Filomat 38:13 (2024), 4791–4804
https://doi.org/10.2298/FIL2413791M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On zero inclusion regions of polynomials and regular functions of a quaternionic variable

Gradimir V. Milovanović[a], Abdullah Mir[b]

[a]*Serbian Academy of Sciences and Arts, 11000 Belgrade, Serbia*
& University of Niš, Faculty of Sciences and Mathematics, P.O. Box 224, 18000 Niš, Serbia
[b]*Department of Mathematics, University of Kashmir, Srinagar, 190006, India*

Abstract. One of the most fundamental problems in numerical mathematics is the estimation of the zeros of a polynomial. This kind of study for polynomials and regular functions of a quaternionic variable has been carried out by many authors in the recent past. In this paper, we build a framework that uses the zero sets of a regular product and the extended Schwarz's lemma to deduce zero inclusion regions of polynomials and regular functions with quaternionic coefficients located on only one side of the powers of the quaternionic variable. The results obtained for this particular subclass of regular functions lead to the generalisation of other results that are known in the relevant literature.

1. Introduction and preliminaries

It is a topic of interest in mathematics as well as in practical domains like physical systems to investigate polynomial zeros and their regional location in the plane using different methods from geometric function theory. A significant portion of the classical content in geometric function theory consists of various methods for finding bounds for the zeros of a polynomial. These techniques are equally significant for developing strategies for using various approaches in contemporary articles. A significant amount of research has been done on the regions that include all the zeros of a polynomial; these regions are typically circular or annular. Eneström and Kakeya [14] were the first to contribute in this direction. They gave a classical solution to the problem when the coefficients of a polynomial are constrained. Following that, several related studies giving the distribution of the zeros of a restricted coefficient polynomial in the plane occurred in the literature, a good overview of which can be found in the comprehensive books of Marden [14] and Milovanović et al. [18].

By $\mathbb{H}$, we denote the noncommutative division ring of quaternions. It consists of elements of the form $q = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the imaginary units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ satisfy

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

2020 *Mathematics Subject Classification*. Primary 30G35; Secondary 16K20

Keywords. Quaternionic polynomials; Schwarz's lemma; Eneström-Kakeya Theorem; Slice regular functions; Zero-sets of a regular product.

Received: 12 September 2023; Accepted: 01 September 2024

Communicated by Vladimir Rakočević

Research of the first author was partly supported by the Serbian Academy of Sciences and Arts (Project Φ-96). Research of the second author was supported by the Science & Engineering Research Board (SERB), Department of Science & Technology, Government of India (No. MTR/2022/000118) and by the National Board for Higher Mathematics (R.P), Department of Atomic Energy, Government of India (No. 02011/19/2022/R & D-II/10212).

Email addresses: gvm@mi.sanu.ac.rs (Gradimir V. Milovanović), drabmir@yahoo.com (Abdullah Mir)

Every element $q = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} \in \mathbb{H}$ is composed by the real part $\mathrm{Re}(q) = x_0$ and the imaginary part $\mathrm{Im}(q) = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$.

The conjugate of q is denoted by $\overline{q}$ and is defined as $\overline{q} = x_0 - x_1\boldsymbol{i} - x_2\boldsymbol{j} - x_3\boldsymbol{k}$ and the norm of q is $|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. The inverse of each non zero element $q \in \mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$.

For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbb{H}; |q| < r\}$. By $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centered at the origin, i.e.,

$$\mathbb{B} = \left\{q = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} : x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1\right\},$$

and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \left\{q = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} : x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

The angle between two quaternions $q_1 = x_0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$ and $q_2 = y_0 + y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k}$ is given by

$$\measuredangle(q_1, q_2) = \cos^{-1}\left(\frac{x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3}{|q_1||q_2|}\right).$$

Notice that if $I \in \mathbb{S}$, then $I^2 = -1$. Thus, for any fixed $I \in \mathbb{S}$, we define

$$\mathbb{C}_I = \{x + Iy : \ x, y \in \mathbb{R}\},$$

which can be identified with a complex plane. The real axis belongs $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and so a real quaternion $q = x_0$ belongs to $\mathbb{C}_I$ for any $I \in \mathbb{S}$. For any non-real quaternion $q \in \mathbb{H} \setminus \mathbb{R}$, there exist, and are unique $x, y \in \mathbb{R}$ with $y > 0$ and $I \in \mathbb{S}$ such that $q = x + Iy$.

We refer the reader to [3], [5], [7]–[9], [13] and the reference therein, for definitions and properties of quaternions and many aspects of the theory of quaternionic regular functions.

The following definition of regularity for functions of a quaternionic variable was introduced in [8] by Gentili and Struppa, who were inspired by a work of Cullen [4] on analytic intrinsic functions of quaternions:

Definition 1.1. Let U be an open set in $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is said to be left slice regular or simply as slice regular if, for every $I \in \mathbb{S}$, its restriction f_I of f to the complex plane $\mathbb{C}_I$ satisfies

$$\overline{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0.$$

Since for all $n \geq 1$ and for all $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)(x + Iy)^n = 0,$$

it follows by definition that the monomial $P(q) = q^n$ is regular. Because addition and right multiplication by a constant preserves regularity, all polynomials of the form

$$T(q) = \sum_{\nu=0}^{n} q^\nu a_\nu, \quad a_\nu \in \mathbb{H}, \ \ \nu = 0, 1, 2, \ldots, n, \tag{1}$$

with coefficients on the right and indeterminate on the left are regular.

