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Collectively fixed points and maximal elements for compact and coercive type maps in topological vector spaces

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Abstract. We present some new collectively fixed point theory for general classes of maps of compact, condensing or coercive type defined on topological vector spaces. As an application we present some new equilibrium results for generalized games.

1. Introduction

In this paper we use results of the author [7] to establish some new collectively fixed point results for very general classes of maps, namely the maps in [1, 2, 3, 4]. The maps considered will be of compact, condensing or coercive type defined on Hausdorff topological vector spaces. These new collectively fixed point theorems will then be rephrased as maximal element type results and will generate existence results for majorized type maps. As an application we present some equilibrium results for generalized games (or abstract economies) and our theory generalizes and complements the theory in the literature; see [4, 5, 6, 7, 8, 9, 10] and the references therein.

First, we define the classes of maps considered in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and let F be a multifunction. We say $F \in HLPY(Z, W)$ [3, 4] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{ int\, S^{-1}(w) : \ w \in W\}$; here $S^{-1}(w) = \{z \in Z : \ w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ [2] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Now we present the results of the author [7] used in this paper. Throughout this paper I will denote an index set.

Theorem 1.1. *Let $\{X_i\}_{i \in I}$ be a family of convex compact sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i \in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also for each $x \in X$ suppose there exists a $j \in I$ with $S_j(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$ (here x_i is the projection of x on X_i).*

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Theorem 1.1 will generate results for compact, condensing or coercive type maps.

Theorem 1.2. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also for each $x \in X$ suppose there exists a $j \in I$ with $S_j(x) \neq \emptyset$. Next assume for each $i \in I$ that there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$ (in fact $x \in K \equiv \prod_{i\in} K_i$).*

Theorem 1.3. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for each $x \in X$ there exists a $j \in I$ with $S_j(x) \neq \emptyset$. Next assume there exists a convex compact set K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Theorem 1.4. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and in addition there exists a map $S_i : X \to X_i$ with $S_i(x) \subseteq F_i(x)$ for $x \in X$, $S_i(x)$ has convex values for $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also for each $x \in X$ suppose there exists a $j \in I$ with $S_j(x) \neq \emptyset$. Next assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

2. Collectively fixed point results

Using the results in Section 1 we establish some collectively fixed point theorems for new classes of maps. We begin with a simple result motivated from the *DKT* maps [2] in the literature. Our first result improves Theorem 1.1.

Theorem 2.1. *Let $\{X_i\}_{i\in I}$ be a family of convex compact sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also for each $x \in X$ suppose there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Proof. For $i \in I$ let $S_i(x) = co\,(T_i(x))$ for $x \in X$. For $i \in I$, first note $S_i(x)$ has convex values for each $x \in X$ and note $S_i(y) \subseteq F_i(y)$ for $y \in X$. In addition for $i \in I$ from [8, Lemma 5.1] we have that $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Finally note if $x \in X$ then there exists a $i_0 \in I$ with $T_{i_0}(x) \neq \emptyset$ and so $\emptyset \neq T_{i_0}(x) \subseteq co\,(T_{i_0}(x)) = S_{i_0}(x)$. Now Theorem 1.1 guarantees that there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$. □

Our next result is motivated from the *HLPY* maps [1, 3, 4] in the literature and our result improves Theorem 1.1 and Theorem 2.1.

Theorem 2.2. *Let $\{X_i\}_{i\in I}$ be a family of convex compact sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $X = \bigcup_{i\in I} \bigcup \{\,int\,T_i^{-1}(w) :\ w \in X_i\}$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Proof. For $i \in I$ let $R_i : X \to X_i$ be given by

$$R_i(y) = \{z_i :\ y \in int\,T_i^{-1}(z_i)\},\ \ y \in X$$

and let $S_i : X \to X_i$ be given by

$$S_i(x) = co\,(R_i(x)) \ \text{ for } \ x \in X.$$

For $i \in I$ first note that $S_i(x)$ has convex values for each $x \in X$. Next note for $i \in I$ that $R_i(x) \subseteq T_i(x)$ for $x \in X$ since if $z_i \in R_i(x)$ then $x \in int\,T_i^{-1}(z_i) \subseteq T_i^{-1}(z_i) = \{w \in X :\ z_i \in T_i(w)\}$ so $z_i \in T_i(x)$ and putting this together yields $R_i(x) \subseteq T_i(x)$. Thus for $i \in I$ we have $S_i(y) = co\,(R_i(y)) \subseteq co\,(T_i(y)) \subseteq F_i(y)$ for $y \in X$.

