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Statistical convergence with respect to power series method on time scales

Kamil Demirci[a], Sevda Yıldız[a], Fadime Dirik[a]

[a]*Department of Mathematics Sinop University Sinop, Turkey*

Abstract. In the present paper, we introduce some new notions, such as convergence in the sense of power series method and strong convergence in the sense of power series method on an arbitrary time scale. Later, with the help of statistical convergence, the notions of P_Δ-density, P_Δ-statistical convergence, P_Δ-statistical Cauchy are defined and investigated their relations. Also, we give the notion of P_Δ-uniform integrability for characterization of P_Δ-statistical and P_Δ-strong convergence.

1. Introduction

In nature and in some engineering problems, discrete and continuous states can sometimes exist. The application of the time scale calculation then makes it convenient to solve these problems. In fact, combining such discrete and continuous states is the fundamental concept behind Hilger's time scale calculus [12]. Additionally, it combines both the theory of difference equations for one variable and the theory of differential equations. The definition range of the functions under consideration can be replaced with any time scale $\mathbb{T}$.

Steinhaus [17] and Fast [10], working independently, introduced the idea of statistical convergence of real number sequences. This convergence method is considered under different names in different spaces. Although time scales have recently had important applications in many areas of mathematics, their applications in additive theory are just beginning. Many reseachers have studied types of convergence on an arbitrary time scale ([1, 15, 22–26]). More recently, Turan Yalçın and Duman [27] introduced the concept of generalized statistical convergence and study regular integral transformations on time scales (see also [30]). Hence they generalized the A-statistical convergence to the time scales where A is a nonnegative regular summability matrix [5]. As it is well known that other interesting convergence methods are power series methods that contains such as Abel and Borel methods and also, P-statistical convergence recently given by Unver and Orhan [29] which is a different type of statistical convergence with power series method. Thus it makes sense to ask: "Do we get some convergence methods and characterization theorems for power series methods or P-statistical convergence from the generalized statistical convergence method on time scales given by Turan Yalçın and Duman ?" The answer is: "No." As we know, the power series methods are not matrix methods. Also, P-statistical convergence and statistical convergence are incompatible. The

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Email addresses: kamild@sinop.edu.tr (Kamil Demirci), sevdaorhan@sinop.edu.tr (Sevda Yıldız), fdirik@sinop.edu.tr (Fadime Dirik)

main motivation and the novetly of the present paper is to generalize the notions of power series methods and P-statistical convergence on an arbitrary time scales, to get meaningful results and charecterizations for these methods.

2. Preliminary Results on Convergence Methods and Time-Scales

In this section, we quickly review some terms and results related to statistical convergence and convergence in the sense of power series method of a sequence x that will be necessary below:

Let E be a subset of $\mathbb{N}$, the set of natural numbers, then *the natural density of* E, denoted by $\delta(E)$, is given by:

$$\delta(E) := \lim_n \frac{1}{n} |\{k \leq n \ : k \in E\}|$$

whenever the limit exists, where $|.|$ denotes the cardinality of the set [14].

A sequence $x = \{x_k\}$ of numbers is *statistically convergen*t to L provided that, for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

that is,

$$E := E_n(\varepsilon) := \{k \leq n : |x_k - L| \geq \varepsilon\}$$

has natural density zero. This is denoted by $st - \lim_k x_k = L$ ([10], [17]). It is worth noting that, every convergent sequence (in the usual sense) is statistically convergent to the same number. However, statistically convergent sequence need not be convergent.

Now, we turn our attention to power series methods.

Let $\{p_k\}$ be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{k=0}^{\infty} p_k t^k$$

has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k x_k$$

exists then we say that $x = \{x_k\}$ is convergent in the sense of power series method to the above limit value ([13], [16]). Note that the method is regular iff $\lim\limits_{0<t \to R^-} \dfrac{p_k t^k}{p(t)} = 0$ for every k (see, e.g. [4]).

Remark 2.1. *It follows immediately from the definition that in case of* $R = 1$, *if* $p_k = 1$ *and* $p_k = \frac{1}{k+1}$, *the power series methods coincide with Abel summability method and logarithmic summability method, respectively. In the case of* $R = \infty$ *and* $p_k = \frac{1}{k!}$, *the power series method coincides with Borel summability method.*

Here and in the sequel, power series method is always assumed to be regular.

Ünver and Orhan [29] have recently introduced P-statistical convergence. It will be of particularly important below that they show statistical convergence and P-statistical convergence are incompatible via striking examples. Many authors study on this method, see for example [7, 9, 18–20, 28]. Now, we recall the next definitions:

Definition 2.2. *[29] Let $E \subset \mathbb{N}_0$. If the limit*

$$\delta_P(E) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k\in E} p_k t^k$$

exists, then $\delta_P(E)$ is called the P-density of E. Note that, from the definition of a power series method and P-density it is obvious that $0 \le \delta_P(E) \le 1$ whenever it exists.

In a similar manner to the natural density, one can give some properties for the P-density:
i) $\delta_P(\mathbb{N}) = 1$,
ii) if $E \subset F$ then $\delta_P(E) \le \delta_P(F)$
iii) if E has $P-$density then $\delta_P(\mathbb{N}/E) = 1 - \delta_P(E)$.

