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Some weighted Hermite–Hadamard type inclusions based on interval-valued convex and co-ordinated convex mappings

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Abstract. In this paper, we establish some Hermite–Hadamard inclusions for interval-valued convex functions and interval-valued co-ordinated convex functions by using interval-valued weighted function. The inclusions established in this work provide generalizations of some results given in earlier works. As special cases, we give some new weighted Hermite–Hadamard type inclusions involving logarithmic function.

1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard see, e.g., [8], [19, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. Hermite–Hadamard inequalities have been established by many mathematicians. We note that Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [5]-[10], [18], [21]).

Interval analysis, which is utilized in mathematics and computer models as one of the ways for resolving interval uncertainty, occupies an important place in the literature. Despite the fact that this theory has a long history dating back to Archimedes' equation of the circle, much research on the subject was not published until the 1950s. In 1966, Ramon E. Moore founder of interval calculus, released the first book [23] on interval analysis. Subsequently, dozens of researchers examined the theory and application of short-term analysis. Recently, thanks to applications, interval analysis is a useful tool in various areas of

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great interest in uncertain data. You can see applications in computer graphics, test and calculation physics, error analysis, robots, and much more. Many authors have recently come to terms with absolute inequality arising from short-term jobs. Sadowska [20] discovered the Hermite–Hadamard inequality for set-valued functions, which are more general form of interval-valued functions.

Jleli and Samet obtained new Hermite–Hadamard type inequalities involving fractional integrals with respect to another function in [11]. In [22], Tunç introduced firstly fractional integrals of a function with respect to the another function for interval-valued functions. Katugompala established a new fractional integration, which generalizes the Riemann–Liouville and Hadamard fractional integrals into a single form. Budak and Agarwal established the Hermite–Hadamard-type inequalities for co-ordinated convex function via generalized fractional integrals, which generalize some important fractional integrals such as the Riemann–Liouville fractional integrals, the Hadamard fractional integrals, and Katugampola fractional integrals in [2]. Budak et al. investigated the Riemann-Liouville integrals for interval-valued functions to obtain Hermite–Hadamard inequality via these integrals in [1]. Kara et al. [12] defined interval-valued left-sided and right-sided generalized fractional double integrals. While many mathematicians have studied the interval-valued convex function, have also considered other type convex functions such as interval-valued *LR*-convex functions. Recently, in [13–15], several researchers extended the concept of interval-valued convexity and defined different kinds of *LR*-convexity for interval-valued functions. They also obtained many Hermite–Hadamard type inequalities for *LR*-interval-valued convex functions.

In 2020, Zabandan [26] established an extensions of Hermite–Hadamard inequality and as a result the author obtain the Hermite–Hadamard inequality for fractional and logarithmical integral. On the other hand, Sarıkaya and Kılıçer [25] obtained an important inequalities for co-ordinated convex functions. In addition to this, the extensions of Hermite–Hadamard type inequalities for Riemann-Liouville fractional integral logarithmic integral are given.

The main goal of this paper is to obtain some Hermite–Hadamard inclusions for interval-valued convex functions and interval-valued co-ordinated convex functions by using interval-valued weighted function. These results will generalize the results obtained in [25] and [26]. The general structure of the study consists of six chapters including an introduction. The remaining part of the paper proceeds as follows: In Section 2, we give the definitions and theorems of interval-valued functions. In Section 3, we summarise the concept of fractional integrals for interval-valued functions with one and two variables. In section 4, we obtain weighted Hermite–Hadamard type inclusions for interval-valued convex functions. The obtained results are provided of the earlier works. We establish important inclusions for co-ordinated convex functions. As special cases of these inclusions we give the extension of Hermite–Hadamard type inclusions Riemann-Liouville fractional integral with logarithmic function in Section 5. Furthermore, some remarks and corollaries are presented in this section. At the end of the paper, some conclusions and further directions of research are discussed in Section 6.