Now for $i \in I$ notice for $y_i \in X_i$ that $R_i^{-1}(y_i) = \{z :\ y_i \in R_i(z)\} = int\,T_i^{-1}(y_i)$ so $R_i^{-1}(y_i)$ is open (in X) and so from [8, Lemma 5.1] we have that $S_i^{-1}(y_i)$ is open (in X).

Now let $x \in X$. Since $X = \bigcup_{i\in I} \bigcup \{ int\, T_i^{-1}(w) : \ w \in X_i\}$ there exists a $i_0 \in I$ with $x \in int\, T_{i_0}^{-1}(w)$ for some $w \in X_{i_0}$ and so $w \in R_{i_0}(x)$ i.e. $R_{i_0}(x) \neq \emptyset$ and as a result $\emptyset \neq R_{i_0}(x) \subseteq co\,(R_{i_0}(x)) = S_{i_0}(x)$. Now Theorem 1.1 guarantees that there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$. □

Remark 2.3. *If in Theorem 2.2 we had $X = \bigcup \{ int\, T_i^{-1}(w) : \ w \in X_i\}$ for each $i \in I$, then we have the following in the proof of Theorem 2.2. Let $x \in X$. Then for each $i \in I$ there exists a $w_i \in X_i$ with $x \in int\, T_i^{-1}(w_i)$ so $R_i(x) \neq \emptyset$ and as a result $S_i(x) \neq \emptyset$ for all $i \in I$. Thus $S_i \in HLPY(X, X_i)$ for all $i \in I$. We refer the reader to [4, 5] for this situation.*

Next we obtain the analogue of Theorem 2.1 and Theorem 2.2 if we use Theorem 1.2 instead of Theorem 1.1. In particular we replace compactness of the sets with compactness of the maps. Our next result improves Theorem 1.2

Theorem 2.4. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also for each $x \in X$ suppose there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Next assume for each $i \in I$ that there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Proof. For $i \in I$ let S_i be as in Theorem 2.1 and note S_i has the same properies as in Theorem 2.1 so we apply Theorem 1.2 to obtain the result. □

Our next result improves Theorem 1.2 and Theorem 2.4.

Theorem 2.5. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $X = \bigcup_{i\in I} \bigcup \{ int\, T_i^{-1}(w) : \ w \in X_i\}$. Also assume for each $i \in I$ that there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Proof. For $i \in I$ let R_i and S_i be as in Theorem 2.2 and note S_i has the same properies as in Theorem 2.2 so we apply Theorem 1.2 to obtain the result. □

Next note if we use Theorem 1.3 we have the following results. In particular we consider condensing type maps.

Theorem 2.6. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and in addition there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also for each $x \in X$ suppose there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Finally assume there exists a convex compact set K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Theorem 2.7. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $X = \bigcup_{i\in I} \bigcup \{ int\, T_i^{-1}(w) : \ w \in X_i\}$. Next assume there exists a convex compact set K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Now we present our two results for coercive type maps. Our first result improves Theorem 1.4.

Theorem 2.8. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also for each $x \in X$ suppose there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Next assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $T_j(x) \cap Y_j \neq \emptyset$ (or, alternatively $co\,(T_j(x)) \cap Y_j \neq \emptyset$). Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Proof. For $i \in I$ let S_i be as in Theorem 2.1 and note S_i has the same properies as in Theorem 2.1. Let $x \in X\backslash K$. We need only consider the case when there exists a $j \in I$ with $T_j(x) \cap Y_j \neq \emptyset$, but here $\emptyset \neq T_j(x) \cap Y_j \subseteq co\,(T_j(x)) \cap Y_j = S_j(x) \cap Y_j$. Now apply Theorem 1.4. □

Our next result improves Theorem 1.5 and Theorem 2.8.