Given two quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ and $g(q) = \sum_{\nu=0}^{\infty} q^\nu b_\nu$ with radii of convergence greater than R, we define the regular product of f and g as the series

$$(f * g)(q) := \sum_{\nu=0}^{\infty} q^\nu c_\nu,$$

where $c_\nu = \sum_{k=0}^{\nu} a_k b_{\nu-k}$ for all ν. Further, as observed in [5] and [8] for each quaternionic power series $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$, there exists a ball

$$B(0,R) = \left\{q \in \mathbb{H} : |q| < R\right\},$$

such that f converges absolutely and uniformly on each compact subset of $B(0,R)$ and where the sum function of f is regular.

Polynomials with quaternionic coefficients located on only one side of the variable were also investigated in [10] and [11]. It is observed (e.g., see [5], [10]) that the zeros of a polynomial of type (1) are either isolated or spherical. This theory of quaternions is by now very well developed in many directions, and we refer the interested reader to [26] for the basic features of quaternionic functions. Nowadays, quaternions are not only part of contemporary mathematical studies such as algebra, analysis, geometry, etc., but they are also widely used in computer graphics, control theory, signal processing, physics, and fluid dynamics.

By using some useful tools from the theory of slice regular functions, Gentili and Stoppato [9, Theorem 3.2] (see also [7]) gave a necessary and sufficient condition for a regular quaternionic power series to have a zero at a point in the form of the following result:

Theorem 1.1. *Let $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ be a given quaternionic power series with radius of convergence R, and let $p \in B(0,R)$. Then $f(p) = 0$ if and only if there exists a quaternionic power series $g(q)$ with radius of convergence R such that*

$$f(q) = (q - p) * g(q).$$

This extends to quaterniomic power series, the theory presented in [13] for polynomials. The following result which completely describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors is given in [13] (see also [7] and [9]).

Theorem 1.2. *For two quaternionic polynomials f and g, their regular product $(f * q)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\left(f(q_0)^{-1} q_0 f(q_0)\right) = 0$.*

Gentili and Struppa [8] introduced a maximum modulus theorem for regular functions, which includes convergent power series and polynomials in the form of the following result.

Theorem 1.3 (Maximum Modulus Theorem). *Let $B = B(0,r)$ be a ball in $\mathbb{H}$ with centre 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is a constant on B.*

Recently, Gardner and Taylor [6] used Theorem 1.3 and extended Schwarz's lemma from the complex to the quaternionic setting as follows:

Theorem 1.4. Let $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ be regular in $|q| \le R$, where the coefficients a_ν, $0 \le \nu < \infty$ and variable q are quaternions. Suppose $f(0) = 0$, then

$$|f(q)| \le \frac{M|q|}{R} \quad \text{for } |q| \le R,$$

where $M = \max\limits_{|q|=R} |f(q)|$.

It is noteworthy that Niven provided the Fundamental Theorem of Algebra (see [23], [24]) for regular polynomials with coefficients in $\mathbb{H}$ from an algebraic perspective. As a result, all polynomial zeros were identified in terms of their factorization, for reference see [25]. Thus it became an interesting perspective to think about the regions containing all the zeros of a regular polynomial of quaternionic variable. Recently, there has been a lot of activity in the study of mathematical objects related to regular functions of a quaternionic variable and zero bounds of polynomials; there are many research papers published in this regard, and different approaches have been taken for different purposes. Most of these recent works deal with the generalizations and extensions of the zero bounds of polynomials with restricted quaternionic coefficients, including various generalizations of the Eneström-Kakeya theorem. To mention here some of them, we refer the interested reader to [2], [6], [15], [16], [17], [20], [22], [27].

This paper aims to establish zero-inclusion regions for a polynomial of type (1) and zero-free regions for some special regular functions of a quaternionic variable with restricted coefficients. The obtained results also lead to several generalizations of various findings from the relevant literature in complex cases.

2. Main results

We first construct a ring-shaped region containing all the zeros of a quaternionic polynomial with coefficients located on only one side of the variable, using the zero sets of a regular product and the extended Schwarz's lemma.

Theorem 2.1. *Let $T(q)=\sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ be a polynomial of degree n with quaternionic coefficients a_{ν}, $\nu=0,1,2,\dots,n$. If $t_1>t_2\ge 0$ can be found such that*

$$\max_{|q|=\varrho}\left|\sum_{\nu=0}^{n+1} q^{n-\nu+2}(t_1t_2a_{\nu}+(t_1-t_2)a_{\nu-1}-a_{\nu-2})\right|\le M_1, \tag{2}$$

and

$$\max_{|q|=\varrho}\left|\sum_{\nu=1}^{n+2} q^{\nu}(t_1t_2a_{\nu}+(t_1-t_2)a_{\nu-1}-a_{\nu-2})\right|\le M_2, \tag{3}$$

for some $\varrho>0$ ($a_{-2}=a_{-1}=a_{n+1}=a_{n+2}=0$), then all the zeros of $T(q)$ lie in the ring

$$\min\left(\frac{t_1t_2|a_0|\varrho}{M_2},\varrho\right)\le|q|\le\max\left(\frac{M_1}{\varrho|a_n|},\frac{1}{\varrho}\right).$$

