \infty$, we infer that the right-hand side of (16) tends to zero by the uniform continuity of A, the boundedness of $\{w_{m_k}\}, \{p_{m_k}\}$ and the limit $\lim_{k\to\infty} \eta_k p_{m_k} = 0$. Consequently, $\langle Ay, y - z\rangle = \liminf_{k\to\infty}\langle Ay, y - w_{m_k}\rangle \ge 0, \forall y \in C$. Using Lemma 2.3 one has $z \in \mathrm{VI}(C, A)$. All in all, $z \in \bigcap_{l=0}^N \mathrm{Fix}(S_l) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \Omega$. □

Theorem 3.5. *Suppose that $\{x_n\}$ is the constructed sequence in Algorithm 3.1, such that $S^n x_n - S^{n+1}x_n \to 0$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle(\rho F - f)x^*, p - x^*\rangle \ge 0,\ \forall p \in \Omega$.*

Proof. Since $\lim_{n\to\infty} \frac{\theta_n}{\beta_n} = 0$ and $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$, we might assume that $\theta_n \le \frac{\beta_n(\ell-\delta)}{2}, \forall n \ge 1$ and $\{\gamma_n\} \subset [a, b] \subset (0, 1)$. Let us show that $P_\Omega(f - \rho F + I)$ is a contraction. In fact, using Lemma 2.9 one has

$$\begin{aligned}\|P_\Omega(f - \rho F + I)x - P_\Omega(f - \rho F + I)y\| &\le \|(I - \rho F)x - (I - \rho F)y\| + \|f(x) - f(y)\| \\ &\le (1 - \ell)\|x - y\| + \delta\|x - y\| = [1 - (\ell - \delta)]\|x - y\|,\ \forall x, y \in H,\end{aligned}$$

which guarantees that $P_\Omega(f - \rho F + I)$ is a contraction. By Banach's Contraction Mapping Principle we obtain that $P_\Omega(f - \rho F + I)$ has the unique fixed point. Say $x^* \in H$, i.e., $x^* = P_\Omega(f - \rho F + I)x^*$. Therefore, $\exists | x^* \in \Omega = \bigcap_{l=0}^N \mathrm{Fix}(S_l) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$ solving the HVI

$$\langle (\rho F - f)x^*, p - x^* \rangle \geq 0, \ \forall p \in \Omega. \tag{17}$$

In what follows, we divide the remainder of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, for $x^* \in \Omega = \bigcap_{l=0}^N \mathrm{Fix}(S_l) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$, we obtain that $x^* = Gx^*$, $x^* = Sx^*$ and $x^* = S_n x^*$, $\forall n \geq 1$, and (10) holds, i.e.,

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - (1 - \mu \frac{\tau_n}{\tau_{n+1}})(\|y_n - w_n\|^2 + \|z_n - y_n\|^2).$$

Since $\lim_{n\to\infty}(1 - \mu \frac{\tau_n}{\tau_{n+1}}) = 1 - \mu > 0$, we might assume that $1 - \mu \frac{\tau_n}{\tau_{n+1}} > 0 \ \forall n \geq 1$. Hence, one gets

$$\|z_n - x^*\| \leq \|w_n - x^*\|, \forall n \geq 1. \tag{18}$$

By the definition of u_n, we get

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \alpha_n \|x_n - x_{n-1}\| = \|x_n - x^*\| + \beta_n \cdot \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\|. \tag{19}$$

Since $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$, one knows that $\exists K_1 > 0$ s.t.

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq K_1 \quad \forall n \geq 1,$$

which together (19), yields

$$\|u_n - x^*\| \leq \|x_n - x^*\| + \beta_n K_1, \ \forall n \geq 1. \tag{20}$$

So, from $\beta_n + \gamma_n < 1$, Lemma 2.9 and (18) it follows that

$$\begin{aligned}
\|z_n - x^*\| &\leq \beta_n \|f(x_n) - f(x^*)\| + \gamma_n \|x_n - x^*\| + (1 - \gamma_n)\|(I - \frac{\beta_n}{1 - \gamma_n}\rho F)Gu_n \\
&\quad - (I - \frac{\beta_n}{1 - \gamma_n}\rho F)x^* + \frac{\beta_n}{1 - \gamma_n}(f - \rho F)x^*\| \\
&\leq \beta_n \delta \|x_n - x^*\| + \gamma_n \|x_n - x^*\| + (1 - \gamma_n - \beta_n \ell)\|u_n - x^*\| + \beta_n \|(f - \rho F)x^*\| \\
&\leq \beta_n \delta(\|x_n - x^*\| + \beta_n K_1) + \gamma_n(\|x_n - x^*\| + \beta_n K_1) \\
&\quad + (1 - \gamma_n - \beta_n \ell)(\|x_n - x^*\| + \beta_n K_1) + \beta_n \|(f - \rho F)x^*\| \\
&\leq \|x_n - x^*\| + \beta_n(K_1 + \|(f - \rho F)x^*\|).
\end{aligned} \tag{21}$$

Noticing $x_{n+1} = (1 - \lambda_n)w_n + \lambda_n S^n z_n$, we infer from (21) that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1 - \lambda_n)\|w_n - x^*\| + \lambda_n(1 + \theta_n)\|z_n - x^*\| \\
&\leq [1 - \beta_n(\ell - \delta)]\|x_n - x^*\| + \beta_n(K_1 + \|(f - \rho F)x^*\|) + \frac{\beta_n(\ell - \delta)}{2} \cdot [\|x_n - x^*\| + \beta_n(K_1 + \|(f - \rho F)x^*\|)] \\
&\leq [1 - \frac{\beta_n(\ell - \delta)}{2}]\|x_n - x^*\| + \frac{\beta_n(\ell - \delta)}{2} \cdot \frac{3(K_1 + \|(f - \rho F)x^*\|)}{\ell - \delta} \\
&\leq \max\{\|x_n - x^*\|, \frac{3(K_1 + \|(f - \rho F)x^*\|)}{\ell - \delta}\}.
\end{aligned}$$

By induction, we obtain $\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, \frac{3(K_1 + \|(f - \rho F)x^*\|)}{\ell - \delta}\}$, $\forall n \geq 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}, \{q_n\}, \{w_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{S_n x_n\}, \{S^n z_n\}$.

Step 2. We show that

$$(\beta_n + \theta_n)K_4 - \|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2 \geq \lambda_n(1 - \mu \frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2),$$

for some $K_4 > 0$.

