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New results and open questions on the theory of convex orbital β-Lipschitz mappings

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Abstract. Based on the impressive feature of convex orbital β-Lipschitz mappings, we are interested in presenting two ways to expand this idea to more generalizations. First, convex orbital β-Lipschitz mappings are extended to quadratic weak orbital β-Lipschitz mappings. Moreover, the new type of decreasing mappings in inner product spaces is presented, and fixed point results for quadratic weak orbital β-Lipschitz mappings in Hilbert spaces are proved with the help of the proposed decreasingness. In a second way, some open question on convex orbital β-Lipschitz mappings is answered in Banach spaces.

1. Introduction and Preliminaries

Let's start by recalling basic notations and definitions in the metrical fixed point theory. First, a fixed point is a point in the domain of the considered self-mapping that is invariant under this self-mapping. Indeed, for a self-mapping T on a nonempty set X, a point $u \in X$ is called a fixed point if $u = Tu$, and the set of fixed points of T is denoted by $\text{Fix}(T)$. In 1922, Banach [2] introduced the most useful metrical fixed point theorem called Banach fixed point theorem as follows:

Theorem 1.1 (Banach fixed point theorem [2]). *Let (X, d) be a complete metric space and T be a self-mapping on X. If T satisfies a Banach contractive condition, i.e., there is $k \in [0, 1)$ such that for every $x, y \in X$, we have*

$$d(Tx, Ty) \leq kd(x, y). \tag{1}$$

Then $|Fix(T)| = 1$. Moreover, for each $x_0 \in X$, the fixed point of T can be approximated by the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

This theorem has many applications to the existence theory in several fields, such as the existence of solutions for differential equations, the existence of solutions for integral equations, the existence and uniqueness of the equilibrium point in a dynamic model, etc. (see in [1, 4, 8, 9, 11, 20] and references therein).

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From the past to present, there are many types of contractive conditions for each self-mapping T on a metric space X. For example, in 1968, Kannan [10] gave the following contractive type: there is a nonnegative real number $c < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq c[d(x, Tx) + d(y, Ty)] \tag{2}$$

for all $x, y \in X$, which is different from the contractive condition (1). Moreover, one of the well-known of contractive conditions called Ćirić-Reich-Rus contractive condition, i.e., there are nonnegative real numbers a, b with $a + 2b < 1$ such that

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] \tag{3}$$

for all $x, y \in X$, was developed in the 1970s (see more details in [5–7, 15–19]). This contractive condition is more generalized than Banach and Kannan types because every Banach contraction mapping or Kannan contraction mapping is also a Ćirić-Reich-Rus contraction mapping.

In 2020, Berinde and Păcurar [3] introduced the new generalization of a Banach contraction mapping named an enriched contraction mapping, which is stated as follows:

Definition 1.2 ([3]). *A self-mapping T on a normed space $(X, \|\cdot\|)$ is said to be an enriched contraction if there are $b \geq 0$ and $\theta \in [0, b+1)$ such that*

$$\|b(x - y) + Tx - Ty\| \leq \theta\|x - y\| \tag{4}$$

for all $x, y \in X$.

One year later, Petruşel and Petruşel [13] gave a new class of self-mappings T on a nonempty closed convex subset Y of a normed space $(X, \|\cdot\|)$ called a convex orbital β-Lipschitz mapping, i.e., there is $\beta > 0$ such that for every $x \in Y$ and $\lambda \in (0, 1]$, we have

$$\|Tx - T((1 - \lambda)x + \lambda Tx)\| \leq \beta\lambda\|x - Tx\|. \tag{5}$$

They also showed that a convex orbital β-Lipschitz mapping is generalized than all above contraction mappings (see in [13]), and proved the existence and uniqueness of the fixed point for the considered mappings by using the help of the fixed point theorem named a graphical contraction principle as follows:

Theorem 1.3 (Graphical contraction principle [14]). *Let T be a self-mapping on a complete metric space (X, d). Supposes that T has a closed graph and T is a k-graphical contraction mapping, i.e., there is $k \in (0, 1)$ such that*

$$d(Tx, T^2x) \leq kd(x, Tx)$$

for all $x \in X$. Then the following assertions hold:

1. *$Fix(T) = Fix(T^n) \neq \emptyset$ for all $n \in \mathbb{N}$;*
2. *T is a $\frac{1}{1-k}$-weakly Picard mapping, i.e., for each $x \in X$, we obtain $\lim_{n\to\infty} T^n x$ converges to some fixed points x^* of T and*

$$d\left(x, \lim_{n\to\infty} T^n x\right) \leq \frac{1}{1-k} d(x, Tx)$$

for all $x \in X$.