Remark 2.1. Let $T(q)=\sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ be a polynomial of degree n (where q is a quaternionic variable) with non negative real coefficients. If for some real numbers t_1 and t_2 with $t_1>t_2\ge 0$,

$$t_1t_2a_{\nu}+(t_1-t_2)a_{\nu-1}-a_{\nu-2}\ge 0,\quad \nu=1,2,\dots,n+1\quad (a_{-1}=a_{n+1}=0),$$

then from (2) for $|q|=\rho=1/t_1$, we get

$$\begin{aligned}\left|\sum_{\nu=0}^{n+1} q^{n-\nu+2}(t_1t_2a_{\nu}+(t_1-t_2)a_{\nu-1}-a_{\nu-2})\right|\\ &\le\sum_{\nu=0}^{n+1}(t_1t_2a_{\nu}+(t_1-t_2)a_{\nu-1}-a_{\nu-2})\,\frac{1}{t_1^{n-\nu+2}}\\ &=a_n=M_1\quad\text{(say)},\end{aligned}$$

and, therefore, from Theorem 2.1, we conclude that all the zeros of $T(q)$ lie in

$$|q|\le\max\left(\frac{M_1}{\varrho|a_n|},\frac{1}{\varrho}\right)=\frac{1}{\varrho}=t_1.$$

which is an earlier result given by Mir and Ahmad [22, Theorem 2.2].

Using Theorem 2.1 we can prove a very general result, which includes several extensions of the well-known Eneström-Kakeya Theorem in quaternionic setting.

Theorem 2.2. *Let $T(q)=\sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ be a polynomial of degree n with quaternionic coefficients a_{ν}, $\nu=0,1,2,\dots,n$. If $t_1>t_2\geq 0$ can be found such that*

$$\max_{|q|=\varrho}\left|\sum_{\nu=0}^{n+1} q^{n-\nu+1}\Big[t_1t_2a_{\nu}+(t_1-t_2)a_{\nu-1}-a_{\nu-2}\Big]\right|\leq M_3, \tag{4}$$

and

$$\max_{|q|=\varrho}\left|\sum_{\nu=1}^{n+2} q^{\nu-1}\Big[t_1t_2a_{\nu}+(t_1-t_2)a_{\nu-1}-a_{\nu-2}\Big]\right|\leq M_4, \tag{5}$$

for some $\varrho>0$ ($a_{-2}=a_{-1}=a_{n+2}=a_{n+1}=0$), then all the zeros of $T(q)$ lie in the ring

$$\min\left(\frac{t_1t_2|a_0|\varrho}{M_4},\varrho\right)\leq |q|\leq \max\left(\frac{M_3}{|a_n|},\frac{1}{\varrho}\right).$$

We now discuss some consequences of Theorem 2.2. Taking $t_2=0$, we get the following result as a special case of Theorem 2.2.

Corollary 2.1. *Let $T(q)=\sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ be a polynomial of degree n with quaternionic coefficients a_{ν}, $\nu=0,1,2,\dots,n$. If for some positive t and ϱ, we have*

$$\max_{|q|=\varrho}\left|q^n ta_0+q^{n-1}(ta_1-a_0)+\cdots+(ta_n-a_{n-1})\right|\leq M,$$

then all the zeros of $T(q)$ lie in

$$|q|\leq \max\left(\frac{M}{|a_n|},\frac{1}{\varrho}\right). \tag{6}$$

Remark 2.2. If $T(q)=\sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ is a polynomial of degree n (where q is a quaternionic variable) with real coefficients and satisfying

$$t^na_n\leq t^{n-1}a_{n-1}\leq\cdots\leq t^{\lambda+1}a_{\lambda+1}\leq t^{\lambda}a_{\lambda}\geq t^{\lambda-1}a_{\lambda-1}\geq\cdots\geq ta_1\geq a_0,$$

where $0\leq\lambda\leq n$ and $t>0$. Then for $\varrho=1/t$, we get from (6) with $a_{-1}=0$, that

$$\max_{|q|=\frac{1}{t}}\left|\sum_{\nu=0}^{n} q^{n-\nu}(ta_{\nu}-a_{\nu-1})\right|\leq\sum_{\nu=0}^{n}\frac{|ta_{\nu}-a_{\nu-1}|}{t^{n-\nu}}=M\quad\text{(say)}.$$

Further note that

$$\frac{1}{\varrho}=t=\left|\sum_{\nu=0}^{n}\frac{ta_{\nu}-a_{\nu-1}}{a_nt^{n-\nu}}\right|\leq\sum_{\nu=0}^{n}\frac{|ta_{\nu}-a_{\nu-1}|}{|a_n|t^{n-\nu}}=\frac{M}{|a_n|}.$$

It follows from Corollary 2.1, that all the zeros of $T(q)$ lie in

$$|q|\leq\sum_{\nu=0}^{n}\frac{|ta_{\nu}-a_{\nu-1}|}{|a_n|t^{n-\nu}}. \tag{7}$$

Now,

$$\begin{aligned}\sum_{\nu=0}^{n}\frac{|ta_{\nu}-a_{\nu-1}|}{|a_n|t^{n-\nu}}&=\sum_{\nu=0}^{\lambda}\frac{ta_{\nu}-a_{\nu-1}}{|a_n|t^{n-\nu}}+\sum_{\nu=\lambda+1}^{n}\frac{a_{\nu-1}-ta_{v}}{|a_n|t^{n-\nu}}\\&=\frac{t}{|a_n|}\left\{\left(\frac{2t^{\lambda}a_{\lambda}}{t^n}-a_n\right)+\frac{1}{t^n}(|a_0|-a_0)\right\}.\end{aligned}$$

Thus, from (7), it follows that all the zeros of $T(q)$ lie in

$$|q| \le \frac{t}{|a_n|}\left\{\left(\frac{2t^{\lambda}a_{\lambda}}{t^n} - a_n\right) + \frac{1}{t^n}\Big(|a_0| - a_0\Big)\right\},$$

which for $\lambda = n$, gives the following generalization of a result of Tripathi [27].