Theorem 2.9. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $X = \bigcup_{i\in I} \bigcup \{\, int\, T_i^{-1}(w) :\ w \in X_i\}$. Next assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $R_j(x) \cap Y_j \neq \emptyset$ (or, alternatively $co\,(R_j(x)) \cap Y_j \neq \emptyset$) where $R_j : X \to X_j$ is given by $R_j(y) = \{z_j :\ y \in int\, T_j^{-1}(z_j)\}$ for $y \in X$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

Proof. For $i \in I$ let S_i be as in Theorem 2.2 and note S_i has the same properies as in Theorem 2.2. Let $x \in X\backslash K$. We need only consider the case when there exists a $j \in I$ with $R_j(x) \cap Y_j \neq \emptyset$, but here $\emptyset \neq R_j(x) \cap Y_j \subseteq co\,(R_j(x)) \cap Y_j = S_j(x) \cap Y_j$. Now apply Theorem 1.4. □

Next we rephrase Theorem 2.1 as a maximal element type result, and then we obtain an existence result for generalized majorized type maps.

Theorem 2.10. *Let $\{X_i\}_{i\in I}$ be a family of convex compact sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for all $i \in I$ that $x_i \notin F_i(x)$ for each $x \in X$. Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Now Theorem 2.1 guarantees a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$, a contradiction. □

Theorem 2.11. *Let $\{X_i\}_{i\in I}$ be a family of convex compact sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $H_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $H_i(w) \subseteq S_i(w)$ for $w \in X$, $S_i(x)$ has convex values for each $x \in X$, $S_i^{-1}(z)$ is open (in X) for each $z \in X_i$ and $w_i \notin S_i(w)$ for each $w \in X$. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in I$*

Proof. Note since S_i has convex values then $S_i(x) = co\,(S_i(x))$ for $x \in X$. Apply Theorem 2.10 with $F_i = T_i = S_i$ (note $co\,T_i = co\,S_i = S_i = F_i$) so there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in I$. Now since $H_j(w) \subseteq S_j(w)$ for $w \in X$ then $H_j(x) = \emptyset$ for all $j \in I$. □

Now we consider Theorem 2.2 and we will reprase it as a maximal element type result, and then we obtain an existence result for generalized majorized type maps.

Theorem 2.12. *Let $\{X_i\}_{i\in I}$ be a family of convex compact sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and*

$$\bigcup_{i\in I} \{z \in X :\ T_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{\, int\, T_i^{-1}(w) :\ w \in X_i\}.$$

Now suppose for all $i \in I$ that $x_i \notin F_i(x)$ for each $x \in X$. Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$.

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Thus

$$X = \bigcup_{i\in I} \{z \in X :\ T_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{\, int\, T_i^{-1}(w) :\ w \in X_i\}.$$

Now Theorem 2.2 guarantees a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$, a contradiction. □

Theorem 2.13. *Let $\{X_i\}_{i\in I}$ be a family of convex compact sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $H_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $H_i(w) \subseteq S_i(w)$ for $w \in X$, $S_i(x)$ has convex values for each $x \in X$, and*

$$\bigcup_{i\in I} \{z \in X :\ S_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{\, int\, S_i^{-1}(w) :\ w \in X_i\}.$$

Also suppose for all $i \in I$ that $x_i \notin S_i(x)$ for each $x \in X$. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in I$.

Proof. Apply Theorem 2.12 with $F_i = T_i = S_i$ (note $co\, T_i = co\, S_i = S_i = F_i$) so there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in I$ and since $H_j(w) \subseteq S_j(w)$ for $w \in X$ then $H_j(x) = \emptyset$ for all $j \in I$. □

The argument above together with Theorem 2.4, Theorem 2.5, Theorem 2.6, and Theorem 2.7 immediately yield the following results. We will consider both maximal element type results and existence results for generalized majorized type maps.

Theorem 2.14. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for all $i \in I$ that $x_i \notin F_i(x)$ for each $x \in X$. Finally assume either (a). for each $i \in I$ that there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$, or (b). there exists a convex compact set K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$, holds. Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Now apply Theorem 2.4 if (a) occurs or Theorem 2.6 if (b) occurs and we have a contradiction. □

Theorem 2.15. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $H_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $H_i(w) \subseteq S_i(w)$ for $w \in X$, $S_i(x)$ has convex values for each $x \in X$, $S_i^{-1}(z)$ is open (in X) for each $z \in X_i$ and $w_i \notin S_i(w)$ for each $w \in X$. Finally assume either (a). for each $i \in I$ that there exists a convex compact set K_i with $S_i(X) \subseteq K_i \subseteq X_i$, or (b). there exists a convex compact set K of X with $S(K) \subseteq K$ where $S(x) = \prod_{i\in I} S_i(x)$ for $x \in X$, holds. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in I$.*

Theorem 2.16. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and*

$$\bigcup_{i\in I} \{z \in X : \ T_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{\, int\, T_i^{-1}(w) : \ w \in X_i\}.$$

Also suppose for all $i \in I$ that $x_i \notin F_i(x)$ for each $x \in X$. Finally assume either (a). for each $i \in I$ that there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$, or (b). there exists a convex compact set K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$, holds. Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$.

