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On better approximation order for the nonlinear Bernstein operator of maximum product kind

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Abstract. Using maximum instead of sum, nonlinear Bernstein operator of maximum product kind is introduced by Bede et al. [2]. The present paper deals with the approximation processes for this operator. The order of approximation for this operator to the function f, can be found in [4] by means of the classical modulus of continuity. Also, in [4], it was indicated that the order of approximation of this operator to the function f under the modulus is $\frac{1}{\sqrt{n}}$ and it could not be improved except for some subclasses of functions. Contrary to this claim, in this paper, we will show that a better order of approximation can be obtained with the help of modulus of continuity.

1. Introduction

Recently, in order to investigate the approximation properties of nonlinear operators, some nonlinear max-product type operators was introduced by Bede et al. [1], [2].

And then, approximation properties of many nonlinear max-product type approximation operators have been investigated (see e.g., in the chronological order, [1], [6], [4], and [3]).

Nonlinear Bernstein operator of max-product kind is defined by Bede et al. [2] using the maximum instead of the sum.

In [4], the approximation and shape preserving properties of Bernstein operator of max-product kind are examined.

The concept of classical modulus of continuity is defined as

$$\omega(f,\delta) = \max\left\{\left|f(x) - f(y)\right|;\ x, y \in I,\ \left|x - y\right| \leq \delta\right\}.$$

The order of approximation for this operator can be found in [4] by means of the modulus of continuity $\omega\left(f; 1/\sqrt{n}\right)$. Also, Bede et al. indicated that the order of approximation under the modulus was $\frac{1}{\sqrt{n}}$ and it could not be improved except for some subclasses of functions (see [4]).

Contrary to this claim, we will show that a better order of approximation can be obtained with the help of modulus of continuity.

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2. The Concept of Nonlinear Maksimum Product Operators

For the proof of the main results, we need some general considerations on the so-called nonlinear operators of maximum product kind.

Therefore, in this part, we will recall basic definitions and theorems about nonlinear operators given in [6], [2] and [5].

Over the set of $\mathbb{R}_+$, we consider the operations $\vee$(maximum) and $''\cdot''$ product. Then $(\mathbb{R}_+, \vee, \cdot)$ has a semirings structure and it is called as Maximum Product algebra.

Let $I \subset \mathbb{R}$ be bounded or unbounded interval, and

$$CB_+ = \{f : I \to \mathbb{R}_+ : f \text{ continuous and bounded on } I\}.$$

Let us take the general form of $L_n : CB_+(I) \to CB_+(I)$, as

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x, x_i) f(x_i) \text{ or } L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i) f(x_i),$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(., x_i) \in CB_+(I)$ and $x_i \in I$, for all i. These operators are nonlinear, positive operators and moreover they satisfy the following pseudo-linearity condition of the form

$$L_n(\alpha f \vee \beta g)(x) = \alpha L_n(f)(x) \vee \beta L_n(g)(x), \forall \alpha, \beta \in \mathbb{R}_+, \ f, g : I \to \mathbb{R}_+.$$

In this section, we present some general results on these kinds of operators which will be used later.

Lemma 2.1. *[6] Let $I \subset \mathbb{R}$ be bounded or unbounded interval,*

$$CB_+ = \{f : I \to \mathbb{R}_+ : f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the following properties:
(i) If $f, g \in CB_+(I)$ satisfy $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in \mathbb{N}$.
(ii) $L_n(f + g) \leq L_n(f) + L_n(g)$ for $f, g \in CB_+(I)$.
Then for all $f, g \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have

$$\left|L_n(f)(x) - L_n(g)(x)\right| \leq L_n\left(\left|f - g\right|\right)(x).$$