Corollary 2.2. *Let $T(q) = \sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ be a polynomial of degree n (where q is a quaternionic variable) with real coefficients and satisfying*

$$t^n a_n \ge t^{n-1}a_{n-1} \ge \cdots \ge ta_1 \ge a_0, \tag{8}$$

for some $t > 0$. Then all the zeros of $T(q)$ lie in

$$|q| \le \frac{t^n a_n + |a_0| - a_0}{t^{n-1}|a_n|}.$$

Remark 2.3. The above Corollary 2.2 extends a result of Joyal et al. [12] from complex to the quaternionic setting when $t = 1$. For $t = 1$, Corollary 2.2 reduces to a result of Tripathi [27] (see also [19, 21]). If, in addition to $t = 1$, we assume $a_0 > 0$ in Corollary 2.2, we recover a recent result obtained by Carney et al. [2, Theorem 8].

If we apply Remark 2.2 with $t = 1$ and $\lambda = 0$ to the polynomial $q^n * T(1/q)$, we get the following result:

Corollary 2.3. *Let $T(q) = \sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ be a polynomial of degree n (where q is a quaternionic variable) with real coefficients and satisfying*

$$a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 > 0,$$

then $T(q)$ does not vanish in

$$|q| < \frac{a_0}{2a_n - a_0}.$$

It is of interest to construct a framework to establish bounds for the zeros of a quaternionic polynomial when the monotonicity of the moduli of its quaternionic coefficients gets flipped at some stage between the first and last coefficients. In this direction, we derive a region consisting of a non-central disc that contains all the zeros of a polynomial with quaternionic coefficients.

Theorem 2.3. *Let $T(q) = \sum_{\nu=0}^{n} q^{\nu}a_{\nu}$ be a polynomial of degree n with quaternionic coefficients a_{ν}, $\nu = 0, 1, 2, \ldots, n$, and satisfying*

$$\varrho|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_{\lambda}| \le |a_{\lambda-1}| \le \cdots \le |a_1| \le |a_0|$$

for some $\varrho \ge 1$ and $0 \le \lambda \le n-1$. Let b be any non-zero quaternion such that $\angle(a_{\nu}, b) \le \theta \le \pi/2$ for some θ and for $\nu = 0, 1, 2, \ldots, n$. Then all the zeros of $T(q)$ lie in

$$|q + \varrho - 1| \le \varrho(\cos\theta + \sin\theta) + \frac{2}{|a_n|}\left\{\Big(|a_0| - |a_{\lambda}|\Big)\cos\theta + \sin\theta\sum_{\nu=0}^{n-1}|a_{\nu}|\right\}.$$

Remark 2.4. Taking $\varrho = 1$ and $\lambda = 0$ in Theorem 2.3, we recover a result of Carney et al. [2, Theorem 10].

Remark 2.5. The essence of Theorem 2.3 lies in its flexibility. It is applicable to a larger class of polynomials of a quaternionic variable and weakens the hypothesis of various known results. For example, in the following cases: a) the monotonic condition of the moduli of the quaternionic coefficients gets flipped at some stage in between the first and last components; b) the monotonic requirement is not met by the moduli of the first two coefficients.

The zero inclusion regions, which are discs centred at the origin, cannot be obtained in these situations by using the result of Carney et al. [2, Theorem 11]. We have constructed a framework to derive generalizations of the aforementioned result of Carney et al. and establish zero inclusion regions consisting of discs that are not centred at the origin. This framework unifies and simplifies the derivation of these generalizations, obtaining new as well as old results in this process.

Finally, we construct a zero-free region in the form of a disc for the relevant sub class of power series regular in the ball $B(0, R)$, where $R > 0$. In this direction, we prove the following result which as a consequence gives the quaternionic analogue of a result due to Aziz and Mohammad [1].

Theorem 2.4. *Let $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ for all $q \in B(0, R)$, with complex coefficients such that $\mathrm{Re}(a_\nu) = \alpha_\nu$, $\mathrm{Im}(a_\nu) = \beta_\nu$ for $\nu = 0, 1, 2, \ldots$. If for some non-negative real numbers ϱ_1 and ϱ_2, we have*

$$\varrho_1 + \alpha_0 \geq t\alpha_1 \geq t^2\alpha_2 \geq \cdots \quad (\alpha_0 > 0)$$

and

$$\varrho_2 + \beta_0 \geq t\beta_1 \geq t^2\beta_2 \geq \cdots \quad (\beta_0 \geq 0),$$

where $0 < t < R$, then $f(q)$ does not vanish in

$$|q| < \frac{t\sqrt{\alpha_0^2 + \beta_0^2}}{2(\varrho_1 + \varrho_2) + (\alpha_0 + \beta_0)}.$$

Taking $\varrho_1 = (\tau_1 - 1)\alpha_0$ and $\varrho_2 = (\tau_2 - 1)\beta_0$, with $\tau_1 \geq 1$ and $\tau_2 \geq 1$ in Theorem 2.4, we get the following result:

Corollary 2.4. *Let $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$, for all $q \in B(0, R)$ with complex coefficients such that $\mathrm{Re}(a_\nu) = \alpha_\nu$, $\mathrm{Im}(a_\nu) = \beta_\nu$ for $\nu = 0, 1, 2, \ldots$. If for some $\tau_1 \geq 1$ and $\tau_2 \geq 1$, we have*

$$0 < \tau_1\alpha_0 \geq t\alpha_1 \geq t^2\alpha_2 \geq \cdots \quad \text{and} \quad 0 \leq \tau_2\beta_0 \geq t\beta_1 \geq t^2\beta_2 \geq \cdots,$$

where $0 < t < R$, then $f(q)$ does not vanish in

$$|q| < \frac{t\sqrt{\alpha_0^2 + \beta_0^2}}{(2\tau_1 - 1)\alpha_0 + (2\tau_2 - 1)\beta_0}.$$

Remark 2.6. Taking $\beta_\nu = 0$ for $\nu = 0, 1, 2, \ldots$, and $\tau_1 = 1$ in Corollary 2.4, we get the quaternionic analogue of a result due to Aziz and Mohammad [1].

Corollary 2.4 provides a range of zero-free regions for power series of a quaternionic variable when the parameters are appropriately chosen. Using $\tau_1 = \tau_2 = 1$ in Corollary 2.4 and noting that $\alpha_0 + \beta_0 \leq \sqrt{2(\alpha_0^2 + \beta_0^2)}$, we get the following result:

Corollary 2.5. *Let $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ be a regular power series in the quaternionic variable q, i.e., $f(q) = \sum_{\nu=0}^{\infty} q^\nu a_\nu$ for all $q \in B(0, R)$, with complex coefficients such that $\mathrm{Re}(a_\nu) = \alpha_\nu$, $\mathrm{Im}(a_\nu) = \beta_\nu$ for $\nu = 0, 1, 2, \ldots$. If for $0 < t < R$, we have*

$$0 < \alpha_0 \geq t\alpha_1 \geq t^2\alpha_2 \geq \cdots \quad \text{and} \quad 0 \leq \beta_0 \geq t\beta_1 \geq t^2\beta_2 \geq \cdots,$$

then $f(q)$ does not vanish in

$$|q| < \frac{t}{\sqrt{2}}.$$

3. Proofs of the main results

Proof of Theorem 2.1. Consider the product

$$
\begin{aligned}
F(q) &= (t_1 - q) * (t_2 + q) * T(q) \\
&= (t_1 t_2 + q(t_1 - t_2) - q^2) * (a_0 + q a_1 + q^2 a_2 + \cdots + q^n a_n) \\
&= t_1 t_2 a_0 + q[t_1 t_2 a_1 + (t_1 - t_2) a_0] + q^2 [t_1 t_2 a_2 + (t_1 - t_2) a_1 - a_0] + \cdots \\
&\quad + q^n [t_1 t_2 a_n + (t_1 - t_2) a_{n-1} - a_{n-2}] + q^{n+1} [(t_1 - t_2) a_n - a_{n-1}] \\
&\quad - q^{n+2} a_n. \qquad (9)
\end{aligned}
$$

Let $G(q) = q^{n+2} * F(1/q) = -a_n + P(q)$, so that

$$|G(q)| \geq |a_n| - |P(q)|, \qquad (10)$$

where

$$
\begin{aligned}
P(q) &= q[(t_1 - t_2) a_n - a_{n-1}] + q^2 [t_1 t_2 a_n + (t_1 - t_2) a_{n-1} - a_{n-2}] + \cdots \\
&\quad + q^n [t_1 t_2 a_2 + (t_1 - t_2) a_1 - a_0] + q^{n+1} [t_1 t_2 a_1 + (t_1 - t_2) a_0] + q^{n+2} t_1 t_2 a_0.
\end{aligned}
$$

Clearly, $P(0) = 0$ and by (2), $|P(q)| \leq M_1$ for $|q| = \varrho$. Therefore, it follows by Theorem 1.4, that

$$|P(q)| \leq \frac{M_1 |q|}{\varrho} \quad \text{for } |q| \leq \varrho,$$

which on using in (10), gives

$$|G(q)| \geq |a_n| - \frac{M_1 |q|}{\varrho}, \quad \text{for } |q| \leq \varrho.$$

Hence, if $|q| < \min(\varrho |a_n| / M_1, \varrho)$, then $G(q) \neq 0$. In other words, all the zeros of $G(q)$ lie in

$$|q| \geq \min\left(\frac{\varrho |a_n|}{M_1}, \varrho\right).$$

As $F(q) = q^{n+2} * G(1/q)$, it follows that all the zeros of $F(q)$ lie in

$$|q| \leq \max\left(\frac{M_1}{\varrho |a_n|}, \frac{1}{\varrho}\right).$$

Since by Theorem 1.1, the only zeros of $F(q) = (t_1 - q) * (t_2 + q) * T(q)$ are $q = t_1$, $q = -t_2$ and the zeros of $T(q)$, it follows that all the zeros of $T(q)$ lie in

$$|q| \leq \max\left(\frac{M_1}{\varrho |a_n|}, \frac{1}{\varrho}\right). \qquad (11)$$