2. Interval-Valued Functions

In this section we recalling some basics definitions, results, notions and properties, which are used throughout the paper. We denote $\mathbb{R}_{\mathcal{I}}^{+}$ the family of all positive intervals of $\mathbb{R}$. The Hausdorff distance between $[\underline{X},\overline{X}]$ and $[\underline{Y},\overline{Y}]$ is defined as

$$d([\underline{X},\overline{X}],[\underline{Y},\overline{Y}]) = \max\left\{\left|\underline{X}-\underline{Y}\right|,\overline{X}-\overline{Y}\right\}.$$

The $(\mathbb{R}_{\mathcal{I}},d)$ is a complete metric space. For more details and basic notations on interval-valued functions see ([24, 27]).

It is remarkable that Moore [23] introduced the Riemann integral for the interval-valued functions. The set of all Riemann integrable interval-valued functions and real-valued functions on $[a,b]$ are denoted by $\mathcal{IR}_{([a,b])}$ and $\mathcal{R}_{([a,b])}$, respectively. The following theorem gives relation between (IR)–integrable and Riemann integrable (R–integrable) (see [24], pp. 131):

Theorem 2.1. *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval–valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$. $F \in \mathcal{IR}_{([a,b])}$ if and only if $\underline{F}(t)$, $\overline{F}(t) \in \mathcal{R}_{([a,b])}$ and*

$$(IR)\int_a^b F(t)dt = \left[(R)\int_a^b \underline{F}(t)dt, (R)\int_a^b \overline{F}(t)dt\right].$$

In [27, 28], Zhao et al. introduced a kind of convex interval–valued function as follows:

Definition 2.2. *Let $h : [c,d] \to \mathbb{R}$ be a non–negative function, $(0,1) \subseteq [c,d]$ and $h \neq 0$. We say that $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is a h–convex interval–valued function, if for all $x, y \in [a,b]$ and $t \in (0,1)$, we have*

$$h(t)F(x) + h(1-t)F(y) \subseteq F(tx + (1-t)y). \tag{2}$$

With $SX(h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$ will show the set of all h–convex interval–valued functions.

The usual notion of convex interval–valued function corresponds to relation (2) with $h(t) = t$, see [20]. Also, if we take $h(t) = t^s$ in (2), then Definition 2.2 gives the other convex interval–valued function defined by Breckner, see [4].
Otherwise, Zhao et al. obtained the following Hermite–Hadamard inequality for interval–valued functions by using h–convex:

Theorem 2.3. *[27] Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ be an interval–valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $F \in \mathcal{IR}_{([a,b])}$, $h : [0,1] \to \mathbb{R}$ be a non–negative function and $h\left(\frac{1}{2}\right) \neq 0$. If $F \in SX(h, [a,b], \mathbb{R}_{\mathcal{I}}^{+})$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq [F(a) + F(b)]\int_0^1 h(t)dt. \tag{3}$$

Remark 2.4. *(i) If $h(t) = t$, then (3) reduces to the following result:*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a) + F(b)}{2}, \tag{4}$$

which is obtained by Sadowka in [20].

(ii) If $h(t) = t^s$, then (3) reduces to the following result:

$$2^{s-1} F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a) + F(b)}{s+1},$$

which is obtained by Gomez et al. in [17].

Theorem 2.5. *[29]Let $F : \Delta \to \mathbb{R}_{\mathcal{I}}$. Then F is called ID-integrable on Δ with ID-integral $U = (ID)\iint_{\Delta} F(t,s)dA$, if for any $\varepsilon > 0$ there exist $\delta > 0$ such that*

$$d\left(S(F, P, \delta, \Delta)\right) < \varepsilon$$

for any $P \in \mathcal{P}(\delta, \Delta)$. The collection of all ID-integrable functions on Δ will be denoted by $\mathcal{ID}_{(\Delta)}$.