In fact, one has

$$w_n - x^* = \beta_n(f(x_n) - f(x^*)) + \gamma_n(x_n - x^*) + (1-\gamma_n)[(I - \frac{\beta_n}{1-\gamma_n}\rho F)Gu_n - (I - \frac{\beta_n}{1-\gamma_n}\rho F)x^*] + \beta_n(f - \rho F)x^*.$$

Using Lemma 2.9 and the convexity of the function $h(s) = s^2$ $\forall s \in \mathbf{R}$, one gets

$$\begin{aligned}\|w_n - x^*\|^2 &\le \|\beta_n(f(x_n) - f(x^*)) + \gamma_n(x_n - x^*) + (1-\gamma_n)[(I - \frac{\beta_n}{1-\gamma_n}\rho F)Gu_n - (I - \frac{\beta_n}{1-\gamma_n}\rho F)x^*]\|^2 \\ &\quad + 2\beta_n\langle (f-\rho F)x^*, w_n - x^*\rangle \\ &\le [\beta_n\delta\|x_n - x^*\| + \gamma_n\|x_n - x^*\| + (1-\beta_n\ell-\gamma_n)\|u_n - x^*\|]^2 + 2\beta_n\langle (f-\rho F)x^*, w_n - x^*\rangle \\ &\le \beta_n\delta\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 + (1-\beta_n\ell-\gamma_n)\|u_n - x^*\|^2 + 2\beta_n\langle (f-\rho F)x^*, w_n - x^*\rangle \\ &\le \beta_n\delta\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 + (1-\beta_n\ell-\gamma_n)\|u_n - x^*\|^2 + \beta_n K_2\end{aligned} \tag{22}$$

where $\sup_{n\ge1} 2\|(f-\rho F)x^*\|\|w_n - x^*\| \le K_2$ for some $K_2 > 0$. Noticing $x_{n+1} = (1-\lambda_n)w_n + \lambda_n S^n z_n$, from (10) we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\lambda_n)\|w_n - x^*\|^2 + \lambda_n\|z_n - x^*\|^2 + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\le \|w_n - x^*\|^2 - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) + \theta_n(2+\theta_n)\|z_n - x^*\|^2.\end{aligned} \tag{23}$$

Also, from (20) we have

$$\begin{aligned}\|u_n - x^*\|^2 &\le \|x_n - x^*\|^2 + \beta_n(2K_1\|x_n - x^*\| + \beta_n K_1^2) \\ &\le \|x_n - x^*\|^2 + \beta_n K_3,\end{aligned} \tag{24}$$

where $\sup_{n\ge1}(2K_1\|x_n - x^*\| + \beta_n K_1^2) \le K_3$ for some $K_3 > 0$. Combining (22)-(24), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\delta\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 + (1-\beta_n\ell-\gamma_n)\|u_n - x^*\|^2 + \beta_n K_2 \\ &\quad - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\le [1-\beta_n(\ell-\delta)]\|x_n - x^*\|^2 - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) \\ &\quad + \beta_n K_3 + \beta_n K_2 + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) + (\beta_n + \theta_n)K_4,\end{aligned}$$

where $\sup_{n\ge1}[K_2 + K_3 + (2+\theta_n)\|z_n - x^*\|^2] \le K_4$. This immediately implies that

$$(\theta_n + \beta_n)K_4 - \|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2 \ge \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2). \tag{25}$$

Step 3. We show that

$$\beta_n(\ell-\delta)[\frac{3K}{\ell-\delta}(\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| + \frac{\theta_n}{3\beta_n}) + \frac{2\langle (f-\rho F)x^*, w_n - x^*\rangle}{\ell-\delta}] + [1-\beta_n(\ell-\delta)]\|x_n - x^*\|^2 \ge \|x_{n+1} - x^*\|^2 \tag{26}$$

for some $K > 0$. In fact, note that

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 + \alpha_n\|x_n - x_{n-1}\|[2\|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\|].$$

Using (18) and (22), one has

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\lambda_n)\|w_n-x^*\|^2+\lambda_n(1+\theta_n)^2\|z_n-x^*\|^2 \le \|w_n-x^*\|^2+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\ &\le \beta_n\delta\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2+(1-\beta_n\ell-\gamma_n)\|u_n-x^*\|^2+2\beta_n\langle(f-\rho F)x^*,w_n-x^*\rangle\\ &\quad+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\ &\le [1-\beta_n(\ell-\delta)]\|x_n-x^*\|^2+\beta_n(\ell-\delta)[\frac{3K}{\ell-\delta}(\frac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|+\frac{\theta_n}{3\beta_n})+\frac{2\langle(f-\rho F)x^*,w_n-x^*\rangle}{\ell-\delta}],\end{aligned}$$

where $\sup_{n\ge1}\{\|x_n-x^*\|,\alpha_n\|x_n-x_{n-1}\|,(2+\theta_n)\|z_n-x^*\|^2\}\le K$ for some $K>0$.

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^*\in\Omega$ of the HVI (17). In fact, putting $\Gamma_n=\|x_n-x^*\|^2$, we demonstrate $\Gamma_n\to0$ $(n\to\infty)$ by considering the following cases.

Case 1. $\exists$ (integer) $\overline{n_0}\ge1$ s.t. $\{\Gamma_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\Gamma_n=\zeta<+\infty$ and $\lim_{n\to\infty}(\Gamma_n-\Gamma_{n+1})=0$. Since $\theta_n\to0$, $\beta_n\to0$, $\Gamma_n-\Gamma_{n+1}\to0$, $1-\mu\frac{\tau_n}{\tau_{n+1}}\to1-\mu$ and $\{\lambda_n\}\subset[\underline{\lambda},\overline{\lambda}]\subset(0,1)$, from (25) one gets

$$\begin{aligned}\limsup_{n\to\infty}\underline{\lambda}(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n-y_n\|^2+\|y_n-z_n\|^2) &\le \limsup_{n\to\infty}\lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n-y_n\|^2+\|y_n-z_n\|^2)\\ &\le \limsup_{n\to\infty}[(\beta_n+\theta_n)K_4+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2]\\ &= \limsup_{n\to\infty}[(\beta_n+\theta_n)K_4+\Gamma_n-\Gamma_{n+1}]=0.\end{aligned}$$

This hence implies that

$$\lim_{n\to\infty}\|w_n-y_n\|=0\quad\text{and}\quad\lim_{n\to\infty}\|y_n-z_n\|=0.$$

Thus, one has

$$\|w_n-z_n\|\le\|w_n-y_n\|+\|y_n-z_n\|\to0\quad(n\to\infty).$$

Since $w_n-x^*=\gamma_n(x_n-x^*)+(1-\gamma_n)(Gu_n-x^*)+\beta_n(f(x_n)-\rho FGu_n)$, we obtain from (3.21) and (3.22) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \|w_n-x^*\|^2+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\ &\le \|\gamma_n(x_n-x^*)+(1-\gamma_n)(Gu_n-x^*)\|^2+2\beta_n\langle f(x_n)-\rho FGu_n,w_n-x^*\rangle\\ &\quad+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\ &\le \gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\|Gu_n-x^*\|^2-\gamma_n(1-\gamma_n)\|x_n-Gu_n\|^2\\ &\quad+2\beta_n\|f(x_n)-\rho FGu_n\|\|w_n-x^*\|+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\ &\le \gamma_n(\|x_n-x^*\|^2+\beta_nK_3)+(1-\gamma_n)(\|x_n-x^*\|^2+\beta_nK_3)-\gamma_n(1-\gamma_n)\|x_n-Gu_n\|^2\\ &\quad+2\beta_n\|f(x_n)-\rho FGu_n\|\|w_n-x^*\|+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\ &= \|x_n-x^*\|^2+\beta_nK_3-\gamma_n(1-\gamma_n)\|x_n-Gu_n\|^2+2\beta_n\|f(x_n)-\rho FGu_n\|\|w_n-x^*\|\\ &\quad+\theta_n(2+\theta_n)\|z_n-x^*\|^2,\end{aligned}\tag{27}$$