Again, from (9), we have

$$|F(q)| \geq t_1 t_2 |a_0| - |H(q)|, \qquad (12)$$

where

$$
\begin{aligned}
|H(q)| &= -q^{n+2} a_n + q^{n+1} \Big[(t_1 - t_2) a_n - a_{n-1}\Big] \\
&\quad + q^n \Big[t_1 t_2 a_n + (t_1 - t_2) a_{n-1} - a_{n-2}\Big] + \cdots + q\Big[t_1 t_2 a_1 + (t_1 - t_2) a_0\Big].
\end{aligned}
$$

Clearly $H(0) = 0$ and by (3), $H(q) \leq M_2$ for $|q| \leq \varrho$. Therefore, it follows by Theorem 1.4, that

$$|H(q)| \leq \frac{M_2|q|}{\varrho} \quad \text{for } |q| \leq \varrho,$$

which on using in (12) gives

$$|F(q)| \geq t_1 t_2 |a_0| - \frac{M_2|q|}{\varrho} \quad \text{for } |q| \leq \varrho.$$

Hence, if $|q| < \min(t_1 t_2 |a_0| \varrho / M_2, \varrho)$, then $F(q) \neq 0$. In other words, all the zeros of $F(q)$ lie in

$$|q| \geq \min\left(\frac{t_1 t_2 |a_0| \varrho}{M_2}, \varrho\right).$$

By Theorem 1.1, the only zeros of $F(q) = (t_1 - q) * (t_2 + q) * T(q)$ are $q = t_1$, $q = -t_2$ and the zeros of $T(q)$, we conclude that all the zeros of $T(q)$ lie in

$$|q| \geq \min\left(\frac{t_1 t_2 |a_0| \varrho}{M_2}, \varrho\right). \tag{13}$$

The desired result follows by combining (11) and (13). □

Proof of Theorem 2.2. From (2) and (4), we have

$$\begin{aligned}
&\max_{|q|=\varrho}\left|\sum_{\nu=0}^{n+1} q^{n-\nu+2}(t_1 t_2 a_\nu + (t_1 - t_2)a_{\nu-1} - a_{\nu-2})\right| \\
&\quad = \varrho \max_{|q|=\varrho}\left|\sum_{\nu=0}^{n+1} q^{n-\nu+1}(t_1 t_2 a_\nu + (t_1 - t_2)a_{\nu-1} - a_{\nu-2})\right| \\
&\quad \leq \varrho M_3 = M_1 \quad \text{(say)}.
\end{aligned}$$

It follows from Theorem 2.1, by replacing M_1 by ϱM_3, that all the zeros of $T(q)$ lie in

$$|q| \leq \max\left(\frac{M_3}{|a_n|}, \frac{1}{\varrho}\right). \tag{14}$$

Next, we have from (3) and (5), that

$$\begin{aligned}
&\max_{|q|=\varrho}\left|\sum_{\nu=1}^{n+2} q^{\nu}(t_1 t_2 a_\nu + (t_1 - t_2)a_{\nu-1} - a_{\nu-2})\right| \\
&\quad = \varrho \max_{|q|=\varrho}\left|\sum_{\nu=1}^{n+2} q^{\nu-1}(t_1 t_2 a_\nu + (t_1 - t_2)a_{\nu-1} - a_{\nu-2})\right| \\
&\quad \leq \varrho M_4 = M_2 \quad \text{(say)}.
\end{aligned}$$

It again follows from Theorem 2.1, by replacing M_2 by ϱM_4, that all the zeros of $T(q)$ lie in

$$|q| \geq \min\left(\frac{t_1 t_2 |a_0|}{M_4}, \varrho\right). \tag{15}$$

The desired result follows by combining (14) and (15).

This completes the proof of Theorem 2.2. □

In the proof of Theorem 2.3, we need the following auxiliary result due to Carney et al. [2, Lemma 12].

Lemma 3.1. *Let* $q_1, q_2 \in \mathbb{H}$, *where* $q_1 = \alpha_1 + \beta_1 \boldsymbol{i} + \gamma_1 \boldsymbol{j} + \delta_1 \boldsymbol{k}$ *and* $q_2 = \alpha_2 + \beta_2 \boldsymbol{i} + \gamma_2 \boldsymbol{j} + \delta_2 \boldsymbol{k}$, $\measuredangle(q_1, q_2) \le 2\theta$ *and* $|q_1| \le |q_2|$. *Then*

$$|q_2 - q_1| \le (|q_2| - |q_1|)\cos\theta + (|q_2| + |q_1|)\sin\theta.$$

Proof of Theorem 2.3. Consider the polynomial

$$\begin{aligned}(1-q) * T(q) &= -q^{n+1}a_n + q^n(a_n - a_{n-1}) + \cdots + q(a_1 - a_0) + a_0 \\ &= -q^n(q + \varrho - 1)a_n + \psi(q),\end{aligned}$$

where

$$\psi(q) = q^n(\varrho a_n - a_{n-1}) + q^{n-1}(a_{n-1} - a_{n-2}) + \cdots + q(a_1 - a_0) + a_0.$$