[29]Let $\Delta = [a,b] \times [c,d]$. If $F : \Delta \to \mathbb{R}_{\mathcal{I}}$ is ID-integrable on Δ, then we have

$$(ID) \iint_{\Delta} F(s,t)dA = (IR) \int_a^b (IR) \int_c^d F(s,t)dsdt.$$

Definition 2.6. *[30] A function $F : \Delta \to \mathbb{R}_{\mathcal{I}}^{+}$ is said to be interval valued co-ordinated convex function if the following inclusion holds*

$$\begin{aligned} & F(tx + (1-t)y, su + (1-s)w) \\ \supseteq \; & tsF(x,u) + t(1-s)F(x,w) + s(1-t)F(y,u) + (1-s)(1-t)F(y,w), \end{aligned} \tag{5}$$

for all $(x,u),(y,w) \in \Delta$ and $s,t \in [0,1]$.

3. Fractional Integrals of Interval-Valued Functions

In this section, we give fractional version of double integral for interval-valued functions and recall some basic definitions of interval-valued integrals.

In [16] Lupulescu defined the following interval-valued left-sided Riemann–Liouville fractional integral.

Definition 3.1. *Let $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ be an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and let $\alpha > 0$. The interval-valued left-sided Riemann–Liouville fractional integral of function F is defined by*

$$J_{a+}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_a^x (x-s)^{\alpha-1} F(t)dt, \quad x > a$$

where Γ is Euler Gamma function.

Based on the definition of Lupulescu, Budak et al. in [3] gave the definition of interval-valued right-sided Riemann–Liouville fractional integral of function F by

$$J_{b-}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_x^b (s-x)^{\alpha-1} F(t)dt, \quad x < b$$

where Γ is Euler Gamma function.

Theorem 3.2. *If $F : [a,b] \to \mathbb{R}_{\mathcal{I}}$ is an interval-valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$, then we have*

$$J_{a+}^{\alpha} F(x) = \left[I_{a+}^{\alpha} \underline{F}(x), I_{a+}^{\alpha} \overline{F}(x)\right],$$

and

$$J_{b-}^{\alpha} F(x) = \left[I_{b-}^{\alpha} \underline{F}(x), I_{b-}^{\alpha} \overline{F}(x)\right].$$

Now we recall the concept of interval-valued double integral given by Zhao et al. in [29]:
In [3], Budak et al. gave the fractional version of Hermite Hadamard type inequalities for interval-valued convex functions as follows:

Theorem 3.3. *If $F : [a,b] \to \mathbb{R}_{\mathcal{I}}^{+}$ is a convex interval–valued function such that $F(t) = \left[\underline{F}(t), \overline{F}(t)\right]$ and $\alpha > 0$, then we have*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} F(b) + J_{b-}^{\alpha} F(a)\right] \supseteq \frac{F(a)+F(b)}{2}. \tag{6}$$

By applying the concepts of Lupulescu [16] and Zhao [29] about interval–valued integrals, Budak et al. [1], define interval-valued Riemann–Liouville double fractional integral of function $F(x,y)$ as follows:

Definition 3.4. *Let $F \in \mathcal{ID}_{(\Delta)}$. The Riemann-Lioville integrals $J^{\alpha,\beta}_{a+,c+}$, $J^{\alpha,\beta}_{a+,d-}$, $+J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$of order $\alpha,\beta>0$ with $a,c\geq 0$ are defined by*

$$J^{\alpha,\beta}_{a+,c+}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}F(t,s)\,dsdt, \quad x>a,\ y>c,$$

$$J^{\alpha,\beta}_{a+,d-}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}F(t,s)\,dsdt, \quad x>a,\ y<d,$$

$$J^{\alpha,\beta}_{b-,c+}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}F(t,s)\,dsdt, \quad x<b,\ y>c,$$

$$J^{\alpha,\beta}_{b-,d-}F(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}(IR)\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}F(t,s)\,dsdt, \quad x<b,\ y<d,$$

respectively.