Remark 2.2. *[4]* 1) *It is easy to see that the nonlinear Bernstein max-product operator satisfy the conditions Lemma 2.1, (i), (ii). In fact, instead of (i) it satisfies the stronger condition*

$$L_n(f \vee g)(x) = L_n(f)(x) \vee L_n(g)(x), \ f, g \in CB_+(I).$$

Indeed, taking in the above equality $f \leq g$, $f, g \in CB_+(I)$, it easily follows $L_n(f)(x) \leq L_n(g)(x)$.
2) *In addition, it is positive homogenous, that is $L_n(\lambda f) = \lambda L_n(f)$ for all $\lambda \geq 0$.*

Corollary 2.3. *[6] Let $L_n : CB_+(I) \to CB_+(I)$, $n \in \mathbb{N}$ be a sequence of operators satisfying the conditions (i), (ii) in Lemma 2.1 and in addition being positive homogenous. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have*

$$\left|f(x) - L_n(f)(x)\right| \leq \left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega(f, \delta) + f(x) \left|L_n(e_0)(x) - 1\right|,$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t - x|$ for all $t \in I$, $x \in I$, and if I is unbounded then we suppose thet there exists $L_n(\varphi_x)(x) \in \mathbb{R}_+ \cup \{\infty\}$, for any $x \in I$, $n \in \mathbb{N}$.

A consequence of Corollary 2.3, we have the following:

Corollary 2.4. *[6] Suppose that in addition to the conditions in Corollary 2.3, the sequence $(L_n)_n$ satisfies $L_n(e_0) = e_0$, for all $n \in \mathbb{N}$. Then for all $f \in CB_+(I)$, $n \in \mathbb{N}$ and $x \in I$ we have*

$$\left|f(x) - L_n(f)(x)\right| \leq \left[1 + \frac{1}{\delta} L_n(\varphi_x)(x)\right] \omega(f, \delta).$$

3. Nonlinear Bernstein Operator of Maximum Product Kind

In the classical Bernstein operator, the sum operator $\sum$ is replaced by the $\bigvee$ maximum operator, and introduced by Bede et al. in [2]. So, nonlinear Bernstein operator of maximum product kind is defined as

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x)\ f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, \tag{1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k \ (1-x)^{n-k}$ and $f \in C[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$.

In [4], the approximation and shape preserving properties of $B_n^{(M)}(f)(x)$ are examinated.

4. Auxiliary Results

Since $B_n^{(M)}(f)(0) - f(0) = 0$ for all n, in this part, we will consider $x > 0$ in the notations, proofs and statements of the all approximation result.

Let us define the following expression similar to [4]. For each $k, j \in \{0,1,...,n\}$ and $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$,

$$M_{k,n,j}(x) := \frac{p_{n,k}(x)\left|\frac{k}{n} - x\right|}{p_{n,j}(x)},\ m_{k,n,j}(x) := \frac{p_{n,k}(x)}{p_{n,j}(x)}.$$

It is clear that if $k \geq j+1$ then we get

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)\left(\frac{k}{n} - x\right)}{p_{n,j}(x)},$$

and if $k \leq j-1$ then we have

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)\left(x - \frac{k}{n}\right)}{p_{n,j}(x)}.$$

Also, for each $k, j \in \{0,1,...,n\}$, $k \geq j+2$, and $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$ let us denote

$$\overline{M}_{k,n,j}(x) := \frac{p_{n,k}(x)\left(\frac{k}{n+1} - x\right)}{p_{n,j}(x)},$$

and for each $k, j \in \{0,1,...,n\}$, $k \leq j-2$, and $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$ let us denote

$$\underline{M}_{k,n,j}(x) := \frac{p_{n,k}(x)\left(x - \frac{k}{n+1}\right)}{p_{n,j}(x)}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

The main result of this part is Lemma 4.3 which is proven by induction method. Notice that, the part of the Lemma 4.3 proof we show is completely different from that of [4].