Definition 2.3. *[29] Let $x = \{x_k\}$ be a sequence. Then x is said to be statistically convergent with respect to power series method (P-statistically convergent) to L if for any $\varepsilon > 0$*

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k\in E_\varepsilon} p_k t^k = 0$$

where $E_\varepsilon = \{k \in \mathbb{N}_0 : |x_k - L| \ge \varepsilon\}$, that is $\delta_P(E_\varepsilon) = 0$ for any $\varepsilon > 0$. This is denoted by $st_P - \lim x_k = L$.

We now give the following example that proofs P-statistical convergence and statistical convergence are incompatible.

Example 2.4. *Let $\{p_k\}$ be defined as follows*

$$p_k = \begin{cases} 1, & k = m^2 \\ 0, & otherwise \end{cases}, \ m = 1, 2, ...,$$

and take the sequence $\{x_k\}$ defined by

$$x_k = \begin{cases} 1, & k = m^2 \\ k, & otherwise \end{cases}, \ m = 1, 2, ...$$

We calculate that, since for any $\varepsilon > 0$,

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k:|x_k-1|\ge\varepsilon} p_k t^k = 0,$$

$\{x_k\}$ is P-statistically convergent to 1. However, the sequence $\{x_k\}$ is not statistically convergent to 1. If we consider the following sequence $\{s_k\}$ defined by

$$s_k = \begin{cases} k, & k = m^2, \\ 1, & otherwise, \end{cases}$$

then $\{s_k\}$ is statistically convergent to 1 but not P-statistically convergent to 1.

Definition 2.5. *A sequence of real numbers $x = \{x_k\}$ is said to be P-statistically bounded if for some $M > 0$ such that $\delta_P(\{k : |x_k| > M\}) = 0$.*

We state now some simple, basic facts about the theory of time scales.

A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$, the set of real numbers. Here and in the sequel, we are interested in time scales such that $\inf \mathbb{T} = j_0$ $(j_0 > 0)$ and $\sup \mathbb{T} = \infty$. For $j \in \mathbb{T}$, the forward and backward jump operators are defined as follows, respectively:

$$\sigma : \mathbb{T} \to \mathbb{T}, \ \sigma(j) := \inf\{s \in \mathbb{T} : s > j\},$$
$$\rho : \mathbb{T} \to \mathbb{T}, \ \rho(j) := \sup\{s \in \mathbb{T} : s < j\},$$

and the graininess function μ is defined by

$$\mu : \mathbb{T} \to [0, \infty), \ \mu(j) = \sigma(j) - j.$$

A closed interval in a time scale $\mathbb{T}$ is given by the notation $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T} = \{j \in \mathbb{T} : a \leq j \leq b\}$. Thus, open intervals and half-open intervals can be given similarly.

Also, the Lebesgue Δ-measure which is given by Guseinov [11], denoted by μ_Δ and it is known that if $a, b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta([a, b)_{\mathbb{T}}) = b - a$, $\mu_\Delta((a, b)_{\mathbb{T}}) = b - \sigma(a)$, $\mu_\Delta((a, b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$, and $\mu_\Delta([a, b]_{\mathbb{T}}) = \sigma(b) - a$.

If A is a Δ-measurable subset of $\mathbb{T}$, then the density of A is defined by Turan and Duman [23] and given by

$$\delta_{\mathbb{T}}(A) := \lim_{j \to \infty} \frac{\mu_\Delta\left(\left\{s \in [j_0, j]_{\mathbb{T}} : s \in A\right\}\right)}{\mu_\Delta\left([j_0, j]_{\mathbb{T}}\right)}$$

(if this limit exists). Now let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then, f is said to be statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\delta_{\mathbb{T}}\left(\left\{j \in \mathbb{T} : \left|f(j) - L\right| \geq \varepsilon\right\}\right) = 0,$$

and denoted by $st_{\mathbb{T}} - \lim_{j \to \infty} f(j) = L$ (see also [15]).

3. Power Series Method and Statistical Type Convergence on Time Scales

We give our main definitions and some of our new results in the present section. Also, we generalize the characterization theorem given by Demirci [8], was studied for A-statistical convergence, to P_Δ-statistical convergence (see also [19]). First of all, let us emphasize that from this point onward, we work under the assumption that $p : \mathbb{T} \to \mathbb{R}$ is non-negative Δ-measurable, also for every $j \in \mathbb{T}$, $p(j)$ is a Lebesgue Δ-integrable function on $\mathbb{T}$ (see for details of Δ-integrability [6, 11]) and $\sup_{t \in (0,R)} p_\Delta(t) < \infty$ where $p_\Delta(t) := \int_{\mathbb{T}} p(j) t^j \Delta j$, $R \in (0, +\infty]$.

Let's give the following definition.

Definition 3.1. *If for a given Δ-measurable and Lebesgue Δ-integrable function f on $\mathbb{T}$*

$$\lim_{0<t \to R^-} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j) t^j f(j) \Delta j = L$$

then we say that f is convergent to L in the sense of power series method and denoted by $P_\Delta - \lim f = L$.