Petruşel and Petruşel [13] also gave many interesting open questions on their fixed point results for convex orbital β-Lipschitz mappings in Hilbert spaces.

Recently, Nithiarayaphaks and Sintunavarat [12] introduced a new approach to generalize the contractive condition by utilizing a double averaged mapping, which is defined by the following definition: for a self-mapping T on a convex subset C of a normed space, and given $\lambda > 0, \gamma \geq 0$ with $\lambda + \gamma \leq 1$, the double averaged mapping of T associated with λ, γ, denoted by $T_{\lambda,\gamma}$, is defined by $T_{\lambda,\gamma}x = (1 - \lambda - \gamma)x + \lambda Tx + \gamma T^2x$ for all $x \in C$. This generalizes the concept of an averaged mapping by taking $\gamma = 0$. However, the authors noted that the double averaged mapping does not possess the same properties as an averaged mapping (as discussed in Remark 3.1 in [12]). To address this, they provided a useful lemma to extend the framework of an averaged mapping using the double averaged mapping as follows:

Lemma 1.4 ([12]). *Let T be a self-mapping on a convex subset C of a normed space $(X, \|\cdot\|)$. Suppose that there are $\lambda > 0, \gamma \geq 0$ with $\lambda + \gamma \leq 1$ and the following condition holds:*

(W1) *there exists a nonnegative real number $k < 1$ such that*

$$\|T_{\lambda,\gamma}x - Tx\| \leq k\|x - Tx\| \tag{6}$$

for all $x \in C$.

Then $Fix(T) = Fix(T_{\lambda,\gamma})$.

Remark 1.5. *It is not difficult to prove that every averaged mapping satisfies (W1). This brings to the fact that the conclusion of $Fix(T) = Fix(T_{\lambda,0})$ does not require any condition.*

Based on the impressive feature of convex orbital β-Lipschitz mappings and double averaged mappings, we are interested to present the ideas of mappings that are more comprehensive than convex orbital β-Lipschitz mappings. Moreover, we will answer the third open question of Petruşel and Petruşel [14] on convex orbital β-Lipschitz mappings in Banach spaces.

2. Quadratic weak orbital β-Lipschitz mappings

This section presents the first way to generalize the idea of convex orbital β-Lipschitz mappings in Hilbert spaces. Furthermore, a fixed point result for the proposed generalized convex orbital β-Lipschitz mapping is proved. Before showing the main theorem in this section, many new ideas are given. We begin with the definition of a generalization of convex orbital β-Lipschitz mappings in Hilbert spaces as follows:

Definition 2.1. *Let C be a nonempty convex subset of a normed space $(X, \|\cdot\|)$. A self-mapping T on C is called a quadratic weak orbital β-Lipschitz mapping if there are $\beta > 0$ and $\gamma \in [0, \frac{1}{1+\beta^2}]$ such that*

$$\lambda^2\|Tx - T(T_{\lambda,\gamma}x)\|^2 + \gamma^2\|T^2x - T^2(T_{\lambda,\gamma}x)\|^2 \leq (\lambda + \gamma)^2\beta^2\|x - T_{\lambda,\gamma}x\|^2 \tag{7}$$

for all $x \in C$ and for all $\lambda > 0$ with $\lambda + \gamma \leq 1$.

Remark 2.2. *It is easy to see that every convex orbital β-Lipschitz mapping is a quadratic weak orbital β-Lipschitz mapping satisfying (7) with $\gamma = 0$.*

It is well-known that if $\lambda = 1$ and $\gamma = 0$, then $T_{\lambda,\gamma} = T$ and so $\text{Fix}(T_{\lambda,\gamma}) = \text{Fix}(T)$. In the other case, it might fail. So Lemma 1.4 is needed.

In the fixed point result of a convex orbital β-Lipschitz mapping in Hilbert spaces in [13], the decreasingness of a mapping is used to prove the existence of a fixed point result. However, it is not enough to prove the fixed point result for quadratic weak orbital β-Lipschitz mappings. So we need one idea of specific decreasing mappings to establish a fixed point result for quadratic weak orbital β-Lipschitz mappings as follows:

Definition 2.3. *Let T be a self-mapping on a nonempty subset C of an inner product space $(X, \langle\cdot,\cdot\rangle)$. Then T is called a strong decreasing mapping if T satisfies the following condition:*

$$\lambda Re(\langle x - y, Tx - Ty\rangle) + (1 - \lambda)\|x - y\|\|T^2x - T^2y\| \leq 0 \tag{8}$$

for all $x, y \in C$ and $\lambda \in (0, 1]$.