Lemma 3.5. *[30] A function $F:\Delta\to\mathbb{R}^+_{\mathcal{I}}$ is interval-valued convex on co-ordinates if and only if there exist two functions $F_x:[c,d]\to\mathbb{R}^+_{\mathcal{I}}$, $F_x(w)=F(x,w)$ and $F_y:[a,b]\to\mathbb{R}^+_{\mathcal{I}}$, $F_y(u)=F(y,u)$ are interval-valued convex.*

4. Weighted Hermite–Hadamard type Inclusions for Interval-Valued Convex Functions

In this section, we will give the following inclusions by using interval-valued convex functions.

Theorem 4.1. *Let $F:[a,b]\to\mathbb{R}^+_I$ be an interval-valued convex function on $[a,b]$ such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$ and $\theta:[0,1]\to\mathbb{R}^+_I$ be an interval-valued convex function such that $\Theta\in\mathcal{IR}_{([0,1])}$. Then the following inclusions hold:*

$$F\left(\frac{a+b}{2}\right)\supseteq\frac{1}{2\Theta(b-a)}(IR)\int_a^b\left[\theta\left(\frac{x-a}{b-a}\right)+\theta\left(\frac{b-x}{b-a}\right)\right]F(x)dx\supseteq\frac{F(a)+F(b)}{2} \tag{7}$$

where $\Theta=(IR)\int_0^1\theta(t)\,dt$.

Proof. Since F is interval-valued convex function on $[a,b]$, by change of variable $x=tb+(1-t)a$ we have

$$\begin{aligned}
&\frac{1}{2\Theta(b-a)}(IR)\int_a^b\left[\theta\left(\frac{x-a}{b-a}\right)+\theta\left(\frac{b-x}{b-a}\right)\right]F(x)\,dx\\
&=\frac{1}{2\Theta(b-a)}(IR)\int_0^1[\theta(t)+\theta(1-t)]F(tb+(1-t)a)(b-a)\,dt\\
&\supseteq\frac{1}{2\Theta}(IR)\int_0^1[\theta(t)+\theta(1-t)](tF(b)+(1-t)F(a))\,dt
\end{aligned}$$

$$= \frac{F(b)}{2\Theta}(IR)\int_0^1 t\left[\theta(t)+\theta(1-t)\right]dt + \frac{F(a)}{2I}(IR)\int_0^1 (1-t)\left[\theta(t)+\theta(1-t)\right]dt.$$

By simple calculation, we see that

$$(IR)\int_0^1 t\left(\theta(t)+\theta(1-t)\right)dt = (IR)\int_0^1 (1-t)\left[\theta(t)+\theta(1-t)\right]dt = (IR)\int_0^1 \theta(t)\,dt = \Theta.$$

So we have

$$\frac{1}{2\Theta(b-a)}(IR)\int_a^b \left[\theta\left(\frac{x-a}{b-a}\right)+\theta\left(\frac{b-x}{b-a}\right)\right]F(x)\,dx \supseteq \frac{F(a)+F(b)}{2}.$$

For proving the second part of the inclusion, considering the convexity of F, we have

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &= F\left(\frac{ta+(1-t)b+(1-t)a+tb}{2}\right) \\ &\supseteq \frac{1}{2}F(ta+(1-t)b)+\frac{1}{2}F((1-t)a+tb). \end{aligned}$$

Multiplying both sides by $\theta(t)$ and integrating on $[0,1]$ we obtain,

$$\begin{aligned} & F\left(\frac{a+b}{2}\right)(IR)\int_0^1 \theta(t)\,dt \\ \supseteq\ & \frac{1}{2}(IR)\int_0^1 \theta(t)F(ta+(1-t)b)\,dt + \frac{1}{2}(IR)\int_0^1 \theta(t)F((1-t)a+tb)\,dt \\ =\ & \frac{1}{2(b-a)}(IR)\int_a^b \left[\theta\left(\frac{x-a}{b-a}\right)+\theta\left(\frac{b-x}{b-a}\right)\right]F(x)\,dx. \end{aligned}$$

Hence

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{2\Theta(b-a)}(IR)\int_a^b \left[\theta\left(\frac{x-a}{b-a}\right)+\theta\left(\frac{b-x}{b-a}\right)\right]F(x)\,dx.$$