which together with $\{\gamma_n\}\subset[a,b]\subset(0,1)$, arrives at

$$\begin{aligned}a(1-b)\|x_n-Gu_n\|^2 &\le \gamma_n(1-\gamma_n)\|x_n-Gu_n\|^2\\ &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\beta_nK_3+2\beta_n\|f(x_n)-\rho FGu_n\|\|w_n-x^*\|+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\ &\le \Gamma_n-\Gamma_{n+1}+\beta_n[K_3+2(\|f(x_n)\|+\|\rho FGu_n\|)\|w_n-x^*\|]+\theta_n(2+\theta_n)\|z_n-x^*\|^2.\end{aligned}$$

Since $\theta_n\to0$, $\beta_n\to0$ and $\Gamma_n-\Gamma_{n+1}\to0$, by the boundedness of $\{w_n\},\{x_n\},\{z_n\},\{Gu_n\}$ one gets $\lim_{n\to\infty}\|x_n-Gu_n\|=0$ and hence

$$\begin{aligned}\|w_n-x_n\| &= \|(1-\gamma_n)(Gu_n-x_n)+\beta_n(f(x_n)-\rho FGu_n)\|\\ &\le \|Gu_n-x_n\|+\beta_n(\|f(x_n)\|+\|\rho FGu_n\|)\to0\quad(n\to\infty).\end{aligned}$$

This immediately yields

$$\|x_n - z_n\| \leq \|x_n - w_n\| + \|w_n - z_n\| \to 0 \quad (n \to \infty). \tag{28}$$

We now claim that $\|u_n - Gu_n\| \to 0$ as $n \to \infty$. In fact, we put $y^* = P_C(x^* - \mu_2 B_2 x^*)$. Note that $v_n = P_C(u_n - \mu_2 B_2 u_n)$ and $q_n = P_C(v_n - \mu_1 B_1 v_n)$. Then $q_n = Gu_n$. By Lemma 2.6 we have

$$\|v_n - y^*\|^2 \leq \|u_n - x^*\|^2 - \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 x^*\|^2,$$

and

$$\|q_n - x^*\|^2 \leq \|v_n - y^*\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2.$$

Combining the last two inequalities, from (24) we get

$$\begin{aligned}\|q_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 x^*\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2 \\ &\leq \|x_n - x^*\|^2 + \beta_n K_3 - \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 x^*\|^2 - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2.\end{aligned}$$

This together with (27), leads to

$$\begin{aligned}\|x_{x+1} - x^*\|^2 &\leq \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n)\|q_n - x^*\|^2 + 2\beta_n\|f(x_n) - \rho F q_n\|\|w_n - x^*\| \\ &\quad + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\leq \gamma_n\|x_n - x^*\|^2 + (1-\gamma_n)\{\|x_n - x^*\|^2 + \beta_n K_3 - \mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 x^*\|^2 \\ &\quad - \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2\} + 2\beta_n\|f(x_n) - \rho F q_n\|\|w_n - x^*\| + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - (1-\gamma_n)[\mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 x^*\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2] \\ &\quad + \beta_n[K_3 + 2\|f(x_n) - \rho F q_n\|\|w_n - x^*\|] + \theta_n(2+\theta_n)\|w_n - x^*\|^2,\end{aligned}$$

which immediately yields

$$\begin{aligned}&(1-\gamma_n)[\mu_2(2\beta - \mu_2)\|B_2 u_n - B_2 x^*\|^2 + \mu_1(2\alpha - \mu_1)\|B_1 v_n - B_1 y^*\|^2] \\ &\qquad \leq \Gamma_n - \Gamma_{n+1} + \beta_n[K_3 + 2\|f(x_n) - \rho F q_n\|\|w_n - x^*\|] + \theta_n(2+\theta_n)\|w_n - x^*\|^2.\end{aligned}$$

Since $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, $\theta_n \to 0$, $\beta_n \to 0$, $\mathit{\Gamma}_n - \mathit{\Gamma}_{n+1} \to 0$ and $1 - \gamma_n \geq 1 - b$, by the boundedness of $\{x_n\}, \{q_n\}, \{w_n\}$, we get

$$\lim_{n\to\infty}\|B_2 u_n - B_2 x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_1 v_n - B_1 y^*\| = 0. \tag{29}$$

Furthermore, observe that

$$\begin{aligned}\|q_n - x^*\|^2 &\leq \langle v_n - y^*, q_n - x^*\rangle + \mu_1\langle B_1 y^* - B_1 v_n, q_n - x^*\rangle \\ &\leq \frac{1}{2}[\|v_n - y^*\|^2 + \|q_n - x^*\|^2 - \|v_n - q_n + x^* - y^*\|^2] + \mu_1\|B_1 y^* - B_1 v_n\|\|q_n - x^*\|.\end{aligned}$$

This ensures that

$$\|q_n - x^*\|^2 \leq \|v_n - y^*\|^2 - \|v_n - q_n + x^* - y^*\|^2 + 2\mu_1\|B_1 y^* - B_1 v_n\|\|q_n - x^*\|.$$

Similarly, we get

$$\|v_n - y^*\|^2 \leq \|u_n - x^*\|^2 - \|u_n - v_n + y^* - x^*\|^2 + 2\mu_2\|B_2 x^* - B_2 u_n\|\|v_n - y^*\|.$$

Combining the last two inequalities, from (24) we get

$$\begin{aligned}\|q_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - v_n + y^* - x^*\|^2 + 2\mu_2\|B_2 x^* - B_2 u_n\|\|v_n - y^*\| \\ &\quad - \|v_n - q_n + x^* - y^*\|^2 + 2\mu_1\|B_1 y^* - B_1 v_n\|\|q_n - x^*\| \\ &\leq \|x_n - x^*\|^2 + \beta_n K_3 - \|u_n - v_n + y^* - x^*\|^2 - \|v_n - q_n + x^* - y^*\|^2 \\ &\quad + 2\mu_2\|B_2 x^* - B_2 u_n\|\|v_n - y^*\| + 2\mu_1\|B_1 y^* - B_1 v_n\|\|q_n - x^*\|.\end{aligned}$$