From the above definition, it is not difficult to see that every strong decreasing mapping is a decreasing mapping, and the converse does not necessarily hold. For instance, if $|C| > 1$ and $T := -I$, where I is an identity mapping, then T is a decreasing mapping, but it is not a strong decreasing.

Theorem 2.4. *Let C be a nonempty closed convex subset of a Hilbert space $(X, \langle\cdot,\cdot\rangle)$ and $T : C \to C$ be a quadratic weak orbital β-Lipschitz mapping having a closed graph. If T is a strong decreasing mapping, then there are $\lambda > 0$ and $\gamma \in \left[0, \frac{1}{1+\beta^2}\right]$ with $\lambda + \gamma < 1$ such that the following conditions hold:*

(f1) $|Fix(T_{\lambda,\gamma})| > 0$;
(f2) *for any given $x_0 \in C$, the iteration $\{x_n\} \subseteq C$ given by*

$$x_n = T_{\lambda,\gamma} x_{n-1} \tag{9}$$

for all $n \in \mathbb{N}$ converges to the fixed point of $T_{\lambda,\gamma}$.

Moreover, if $T_{\lambda,\gamma}$ satisfies (W1), then the following conditions hold:

(F1) $|Fix(T)| = 1$;
(F2) *for any given $x_0 \in C$, the iteration $\{x_n\} \subseteq C$ given by*

$$x_n = T_{\lambda,\gamma} x_{n-1} \tag{10}$$

for all $n \in \mathbb{N}$ converges to the unique fixed point of T.

Proof. Since T is a quadratic weak orbital β-Lipschitz mapping, there are β, γ such that (7) holds. For each $\lambda > 0$ with $\lambda + \gamma < 1$, we have $T_{\lambda,\gamma}$ is a self-mapping on C and it has a closed graph since C is a convex subset of X and T is a closed graph mapping.

Now, for each $\lambda > 0$ with $\lambda + \gamma < 1$ and for each $x, y \in C$, we have

$$\begin{aligned}
\|T_{\lambda,\gamma}x - T_{\lambda,\gamma}y\|^2 &= \|(1-\lambda-\gamma)(x-y) + \lambda(Tx - Ty) + \gamma(T^2x - T^2y)\|^2 \\
&\leq (1-\lambda-\gamma)^2\|x-y\|^2 + \lambda^2\|Tx-Ty\|^2 + \gamma^2\|T^2x - T^2y\|^2 \\
&\quad +2(1-\lambda-\gamma)\Big[\mathrm{Re}(\langle x-y, \lambda(Tx-Ty)\rangle) + \mathrm{Re}(\langle x-y, \gamma(T^2x - T^2y)\rangle)\Big] \\
&\quad +2\lambda\gamma\mathrm{Re}(\langle Tx-Ty, T^2x - T^2y\rangle) \\
&\leq (1-\lambda-\gamma)^2\|x-y\|^2 + \lambda^2\|Tx-Ty\|^2 + \gamma^2\|T^2x - T^2y\|^2 \\
&\quad +2(1-\lambda-\gamma)\Big[\lambda\mathrm{Re}(\langle x-y, Tx-Ty\rangle) + \gamma\|x-y\|\|T^2x - T^2y\|\Big] \\
&\leq (1-\lambda-\gamma)^2\|x-y\|^2 + \lambda^2\|Tx-Ty\|^2 + \gamma^2\|T^2x - T^2y\|^2 \\
&\quad +2(1-\lambda-\gamma)\Big[\lambda\mathrm{Re}(\langle x-y, (Tx-Ty)\rangle) + (1-\lambda)\|x-y\|\|T^2x - T^2y\|\Big].
\end{aligned} \tag{11}$$

Since T is a strong decreasing mapping, Inequality (11) implies that

$$\|T_{\lambda,\gamma}x - T_{\lambda,\gamma}y\|^2 \leq (1-\lambda-\gamma)^2\|x-y\|^2 + \lambda^2\|Tx-Ty\|^2 + \gamma^2\|T^2x - T^2y\|^2 \tag{12}$$