For $|q| = 1$, we get on using the given hypothesis and Lemma 3.1, that

$$\begin{aligned}|\psi(q)| &= \left| q^n(\varrho a_n - a_{n-1}) + \sum_{\nu=1}^{n-1} q^\nu (a_\nu - a_{\nu-1}) + a_0 \right| \\ &\le (\varrho|a_n| - |a_{n-1}|)\cos\theta + (\varrho|a_n| + |a_{n-1}|)\sin\theta \\ &\quad + \sum_{\nu=1}^{n-1}\left\{ \Big||a_\nu| - |a_{\nu-1}|\Big| \cos\theta + (|a_\nu| + |a_{\nu-1}|)\sin\theta \right\} + |a_0| \\ &= (\varrho|a_n| - |a_{n-1}|)\cos\theta + (\varrho|a_n| + |a_{n-1}|)\sin\theta + \sum_{\nu=1}^{\lambda}(|a_{\nu-1}| - |a_\nu|)\cos\theta \\ &\quad + \sum_{\nu=\lambda+1}^{n-1}(|a_\nu| - |a_{\nu-1}|)\cos\theta + \sum_{\nu=1}^{n-1}(|a_\nu| + |a_{\nu-1}|)\sin\theta + |a_0| \\ &= |a_0|(1 - \cos\theta - \sin\theta) + 2(|a_0| - |a_\lambda|)\cos\theta + 2\sin\theta \sum_{\nu=0}^{n-1}|a_\nu| \\ &\quad + \varrho|a_n|(\cos\theta + \sin\theta),\end{aligned}$$

i.e.,

$$|\psi(q)| \le 2(|a_0| - |a_\lambda|)\cos\theta + 2\sin\theta \sum_{\nu=0}^{n-1}|a_\nu| + \varrho|a_n|(\cos\theta + \sin\theta),$$

since $\theta \in [0, \pi/2]$. Notice that, we have

$$\max_{|q|=1}\left| q^n * \psi\left(\frac{1}{q}\right)\right| = \max_{|q|=1}\left| q^n \psi\left(\frac{1}{q}\right)\right| = \max_{|q|=1}\left| \psi\left(\frac{1}{q}\right)\right| = \max_{|q|=1}\left|\psi(q)\right|,$$

it is clear that $q^n * \psi(1/q)$ has the same bound on $|q| = 1$ as ψ, that is

$$\begin{aligned}\left| q^n * \psi\left(\frac{1}{q}\right)\right| &= \left| q^n \psi\left(\frac{1}{q}\right)\right| \\ &\le 2(|a_0| - |a_\lambda|)\cos\theta + 2\sin\theta \sum_{\nu=0}^{n-1}|a_\nu| + \varrho|a_n|(\cos\theta + \sin\theta) \quad \text{for } |q| = 1.\end{aligned}$$

Since $q^n * \psi(1/q)$ is a polynomial and so is regular in $|q| \le 1$, it follows by Theorem 1.3, that

$$\left|q^n \psi\left(\frac{1}{q}\right)\right| \le 2(|a_0| - |a_\lambda|)\cos\theta + 2\sin\theta \sum_{\nu=0}^{n-1} |a_\nu| + \varrho|a_n|(\cos\theta + \sin\theta)$$

for $|q| \le 1$. Hence,

$$\left|\psi\left(\frac{1}{q}\right)\right| \le \frac{1}{|q|^n}\left\{2(|a_0| - |a_\lambda|)\cos\theta + 2\sin\theta \sum_{\nu=0}^{n-1} |a_\nu| + \varrho|a_n|(\cos\theta + \sin\theta)\right\}$$

for $|q| \le 1$. Equivalently, for $|q| \ge 1$, we have

$$\left|\psi(q)\right| \le \left\{2(|a_0| - |a_\lambda|)\cos\theta + 2\sin\theta \sum_{\nu=0}^{n-1} |a_\nu| + \varrho|a_n|(\cos\theta + \sin\theta)\right\}|q|^n. \tag{16}$$

For $|q| \ge 1$, we get on using (16), that

$$\begin{aligned}\left|(1-q) * T(q)\right| &= \left|\psi(q) - q^n(q + \varrho - 1)a_n\right| \\ &\ge |q|^n|a_n|q + \varrho - 1| - |\psi(q)| \\ &\ge |q|^n|a_n|\Big\{|q + \varrho - 1| - \frac{1}{|a_n|}\Big[2(|a_0| - |a_\lambda|)\cos\theta + 2\sin\theta \sum_{\nu=0}^{n-1} |a_\nu|\Big] \\ &\qquad + \varrho(\cos\theta + \sin\theta)\Big\}.\end{aligned}$$

Hence, if

$$\left|q + \varrho - 1\right| > \frac{2}{|a_n|}\Big[(|a_0| - |a_\lambda|)\cos\theta + \sin\theta \sum_{\nu=0}^{n-1} |a_\nu|\Big] + \varrho(\cos\theta + \sin\theta),$$

then $|(1-q) * T(q)| > 0$, that is $(1-q) * T(q) \ne 0$. Further, notice that

$$\left|q + \varrho - 1\right| > \frac{2}{|a_n|}\Big[(|a_0| - |a_\lambda|)\cos\theta + \sin\theta \sum_{\nu=0}^{n-1} |a_\nu|\Big] + \varrho(\cos\theta + \sin\theta),$$

implies that $|q| > 1$, since $\theta \in [0, \pi/2]$.