Theorem 2.17. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $H_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $H_i(w) \subseteq S_i(w)$ for $w \in X$, $S_i(x)$ has convex values for each $x \in X$, and*

$$\bigcup_{i\in I} \{z \in X : \ S_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{\, int\, S_i^{-1}(w) : \ w \in X_i\}.$$

Also suppose for all $i \in I$ that $x_i \notin S_i(x)$ for each $x \in X$. Finally assume either (a). for each $i \in I$ that there exists a convex compact set K_i with $S_i(X) \subseteq K_i \subseteq X_i$, or (b). there exists a convex compact set K of X with $S(K) \subseteq K$ where $S(x) = \prod_{i\in I} S_i(x)$ for $x \in X$, holds. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in I$.

Next we use our coercive type results (Theorem 2.8 and Theorem 2.9) to immediately rephrase as a maximal element type result and an existence result for general majorized type maps.

Theorem 2.18. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for all $i \in I$ that $x_i \notin F_i(x)$ for each $x \in X$. Finally assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $T_j(x) \cap Y_j \neq \emptyset$ (or, alternatively $co\,(T_j(x)) \cap Y_j \neq \emptyset$). Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$.*

Theorem 2.19. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $H_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $H_i(w) \subseteq S_i(w)$ for $w \in X$, $S_i(x)$ has convex values for each $x \in X$, $S_i^{-1}(z)$ is open (in X) for each $z \in X_i$ and $w_i \notin S_i(w)$ for each $w \in X$. Finally assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$ (or, alternatively $H_j(x) \cap Y_j \neq \emptyset$). Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in I$.*

Proof. Apply Theorem 2.18 with $F_i = T_i = S_i$ (note $co\,T_i = co\,S_i = S_i = F_i = T_i$) so there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in I$ and since $H_j(w) \subseteq S_j(w)$ for $w \in X$ then $H_j(x) = \emptyset$ for all $j \in I$. □

Remark 2.20. *We note here, that in fact, Theorem 2.11, Theorem 2.13, and Theorem 2.19 were proved in [7].*

Theorem 2.21. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and*

$$\bigcup_{i\in I} \{z \in X : \ T_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{\, int\, T_i^{-1}(w) : \ w \in X_i\}.$$

Also suppose for all $i \in I$ that $x_i \notin F_i(x)$ for each $x \in X$. Finally assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $R_j(x) \cap Y_j \neq \emptyset$ (or, alternatively $co\,(R_j(x)) \cap Y_j \neq \emptyset$) where $R_j : X \to X_j$ is given by $R_j(y) = \{z_j : \ y \in int\, T_j^{-1}(z_j)\}$ for $y \in X$. Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$.

Theorem 2.22. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each lying in a Hausdorff topological vector space. For each $i \in I$ suppose $H_i : X \equiv \prod_{i\in I} X_i \to X_i$ and there exists a map $S_i : X \to X_i$ with $H_i(w) \subseteq S_i(w)$ for $w \in X$, $S_i(x)$ has convex values for each $x \in X$, and*

$$\bigcup_{i\in I} \{z \in X : \ S_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{\, int\, S_i^{-1}(w) : \ w \in X_i\}.$$

Also suppose for all $i \in I$ that $x_i \notin S_i(x)$ for each $x \in X$. Finally assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $\theta_j(x) \cap Y_j \neq \emptyset$ where $\theta_j : X \to X_j$ is given by $\theta_j(y) = \{z_j : \ y \in int\, S_j^{-1}(z_j)\}$ for $y \in X$. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in I$.

Now we use our maximal element results to establish some equilibrium results for generalized games (or abstract economies). A generalized game is given by $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ where I is a set of players (agents), X_i is a nonempty subset of a Hausdorff topological vector space E_i, A_i, $B_i : X \equiv \prod_{i\in I} X_i \to E_i$ are constraint correspondences and $P_i : X \to E_i$ is a preference correspondence. An equilibrium of Γ is a point $x \in X$ such that for each $i \in I$ we have $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$.