Lemma 4.1. *[4] Let* $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$*:*

(i) *For all* $k, j \in \{0, 1, ..., n\}$, $k \geq j + 2$, *one has*

$$\overline{M}_{k,n,j}(x) \leq M_{k,n,j}(x) \leq 3\underline{M}_{k,n,j}(x),$$

(ii) *For all* $k, j \in \{0, 1, ..., n\}$, $k \leq j - 2$ *, one has*

$$M_{k,n,j}(x) \leq \underline{M}_{k,n,j}(x) \leq 6M_{k,n,j}(x).$$

Lemma 4.2. *[4] For all* $k, j \in \{0, 1, ..., n\}$ *and* $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$*, one has*

$$m_{k,n,j}(x) \leq 1.$$

Lemma 4.3. *Let* $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$ *and* $\alpha \in \{2, 3, ...\}$

(i) *If* $k \in \{j+2, j+3, ..., n-1\}$ *such that* $k - (k+1)^{1/\alpha} \geq j$*, then we have*

$$\overline{M}_{k,n,j}(x) \geq \overline{M}_{k+1,n,j}(x).$$

(ii) *If* $k \in \{1, 2, ..., j-2\}$ *such that* $k + (k)^{1/\alpha} \leq j$*, then we get*

$$\underline{M}_{k,n,j}(x) \geq \underline{M}_{k-1,n,j}(x).$$

Proof. (i) From the case (i) of Lemma 3.2 in [4], we can write

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} \geq \frac{k+1}{n-k}\frac{n-j}{j+1}\frac{k-j-1}{k-j}.$$

After this point we will use a different proof tecnique from [4].

By the induction method, let's show that, the following inequality

$$\frac{k+1}{j+1}\frac{k-j-1}{k-j} \geq 1 \tag{2}$$

holds for $k - (k+1)^{1/\alpha} \geq j$.

For $\alpha = 2$, this inequality becomes as demonstrated case in (i) of Lemma 3.2 in [4]. So, we obtain the inequality (2) is correct for $\alpha = 2$.

Now, we assume that the inequalty (2) is provided for $\alpha - 1$. It follows

$$\frac{k+1}{j+1}\frac{k-j-1}{k-j} \geq 1$$

holds for $k - (k+1)^{1/(\alpha-1)} \geq j$. This means that

$$(k-j)^{\alpha} \geq (k+1)(k-j).$$

Since $k \geq j+2$, $(k-j)^{\alpha} \geq (k+1)(k-j) \geq k+1$ is true for α, hence, for arbitrary $\alpha = 2, 3, ...$, the inequality (2) is provided when $k - (k+1)^{1/\alpha} \geq j$. So we obtain,

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} \geq \frac{k+1}{n-k}\frac{n-j}{j+1}\frac{k-j-1}{k-j} \geq 1.$$

(ii) From the case (ii) of Lemma 3.2 in [4], we can write

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} \geq \frac{n-k+1}{k}\frac{j}{n+1-j}\frac{j-k}{j-k+1}.$$

After this point we will use the our proof technique again. Same as proof of (*i*), using the induction method, let's show that the following inequality

$$\frac{j}{k}\frac{j-k}{j-k+1} \geq 1 \tag{3}$$

holds for $k+(k)^{1/\alpha} \leq j$.

Similar to (*i*), for $\alpha = 2$, this inequality becomes as demonstrated in case (*ii*) of Lemma 3.2 in [4]. So, we obtain the inequality (3) is satisfied.

Now, we assume that (3) is correct for $\alpha - 1$. Since

$$\frac{j}{k}\frac{j-k}{j-k+1} \geq 1$$

for $k+(k)^{1/(\alpha-1)} \leq j$, we obtain $k\left(j-k\right) \leq \left(j-k\right)^{\alpha}$.

Since $k \leq j-2$, $k \leq k(j-k) \leq \left(j-k\right)^{\alpha}$ is true for $\alpha > 1$, then the desired inequality is provided for $k+(k)^{1/\alpha} \leq j$. So we obtain,

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} \geq \frac{n-k-1}{k}\frac{j}{n+1-j}\frac{j-k}{j-k+1} \geq 1,$$

which gives the desired result.