By Theorem 1.2, the only zeros of $(1-q) * T(q) = 0$ are $q = 1$ and the zeros of $T(q)$, therefore $T(q) \ne 0$ for

$$\left|q + \varrho - 1\right| > \frac{2}{|a_n|}\Big[(|a_0| - |a_\lambda|)\cos\theta + \sin\theta \sum_{\nu=0}^{n-1} |a_\nu|\Big] + \varrho(\cos\theta + \sin\theta).$$

In other words, all the zeros of $T(q)$ lie in

$$\left|q + \varrho - 1\right| \le \frac{2}{|a_n|}\Big[(|a_0| - |a_\lambda|)\cos\theta + \sin\theta \sum_{\nu=0}^{n-1} |a_\nu|\Big] + \varrho(\cos\theta + \sin\theta).$$

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. Consider the power series

$$\begin{aligned} F(q) &= (t-q)*f(q) = (t-q)*(a_0 + qa_1 + q^2a_2 + \cdots) \\ &= ta_0 - q\sum_{\nu=1}^{\infty} q^{\nu-1}(a_{\nu-1} - ta_\nu), \\ &= ta_0 + q(\varrho_1 + \boldsymbol{i}\varrho_2) - q\Big[(\varrho_1 + \alpha_0 - t\alpha_1) + \boldsymbol{i}(\varrho_2 + \beta_0 - t\beta_1) \\ &\quad + \sum_{\nu=2}^{\infty} q^{\nu-1}\Big\{(\alpha_{\nu-1} - t\alpha_\nu) + \boldsymbol{i}(\beta_{\nu-1} - t\beta_\nu)\Big\}\Big] \\ &= ta_0 + q(\varrho_1 + \boldsymbol{i}\varrho_2) - \psi(q), \end{aligned}$$

where

$$\psi(q) = q\Big\{(\varrho_1 + \alpha_0 - t\alpha_1) + \boldsymbol{i}(\varrho_2 + \beta_0 - t\beta_1) + \sum_{\nu=2}^{\infty} q^{\nu-1}\Big[(\alpha_{\nu-1} - t\alpha_\nu) + \boldsymbol{i}(\beta_{\nu-1} - t\beta_\nu)\Big]\Big\}.$$

Since the series f is absolutely convergent in $B(0,R)$ and $0 < t < R$, the series F is also convergent. For $|q| = t$, we get on using the given hypothesis, that

$$\begin{aligned} |\psi(q)| &\le t\Big[|(\varrho_1 + \alpha_0 - t\alpha_1) + \boldsymbol{i}(\varrho_2 + \beta_0 - t\beta_1)| \\ &\qquad + \sum_{\nu=2}^{\infty} t^{\nu-1}|(\alpha_{\nu-1} - t\alpha_\nu) + \boldsymbol{i}(\beta_{\nu-1} - t\beta_\nu)|\Big] \\ &\le t\Big[|(\varrho_1 + \alpha_0 - t\alpha_1) + (\varrho_2 + \beta_0 - t\beta_1)| \\ &\qquad + \sum_{\nu=2}^{\infty} t^{\nu-1}|(\alpha_{\nu-1} - t\alpha_\nu) + (\beta_{\nu-1} - t\beta_\nu)|\Big] \\ &= t(\varrho_1 + \alpha_0 + \varrho_2 + \beta_0). \end{aligned}$$

Since $\psi(0) = 0$ and $\psi(q)$ is regular in $|q| \le t$, it follows by Theorem 1.4, that

$$|\psi(q)| \le (\varrho_1 + \alpha_0 + \varrho_2 + \beta_0)|q| \quad \text{for} \ \ |q| \le t.$$

This implies

$$\begin{aligned} |F(q)| &= |ta_0 + q(\varrho_1 + \boldsymbol{i}\varrho_2) - \psi(q)| \\ &\ge |ta_0 + q(\varrho_1 + \boldsymbol{i}\varrho_2)| - |\psi(q)| \\ &\ge t|a_0| - (\varrho_1 + \varrho_2)|q| - (\varrho_1 + \alpha_0 + \varrho_2 + \beta_0)|q| \quad \text{for} \ \ |q| \le t. \end{aligned}$$

Hence, if

$$|q| < \min\left(\frac{t|a_0|}{2(\varrho_1 + \varrho_2) + \alpha_0 + \beta_0}, t\right),$$

then $F(q) \neq 0$. In other words, all the zeros of $F(q)$ lie in

$$|q| \ge \min\left(\frac{t|a_0|}{2(\varrho_1 + \varrho_2) + \alpha_0 + \beta_0}, t\right). \tag{17}$$

Since $\alpha_0 > 0$ and $\beta_0 \geq 0$, it is easy to see that

$$\frac{t|a_0|}{2(\varrho_1 + \varrho_2) + \alpha_0 + \beta_0} \leq t,$$

which on using in (17), implies that $F(q)$ does not vanish in

$$|q| < \frac{t|a_0|}{2(\varrho_1 + \varrho_2) + \alpha_0 + \beta_0}. \tag{18}$$

By Theorem 1.1, the only zeros of $F(q)$ are $q = t$ and the zeros of $f(q)$, it follows that $f(q)$ does not vanish in the disc defined by (18).

This completes the proof of Theorem 2.4. □

4. Conclusion

The classical and fundamental approaches dealing with the regional location of zeros in regular functions have their own intrinsic value in geometric function theory. They play an equally significant role in contemporary studies that address these kinds of issues. In this paper, we study the properties of zeros of quaternionic polynomials and regular functions with coefficients located on only one side of the quaternionic variable. We obtain zero free regions for this subclass of regular functions and extend some important results for polynomials in the quaternionic variable to the case of power series.

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