In our first result, we can use either Theorem 2.10, Theorem 2.14 or Theorem 2.18, but since the analysis is essentially the same using either of these theorems we will just state and prove the result using Theorem 2.18.

Theorem 2.23. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be a generalized game, i.e. $\{X_i\}_{i\in I}$ is a family of convex sets each lying in a Hausdorff topological vector space E_i and for each $i \in I$ the constraint correspondences A_i, $B_i : X \equiv \prod_{i\in I} X_i \to E_i$ and the preference correspondence $P_i : X \to E_i$. Also for each $i \in I$ suppose $cl\,B_i\,(\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous (here $CK(X_i)$ denotes the family of nonempty convex compact subsets of X_i) and assume the following conditions hold for each $i \in I$:*

$$\begin{cases} A_i : X \to X_i \ \ \textit{has nonempty convex values and} \\ A_i^{-1}(x) \ \ \textit{is open (in X) for each} \ \ x \in X_i \end{cases} \tag{2.1}$$

$$A_i(x) \subseteq \overline{B_i}(x) \ \ \textit{for} \ \ x \in X \tag{2.2}$$

$$\begin{cases} \textit{there exist maps} \ \ S_i,\ \phi_i : X \to X_i \ \ \textit{with} \ \ co\,(S_i(z)) \subseteq \phi_i(z) \ \ \textit{for} \ z \in X, \\ S_i^{-1}(z) \ \ \textit{is open (in X) for each} \ \ z \in X_i \ \ \textit{and} \ \ x_i \notin \phi_i(x) \ \ \textit{for} \ \ x \in X \end{cases} \tag{2.3}$$

and

$$S_i(x) \subseteq A_i(x) \ \ \textit{for} \ \ x \in X. \tag{2.4}$$

In addition, suppose

$$\text{(2.5)} \qquad \begin{cases} \text{if } x \in X \text{ with } x_i \in \overline{B_i}(x) \text{ and } S_i(x) = \emptyset \\ \text{for a } i \in I, \text{ then } A_i(x) \cap P_i(x) = \emptyset. \end{cases}$$

Next suppose there exists a compact subset K of X and for each $i \in I$ a convex compact set Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$. Then there exists an equilibrium point of Γ i.e. there exists a $x \in X$ with $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in I$.

Remark 2.24. *(i). Note a particular case of ϕ_i in* (2.3) *is $\phi_i = A_i \cap P_i$ if the appropriate assumptions are satisfied.*

(ii). One could write (2.5) *as: if $x \in X$ and $x_i \in \overline{B_i}(x)$ and $S_i(x) = \emptyset$ for all $i \in I$, then $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in I$.*

Proof. For $i \in I$ let $M_i = \{x \in X : \ x \notin \overline{B_i}(x)\}$ and note M_i is open in X since $\overline{B_i} : X \to CK(X_i)$ is upper semicontinuous. Let $F_i : X \to X_i$ and $T_i : X \to X_i$ be given by

$$F_i(x) = \begin{cases} \phi_i(x), & x \notin M_i \\ A_i(x), & x \in M_i \end{cases} \quad \text{and} \quad T_i(x) = \begin{cases} S_i(x), & x \notin M_i \\ A_i(x), & x \in M_i. \end{cases}$$

Note if $i \in I$ then $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ (note if $x \in M_i$ then $co\,(T_i(x)) = co\,(A_i(x)) = A_i(x) = F_i(x)$ from (2.1) whereas if $x \notin M_i$ then $co\,(T_i(x)) = co\,(S_i(x)) \subseteq \phi_i(x) = F_i(x)$ from (2.3)) and also note for $y \in X_i$ we have

$$\begin{aligned} T_i^{-1}(y) &= \{z \in M_i : \ y \in T_i(z)\} = \{z \in M_i : \ y \in A_i(z)\} \cup \{z \in X \backslash M_i : \ y \in S_i(z)\} \\ &= [M_i \cap \{z \in X : \ y \in A_i(z)\}] \cup [(X \backslash M_i) \cap \{z \in X : \ y \in S_i(z)\}] \\ &= \left[M_i \cap A_i^{-1}(y)\right] \cup \left[(X \backslash M_i) \cap S_i^{-1}(y)\right] \\ &= \left[M_i \cup S_i^{-1}(y)\right] \cap A_i^{-1}(y) \end{aligned}$$