This together with (27), arrives at

$$\begin{aligned}\|x_{x+1}-x^*\|^2 &\le \gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\|q_n-x^*\|^2+2\beta_n\|f(x_n)-\rho Fq_n\|\|w_n-x^*\|+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\
&\le \gamma_n\|x_n-x^*\|^2+(1-\gamma_n)\{\|x_n-x^*\|^2+\beta_nK_3-\|u_n-v_n+y^*-x^*\|^2\\
&\quad -\|v_n-q_n+x^*-y^*\|^2+2\mu_2\|B_2x^*-B_2u_n\|\|v_n-y^*\|+2\mu_1\|B_1y^*-B_1v_n\|\|q_n-x^*\|\}\\
&\quad +2\beta_n\|f(x_n)-\rho Fq_n\|\|w_n-x^*\|+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\
&\le \|x_n-x^*\|^2-(1-\gamma_n)[\|u_n-v_n+y^*-x^*\|^2+\|v_n-q_n+x^*-y^*\|^2]\\
&\quad +2\mu_2\|B_2x^*-B_2u_n\|\|v_n-y^*\|+2\mu_1\|B_1y^*-B_1v_n\|\|q_n-x^*\|\\
&\quad +\beta_n[K_3+2\|f(x_n)-\rho Fq_n\|\|w_n-x^*\|]+\theta_n(2+\theta_n)\|w_n-x^*\|^2,\end{aligned}$$

which immediately leads to

$$\begin{aligned}(1-\gamma_n)[\|u_n-v_n+y^*-x^*\|^2+\|v_n-q_n+x^*-y^*\|^2] \le \Gamma_n-\Gamma_{n+1}&+\beta_n[K_3+2\|f(x_n)-\rho Fq_n\|\|w_n-x^*\|]\\
&+2\mu_2\|B_2x^*-B_2u_n\|\|v_n-y^*\|+2\mu_1\|B_1y^*-B_1v_n\|\|q_n-x^*\|\\
&+\theta_n(2+\theta_n)\|w_n-x^*\|^2.\end{aligned}$$

Since $\theta_n\to 0$, $\beta_n\to 0$, $\Gamma_n-\Gamma_{n+1}\to 0$ and $1-\gamma_n\ge 1-b$, by the boundedness of $\{x_n\},\{q_n\},\{v_n\},\{w_n\}$, from (29) we deduce that $\lim_{n\to\infty}\|u_n-v_n+y^*-x^*\|=0$ and $\lim_{n\to\infty}\|v_n-q_n+x^*-y^*\|=0$. Thus, we get $\|u_n-q_n\|\le\|u_n-v_n+y^*-x^*\|+\|v_n-q_n+x^*-y^*\|\to 0$ $(n\to\infty)$. So it follows that

$$\lim_{n\to\infty}\|u_n-Gu_n\|=\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{30}$$

On the other hand, from (18), (22) and (24) it follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= (1-\lambda_n)\|w_n-x^*\|^2+\lambda_n\|S^nz_n-x^*\|^2-\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&\le (1-\lambda_n)\|w_n-x^*\|^2+\lambda_n(1+\theta_n)^2\|z_n-x^*\|^2-\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&\le \|w_n-x^*\|^2+\theta_n(2+\theta_n)\|z_n-x^*\|^2-\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&\le \beta_n\delta\|x_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2+(1-\beta_n\ell-\gamma_n)\|u_n-x^*\|^2+\beta_nK_2\\
&\quad +\theta_n(2+\theta_n)\|z_n-x^*\|^2-\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&\le \beta_n\delta(\|x_n-x^*\|^2+\beta_nK_3)+\gamma_n(\|x_n-x^*\|^2+\beta_nK_3)+(1-\beta_n\ell-\gamma_n)(\|x_n-x^*\|^2\\
&\quad +\beta_nK_3)+\beta_nK_2+\theta_n(2+\theta_n)\|z_n-x^*\|^2-\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&= [1-\beta_n(\ell-\delta)](\|x_n-x^*\|^2+\beta_nK_3)+\beta_nK_2+\theta_n(2+\theta_n)\|z_n-x^*\|^2\\
&\quad -\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&\le \|x_n-x^*\|^2+\beta_nK_3+\beta_nK_2+\theta_n(2+\theta_n)\|z_n-x^*\|^2-\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&\le \|x_n-x^*\|^2+(\beta_n+\theta_n)K_4-\lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2,\end{aligned}$$

which together with $\{\lambda_n\}\subset[\underline{\lambda},\overline{\lambda}]\subset(0,1)$, arrives at

$$\begin{aligned}\underline{\lambda}(1-\overline{\lambda})\|w_n-S^nz_n\|^2 &\le \lambda_n(1-\lambda_n)\|w_n-S^nz_n\|^2\\
&\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+(\beta_n+\theta_n)K_4\\
&= \Gamma_n-\Gamma_{n+1}+(\beta_n+\theta_n)K_4.\end{aligned}$$

Since $\Gamma_n-\Gamma_{n+1}\to 0$, $\beta_n\to 0$ and $\theta_n\to 0$, one obtains $\lim_{n\to\infty}\|w_n-S^nz_n\|=0$. Accordingly, one has

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(1-\lambda_n)(w_n-x_n)+\lambda_n(S^nz_n-x_n)\|\\
&= \|(1-\lambda_n)(w_n-x_n)+\lambda_n(S^nz_n-w_n+w_n-x_n)\|\\
&= \|w_n-x_n+\lambda_n(S^nz_n-w_n)\|\\
&\le \|w_n-x_n\|+\|S^nz_n-w_n\|\to 0\quad (n\to\infty),\end{aligned} \tag{31}$$

and

$$\begin{aligned}\|x_n - S^n x_n\| &\le \|x_n - w_n\| + \|w_n - S^n z_n\| + \|S^n z_n - S^n x_n\| \\ &\le \|x_n - w_n\| + \|w_n - S^n z_n\| + (1+\theta_n)\|z_n - x_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{32}$$

Also, by the boundedness of $\{x_n\}$, we deduce that $\exists\{x_{n_k}\} \subset \{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle(f-\rho F)x^*, x_{n_k} - x^*\rangle.$$

According to the reflexivity of H and boundedness of $\{x_n\}$, we might assume that $x_{n_k} \rightharpoonup \tilde{x}$. So it follows that

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle(f-\rho F)x^*, x_{n_k} - x^*\rangle = \langle(f-\rho F)x^*, \tilde{x} - x^*\rangle. \tag{33}$$

Note that $x_n - x_{n+1} \to 0$, $x_n - u_n \to 0$, $x_n - z_n \to 0$, $u_n - Gu_n \to 0$ and $x_n - S^n x_n \to 0$ (due to (28), (30)-(32)). Since $S^n x_n - S^{n+1}x_n \to 0$ and $x_{n_k} \rightharpoonup \tilde{x}$, by Lemma 3.4 we get $\tilde{x} \in \Omega$. Thus, using (17) and (33) one has

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*, x_n - x^*\rangle = \langle(f-\rho F)x^*, \tilde{x} - x^*\rangle \le 0, \tag{34}$$

which immediately leads to

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*, w_n - x^*\rangle \le \limsup_{n\to\infty}[\|(f-\rho F)x^*\|\|w_n - x_n\| + \langle(f-\rho F)x^*, x_n - x^*\rangle] \le 0. \tag{35}$$

It is not hard to check that $\{\beta_n(\ell-\delta)\} \subset [0,1]$, $\sum_{n=1}^{\infty}\beta_n(\ell-\delta) = \infty$, and

$$\limsup_{n\to\infty}[\frac{3K}{\ell-\delta}(\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| + \frac{\theta_n}{3\beta_n}) + \frac{2\langle(f-\rho F)x^*, w_n - x^*\rangle}{\ell-\delta}] \le 0.$$

Therefore, applying Lemma 2.4 to (26), we have $\lim_{n\to 0}\|x_n - x^*\|^2 = 0$.