□

Lemma 4.4. *[4] One has*

$$\bigvee_{k=0}^{\infty} p_{n,k}\left(x\right) = p_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n}, \frac{j+1}{n}\right],\ j = 0, 1, ..., n,$$

where $p_{n,k}\left(x\right) = \binom{n}{k}x^{k}\left(1-x\right)^{n-k}$.

5. Appoximation Results

The main aim of this section is to obtain a better order of approximation for the operators $B_n^{(M)}(f)(x)$ to the function f by means of the modulus of continuity. According to the following theorem we can say that the order of approximation can be improved when the α is big enough. Moreover if we choose as $\alpha = 2$, these approximation results turn out to be the results in [4].

Theorem 5.1. *If* $f : [0,1] \to \mathbb{R}_+$ *is continuous, then we have the following order of approximation for the operators (1) to the function* f *by means of the modulus of continuity:*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 12\,\omega\left(f; \frac{1}{(n+1)^{1-\frac{1}{\alpha}}}\right), \text{ for all } n \in \mathbb{N}, x \in [0,1],$$

where $\alpha = 2, 3, \ldots$.

Proof. Since nonlinear max-product Bernstein operators satisfy the conditions in Corollary 2.4, for any $x \in [0,1]$, we get

$$\left|B_n^{(M)}\left(f\right)\left(x\right) - f\left(x\right)\right| \leq \left[1 + \frac{1}{\delta_n}B_n^{(M)}\left(\varphi_x\right)\left(x\right)\right]\omega\left(f,\delta\right), \tag{4}$$

where $\varphi_x(t) = |t - x|$. At this point let us denote

$$\begin{aligned} E_n(x) \quad : \quad &= B_n^{(M)}(\varphi_x)(x) \\ &= \frac{\bigvee_{k=0}^{n} p_{n,k}(x) \left| \frac{k}{n} - x \right|}{\bigvee_{k=0}^{n} p_{n,k}(x)}, \; x \in [0,1]. \end{aligned}$$

Let $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$, where $j \in \{0, 1, ..., n\}$ is fixed, arbitary. By Lemma 4.4 we easily obtain

$$E_n(x) = \max_{k=0,1,...,n} \left\{ M_{k,n,j}(x) \right\}, \; x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right].$$

Firstly let's examinate for $j = 0$, where $x \in \left[0, \frac{1}{n+1}\right]$. We have

$$M_{k,n,0}(x) \leq \frac{1}{n+1}.$$

So, we find an upper estimate for any $k = 0, 1, ..., n$, $E_n(x) \leq \frac{1}{n+1}$ when $j = 0$.

Now, it remains to find an upper estimate for each $M_{k,n,j}(x)$ when $j = 1, 2, ..., n$, is fixed, $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$, $k \in \{0, 1, ..., n\}$ and $\alpha = 2, 3, ...$. Indeed we will prove that

$$M_{k,n,j}(x) \leq \frac{6}{(n+1)^{1-\frac{1}{\alpha}}} \tag{5}$$

for all $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$, $k = 0, 1, 2, ..., n$ which directly will implies that

$$E_n(x) \leq \frac{6}{(n+1)^{1-\frac{1}{\alpha}}}, \text{ for all } x \in [0,1], \; n \in \mathbb{N}$$

and taking $\delta_n = \frac{6}{(n+1)^{1-\frac{1}{\alpha}}}$ in (4) we obtain the estimate in the statement immediately.

So, in order to completing the proof of (5), we consider the following cases:
1) $k \in \{j-1, j, j+1\}$,
2) $k \geq j+2$,
and
3) $k \leq j-2$.

Case 1). The following is obtained, similar to Case 1 in the proof of Theorem 4.1 in [4].
If $k = j$, then we get, $M_{j,n,j}(x) \leq \frac{1}{n+1}$.
If $k = j+1$, then then we get, $M_{j+1,n,j}(x) \leq \frac{3}{n+1}$.
If $k = j-1$, then, we get, $M_{j-1,n,j}(x) \leq \frac{2}{n+1}$.