With the help of this definition, let us give the lemma showing that the method is regularity as below:

Lemma 3.2. *The power series method is regular for bounded functions on a time scale $\mathbb{T}$ provided that, for every finite $M \in \mathbb{T}$,*

$$\lim_{0<t \to R^-} \frac{1}{p_\Delta(t)} \int_{[j_0, M]_{\mathbb{T}}} p(j) t^j \Delta j = 0. \tag{1}$$

Proof. *Assume that $\lim_{j\to\infty} f(j) = L$, we might record that, for every $\varepsilon > 0$, there exists a $M \in \mathbb{T}$ such that $\left|f(j) - L\right| < \varepsilon$ for every $j > M$ with $j \in \mathbb{T}$. Then,*

$$\begin{aligned}
&\left|\frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j f(j)\, \Delta j - L\right| \\
\leq\ & \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j \left|f(j) - L\right| \Delta j \\
=\ & \frac{1}{p_\Delta(t)} \left[\int_{[j_0,M]_{\mathbb{T}}} p(j)\, t^j \left|f(j) - L\right| \Delta j + \int_{[j_0,M]_{\mathbb{T}}^c} p(j)\, t^j \left|f(j) - L\right| \Delta j\right] \\
\leq\ & \frac{1}{p_\Delta(t)} \left[\int_{[j_0,M]_{\mathbb{T}}} p(j)\, t^j \left|f(j) - L\right| \Delta j + \varepsilon \int_{\mathbb{T}} p(j)\, t^j \Delta j\right] \\
=\ & \frac{1}{p_\Delta(t)} \int_{[j_0,M]_{\mathbb{T}}} p(j)\, t^j \left|f(j) - L\right| \Delta j + \varepsilon \leq M_1 \frac{1}{p_\Delta(t)} \int_{[j_0,M]_{\mathbb{T}}} p(j)\, t^j \Delta j + \varepsilon.
\end{aligned}$$

We get immediately from the (1) that

$$\lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j f(j)\, \Delta j = L,$$

this completes the proof. □

Definition 3.3. *For a given Δ-measurable and Lebesgue Δ-integrable function f on $\mathbb{T}$*

$$\lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j \left|f(j) - L\right| \Delta j = 0$$

then we say that f is strongly convergent to L in the sense of power series method (P_Δ-strongly convergent).

Throughout this paper, we work under the assumption that the power series method on a time scale satisfying the condition (1). Hence, the method is regular for bounded functions on $\mathbb{T}$.

Definition 3.4. *Let E be a Δ-measurable subset of $\mathbb{T}$. P_Δ-density of E in $\mathbb{T}$ is defined by*

$$\delta_{P_\Delta}(E) := \lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_E p(j)\, t^j \Delta j$$

if the limit exists.

Remark 3.5. *We should note that the case of $\mathbb{T} = \mathbb{N}_0$, Definition 3.4 reduces to the concept of P-density in Definition 2.2.*

Lemma 3.6. (i) $\delta_{P_\Delta}(\mathbb{T}) = 1$.

(ii) *For any Δ-measurable subset E of $\mathbb{T}$, $0 \leq \delta_{P_\Delta}(E) \leq 1$.*

(iii) *If E is Δ-measurable subset of $\mathbb{T}$ and $\delta_{P_\Delta}(E)$ exists, then $\delta_{P_\Delta}(E^c)$ exists and $\delta_{P_\Delta}(E) + \delta_{P_\Delta}(E^c) = 1$,* *since E is Δ-measurable so is E^c is Δ-measurable. On the other hand $E \cup E^c = \mathbb{T}$ and*

$$\begin{aligned} 1 &= \lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j \Delta j \\ &= \lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \left[\int_E p(j)\, t^j \Delta j + \int_{E^c} p(j)\, t^j \Delta j \right]. \end{aligned}$$

Assume that E and F are Δ-measurable subsets of $\mathbb{T}$ and $\delta_{P_\Delta}(E)$, $\delta_{P_\Delta}(F)$ exist.
(iv) If $E \subseteq F$, then $\delta_{P_\Delta}(E) \le \delta_{P_\Delta}(F)$.
(v) $\delta_{P_\Delta}(E \cup F) \le \delta_{P_\Delta}(E) + \delta_{P_\Delta}(F)$.
(vi) If E is bounded, then $\delta_{P_\Delta}(E) = 0$,
for a sufficiently large $M \in \mathbb{T}$ we can write $E \subseteq [j_0, M]_{\mathbb{T}}$, then using the condition (1), we get

$$0 \le \lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_E p(j)\, t^j \Delta j \le \lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_{[j_0,M]_{\mathbb{T}}} p(j)\, t^j \Delta j = 0.$$

(vii) If $\delta_{P_\Delta}(E) = \delta_{P_\Delta}(F) = 1$, then $\delta_{P_\Delta}(E \cup F) = \delta_{P_\Delta}(E \cap F) = 1$.

Now, we give the concept of P_Δ-statistical convergence (statistical convergence with respect to power series method on a time scale) via the concept of P_Δ-density.

Definition 3.7. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. We say that f is P_Δ-statistically convergent to a number L if, for every $\varepsilon > 0$,*

$$\delta_{P_\Delta}\left(\left\{ j \in \mathbb{T} : \left| f(j) - L \right| \ge \varepsilon \right\}\right) = 0,$$

holds, i.e.,

$$\lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_{\{ j\in\mathbb{T}: |f(j)-L| \ge \varepsilon \}} p(j)\, t^j \Delta j = 0 \tag{2}$$

and denoted by $st_{P_\Delta} - \lim f(j) = L$.