The proof is completed. □

Remark 4.2. *If we choose* $\theta(t) = [t,t]$, *on* $[0,1]$ *in Theorem 4.1, then the inclusions (7) reduce to Sadowska's Hermite Hadamard inclusions (3).*

Remark 4.3. *If we choose* $\theta(t) = \left[t^{\alpha-1}, t^{\alpha-1}\right]$ $(\alpha > 0)$ *on* $[0,1]$ *in Theorem 4.1, then the inclusions (7) reduce to the Riemann-Liouville fractional integral inclusions (6).*

Corollary 4.4. *Under assumption of Theorem 4.1 with* $\theta(t) = \left[(-\ln t)^{\alpha-1}, (-\ln t)^{\alpha-1}\right]$ $(\alpha > 0)$ *on* $[0,1]$, *we get the following inclusions*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{2\Gamma(\alpha)(b-a)}(IR)\int_a^b \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1} + \left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right]F(x)dx \supseteq \frac{F(a)+F(b)}{2}. \tag{8}$$

Corollary 4.5. *Let $F:[a,b]\to\mathbb{R}_I^+$ be interval-valued convex on $[a,b]$ and $F\in\mathcal{IR}_{([a,b])}$. Then, one has the inclusions:*

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &\supseteq \frac{1}{(b-a)}(IR)\int_a^b F(x)dx \\ &\supseteq \frac{1}{2(b-a)}(IR)\int_a^b \left[\ln\left(\frac{b-a}{b-x}\right)\right]+\left[\ln\left(\frac{b-a}{x-a}\right)\right]F(x)dx \\ &\supseteq \frac{F(a)+F(b)}{2}. \end{aligned} \tag{9}$$

Proof. From (3), we have

$$\begin{aligned} & F\left(\frac{a+b}{2}\right) \\ \supseteq\ & \frac{1}{(b-a)}(IR)\int_a^b F(x)\,dx \\ =\ & \frac{1}{(b-a)}\left[(IR)\int_a^{\frac{a+b}{2}} F(x)\,dx+(IR)\int_{\frac{a+b}{2}}^b F(x)\,dx\right]. \end{aligned} \tag{10}$$

By change of variable $x=\frac{a+t}{2}$ in the first integral of right side of (10), using inclusion (3), we get

$$\begin{aligned} (IR)\int_a^{\frac{a+b}{2}} F(x)\,dx &= \frac{1}{2}(IR)\int_a^b F\left(\frac{a+t}{2}\right)dt \\ &\supseteq \frac{1}{2}(IR)\int_a^b\left(\frac{1}{(t-a)}(IR)\int_a^t F(x)\,dx\right)dt \\ &= \frac{1}{2}(IR)\int_a^b\left((IR)\int_x^b \frac{1}{(t-a)}dt\right)F(x)\,dx \\ &= \frac{1}{2}(IR)\int_a^b\left(\ln\frac{b-a}{x-a}\right)F(x)\,dx. \end{aligned} \tag{11}$$

By similar way, using change of variable $x=\frac{b+t}{2}$ in the other integrals of right side of (10), respectively, we have

$$\begin{aligned} (IR)\int_{\frac{a+b}{2}}^b F(x)\,dx &= \frac{1}{2}(IR)\int_a^b F\left(\frac{b+t}{2}\right)dt \\ &\supseteq \frac{1}{2}(IR)\int_a^b\left(\frac{1}{(b-t)}(IR)\int_t^b F(x)\,dx\right)dt \\ &= \frac{1}{2}(IR)\int_a^b\left(\ln\frac{b-a}{b-x}\right)F(x)\,dx. \end{aligned} \tag{12}$$