Case 2). Subcase a). Assume first that $k - (k+1)^{\frac{1}{\alpha}} < j$. Since Lemma 4.2, we get

$$\begin{aligned} \overline{M}_{k,n,j}(x) &\leq \frac{k}{n+1} - x \leq \frac{k}{n+1} - \frac{j}{n+1} \\ &\leq \frac{k}{n+1} - \frac{k-(k+1)^{\frac{1}{\alpha}}}{n+1} = \frac{(k+1)^{\frac{1}{\alpha}}}{n+1} \\ &= \left(\frac{k+1}{n+1}\right)^{\frac{1}{\alpha}} \frac{1}{(n+1)^{1-\frac{1}{\alpha}}} \leq \frac{1}{(n+1)^{1-\frac{1}{\alpha}}}. \end{aligned}$$

Subcase *b*). Assume now that $k-(k+1)^{\frac{1}{\alpha}} \geq j$. For the function $h(x) := x-(x+1)^{\frac{1}{\alpha}}$, we have $h'(x) = 1-\left(1/\alpha\,(x+1)^{1-\frac{1}{\alpha}}\right) > 0$. Thus we can say that the function $h(x)$ is nondecreasing on the interval $[0,1]$, it follows that there exists a maximum value $\bar{k} \in \{1,2,...,n\}$ satisfying the inequality $\bar{k}-\left(\bar{k}+1\right)^{\frac{1}{\alpha}} < j$. Then, for $k_1 = \bar{k}+1$, we have $k_1-(k_1+1)^{\frac{1}{\alpha}} \geq j$ and

$$\begin{aligned}\overline{M}_{\bar{k}+1,n,j}(x) &\leq \frac{\bar{k}+1}{n+1}-x \leq \frac{\bar{k}+1}{n+1}-\frac{j}{n+1}\\ &\leq \frac{\bar{k}+1}{n+1}-\frac{\bar{k}-\left(\bar{k}+1\right)^{\frac{1}{\alpha}}}{n+1} = \frac{\left(\bar{k}+1\right)^{\frac{1}{\alpha}}+1}{n+1}\\ &= \left(\frac{\bar{k}+1}{n+1}\right)^{\frac{1}{\alpha}}\frac{1}{(n+1)^{1-\frac{1}{\alpha}}}+\frac{1}{n+1}\\ &\leq \frac{1}{(n+1)^{1-\frac{1}{\alpha}}}+\frac{1}{(n+1)^{1-\frac{1}{\alpha}}} \leq \frac{2}{(n+1)^{1-\frac{1}{\alpha}}}.\end{aligned}$$

Also, we have $k_1 \geq j+2$. Indeed, this is a consequence of the fact that g is nondecreasing on the interval $[0,1]$ and $g(j+1) < j$.

By Lemma 4.3 (*i*), it follows that

$$\overline{M}_{\bar{k}+1,n,j}(x) \geq \overline{M}_{\bar{k}+2,n,j}(x) \geq ... \geq \overline{M}_{n,n,j}(x) .$$

Hence, we obtain $\overline{M}_{k,n,j}(x) \leq \frac{2}{(n+1)^{1-\frac{1}{\alpha}}}$ for any $k \in \left\{\bar{k}+1, \bar{k}+2, ..., n\right\}$.

We thereby obtain in both subcases, by Lemma 4.1, (*i*) too, $M_{k,n,j}(x) \leq 6\,\frac{1}{n^{1-\frac{1}{\alpha}}}$.