for all $x, y \in C$. Letting $y = T_{\lambda,\gamma}x$ in (12), we get

$$\begin{aligned}
\|T_{\lambda,\gamma}x - T^2_{\lambda,\gamma}x\|^2 &\leq (1-\lambda-\gamma)^2\|x - T_{\lambda,\gamma}x\|^2 + \lambda^2\|Tx - T(T_{\lambda,\gamma}x)\|^2 + \gamma^2\|T^2x - T^2(T_{\lambda,\gamma}x)\|^2 \\
&\leq (1-\lambda-\gamma)^2\|x - T_{\lambda,\gamma}x\|^2 + (\lambda+\gamma)^2\beta^2\|x - T_{\lambda,\gamma}x\|^2 \\
&= [(1-\lambda-\gamma)^2 + (\lambda+\gamma)^2\beta^2]\|x - T_{\lambda,\gamma}x\|^2
\end{aligned} \tag{13}$$

for all $x \in C$. Since (13) holds for any $\lambda > 0$ with $\lambda + \gamma < 1$, we can take $\lambda := \frac{1}{1+\beta^2} - \gamma \in (0,1)$ in (13). Then we obtain

$$\|T_{\lambda,\gamma}x - T^2_{\lambda,\gamma}x\|^2 \leq k^2\|x - T_{\lambda,\gamma}x\|^2$$

for all $x \in C$, where $k := \sqrt{\frac{\beta^2}{1+\beta^2}} \in [0,1)$. This implies that

$$\|T_{\lambda,\gamma}x - T^2_{\lambda,\gamma}x\| \leq k\|x - T_{\lambda,\gamma}x\| \tag{14}$$

for all $x \in C$, that is, $T_{\lambda,\gamma}$ is a k-graphical contraction mapping. Therefore, Theorem 1.3 implies that (f1) and (f2) hold.

Finally, we will show that (F1) and (F2) hold. Assume that $T_{\lambda,\gamma}$ satisfies (W1). By Lemma 1.4, we get $|\mathrm{Fix}(T)| = |\mathrm{Fix}(T_{\lambda,\gamma})| > 0$. Next, we suppose that T has more than one fixed points says u, v with $u \neq v$. Letting $x = u$ and $y = v$ in (12), we get

$$\begin{aligned} \|u-v\|^2 &\leq [(1-\lambda-\gamma)^2+\lambda^2+\gamma^2]\|u-v\|^2 \\ 1 &\leq (1-\lambda-\gamma)^2+\lambda^2+\gamma^2 \\ \lambda\gamma &\leq (\lambda+\gamma)(\lambda+\gamma-1), \end{aligned}$$

which is a contradiction since $(\lambda+\gamma)(\lambda+\gamma-1) < 0$. Hence, T has a unique fixed point, that is, (F1) holds. For (F2), it follows from (f2) and (F1) since $|\mathrm{Fix}(T)| = |\mathrm{Fix}(T_{\lambda,\gamma})|$. This completes the proof. □

Based on the fact in Remarks 2.2 and 1.5 together with Theorem 2.5, we get the following fixed result for convex orbital β-Lipschitz mappings in Hilbert spaces:

Corollary 2.5 ([13]). *Let C be a nonempty closed convex subset of a Hilbert space $(X, \langle\cdot,\cdot\rangle)$ and $T : C \to C$ be a convex orbital β-Lipschitz mapping having a closed graph. If T is decreasing mapping, then the following conditions hold:*

(F1) $|Fix(T)| = 1$;

(F2) *there is $\lambda \in (0,1)$ and for any given $x_0 \in C$, the iteration $\{x_n\} \subseteq C$ given by*

$$x_n = (1-\lambda)x_{n-1} + Tx_{n-1} \tag{15}$$

for all $n \in \mathbb{N}$ converges to the unique fixed point of T.

3. The open question answering on convex orbital β-Lipschitz mappings

Based on the third open question in [13], we will extend a fixed point result for convex orbital β-Lipschitz mappings from Hilbert spaces to Banach spaces. In order to show this, we need to introduce a second generalization of a convex orbital β-Lipschitz mapping as follows.

Definition 3.1. *Let C be a nonempty convex subset of a normed space $(X, \|\cdot\|)$. A self-mapping T on C is called a weak convex orbital β-Lipschitz mapping if there are $\beta > 0$ and $\gamma \in [0,1]$ such that*

$$\lambda\|Tx - T(T_{\lambda,\gamma}x)\| + \gamma\|T^2x - T^2(T_{\lambda,\gamma}x)\| \leq (\lambda+\gamma)\beta\|x - T_{\lambda,\gamma}x\| \tag{16}$$

for all $x \in C$ and for all $\lambda > 0$ with $\lambda + \gamma \leq 1$.

Remark 3.2. *It is easy to see that every convex orbital β-Lipschitz mapping satisfies the weak convex orbital β-Lipschitz condition (16) with $\gamma = 0$.*

The following result is a fixed point theorem for weak convex orbital β-Lipschitz mappings in Banach spaces, which is a an answer of a third open question in [13].