By substituting the inclusions (11) and (12) in (10), and using the last inclusion of (8) for $\alpha=2$, we get

$$\begin{aligned} & F\left(\frac{a+b}{2}\right) \\ \supseteq\ & \frac{1}{(b-a)}(IR)\int_a^b F(x)\,dx \end{aligned}$$

$$\supseteq \frac{1}{2(b-a)}\left\{(IR)\int_a^b \left(\ln\frac{b-a}{x-a}\right)F(x)\,dx + (IR)\int_a^b \left(\ln\frac{b-a}{b-x}\right)F(x)\,dx\right\}$$

$$\supseteq \frac{F(a)+F(b)}{2}$$

which proves the inclusion (9). □

5. Weighted Hermite–Hadamard type Inclusions for Interval-Valued Co-ordinated Convex Functions

Throughout this section, we will use the following symbols

$$\Upsilon_1(x) = \upsilon_1\left(\frac{b-x}{b-a}\right) + \upsilon_1\left(\frac{x-a}{b-a}\right),$$

$$\Upsilon_2(y) = \upsilon_2\left(\frac{d-y}{d-c}\right) + \upsilon_2\left(\frac{y-c}{d-c}\right)$$

and

$$\Psi_{\upsilon_1} = (IR)\int_0^1 \upsilon_1(t)\,dt \text{ and } \Psi_{\upsilon_2} = (IR)\int_0^1 \upsilon_2(s)\,ds.$$

In this section, we will give the inclusions for interval-valued co-ordinated convex functions.

Theorem 5.1. *Let $F : \Delta \subset \mathbb{R}^2 \to \mathbb{R}_I^+$ be interval-valued co-ordinated convex on Δ in $\mathbb{R}^2$ and $F \in \mathcal{ID}_{(\Delta)}$. Let $\upsilon_1, \upsilon_2 : [0,1] \to \mathbb{R}_I^+$ be two functions such that $\upsilon_1, \upsilon_2 \in \mathcal{IR}_{([0,1])}$. Then, one has the inclusions:*

$$F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \supseteq \frac{1}{4\Psi_{\upsilon_1}\Psi_{\upsilon_2}(b-a)(d-c)}(IR)\int_a^b\int_c^d \Upsilon_1(x)\Upsilon_2(y)F(x,y)\,dydx \tag{13}$$
$$\supseteq \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.$$

Proof. According to (5) with $x = ta + (1-t)b$, $y = (1-t)a + tb$, $u = sc + (1-s)d$, $w = (1-s)c + sd$ and $t = s = \frac{1}{2}$, we find that

$$F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$
$$\supseteq \frac{1}{4}\left[F(ta+(1-t)b, sc+(1-s_1)d) + F(ta+(1-t)b, (1-s)c+sd)\right. \tag{14}$$
$$\left. + F((1-t)a+tb, sc+(1-s)d) + F((1-t)a+tb, (1-s)c+sd)\right].$$

Thus, multiplying both sides of (14) by $\upsilon_1(t)\upsilon_2(s)$, then by integrating with respect to (t,s) on $[0,1]\times[0,1]$, we obtain

$$F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)(IR)\int_0^1\int_0^1 \upsilon_1(t)\upsilon_2(s)\,dsdt$$

$$\supseteq \frac{1}{4}\left[(IR)\int_0^1\int_0^1 \upsilon_1(t)\,\upsilon_2(s)\,[F(ta+(1-t)b, sc+(1-s)d)+F(ta+(1-t)b,(1-s)c+sd)]\,dsdt\right.$$
$$\left.+(IR)\int_0^1\int_0^1 \upsilon_1(t)\,\upsilon_2(s)\,[F((1-t)a+tb, sc+(1-s)d)+F((1-t)a+tb,(1-s)c+sd)]\,dsdt\right].$$