(note, see (2.4), $S_i^{-1}(y) \subseteq A_i^{-1}(y)$) which is open in X. Next, we show for $i \in I$ that $x_i \notin F_i(x)$ for $x \in X$. To see this fix $i \in I$ and $x \in X$. First consider $x \in M_i$ and then $x_i \notin \overline{B_i}(x)$ so $x_i \notin A_i(x)$ from (2.2), i.e. $x_i \notin F_i(x)$ if $x \in M_i$. Next, suppose $x \notin M_i$ and then $x_i \notin \phi_i(x) = F_i(x)$ from (2.3). Consequently $x_i \notin F_i(x)$ for $x \in X$ and $i \in I$. Now let K and Y_i be as in the statement of Theorem 2.23. If $x \in X \backslash K$ then there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$, so if $x \in X \backslash K$ and $x \notin M_j$ then $T_j(x) \cap Y_j = S_j(x) \cap Y_j \neq \emptyset$ whereas if $x \in X \backslash K$ and $x \in M_j$ then $\emptyset \neq S_j(x) \cap Y_j \subseteq A_j(x) \cap Y_j = T_j(x) \cap Y_j$ from (2.4).

Then all the conditions in Theorem 2.18 are satisfied so there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$. Now since A_i has nonempty values for each $i \in I$, then for each $i \in I$ we have $x \notin M_i$ so $x \notin M_i$ with $T_i(x) = \emptyset$ for all $i \in I$ i.e. $x_i \in \overline{B_i}(x)$ and $S_i(x)(= T_i(x)) = \emptyset$ for $i \in I$. Now from (2.5) we have $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in I$, so x is an equilibrium point of Γ. □

In our next result we can use either Theorem 2.12, Theorem 2.16 or Theorem 2.21 but since the analysis is essentially the same using either of these theorems we will just state and prove the result using Theorem 2.16.

Theorem 2.25. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be a generalized game (as described in Theorem 2.23) and for each $i \in I$ suppose $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous. Also for $i \in I$ assume* (2.1) *and* (2.2) *hold and in addition assume*

$$\text{(2.6)} \qquad \begin{cases} \text{there exist maps } S_i,\ \phi_i : X \to X_i \text{ with } co\,(S_i(z)) \subseteq \phi_i(z) \text{ for } z \in X, \\ \bigcup_{j \in I} \{x \in X : \ S_j(x) \neq \emptyset\} = \bigcup_{i \in I} \bigcup \{ int\, S_i^{-1}(w) : \ w \in X_i\} \\ \text{and } x_i \notin \phi_i(x) \text{ for } x \in X \end{cases}$$

and suppose (2.4) *and* (2.5) *hold. Finally assume there exists a convex compact set K of X with $A(K) \subseteq K$ and $\phi(K) \subseteq K$ where $A(x) = \prod_{i \in I} A_i(x)$ and $\phi(x) = \prod_{i \in I} \phi_i(x)$ for $x \in X$. Then there exists an equilibrium point of Γ.*

Proof. For $i \in I$ let M_i, F_i and T_i be as in Theorem 2.23 and as in Theorem 2.23 we have $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$, $x_i \notin F_i(x)$ for $x \in X$, and for $y \in X_i$ we have

$$T_i^{-1}(y) = \left[M_i \cup S_i^{-1}(y)\right] \cap A_i^{-1}(y).$$

Now, we show $\bigcup_{j\in I}\{x \in X : T_j(x) \neq \emptyset\} = \bigcup_{i\in I} \bigcup\{\,int\,T_i^{-1}(w) : w \in X_i\}$. To see this let $i \in I$ and $x \in X$ with $T_i(x) \neq \emptyset$. First, consider $x \notin M_i$. Then $S_i(x) = T_i(x) \neq \emptyset$ and since $\bigcup_{j\in I}\{x \in X : S_j(x) \neq \emptyset\} = \bigcup_{i\in I} \bigcup\{\,int\,S_i^{-1}(w) : w \in X_i\}$ then there exists a $j \in I$ and a $w \in X_j$ with $x \in int\,S_j^{-1}(w)$. Then we have immediately that $x \in M_j \cup int\,S_j^{-1}(w)$ and also $x \in A_j^{-1}(w)$ (note $int\,S_j^{-1}(z) \subseteq S_j^{-1}(z) \subseteq A_j^{-1}(z)$ for $z \in X$). Thus $x \in [M_j \cup int\,S_j^{-1}(w)] \cap A_j^{-1}(w)$. Now $[M_j \cup int\,S_j^{-1}(w)] \cap A_j^{-1}(w)$ is open in X and