Case 3). Subcase *a*). Assume first that $k+k^{\frac{1}{\alpha}} \geq j$. Then we obtain

$$\begin{aligned}\underline{M}_{k,n,j}(x) &\leq x-\frac{k}{n+1} \leq \frac{j+1}{n+1}-\frac{k}{n+1}\\ &\leq \frac{k+k^{\frac{1}{\alpha}}+1}{n+1}-\frac{k}{n+1} = \frac{k^{\frac{1}{\alpha}}+1}{n+1} \leq \frac{n^{\frac{1}{\alpha}}+1}{n+1}\\ &\leq \left(\frac{n}{n+1}\right)^{\frac{1}{\alpha}}\frac{1}{(n+1)^{1-\frac{1}{\alpha}}}+\frac{1}{(n+1)^{1-\frac{1}{\alpha}}}\\ &\leq \frac{2}{(n+1)^{1-\frac{1}{\alpha}}}.\end{aligned}$$

Subcase *b*). Assume now that $k+k^{\frac{1}{\alpha}} < j$. Let $\tilde{k} \in \{0,1,2,...,n\}$ be the minimum value such that $\tilde{k}+\left(\tilde{k}\right)^{\frac{1}{\alpha}} \geq j$. Then $k_2 = \tilde{k}-1$ satisfies $k_2+(k_2)^{\frac{1}{\alpha}} < j$ and

$$\begin{aligned}\underline{M}_{\tilde{k}-1,n,j}(x) &\leq x-\frac{\tilde{k}-1}{n+1} \leq \frac{j+1}{n+1}-\frac{\tilde{k}-1}{n+1}\\ &\leq \frac{\tilde{k}+(\tilde{k})^{\frac{1}{\alpha}}+1}{n+1}-\frac{\tilde{k}-1}{n+1}\\ &= \frac{(\tilde{k})^{\frac{1}{\alpha}}+2}{n+1} \leq \frac{n^{\frac{1}{\alpha}}+2}{n+1}\\ &\leq 3\left(\frac{n}{n+1}\right)^{\frac{1}{\alpha}}\frac{1}{(n+1)^{1-\frac{1}{\alpha}}}\\ &\leq \frac{3}{(n+1)^{1-\frac{1}{\alpha}}}.\end{aligned}$$

Also, because of $j \geq 2$, we have $k_2 \leq j-2$.

By Lemma 4.1, (*ii*), it follows that $\underline{M}_{\tilde{k}-1,n,j}(x) \geq \underline{M}_{\tilde{k}-2,n,j}(x) \geq ... \geq \underline{M}_{0,n,j}(x)$. Thus we obtain $\underline{M}_{k,n,j}(x) \leq \frac{3}{(n+1)^{1-\frac{1}{\alpha}}}$ for any $k \leq j-2$ and $x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right]$.

Therefore, in both subcases, by Lemma 4.1, (*ii*) , we get

$$M_{k,n,j}(x) \leq \frac{3}{(n+1)^{1-\frac{1}{\alpha}}}. \tag{6}$$

If we use

$$\max\left\{\frac{1}{n}, \frac{1}{n+1}, \frac{2}{n+1}, \frac{3}{n+1}, \frac{3}{(n+1)^{1-\frac{1}{\alpha}}}, \frac{6}{(n+1)^{1-\frac{1}{\alpha}}}\right\} = \frac{6}{(n+1)^{1-\frac{1}{\alpha}}}$$

in (6), we have desired result. □

Remark 5.2. *In [4], the order of approximation for nonlinear max-product Bernstein operators was found as* $1/\sqrt{n}$ *by means of modulus of continuity and the authors claim that this order of approximation cannot be improving except for some subclasses of functions such as concav functions. In addition, in [7], the order of approximation can be found as* $1/n$*, for the special sequence of the function* $f_n(x) = f(x) + n$*. However, due to Theorem 5.1, we proved that the order of approximation is* $1/(n+1)^{1-\frac{1}{\alpha}}$ *. For big enough* α *,* $1/(n+1)^{1-\frac{1}{\alpha}}$ *tends to* $1/(n+1)$*. As a result, since* $1 - \frac{1}{\alpha} \geq \frac{1}{2}$ *for* $\alpha = 2, 3, ...,$ *this selection of* α *improving the order of approximation.*

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