Remark 3.8. *In the case $\mathbb{T} = \mathbb{N}_0$, statistical convergence with respect to power series method on a time scale is reduced to the classical statistical convergence in the sense of power series method, which is given by Definition 2.3.*

Proposition 3.9. *The st_{P_Δ}-limit of a function $f : \mathbb{T} \to \mathbb{R}$ is unique.*

Proof. Let $st_{P_\Delta} - \lim f(j) = L_1$ and $st_{P_\Delta} - \lim f(j) = L_2$. Thanks to Theorem 3.14-(*iv*), we may write that, for every $\varepsilon > 0$, there exist subsets E_1, E_2 of $\mathbb{T}$ such that $\delta_{P_\Delta}(E_1) = \delta_{P_\Delta}(E_2) = 1$ and $\lim_{j\to\infty (j\in E_1)} f(j) = L_1$, $\lim_{j\to\infty (j\in E_2)} f(j) = L_2$. Hence, we get $\left| f(j) - L_1 \right| < \frac{\varepsilon}{2}$, for $j \in E_1$ and $\left| f(j) - L_2 \right| < \frac{\varepsilon}{2}$, for $j \in E_2$. Then, for every $j \in E_1 \cap E_2$ one has

$$|L_1 - L_2| \le \left| f(j) - L_1 \right| + \left| f(j) - L_2 \right| < \varepsilon,$$

thus $L_1 = L_2$. □

We can give the linearity of P_Δ-statistical convergence to the following proposition:

Proposition 3.10. *Let $f, g : \mathbb{T} \to \mathbb{R}$. If $st_{P_\Delta} - \lim f(j) = L_1$ and $st_{P_\Delta} - \lim g(j) = L_2$, then the following statements hold:*
(i) $st_{P_\Delta} - \lim \{f(j) + g(j)\} = L_1 + L_2$,
(ii) $st_{P_\Delta} - \lim cf(j) = cL_1$ $(c \in \mathbb{R})$.

Let us give the proposition that gives the relationship between classical convergence on a time scale and P_Δ-statistical convergence.

Proposition 3.11. *If $f : \mathbb{T} \to \mathbb{R}$ with $\lim f(j) = L$, then $st_{P_\Delta} - \lim f(j) = L$.*

Proof. *Let $\lim f(j) = L$. For a given $\varepsilon > 0$, we can find a $j_0 \in \mathbb{T}$ such that, for every $j > j_0$, $|f(j) - L| < \varepsilon$ holds. Thanks to Lemma 3.6-(iii) and (vi), the set*

$$\left\{ j \in \mathbb{T} : \left| f(j) - L \right| \geq \varepsilon \right\}$$

has P_Δ-density zero. By the Definition 3.7, we get $st_{P_\Delta} - \lim f(j) = L$. □

Now, we can express the following characterization about P_Δ-statistical convergence.

Theorem 3.12. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. Then, $st_{P_\Delta} - \lim f(j) = L$ if and only if there exists a Δ-measurable subset E of $\mathbb{T}$ such that $\delta_{P_\Delta}(E) = 1$ and $\lim\limits_{j \to \infty (j \in E)} f(j) = L$.*

Proof. *Necessity: Let $st_{P_\Delta} - \lim f(j) = L$. Define the following sets:*

$$\begin{aligned} S_r^1 &:= \left\{ j \in \mathbb{T} : \left| f(j) - L \right| \geq \frac{1}{r} \right\}, \\ S_r^2 &:= \left\{ j \in \mathbb{T} : \left| f(j) - L \right| < \frac{1}{r} \right\}, \end{aligned}$$

$r = 1, 2, \ldots$. Observe that, $\delta_{P_\Delta}\left(S_r^1\right) = 0$ and $\delta_{P_\Delta}\left(S_r^2\right) = 1$, $r = 1, 2, \ldots$. Also, $S_1^2 \supset S_2^2 \supset \ldots$. If we show that $\lim f(j) = L$ for every $j \in S_r^2$ then we get the desired result. Now, suppose that f is not convergent to L. Hence, there is $\varepsilon > 0$ such that, $|f(j) - L| \geq \varepsilon$ for infinitely many terms. Put

$$S_\varepsilon := \left\{ j \in \mathbb{T} : \left| f(j) - L \right| < \varepsilon \right\} \text{ and } \varepsilon > \frac{1}{r},$$

$r = 1, 2, \ldots$. Then $\delta_{P_\Delta}(S_\varepsilon) = 0$ and $S_r^2 \subset S_\varepsilon$. Therefore $\delta_{P_\Delta}\left(S_r^2\right) = 0$ which is a contradiction. Hence, f is convergent to L.

Sufficiency: From the hypothesis, there exists $J \in \mathbb{T}$ such that, for every $j > J$ with $j \in E$, $|f(j) - L| < \varepsilon$ holds. Now

$$E_\varepsilon = \left\{ j \in \mathbb{T} : \left| f(j) - L \right| \geq \varepsilon \right\} \subset \mathbb{T} \setminus \left(E \cap [J, \infty)_{\mathbb{T}} \right),$$

then we can easily see that $\delta_{P_\Delta}\left(E \cap [J, \infty)_{\mathbb{T}}\right) = 1$ and finally we get $\delta_{P_\Delta}(E_\varepsilon) = 0$ whence the result. □

Now, we will generalize a statistically Cauchy sequence with respect to power series method to an arbitrary time scale.