Theorem 3.3. *Let C be a closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a weak convex orbital β-Lipschitz mapping with has a closed graph. If $\beta < 1$, then the following assertions hold:*

(ff1) $|Fix(T_{\lambda,\gamma})| > 0$ *for all $\lambda > 0$ with $\lambda + \gamma \leq 1$;*

(ff2) *for each $\lambda > 0$ with $\lambda + \gamma < 1$ and for any given $x_0 \in C$, the iteration $\{x_n\} \subseteq C$ given by*

$$x_n = T_{\lambda,\gamma}x_{n-1} \tag{17}$$

for all $N \in \mathbb{N}$ converges to the fixed point of $T_{\lambda,\gamma}$.

Moreover, if $T_{\lambda,\gamma}$ satisfies (W1) for some $\lambda > 0$ with $\lambda + \gamma \leq 1$, then the following assertions hold:

(FF1) $|Fix(T)| > 0$;

(FF2) *for any given $x_0 \in C$, the iteration $\{x_n\} \subseteq C$ given by*

$$x_n = T_{\lambda,\gamma} x_{n-1} \tag{18}$$

for all $n \in \mathbb{N}$ converges to the fixed point of T.

Proof. From the assumption, there are $\beta \in (0,1)$ and $\gamma \in [0,1]$ such that T satisfies (16) for every $\lambda > 0$ with $\lambda + \gamma \leq 1$. Then for each $x \in C$, we have

$$\begin{aligned} \|T_{\lambda,\gamma}x - T^2_{\lambda,\gamma}x\| &= \|(1-\lambda-\gamma)(x - T_{\lambda,\gamma}x) + \lambda(Tx - T(T_{\lambda,\gamma}x)) + \gamma(T^2x - T^2(T_{\lambda,\gamma}x))\| \\ &\leq (1-\lambda-\gamma)\|x - T_{\lambda,\gamma}x\| + \lambda\|Tx - T(T_{\lambda,\gamma}x)\| + \gamma\|T^2x - T^2(T_{\lambda,\gamma}x)\| \\ &\leq (1-\lambda-\gamma)\|x - T_{\lambda,\gamma}x\| + (\lambda+\gamma)\beta\|x - T_{\lambda,\gamma}x\| \\ &= [1-\lambda-\gamma+(\lambda+\gamma)\beta]\|x - T_{\lambda,\gamma}x\| \end{aligned} \tag{19}$$

for all $\lambda > 0$ with $\lambda + \gamma \leq 1$. Due to $1 - \lambda - \gamma + (\lambda + \gamma)\beta \in (0,1)$ and we know that $T_{\lambda,\gamma}$ has a closed graph, we can apply the graphical contraction principle to conclude that (ff1) and (ff2) hold.

Finally, (FF1) is proved by Lemma 1.4 and (ff1). For (FF2), we can use (ff2) and the conclusion of Lemma 1.4. This completes the proof. □

Remark 3.4. *If we take $\gamma = 0$ in Theorem 3.3, we get a result of a convex orbital β-Lipschitz mapping in a Banach space, which is an answer of a third open question in [13].*

4. Conclusion and Open Problems

In this paper, we gave two ways to generalize the idea of a convex orbital β-Lipschitz mapping by using the concept of a double average mapping. In Theorem 3.3, we extended the result in [13] from Hilbert spaces to Banach spaces, which answers one of the open questions in [13]. In [13], we know that the class of convex orbital β-Lipschitz mappings covering many types of mappings, including Banach contraction mappings, Kannan contraction mappings, Ciríc-Reich-Rus contraction mappings, weak contraction mappings in the sense of Berinde, nonexpansive mappings, enriched contraction mappings, and Lipschitz mappings. Therefore, from Remarks 2.2 and 3.2, our fixed point results for quadratic weak orbital β-Lipschitz mappings and weak convex orbital β-Lipschitz mappings are generalizations of many famous fixed point results in the fixed point theory.

In [13], Petruşel and Petruşel have already proved that the fixed point equation of a convex orbital β-Lipschitz mapping in a Hilbert space is well-posed and Ulam-Hyers stable. It would be interesting to investigate under which conditions the fixed point equation for the mappings in Definition 2.1 (or 3.1) is Ulam-Hyers stable and well-posed when $\gamma > 0$?

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Author contributions

All authors have contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflict of interest

The authors declare that they have no competing interests.

Data Availability Statementst

The authors declare that all data supporting the findings of this study are available within the article.

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