Case 2. $\exists\{\Gamma_{n_m}\} \subset \{\Gamma_n\}$ s.t. $\Gamma_{n_m} < \Gamma_{n_m+1}$ $\forall m \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\hbar : \mathcal{N} \to \mathcal{N}$ by $\hbar(n) := \max\{m \le n : \Gamma_m < \Gamma_{m+1}\}$. In the light of Lemma 2.8, we obtain $\Gamma_{\hbar(n)} \le \Gamma_{\hbar(n)+1}$ and $\Gamma_n \le \Gamma_{\hbar(n)+1}$. Hence, from (25) we have

$$\begin{aligned}&\limsup_{n\to\infty}\underline{\lambda}(1-\mu\frac{\tau_{\hbar(n)}}{\tau_{\hbar(n)+1}})(\|w_{\hbar(n)} - y_{\hbar(n)}\|^2 + \|y_{\hbar(n)} - z_{\hbar(n)}\|^2) \\ &\le \limsup_{n\to\infty}\lambda_{\hbar(n)}(1-\mu\frac{\tau_{\hbar(n)}}{\tau_{\hbar(n)+1}})(\|w_{\hbar(n)} - y_{\hbar(n)}\|^2 + \|y_{\hbar(n)} - z_{\hbar(n)}\|^2) \\ &\le \limsup_{n\to\infty}(\Gamma_{\hbar(n)} - \Gamma_{\hbar(n)+1} + (\beta_{\hbar(n)} + \theta_{\hbar(n)})K_4) = 0.\end{aligned}$$

This hence arrives at $\lim_{n\to\infty}\|w_{\hbar(n)} - y_{\hbar(n)}\| = 0$ and $\lim_{n\to\infty}\|y_{\hbar(n)} - z_{\hbar(n)}\| = 0$. Using the same inferences as in the proof of Case 1, we deduce that $\lim_{n\to\infty}\|x_{\hbar(n)} - z_{\hbar(n)}\| = 0$,

$$\lim_{n\to\infty}\|u_{\hbar(n)} - Gu_{\hbar(n)}\| = \lim_{n\to\infty}\|x_{\hbar(n)} - u_{\hbar(n)}\| = \lim_{n\to\infty}\|x_{\hbar(n)+1} - x_{\hbar(n)}\| = 0,$$

$$\lim_{n\to\infty}\|x_{\hbar(n)} - S^{\hbar(n)}x_{\hbar(n)}\| = 0 \quad \text{and} \quad \limsup_{n\to\infty}\langle(f-\rho F)x^*, w_{\hbar(n)} - x^*\rangle \le 0.$$

On the other hand, from (26) we obtain

$$\begin{aligned}\beta_{\hbar(n)}(\ell-\delta)\Gamma_{\hbar(n)} &\le \Gamma_{\hbar(n)} - \Gamma_{\hbar(n)+1} + \beta_{\hbar(n)}(\ell-\delta)[\frac{3K}{\ell-\delta}(\frac{\alpha_{\hbar(n)}}{\beta_{\hbar(n)}}\|x_{\hbar(n)} - x_{\hbar(n)-1}\| + \frac{\theta_{\hbar(n)}}{3\beta_{\hbar(n)}}) + \frac{2\langle(f-\rho F)x^*, w_{\hbar(n)} - x^*\rangle}{\ell-\delta}] \\ &\le \beta_{\hbar(n)}(\ell-\delta)[\frac{3K}{\ell-\delta}(\frac{\alpha_{\hbar(n)}}{\beta_{\hbar(n)}}\|x_{\hbar(n)} - x_{\hbar(n)-1}\| + \frac{\theta_{\hbar(n)}}{3\beta_{\hbar(n)}}) + \frac{2\langle(f-\rho F)x^*, w_{\hbar(n)} - x^*\rangle}{\ell-\delta}],\end{aligned}$$

which hence yields

$$\limsup_{n\to\infty}\Gamma_{\hbar(n)} \le \limsup_{n\to\infty}[\frac{3K}{\ell-\delta}(\frac{\alpha_{\hbar(n)}}{\beta_{\hbar(n)}}\|x_{\hbar(n)} - x_{\hbar(n)-1}\| + \frac{\theta_{\hbar(n)}}{3\beta_{\hbar(n)}}) + \frac{2\langle(f-\rho F)x^*, w_{\hbar(n)} - x^*\rangle}{\ell-\delta}] \le 0.$$

Thus, $\lim_{n\to\infty}\|x_{\hbar(n)}-x^*\|^2=0$. Also, note that

$$\begin{aligned}\|x_{\hbar(n)+1}-x^*\|^2-\|x_{\hbar(n)}-x^*\|^2&=2\langle x_{\hbar(n)+1}-x_{\hbar(n)},x_{\hbar(n)}-x^*\rangle+\|x_{\hbar(n)+1}-x_{\hbar(n)}\|^2\\&\le 2\|x_{\hbar(n)+1}-x_{\hbar(n)}\|\|x_{\hbar(n)}-x^*\|+\|x_{\hbar(n)+1}-x_{\hbar(n)}\|^2\to 0\quad(n\to\infty).\end{aligned}$$

Thanks to $\Gamma_n\le\Gamma_{\hbar(n)+1}$, we get

$$\|x_n-x^*\|^2\le\|x_{\hbar(n)+1}-x^*\|^2\le\|x_{\hbar(n)}-x^*\|^2+2\|x_{\hbar(n)+1}-x_{\hbar(n)}\|\|x_{\hbar(n)}-x^*\|+\|x_{\hbar(n)+1}-x_{\hbar(n)}\|^2\to 0\quad(n\to\infty).$$

That is, $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

Theorem 3.6. *Suppose that $S:H\to H$ is nonexpansive and $\{x_n\}$ is the sequence constructed by the modified version of Algorithm 3.1, i.e., for any initial $x_0,x_1\in H$,*

$$\begin{cases}u_n=S_nx_n+\alpha_n(S_nx_n-S_nx_{n-1}),\\ v_n=P_C(u_n-\mu_2B_2u_n),\\ q_n=P_C(v_n-\mu_1B_1v_n),\\ w_n=\beta_nf(x_n)+\gamma_nx_n+((1-\gamma_n)I-\beta_n\rho F)q_n,\\ y_n=P_C(w_n-\tau_nAw_n),\\ z_n=P_{C_n}(w_n-\tau_nAy_n),\\ x_{n+1}=(1-\lambda_n)w_n+\lambda_nSz_n,\forall n\ge 1,\end{cases}$$

and α_n, τ_n and C_n are chosen as in Algorithm 3.1. Then $x_n\to x^*\in\Omega$, which is the unique solution to the HVI: $\langle(\rho F-f)x^*,p-x^*\rangle\ge 0,\ \forall p\in\Omega$.