Definition 3.13. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. We say that f is P_Δ-statistical Cauchy if there exists $J \in \mathbb{T}$ such that, for every $\varepsilon > 0$,*

$$\lim_{0 < t \to R^-} \frac{1}{p_\Delta(t)} \int\limits_{\{j \in \mathbb{T} : |f(j) - f(J)| \geq \varepsilon\}} p(j)\, t^j \Delta j = 0.$$

The theorem that will give the following characterization of P_{Δ}-statistical convergence and P_{Δ}-statistical Cauchy is as follows:

Theorem 3.14. *Let $f:\mathbb{T}\rightarrow\mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. Then the following expressions are equivalent:*
(i) $st_{P_{\Delta}}-\lim f(j)=L$;
(ii) *f is P_{Δ}-statistical Cauchy on $\mathbb{T}$,*
(iii) *there exists a Δ-measurable function g such that $\lim\limits_{j\rightarrow\infty} g(j)=L$ and $\delta_{P_{\Delta}}(\{j\in\mathbb{T}:f(j)\neq g(j)\})=0$, i.e.,*

$$f(j)=g(j) \text{ for } P_{\Delta}\text{-almost all } j \text{ (for short } P_{\Delta}\text{-a.a. } j).$$

Proof. $(i)\Longrightarrow(ii)$: *Let $st_{P_{\Delta}}-\lim f(j)=L$ and $\varepsilon>0$. Then we can write*

$$\left\{j\in\mathbb{T}:\left|f(j)-L\right|\geq\frac{\varepsilon}{2}\right\}$$

has P_{Δ}-density zero. We can choose $J\in\mathbb{T}$ such that $\left|f(J)-L\right|<\frac{\varepsilon}{2}$ holds. Hence, thanks to the inequality

$$\left|f(j)-f(J)\right|\leq\left|f(j)-L\right|+\left|f(J)-L\right| \tag{3}$$

and from Lemma 3.6, we can get that

$$\left\{j\in\mathbb{T}:\left|f(j)-f(J)\right|\geq\varepsilon\right\}$$

has P_{Δ}-density zero and f is P_{Δ}-statistical Cauchy on $\mathbb{T}$.

$(ii)\Longrightarrow(iii)$: *Let f be P_{Δ}-statistical Cauchy on $\mathbb{T}$. Put $j_1\in\mathbb{T}$ and define an interval $G=[f(j_1)-1,f(j_1)+1]$ with*

$$\delta_{P_{\Delta}}(\{j\in\mathbb{T}:f(j)\notin G\})=0.$$

Also, similarly, put $j^{*}\in\mathbb{T}$ and define an interval $G^{*}=\left[f(j^{*})-\frac{1}{2},f(j^{*})+\frac{1}{2}\right]$ with

$$\delta_{P_{\Delta}}(\{j\in\mathbb{T}:f(j)\notin G^{*}\})=0.$$

Then, we can write

$$\begin{aligned}&\{j\in\mathbb{T}:f(j)\notin G\cap G^{*}\}\\ =\;&\{j\in\mathbb{T}:f(j)\notin G\}\cup\{j\in\mathbb{T}:f(j)\notin G^{*}\}\end{aligned}$$

and, by Lemma 3.6-(v),

$$\delta_{P_{\Delta}}(\{j\in\mathbb{T}:f(j)\notin G\cap G^{*}\})=0.$$

Therefore, f is in the closed interval $G_1=G\cap G^{*}$ for P_{Δ}-a.a. j. It is obvious that the lenght of the interval G_1 is less than or equal to 1. Now, put $j_2\in\mathbb{T}$ and define an interval $G^{**}=\left[f(j_2)-\frac{1}{4},f(j_2)+\frac{1}{4}\right]$ with

$$\delta_{P_{\Delta}}(\{j\in\mathbb{T}:f(j)\notin G^{**}\})=0.$$

By the preceding argument, the closed interval $G_2=G_1\cap G^{**}$ contains f P_{Δ}-a.a. j and the lenght of the interval G_2 is less than or equal to $\frac{1}{2}$.Continuing inductively, we can construct a sequence $\{G_k\}_{k=1}^{\infty}$ of closed invertals such that for each k, $G_k\supseteq G_{k+1}$, the lenght of G_k is less than or equal 2^{1-k} and

$$\delta_{P_{\Delta}}(\{j\in\mathbb{T}:f(j)\notin G_k\})=0.$$

Thanks to properties of the intersection of closed intervals, there is a number L with $\cup_{k=1}^{\infty} G_k = \{L\}$. *Using the fact that the* P_Δ*-density of* $\{j \in \mathbb{T} : f(j) \notin G_k\}$ *is equal to zero, we can write*

$$\frac{1}{p_\Delta(t)} \int\limits_{\{j \in \mathbb{T} : f(j) \notin G_k\}} p(j)\, t^j \Delta j < \frac{1}{k}. \tag{4}$$

Let us consider the following function $g : \mathbb{T} \to \mathbb{R}$ *with*

$$g(j) = \begin{cases} L, & f(j) \notin G_k \\ f(j), & \text{otherwise.} \end{cases}$$

Observe that, g is Δ*-measurable and* $\lim g(j) = L$; *either* $g(j) = L$ *or* $g(j) = f(j) \in G_k$ *and* $|g(j) - L| \leq 2^{1-k}$.

Finally, consider

$$\{j \in \mathbb{T} : f(j) \neq g(j)\} \subset \{j \in \mathbb{T} : f(j) \notin G_k\}.$$

Thanks to (4), we get

$$\frac{1}{p_\Delta(t)} \int\limits_{\{j \in \mathbb{T} : f(j) \neq g(j)\}} p(j)\, t^j \Delta j < \frac{1}{k}$$

which gives $\delta_{P_\Delta}(\{j \in \mathbb{T} : f(j) \neq g(j)\}) = 0$, *that is,* $f(j) = g(j)$ *for* P_Δ*-almost all j.*

$(iii) \Longrightarrow (i)$: *For a given* $\varepsilon > 0$, *we have*

$$\begin{aligned} & \left\{j \in \mathbb{T} : \left|f(j) - L\right| \geq \varepsilon\right\} \\ \subset \; & \{j \in \mathbb{T} : f(j) \neq g(j)\} \cup \left\{j \in \mathbb{T} : \left|g(j) - L\right| \geq \varepsilon\right\}. \end{aligned}$$

From this inclusion and using the hypothesis, we get $st_{P_\Delta} - \lim f(j) = L$. □

It is easy to see that if we choose $\mathbb{T} = \mathbb{N}_0$, then we get Theorem 2 of [19] which is the discrete version of above theorem.