$$[M_j \cup int\,S_j^{-1}(w)] \cap A_j^{-1}(w) \subseteq [M_j \cup S_j^{-1}(w)] \cap A_j^{-1}(w) = T_j^{-1}(w)$$

so

$$x \in [M_j \cup int\,S_j^{-1}(w)] \cap A_j^{-1}(w) \subseteq int\,T_j^{-1}(w)$$

i.e. $x \in \bigcup_{i\in I} \bigcup\{\,int\,T_i^{-1}(w) : w \in X_i\}$. Next, consider $x \in M_i$. Then $T_i(x) = A_i(x) \neq \emptyset$ so there exists a $y \in X_i$ with $y \in A_i(x)$ so $x \in A_i^{-1}(y)$. Now $x \in M_i$ so $x \in M_i \cup int\,S_i^{-1}(y)$ and thus $x \in [M_i \cup int\,S_i^{-1}(y)] \cap A_i^{-1}(y)$ and the argument above gives

$$x \in [M_i \cup int\,S_i^{-1}(y)] \cap A_i^{-1}(y) \subseteq int\,T_i^{-1}(y)$$

i.e. $x \in \bigcup_{i\in I} \bigcup\{\,int\,T_i^{-1}(w) : w \in X_i\}$. As a result

$$\bigcup_{i\in I}\{z \in X : T_i(z) \neq \emptyset\} = \bigcup_{i\in I} \bigcup\{\,int\,T_i^{-1}(w) : w \in X_i\}.$$

Finally note $F(K) \subseteq K$ (here $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$) since $A(K) \subseteq K$ and $\phi(K) \subseteq K$. Thus all the conditions in Theorem 2.16 are satisfied so there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in I$. Now since A_i has nonempty values for each $i \in I$, then for each $i \in I$ we have $x \notin M_i$ so $x \notin M_i$ with $T_i(x) = \emptyset$ for all $i \in I$ i.e. $x_i \in \overline{B_i}(x)$ and $S_i(x)(= T_i(x)) = \emptyset$ for $i \in I$. Now (2.5) guarantees that x is an equilibrium point of Γ. □

We can also use our results for majorized type maps to obtain some equilibrium points of Γ. We will consider Theorem 2.17 and Theorem 2.19 to illustrate the ideas (the argument using the other theorems is essentially the same).

Theorem 2.26. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be a generalized game (as described in Theorem 2.23) and for each $i \in I$ suppose $cl\,B_i\,(\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous. Also for $i \in I$ assume* (2.1) *and* (2.2) *hold and in addition assume*

$$(2.7)\qquad \begin{cases} \text{there exist maps } S_i,\ \phi_i : X \to X_i \text{ with } \phi_i(z)) \subseteq S_i(z) \\ \text{for } z \in X,\ S_i(x) \text{ is convex valued for each } x \in X, \\ \bigcup_{j\in I}\{x \in X : S_j(x) \neq \emptyset\} = \bigcup_{i\in I} \bigcup\{\,int\,S_i^{-1}(w) : w \in X_i\} \\ \text{and } x_i \notin S_i(x) \text{ for } x \in X \end{cases}$$

and suppose (2.4) *holds. In addition, assume*

$$(2.8)\qquad \begin{cases} \text{if } x \in X \text{ with } x_i \in \overline{B_i}(x) \text{ and } \phi_i(x) = \emptyset \\ \text{for a } i \in I, \text{ then } A_i(x) \cap P_i(x) = \emptyset. \end{cases}$$

Next assume there exists a convex compact set K of X with $A(K) \subseteq K$ and $S(K) \subseteq K$ where $A(x) = \prod_{i\in I} A_i(x)$ and $S(x) = \prod_{i\in I} S_i(x)$ for $x \in X$. Then there exists an equilibrium point of Γ.