Proof. We divide the proof of the theorem into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, using the same arguments as in Step 1 of the proof of Theorem 3.1, we obtain the desired assertion.

Step 2. We show that

$$\beta_nK_4+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\ge\lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n-y_n\|^2+\|y_n-z_n\|^2)$$

for some $K_4>0$. In fact, using the same arguments as in Step 2 of the proof of Theorem 3.5, we obtain the desired assertion.

Step 3. We show that

$$\beta_n(\ell-\delta)[\frac{3K\alpha_n}{(\ell-\delta)\beta_n}\|x_n-x_{n-1}\|+\frac{2\langle(f-\rho F)x^*,w_n-x^*\rangle}{\ell-\delta}]+[1-\beta_n(\ell-\delta)]\|x_n-x^*\|^2\ge\|x_{n+1}-x^*\|^2$$

for some $K>0$. In fact, using the same arguments as in Step 3 of the proof of Theorem 3.5, we obtain the desired assertion.

Step 4. We prove that $\{x_n\}$ converges strongly to the unique solution $x^*\in\Omega$ of the HVI (17), with $S_0=S$ a nonexpansive mapping. In fact, from Step 3, one obtains

$$\beta_n(\ell-\delta)[\frac{3K\alpha_n}{(\ell-\delta)\beta_n}\|x_n-x_{n-1}\|+\frac{2\langle(f-\rho F)x^*,w_n-x^*\rangle}{\ell-\delta}]+[1-\beta_n(\ell-\delta)]\|x_n-x^*\|^2\ge\|x_{n+1}-x^*\|^2. \tag{36}$$

Putting $\Gamma_n=\|x_n-x^*\|^2$, we demonstrate $\Gamma_n\to 0$ $(n\to\infty)$ by considering the following cases.

Case 1. Assume that ∃ (integer) $\overline{n_0}\ge 1$ s.t. $\{\Gamma_n\}$ is nonincreasing. Then the limit $\lim_{n\to\infty}\Gamma_n=\zeta<+\infty$ and $\lim_{n\to\infty}(\Gamma_n-\Gamma_{n+1})=0$.

Utilizing Step 2, $\Gamma_n-\Gamma_{n+1}\to 0$, $\beta_n\to 0$, $1-\mu\frac{\tau_n}{\tau_{n+1}}\to 1-\mu$ and $\{\lambda_n\}\subset[\underline{\lambda},\overline{\lambda}]\subset(0,1)$, one has

$$\begin{aligned}\limsup_{n\to\infty}\underline{\lambda}(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n-y_n\|^2+\|y_n-z_n\|^2)&\le\limsup_{n\to\infty}\lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n-y_n\|^2+\|y_n-z_n\|^2)\\&\le\limsup_{n\to\infty}(\Gamma_n-\Gamma_{n+1}+\beta_nK_4)=0,\end{aligned}$$

which immediately yields

$$\lim_{n\to\infty} \|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{37}$$

Thus, one has

$$\|w_n - z_n\| \le \|w_n - y_n\| + \|y_n - z_n\| \to 0 \quad (n \to \infty). \tag{38}$$

Utilizing the same inferences as in Case 1 of the proof of Theorem 3.5, we conclude that $\lim_{n\to\infty} \|x_n - z_n\| = 0$,

$$\lim_{n\to\infty} \|u_n - Gu_n\| = \lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{39}$$

and

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \langle (f - \rho F)x^*, w_n - x^* \rangle \le 0. \tag{40}$$

As a result, applying Lemma 2.4 to (36), one derives $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$.

Case 2. Assume that $\exists \{\Gamma_{n_m}\} \subset \{\Gamma_n\}$ s.t. $\Gamma_{n_m} < \Gamma_{n_m+1}$ $\forall m \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. Let the mapping $\hbar : \mathcal{N} \to \mathcal{N}$ be defined as $\hbar(n) := \max\{m \le n : \Gamma_m < \Gamma_{m+1}\}$. From Lemma 2.8, one has $\Gamma_{\hbar(n)} \le \Gamma_{\hbar(n)+1}$ and $\Gamma_n \le \Gamma_{\hbar(n)+1}$. In the rest of the proof, utilizing the same inferences as in Case 2 of the proof of Theorem 3.5, we obtain the desired result. This completes the proof. □

Next, we formulate another modified inertial-like subgradient extragradient algorithm below.

Algorithm 3.7. **Initialization:** *Given $\tau_1 > 0$, $\alpha > 0$, $\mu \in (0, 1)$. Let $x_0, x_1 \in H$ be arbitrary and choose α_n s.t.*

$$0 \le \alpha_n \le \widetilde{\alpha}_n := \begin{cases} \min\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, & if\, x_n \ne x_{n-1}, \\ \alpha, & otherwise. \end{cases}$$

Iterative Steps: *Calculate x_{n+1} as follows:*
Step 1. Set $u_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1})$, and compute

$$\begin{cases} v_n = P_C(x_n - \mu_2 B_2 x_n), \\ q_n = P_C(v_n - \mu_1 B_1 v_n). \end{cases}$$

Step 2. Compute

$$\begin{cases} w_n = \beta_n f(x_n) + \gamma_n u_n + ((1 - \gamma_n)I - \beta_n \rho F)q_n, \\ y_n = P_C(w_n - \tau_n A w_n). \end{cases}$$

Step 3. Compute

$$x_{n+1} = (1 - \lambda_n)w_n + \lambda_n S^n P_{C_n}(w_n - \tau_n A y_n)$$

where

$$C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \le 0\}.$$

Step 4. Update

$$\tau_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A w_n - A y_n, z_n - y_n \rangle}, \tau_n\}, & if\ \langle A w_n - A y_n, z_n - y_n \rangle > 0, \\ \tau_n, & otherwise, \end{cases}$$

where $z_n = P_{C_n}(w_n - \tau_n A y_n)$. Again set $n := n + 1$ and go to Step 1.

It is worth pointing out that Lemmas 3.2, 3.3 and 3.4 are still valid for Algorithm 3.7.

Theorem 3.8. *Suppose that $\{x_n\}$ is the constructed sequence in Algorithm 3.7, such that $S^n x_n - S^{n+1}x_n \to 0$. Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle(\rho F - f)x^*, p - x^*\rangle \geq 0,\ \forall p \in \Omega$.*

Proof. Noticing $\lim_{n\to\infty}\frac{\theta_n}{\beta_n} = 0$ and $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$, we might assume that $\theta_n \leq \frac{\beta_n(\ell-\delta)}{2}, \forall n \geq 1$ and $\{\gamma_n\} \subset [a,b] \subset (0,1)$. Using the same arguments as in the proof of Theorem 3.5, we obtain that there exists the unique solution $x^* \in \Omega = \bigcap_{l=0}^N \mathrm{Fix}(S_l) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C,A)$ to the VIP (17).