Lemma 3.15. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a* Δ*-measurable function on* $\mathbb{T}$. *If* $st_{P_\Delta} - \lim f(j) = L$ *and f is bounded, then we get*

$$\lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int\limits_{\mathbb{T}} p(j)\, t^j f(j)\, \Delta j = L.$$

Lemma 3.16. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a* Δ*-measurable function on* $\mathbb{T}$ *and* $st_{P_\Delta} - \lim f(j) = L$. *If* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous function at L, then we get*

$$st_{P_\Delta} - \lim g(f(j)) = g(L).$$

Proof. *Let* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous function at L. Then, we can write that, for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $|g(i) - g(L)| < \varepsilon$ *whenever* $|i - L| < \delta$. *Observe that* $|g(i) - g(L)| \geq \varepsilon$ *implies* $|i - L| \geq \delta$, *and therefore* $|g(f(j)) - g(L)| \geq \varepsilon$ *implies* $|f(j) - L| \geq \delta$. *Hence, we have*

$$\left\{j \in \mathbb{T} : \left|g(f(j)) - g(L)\right| \geq \varepsilon\right\} \subset \left\{j \in \mathbb{T} : \left|f(j) - L\right| \geq \delta\right\},$$

then we can easily see that

$$\delta_{P_\Delta}\left(\left\{j \in \mathbb{T} : \left|g(f(j)) - g(L)\right| \geq \varepsilon\right\}\right) = 0$$

which completes the proof. □

Now, we can give the following theorem which is the generalization of given before.

Theorem 3.17. *For a* Δ*-measurable function* $f:\mathbb{T}\to\mathbb{R}$,

$$st_{P_\Delta}-\lim f(j)=L \tag{5}$$

a necessary and sufficient condition, for every $\kappa\in\mathbb{R}$,

$$\lim_{0<t\to R^-}\frac{1}{p_\Delta(t)}\int_{\mathbb{T}}p(j)\,t^j e^{i\kappa f(j)}\Delta j=e^{i\kappa L}. \tag{6}$$

Proof. Let $st_{P_\Delta}-\lim f(j)=L$. Since, for a fixed $\kappa\in\mathbb{R}$, $g(\kappa)=e^{i\kappa f(j)}$ is continuous, thanks to Lemma 3.16, we get

$$st_{P_\Delta}-\lim e^{i\kappa f(j)}=e^{i\kappa L}$$

and since $g(\kappa)$ is bounded, thanks to Lemma 3.15, we immediately get the equality (6). Conversely, assume that the equality (6) holds. Following [21] (see also [8, 23]), let us define the following continuous function

$$H(u)=\begin{cases}0, & u\le -1 \text{ or } u\ge 1,\\ 1+u, & -1<u<0,\\ 1-u, & 0\le u<1.\end{cases}$$

It follows immediately from the inverse Fourier transformation that we have

$$H(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa}{2}}{\frac{\kappa}{2}}\right)^2 e^{i\kappa y}d\kappa \text{ for } u\in\mathbb{R}. \tag{7}$$

It is enough to prove equality (6) for the case in which $L=0$. Hence, by hypothesis, for every $\kappa\in\mathbb{R}$,

$$\lim_{0<t\to R^-}\frac{1}{p_\Delta(t)}\int_{\mathbb{T}}p(j)\,t^j e^{i\kappa f(j)}\Delta j=1. \tag{8}$$

Let $\varepsilon>0$ and $E:=\left\{j\in\mathbb{T}:\left|f(j)\right|\ge\varepsilon\right\}$. Then, from equality (7), we can write

$$H\left(\frac{f(j)}{\varepsilon}\right)=\frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2 e^{i\kappa f(j)}d\kappa.$$

Thus

$$\begin{aligned}&\frac{1}{p_\Delta(t)}\int_{\mathbb{T}}p(j)\,t^j H\left(\frac{f(j)}{\varepsilon}\right)\Delta j\\ =\ &\frac{\varepsilon}{2\pi}\frac{1}{p_\Delta(t)}\int_{\mathbb{T}}p(j)\,t^j\left\{\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2 e^{i\kappa f(j)}d\kappa\right\}\Delta j.\end{aligned}$$

Note that (7) is an absolutely convergent integral. By the time scale version of Fubini theorem (see [2, 3]) we can get

$$\begin{aligned}&\frac{1}{p_\Delta(t)}\int_{\mathbb{T}}p(j)\,t^j H\left(\frac{f(j)}{\varepsilon}\right)\Delta j\\ =\ &\frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2\left\{\frac{1}{p_\Delta(t)}\int_{\mathbb{T}}p(j)\,t^j e^{i\kappa f(j)}\Delta j\right\}d\kappa.\end{aligned}$$

Since $g(\kappa)$ is bounded, there exists a finite constants B such that, for every $\kappa \in \mathbb{R}$ and $j \in \mathbb{T}$,

$$\left| \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j e^{i\kappa f(j)} \Delta j \right| \leq B \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j \Delta j = B.$$