Using the change of the variable, we get

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)(IR)\int_0^1\int_0^1 \upsilon_1(t)\,\upsilon_2(s)\,dsdt$$
$$\supseteq \frac{1}{4(b-a)(d-c)}\left\{(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{b-x}{b-a}\right)\upsilon_2\left(\frac{d-y}{d-c}\right)F(x,y)\,dydx\right.$$
$$+(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{b-x}{b-a}\right)\upsilon_2\left(\frac{y-c}{d-c}\right)F(x,y)\,dydx$$
$$+(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{x-a}{b-a}\right)\upsilon_2\left(\frac{d-y}{d-c}\right)F(x,y)\,dydx$$
$$\left.+(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{x-a}{b-a}\right)\upsilon_2\left(\frac{y-c}{d-c}\right)F(x,y)\,dydx\right\},$$

which completes the proof of the first inclusion. For the proof of the second inclusion in (13), we first note that if F is an interval-valued co-ordinated convex on Δ, then we have

$$F(ta+(1-t)b, sc+(1-s)d) \supseteq tsF(a,c)+s(1-t)F(b,c)+t(1-s)F(a,d)+(1-t)(1-s)F(b,d),$$
$$F(ta+(1-t)b,(1-s)c+sd) \supseteq t(1-s)F(a,c)+(1-t)(1-s)F(b,c)+tsF(a,d)+(1-t)sF(b,d),$$
$$F((1-t)a+tb, sc+(1-s)d) \supseteq (1-t)sF(a,c)+stF(b,c)+(1-t)(1-s)F(a,d)+t(1-s)F(b,d),$$

and

$$F((1-t)a+tb,(1-s)c+sd) \supseteq (1-t)(1-s)F(a,c)+t(1-s)F(b,c)+(1-t)sF(a,d)+tsF(b,d).$$

By adding these inclusions, we have

$$\begin{aligned}&F(ta+(1-t)b, sc+(1-s)d)+F(ta+(1-t)b,(1-s)c+sd)\\&+F((1-t)a+tb, sc+(1-s)d)+F((1-t)a+tb,(1-s)c+sd)\\ \supseteq\ &F(a,c)+F(b,c)+F(a,d)+F(b,d).\end{aligned} \tag{15}$$

Then, multiplying both sides of (15) by $\upsilon_1(t)\,\upsilon_2(s)$ and integrating with respect to (t,s) over $[0,1]\times[0,1]$, we get

$$(IR)\int_0^1\int_0^1 \upsilon_1(t)\,\upsilon_2(s)\,[F(ta+(1-t)b, sc+(1-s)d)+F(ta+(1-t)b,(1-s)c+sd)$$
$$+F((1-t)a+tb, sc+(1-s)d)+F((1-t)a+tb,(1-s)c+sd)]\,dsdt$$
$$\supseteq [F(a,c)+F(b,c)+F(a,d)+F(b,d)]\,(IR)\int_0^1\int_0^1 \upsilon_1(t)\,\upsilon_2(s)\,dsdt.$$

Here, using the change of the variable we have

$$\begin{aligned}
&\frac{1}{4(b-a)(d-c)}\left\{(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{b-x}{b-a}\right)\upsilon_2\left(\frac{d-y}{d-c}\right)F(x,y)\,dydx\right.\\
&+(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{b-x}{b-a}\right)\upsilon_2\left(\frac{y-c}{d-c}\right)F(x,y)\,dydx\\
&+(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{x-a}{b-a}\right)\upsilon_2\left(\frac{d-y}{d-c}\right)F(x,y)\,dydx\\
&\left.+(IR)\int_a^b\int_c^d \upsilon_1\left(\frac{x-a}{b-a}\right)\upsilon_2\left(\frac{y-c}{d-c}\right)F(x,y)\,dydx\right\}\\
\supseteq\;&\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}(IR)\int_0^1\int_0^1 \upsilon_1(t)\,\upsilon_2(s)\,dsdt.
\end{aligned}$$

The proof is completed. □

Remark 5.2. *If we choose* $\upsilon_1(t)=[t,t]$, $\upsilon_2(s)=[s,s]$ *on* $[0,1]$ *in Theorem 5.1, then the inclusions (7) become the inclusions*

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\supseteq\frac{1}{(b-a)(d-c)}(IR)\int_a^b\int_c^d F(x,y)\,dydx\supseteq\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4},\tag{16}$$

which are given by Kara et al. in [12].