Next we show the conclusion of the theorem. To the aim, we divide the remainder of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, utilizing the same arguments as in Step 1 of the proof of Theorem 3.5, one obtains that inequalities (18)-(20) hold. So, from $\beta_n + \gamma_n < 1$, Lemma 2.9 and (20) it follows that

$$\begin{aligned}\|z_n - x^*\| &\leq \beta_n\delta\|x_n - x^*\| + \gamma_n\|u_n - x^*\| + (1-\gamma_n-\beta_n\ell)\|x_n - x^*\| + \beta_n\|(f-\rho F)x^*\| \\ &\leq \beta_n\delta(\|x_n - x^*\| + \beta_n K_1) + \gamma_n(\|x_n - x^*\| + \beta_n K_1) + (1-\gamma_n-\beta_n\ell) \\ &\quad\times(\|x_n - x^*\| + \beta_n K_1) + \beta_n\|(f-\rho F)x^*\| \\ &\leq \|x_n - x^*\| + \beta_n(K_1 + \|(f-\rho F)x^*\|).\end{aligned}$$

Owing to $x_{n+1} = (1-\lambda_n)w_n + \lambda_n S^n z_n$, from the last inequality one gets

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \|w_n - x^*\| + \theta_n\|w_n - x^*\| \\ &\leq [1 - \frac{\beta_n(\ell-\delta)}{2}]\|x_n - x^*\| + \frac{\beta_n(\ell-\delta)}{2}\cdot\frac{3(K_1 + \|(f-\rho F)x^*\|)}{\ell-\delta} \\ &\leq \max\{\|x_n - x^*\|, \frac{3(K_1 + \|(f-\rho F)x^*\|)}{\ell-\delta}\}.\end{aligned}$$

By induction, we obtain $\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, \frac{3(K_1+\|(f-\rho F)x^*\|)}{\ell-\delta}\}, \forall n \geq 1$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{q_n\}, \{u_n\}, \{w_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{S^n z_n\}$.

Step 2. We show that

$$(\theta_n + \beta_n)K_4 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \geq \lambda_n(1 - \mu\frac{\tau_n}{\tau_{n+1}})(\|z_n - y_n\|^2 + \|y_n - w_n\|^2)$$

for some $K_4 > 0$. Utilizing the similar arguments to those of (22), one gets

$$\beta_n K_2 + (1-\beta_n\ell-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 + \beta_n\delta\|x_n - x^*\|^2 \geq \|w_n - x^*\|^2, \tag{41}$$

where $\sup_{n\geq 1} 2\|(f-\rho F)x^*\|\|w_n - x^*\| \leq K_2$ for some $K_2 > 0$. Using the same arguments as those of (23) and (24), we have

$$\theta_n(2+\theta_n)\|z_n - x^*\|^2 - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) + \|w_n - x^*\|^2 \geq \|x_{n+1} - x^*\|^2, \tag{42}$$

and

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \beta_n K_3, \tag{43}$$

where $\sup_{n\geq 1}(2K_1\|x_n - x^*\| + \beta_n K_1^2) \leq K_3$ for some $K_3 > 0$. From (41)-(43), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\delta\|x_n - x^*\|^2 + \gamma_n(\|x_n - x^*\|^2 + \beta_n K_3) + (1-\beta_n\ell-\gamma_n)\|x_n - x^*\|^2 + \beta_n K_2 \\ &\quad - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\leq [1-\beta_n(\ell-\delta)]\|x_n - x^*\|^2 - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) \\ &\quad + \beta_n K_3 + \beta_n K_2 + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) + (\theta_n + \beta_n)K_4,\end{aligned}$$

where $\sup_{n\geq 1}[K_2 + K_3 + (2+\theta_n)\|z_n - x^*\|^2] \leq K_4$. This immediately implies that

$$\lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (\theta_n + \beta_n)K_4. \tag{44}$$

Step 3. We show that

$$\beta_n(\ell-\delta)[\frac{3K}{\ell-\delta}(\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| + \frac{\theta_n}{3\beta_n}) + \frac{2\langle (f-\rho F)x^*, w_n - x^*\rangle}{\ell-\delta}] + [1-\beta_n(\ell-\delta)]\|x_n - x^*\|^2 \geq \|x_{n+1} - x^*\|^2$$

for some $K > 0$. In fact, one has

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \alpha_n\|x_n - x_{n-1}\|[2\|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\|]. \tag{45}$$

Combining (18), (41) and (45), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\delta\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 + (1-\beta_n\ell - \gamma_n)\|x_n - x^*\|^2 + 2\beta_n\langle (f-\rho F)x^*, w_n - x^*\rangle \\ &\quad + \theta_n(2+\theta_n)\|z_n - x^*\|^2 \\ &\leq [1-\beta_n(\ell-\delta)]\|x_n - x^*\|^2 + \beta_n(\ell-\delta)[\frac{3K}{\ell-\delta}(\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| + \frac{\theta_n}{3\beta_n}) + \frac{2\langle (f-\rho F)x^*, w_n - x^*\rangle}{\ell-\delta}],\end{aligned} \tag{46}$$

where $\sup_{n\geq 1}\{\|x_n - x^*\|, \alpha_n\|x_n - x_{n-1}\|, (2+\theta_n)\|z_n - x^*\|^2\} \leq K$ for some $K > 0$.

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (17). In fact, utilizing the same arguments as in Step 4 of the proof of Theorem 3.5, we obtain the desired assertion. □

Theorem 3.9. *Suppose that $S: H \to H$ is nonexpansive and $\{x_n\}$ is the sequence constructed by the modified version of Algorithm 3.7, i.e., for any initial $x_0, x_1 \in H$,*

$$\begin{cases} u_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1}), \\ v_n = P_C(x_n - \mu_2 B_2 x_n), \\ q_n = P_C(v_n - \mu_1 B_1 v_n), \\ w_n = \beta_n f(x_n) + \gamma_n u_n + ((1-\gamma_n)I - \beta_n\rho F)q_n, \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = (1-\lambda_n)w_n + \lambda_n S z_n, \forall n \geq 1, \end{cases}$$

and α_n, τ_n and C_n are chosen as in Algorithm 3.7. Then $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle (\rho F - f)x^*, p - x^*\rangle \geq 0, \forall p \in \Omega$.

Proof. We divide the proof of the theorem into several steps.

Step 1. We show that $\{x_n\}$ is bounded. In fact, utilizing the same arguments as in Step 1 of the proof of Theorem 3.6, we obtain the desired assertion.

Step 2. We show that

$$\theta_n k_4 - \|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2 \geq \lambda_n(1-\mu\frac{\tau_n}{\tau_{n+1}})(\|w_n - y_n\|^2 + \|y_n - z_n\|^2)$$

for some $K_4 > 0$. In fact, utilizing the same arguments as in Step 2 of the proof of Theorem 3.6, we obtain the desired assertion.