Hence, thanks to Lebesgue Dominated Convergence Theorem and if we consider equality (7) and (8), we get

$$\begin{aligned} & \lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j H\left(\frac{f(j)}{\varepsilon}\right) \Delta j \\ = & \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2 \left\{ \lim_{0<t\to R^-} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j)\, t^j e^{i\kappa f(j)} \Delta j \right\} d\kappa \\ = & \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\kappa\varepsilon}{2}}{\frac{\kappa\varepsilon}{2}}\right)^2 d\kappa = H(0) = 1, \end{aligned} \tag{9}$$

and it follows immediately from the definition of H that we may also have

$$\begin{aligned} & \int_{\mathbb{T}} p(j)\, t^j H\left(\frac{f(j)}{\varepsilon}\right) \Delta j \\ = & \int_{\{j\in\mathbb{T}:|f(j)|<\varepsilon\}} p(j)\, t^j H\left(\frac{f(j)}{\varepsilon}\right) \Delta j \\ = & \int_{\left\{j\in\mathbb{T}:-1<\frac{f(j)}{\varepsilon}<0\right\}} p(j)\, t^j H\left(\frac{f(j)}{\varepsilon}\right) \Delta j + \int_{\left\{j\in\mathbb{T}:0\leq\frac{f(j)}{\varepsilon}<1\right\}} p(j)\, t^j H\left(\frac{f(j)}{\varepsilon}\right) \Delta j \\ = & \int_{\left\{j\in\mathbb{T}:-1<\frac{f(j)}{\varepsilon}<0\right\}} p(j)\, t^j \Delta j + \int_{\left\{j\in\mathbb{T}:-1<\frac{f(j)}{\varepsilon}<0\right\}} p(j)\, t^j \frac{f(j)}{\varepsilon} \Delta j \\ & + \int_{\left\{j\in\mathbb{T}:0\leq\frac{f(j)}{\varepsilon}<1\right\}} p(j)\, t^j \Delta j - \int_{\left\{j\in\mathbb{T}:0\leq\frac{f(j)}{\varepsilon}<1\right\}} p(j)\, t^j \frac{f(j)}{\varepsilon} \Delta j \\ \leq & \int_{\{j\in\mathbb{T}:|f(j)|<\varepsilon\}} p(j)\, t^j \Delta j = 1 - \int_{\{j\in\mathbb{T}:|f(j)|\geq\varepsilon\}} p(j)\, t^j \Delta j. \end{aligned}$$

Then,

$$\int_{\{j\in\mathbb{T}:|f(j)|\geq\varepsilon\}} p(j)\, t^j \Delta j \leq 1 - \int_{\mathbb{T}} p(j)\, t^j H\left(\frac{f(j)}{\varepsilon}\right) \Delta j.$$

Hence, using (9) and taking limit $0 < t \to R^-$, we get

$$\lim_{0<t\to R^-} \int_{\{j\in\mathbb{T}:|f(j)|\geq\varepsilon\}} p(j)\, t^j \Delta j = 0$$

hence the result is very simple from above equation. □

4. Strong Convergence and Uniform Integrability via Power Series Method

This last section devoted to the characterization of P_Δ-statistical and P_Δ-strong convergence. For this reason, we first introduce the following concept.

Definition 4.1. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. We say that f is P_Δ-uniformly integrable if for any $\varepsilon > 0$ there exist $t_0 \in (0, R)$ and $A > 0$ such that*

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_{\{j \in \mathbb{T}: |f(j)| \ge a\}} |f(j)| \, p(j) \, t^j \Delta j < \varepsilon$$

whenever $a \ge A$.

We now begin the characterization of P_Δ-uniform integrability.

Theorem 4.2. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. Then the followings are equivalent:*

(i) f is P_Δ-uniformly integrable,

(ii) there exists $t_0 \in (0, R)$ such that; $(*)$ $\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} |f(j)| \, p(j) \, t^j \Delta j < \infty$, $(**)$ for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any subset E, Δ-measurable subset of $\mathbb{T}$, with

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_E p(j) \, t^j \Delta j < \delta, \tag{10}$$

we have

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_E |f(j)| \, p(j) \, t^j \Delta j < \varepsilon. \tag{11}$$

Proof. $(i) \Longrightarrow (ii)$: *Let $\varepsilon > 0$. Then, from the hypothesis, there exist $t_0 \in (0, R)$ and $A > 0$ such that*

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_{\{j \in \mathbb{T}: |f(j)| \ge a\}} |f(j)| \, p(j) \, t^j \Delta j < \frac{\varepsilon}{2}$$

whenever $a \ge A$. Hence, we get

$$\begin{aligned}
& \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_{\mathbb{T}} |f(j)| \, p(j) \, t^j \Delta j \\
\le \; & \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_{\{j \in \mathbb{T}: |f(j)| < A\}} |f(j)| \, p(j) \, t^j \Delta j + \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_{\{j \in \mathbb{T}: |f(j)| \ge A\}} |f(j)| \, p(j) \, t^j \Delta j \\
< \; & A \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int\limits_{\mathbb{T}} p(j) \, t^j \Delta j + \frac{\varepsilon}{2} = A + \frac{\varepsilon}{2}
\end{aligned}$$

which proves $(*)$ *. Now, take* $\delta = \frac{\varepsilon}{2A}$ *and given* E, Δ*-measurable subset of* $\mathbb{T}$*, with (10), we have*