Proof. For $i \in I$ let M_i, F_i and T_i be as in Theorem 2.23 and note from (2.7) that $F_i(x) \subseteq T_i(x)$ for $x \in X$, and $T_i(x)$ is convex valued for each $x \in X$ (note $S_i(x)$ and $A_i(x)$ are convex valued for each $x \in X$). For $i \in I$ and $y \in X_i$ we have (see Theorem 2.23)

$$T_i^{-1}(y) = \left[M_i \cup S_i^{-1}(y)\right] \cap A_i^{-1}(y)$$

and also the argument in Theorem 2.25 gives

$$\bigcup_{j\in I} \{x \in X : T_j(x) \neq \emptyset\} = \bigcup_{i\in I} \bigcup \{ int\, T_i^{-1}(w) : w \in X_i\}.$$

Next, note for $i \in I$ that $x_i \notin T_i(x)$ for $x \in X$. To see this fix $i \in I$ and $x \in X$. First consider $x \in M_i$ and then $x_i \notin \overline{B_i}(x)$ so $x_i \notin A_i(x)$ from (2.2) i.e. $x_i \notin T_i(x)$ if $x \in M_i$. Next, suppose $x \notin M_i$ and then $x_i \notin S_i(x) = T_i(x)$ from (2.7). Consequently $x_i \notin T_i(x)$ for $x \in X$ and $i \in I$. Finally note $T(K) \subseteq K$ (here $T(x) = \prod_{i\in I} T_i(x)$ for $x \in X$) since $A(K) \subseteq K$ and $S(K) \subseteq K$. Thus all the conditions in Theorem 2.17 are satisfied so there exists a $x \in X$ with $F_i(x) = \emptyset$ for all $i \in I$. Now since A_i has nonempty values for each $i \in I$, then for each $i \in I$ we have $x \notin M_i$ so $x \notin M_i$ with $\phi_i(x)(= F_i(x)) = \emptyset$ for all $i \in I$. Now (2.8) guarantees that x is an equilibrium point of Γ. □

Theorem 2.27. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be a generalized game (as described in Theorem 2.23) and for each $i \in I$ suppose $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous. Also for $i \in I$ assume* (2.1) *and* (2.2) *hold and in addition assume*

$$(2.9)\qquad \begin{cases} \text{there exist maps } S_i,\ \phi_i : X \to X_i \text{ with } \phi_i(z)) \subseteq S_i(z) \text{ for } z \in X, \\ S_i(x) \text{ is convex valued for each } x \in X,\ S_i^{-1}(z) \text{ is open (in } X) \\ \text{for each } z \in X_i \text{ and } x_i \notin S_i(x) \text{ for } x \in X \end{cases}$$

and suppose (2.4) *and* (2.8) *hold. Next suppose there exists a compact subset K of X and for each $i \in I$ a convex compact set Y_i of X_i such that for each $x \in X\backslash K$ there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$. Then there exists an equilibrium point of* Γ.

Proof. For $i \in I$ let M_i, F_i and T_i be as in Theorem 2.23 and note (see Theorem 2.23 and Theorem 2.26) that $F_i(x) \subseteq T_i(x)$ for $x \in X$, $T_i(x)$ is convex valued for each $x \in X$ and for $y \in X_i$ we have $T_i^{-1}(y) = \left[M_i \cup S_i^{-1}(y)\right] \cap A_i^{-1}(y)$ so from (2.9) we have that $T_i^{-1}(y)$ is open (in X), and $x_i \notin T_i(x)$ for $x \in X$. Now let K and Y_i be as in the statement of Theorem 2.27. If $x \in X\backslash K$ then there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$, so if $x \in X\backslash K$ and $x \notin M_j$ then $T_j(x) \cap Y_j = S_j(x) \cap Y_j \neq \emptyset$ whereas if $x \in X\backslash K$ and $x \in M_j$ then $\emptyset \neq S_j(x) \cap Y_j \subseteq A_j(x) \cap Y_j = T_j(x) \cap Y_j$ from (2.4). Thus all the conditions in Theorem 2.19 are satisfied so there exists a $x \in X$ with $F_i(x) = \emptyset$ for all $i \in I$. Now since A_i has nonempty values for each $i \in I$, then for each $i \in I$ we have $x \notin M_i$ so $x \notin M_i$ with $\phi_i(x)(= F_i(x)) = \emptyset$ for all $i \in I$. Now (2.8) guarantees that x is an equilibrium point of Γ. □

Remark 2.28. *Theorem 2.27 extends and complements results in [6].*

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