Step 3. We show that

$$\beta_n(\ell-\delta)[\frac{3K}{\ell-\delta}\cdot\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| + \frac{2\langle (f-\rho F)x^*, w_n - x^*\rangle}{\ell-\delta}] + [1-\beta_n(\ell-\delta)]\|x_n - x^*\|^2 \geq \|x_{n+1} - x^*\|^2$$

for some $K > 0$. In fact, utilizing the same arguments as in Step 3 of the proof of Theorem 3.6, we obtain the desired assertion.

Step 4. We show that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (17), with $S_0 = S$ a nonexpansive mapping. In fact, utilizing the same arguments as in Step 4 of the proof of Theorem 3.6, we obtain the desired assertion. This completes the proof. □

4. Examples

In this section, our main results are exploited to solve the VIP, GSVI and CFPP in an illustrated example. Put $\tau_1 = \alpha = \mu = \frac{1}{2}$, $\lambda_n = \frac{1}{3}$, $\beta_n = \frac{1}{3(n+1)}$, $\gamma_n = \frac{n}{3(n+1)}$ and $\epsilon_n = \frac{1}{3(n+1)^2}$.

We next provide an example of Lipschitz continuous and pseudomonotone mapping A, two inverse-strongly monotone mappings $B_i, i = 1,2$, asymptotically nonexpansive mapping S, and nonexpansive mapping S_1 satisfying $\Omega = \bigcap_{i=0}^{1} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C,A) \neq \emptyset$ with $S_0 := S$.

Let $C = [-1,1]$ and $H = \mathbf{R}$ with the inner product $\langle a,b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The initial points x_0, x_1 are randomly chosen in H. Take $\rho = 2$ and $f(x) = F(x) = \frac{1}{2}x, \forall x \in H$. Then $\delta = \kappa = \eta = \frac{1}{2}$, $\rho = 2 \in (0, \frac{2\eta}{\kappa^2}) = (0,4)$ and $\delta = \frac{1}{2} < \ell = 1 - \sqrt{1-\rho(2\eta - \rho\kappa^2)} = 1$.

For $i = 1,2$, let $A, B_i : H \to H$, and $S, S_1 : H \to H$ be defined as, for all $x \in H$,

$$\begin{cases} Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}, \\ B_i x := x - \frac{1}{2}\sin x, \\ Sx := \frac{3}{4}\sin x, \\ S_1 x := \sin x. \end{cases}$$

Let us show that A is pseudomonotone and Lipschitz continuous. In fact, for all $x, y \in H$ we have

$$\begin{aligned} \|Ax - Ay\| &\le \frac{|\|y\| - \|x\||}{(1+\|x\|)(1+\|y\|)} + \frac{|\|\sin y\| - \|\sin x\||}{(1+\|\sin x\|)(1+\|\sin y\|)} \\ &\le \frac{\|y-x\|}{(1+\|x\|)(1+\|y\|)} + \frac{\|\sin y - \sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)} \\ &\le \|x-y\| + \|\sin x - \sin y\| \le 2\|x-y\|. \end{aligned}$$

This ensures that A is of Lipschitz continuity on H. Also, we claim that A is pseudomonotone. Actually, it is easy to check that for all $x, y \in H$,

$$\langle Ax, y-x\rangle = (\frac{1}{1+|\sin x|} - \frac{1}{1+|x|})(y-x) \ge 0 \Rightarrow \langle Ay, y-x\rangle = (\frac{1}{1+|\sin y|} - \frac{1}{1+|y|})(y-x) \ge 0.$$

In the meantime, for $i = 1,2$, B_i is $\frac{2}{9}$-inverse-strongly monotone with $\alpha = \beta = \frac{2}{9}$, since for all $x, y \in H$ we deduce that $\|B_i x - B_i y\| \le \frac{3}{2}\|x-y\|$ and

$$\langle B_i x - B_i y, x-y\rangle = \|x-y\|^2 - \frac{1}{2}\langle \sin x - \sin y, x-y\rangle \ge \frac{1}{2}\|x-y\|^2.$$

It is clear that $G(0) = P_C(I - \frac{2}{9}B_1)P_C(I - \frac{2}{9}B_2)0 = P_C(I - \frac{2}{9}B_1)0 = 0$, and hence $0 \in \mathrm{Fix}(G)$. Moreover, it is easy to verify that S is asymptotically nonexpansive with $\theta_n = (\frac{3}{4})^n, \forall n \ge 1$, such that $\|S^{n+1}x_n - S^n x_n\| \to 0$ as $n \to \infty$. In fact, note that

$$\|S^n x - S^n y\| \le \frac{3}{4}\|S^{n-1}x - S^{n-1}y\| \le \cdots \le (\frac{3}{4})^n\|x-y\| \le (1+\theta_n)\|x-y\|,$$

and

$$\|S^{n+1}x_n - S^n x_n\| \le (\frac{3}{4})^{n-1}\|S^2 x_n - Sx_n\| = (\frac{3}{4})^{n-1}\|\frac{3}{4}\sin(Sx_n) - \frac{3}{4}\sin x_n\| \le 2(\frac{3}{4})^n \to 0.$$

It is clear that $\mathrm{Fix}(S) = \{0\}$ and

$$\lim_{n\to\infty} \frac{\theta_n}{\beta_n} = \lim_{n\to\infty} \frac{(3/4)^n}{1/3(n+1)} = 0.$$

In addition, it is clear that S_1 is nonexpansive and $\mathrm{Fix}(S_1) = \{0\}$. Therefore, $\Omega = \bigcap_{i=0}^{1} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \{0\} \neq \emptyset$. In this case, Algorithm 3.1 can be rewritten below:

$$\begin{cases} u_n = S_1 x_n + \alpha_n (S_1 x_n - S_1 x_{n-1}), \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ q_n = P_C(v_n - \mu_1 B_1 v_n), \\ w_n = \frac{1}{3(n+1)} \cdot \frac{1}{2} x_n + \frac{n}{3(n+1)} x_n + \frac{2}{3} q_n, \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{2}{3} w_n + \frac{1}{3} S^n z_n, \ \forall n \geq 1 \end{cases} \tag{47}$$

and α_n, τ_n and C_n are chosen as in Algorithm 3.1. Then, by Theorem 3.5, we know that $\{x_n\}$ converges to $0 \in \Omega = \bigcap_{i=0}^{1} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$.

In particular, since $Sx := \frac{3}{4} \sin x$ is also nonexpansive, we consider the modified version of Algorithm 3.1, that is,

$$\begin{cases} u_n = x_n + \alpha_n (x_n - x_{n-1}), \\ v_n = P_C(u_n - \mu_2 B_2 u_n), \\ q_n = P_C(v_n - \mu_1 B_1 v_n), \\ w_n = \frac{1}{3(n+1)} \cdot \frac{1}{2} x_n + \frac{n}{3(n+1)} x_n + \frac{2}{3} q_n, \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{2}{3} w_n + \frac{1}{3} S z_n, \ \forall n \geq 1 \end{cases} \tag{48}$$

and α_n, τ_n and C_n are chosen as above. Then, by Theorem 3.6, we know that $\{x_n\}$ converges to $0 \in \Omega = \bigcap_{i=0}^{1} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$.

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