$$\begin{aligned}
&\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_E |f(j)| \, p(j) \, t^j \Delta j \\
\le\ & \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in E : |f(j)| < A\}} |f(j)| \, p(j) \, t^j \Delta j + \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in E : |f(j)| \ge A\}} |f(j)| \, p(j) \, t^j \Delta j \\
\le\ & A \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_E p(j) \, t^j \Delta j + \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T} : |f(j)| \ge A\}} |f(j)| \, p(j) \, t^j \Delta j \\
<\ & A \frac{\varepsilon}{2A} + \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T} : |f(j)| \ge A\}} |f(j)| \, p(j) \, t^j \Delta j < \varepsilon
\end{aligned}$$

which gives $(**)$ *.*

$(ii) \Longrightarrow (i)$ *: There exists* $t_0 \in (0, R)$ *such that* $(*)$ *and* $(**)$ *hold. From* $(*)$ *let* $B := \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} |f(j)| \, p(j) \, t^j \Delta j < \infty$*. By* $(**)$*, given* $\varepsilon > 0$ *there exists* $\delta > 0$ *with (10) implies (11). Hence, take* $A = \frac{B}{\delta}$ *and consider the set* $E = E(a) := \{ j \in \mathbb{T} : |f(j)| \ge a \}$*. Then, we get for any fixed* $a \ge A$ *that*

$$\frac{1}{p_\Delta(t)} \int_E p(j) \, t^j \Delta j \le \frac{1}{a} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} |f(j)| \, p(j) \, t^j \Delta j.$$

So, taking supremum over $t \in [t_0, R)$*, we obtain that*

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_E p(j) \, t^j \Delta j \le \frac{B}{a} \le \frac{B}{A} = \delta.$$

Which gives for the set E*, thanks to* $(**)$*, that*

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T} : |f(j)| \ge a\}} |f(j)| \, p(j) \, t^j \Delta j < \varepsilon$$

for $a \ge A$ *whence the result.* □

With the following theorem, we show that the condition for f, for characterizing P_Δ-strong convergence via P_Δ-statistical convergence, is P_Δ-uniform integrable.

Theorem 4.3. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a* Δ*-measurable function on* $\mathbb{T}$*. Then* f *is* P_Δ*-strongly convergent to zero if and only if* f *is* P_Δ*-statistically convergent to zero and* P_Δ*-uniformly integrable.*

Proof. *Let* f *be* P_Δ*-strongly convergent to zero. Then we can write that*

$$\lim_{0 < t \to R^-} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j) \, t^j |f(j)| \Delta j = 0. \tag{12}$$

Also, for any $\varepsilon > 0$*, we get*

$$\frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T} : |f(j)| \ge \varepsilon\}} p(j) \, t^j \Delta j \le \frac{1}{\varepsilon} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j) \, t^j |f(j)| \Delta j. \tag{13}$$

Then, thanks to (12) and (13) that (2) is satisfied. Hence, f is P_Δ-statistically convergent to zero. Also, from P_Δ-strong convergence of f, we can write, for any $\varepsilon > 0$, there exists $t_0 \in (0, R)$ such that

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} |f(j)| \, p(j) \, t^j \Delta j \le \varepsilon.$$

Therefore, for any $A > 0$ and $a \ge A$, we have

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| \ge a\}} |f(j)| \, p(j) \, t^j \Delta j \le \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} |f(j)| \, p(j) \, t^j \Delta j \le \varepsilon.$$

Hence, f is P_Δ-uniformly integrable.

Conversely, let f be P_Δ-statistically convergent to zero and P_Δ-uniformly integrable. Then there exist $t_0 \in (0, R)$ and $A > 0$ such that

$$\sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| \ge a\}} |f(j)| \, p(j) \, t^j \Delta j < \frac{\varepsilon}{3}$$

whenever $a \ge A$ and

$$\frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| \ge \frac{\varepsilon}{3}\}} p(j) \, t^j \Delta j < \frac{\varepsilon}{3A}.$$

For $t_0 \le t < R$, we obtain

$$\begin{aligned}
& \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| \le A\}} |f(j)| \, p(j) \, t^j \Delta j \\
\le \; & \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: \frac{\varepsilon}{3} \le |f(j)| \le A\}} |f(j)| \, p(j) \, t^j \Delta j + \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| \le \min\{A, \frac{\varepsilon}{3}\}\}} |f(j)| \, p(j) \, t^j \Delta j \\
\le \; & A \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| \ge \frac{\varepsilon}{3}\}} p(j) \, t^j \Delta j + \frac{\varepsilon}{3} \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j) \, t^j \Delta j < A \frac{\varepsilon}{3A} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\end{aligned}$$

Using the above inequality and for $t_0 \le t < R$, we have

$$\begin{aligned}
& \frac{1}{p_\Delta(t)} \int_{\mathbb{T}} p(j) \, t^j \, |f(j)| \, \Delta j \\
= \; & \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| \le A\}} p(j) \, t^j \, |f(j)| \, \Delta j + \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| > A\}} p(j) \, t^j \, |f(j)| \, \Delta j \\
< \; & \frac{2\varepsilon}{3} + \sup_{t_0 \le t < R} \frac{1}{p_\Delta(t)} \int_{\{j \in \mathbb{T}: |f(j)| > A\}} p(j) \, t^j \, |f(j)| \, \Delta j < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

hence f is P_Δ-strongly convergent to zero. □

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