Remark 5.3. *If we choose* $\upsilon_1(t)=\left[t^{\alpha-1},t^{\alpha-1}\right]$ $(\alpha>0)$, $\upsilon_2(s)=\left[s^{\beta-1},s^{\beta-1}\right]$ $(\beta>0)$ *on* $[0,1]$ *in Theorem 5.1, then the inclusions (7) become the fractional integral inclusions*

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\supseteq\;&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a+,c+}F(b,d)+J^{\alpha,\beta}_{a+,d-}F(b,c)+J^{\alpha,\beta}_{b-,c+}F(a,d)+J^{\alpha,\beta}_{b-,d-}F(a,c)\right]\\
\supseteq\;&\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}
\end{aligned}$$

which are proved by Kara et al. in [12].

Corollary 5.4. *Under assumption of Theorem 5.1 with* $\upsilon_1(t)=\left[(-\ln t)^{\alpha-1},(-\ln t)^{\alpha-1}\right]$, $(\alpha>0)$, $\upsilon_2(s)=\left[(-\ln s)^{\beta-1},(-\ln s)^{\beta-1}\right]$ (β *on* $[0,1]$, *we get the following inclusions*

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\supseteq\;&\frac{1}{4\Gamma(\alpha)\Gamma(\beta)(b-a)(d-c)}\\
&\times(IR)\int_a^b\int_c^d\left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1}+\left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right]\\
&\times\left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1}+\left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right]F(x,y)\,dydx
\end{aligned}\tag{17}$$

$$\supseteq \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.$$

Corollary 5.5. *Let $F:\Delta\subset\mathbb{R}^2\to\mathbb{R}_I^+$ be interval-valued co-ordinated convex on Δ in $\mathbb{R}^2$ with $0\le a<b,\ 0\le c<d$ and $F\in\mathcal{ID}_{(\Delta)}$. Then, one has the inclusions:*

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (18)\\
\supseteq\ & \frac{1}{(b-a)(d-c)}(IR)\int_a^b\int_c^d F(x,y)\,dydx\\
\supseteq\ & \frac{1}{4(b-a)(d-c)}\\
&\times(IR)\int_a^b\int_c^d\left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]+\left[\ln\left(\frac{b-a}{x-a}\right)\right]\right]\\
&\times\left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]+\left[\ln\left(\frac{d-c}{y-c}\right)\right]\right]F(x,y)\,dydx\\
\supseteq\ & \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.
\end{aligned}$$

Proof. From (16), we have

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (19)\\
\supseteq\ & \frac{1}{(b-a)(d-c)}(IR)\int_a^b\int_c^d F(x,y)\,dydx\\
=\ & \frac{1}{(b-a)(d-c)}\left\{(IR)\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} F(x,y)\,dydx+(IR)\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d F(x,y)\,dydx\right.\\
&\left.+(IR)\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} F(x,y)\,dydx+(IR)\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d F(x,y)\,dydx\right\}.
\end{aligned}$$

By change of variable $x=\frac{a+t}{2}$ and $y=\frac{c+s}{2}$ in the first integral of right side of (19), using inclusion (16) and from Fubini Theorem, we get

$$\begin{aligned}
&(IR)\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}} F(x,y)\,dydx \qquad (20)\\
=\ & \frac{1}{4}(IR)\int_a^b\int_c^d F\left(\frac{a+t}{2},\frac{c+s}{2}\right)dsdt\\
\supseteq\ & \frac{1}{4}(IR)\int_a^b\int_c^d\left(\frac{1}{(t-a)(s-c)}(IR)\int_a^t\int_c^s F(x,y)\,dydx\right)dsdt\\
=\ & \frac{1}{4}(IR)\int_a^b\int_c^d\left((IR)\int_x^b\int_y^d\frac{1}{(t-a)(s-c)}dsdt\right)F(x,y)\,dydx
\end{aligned}$$

$$= \frac{1}{4}(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{y-c}\right)F(x,y)\,dydx.$$

By similar way, using change of variable $x=\frac{a+t}{2}$ and $y=\frac{d+s}{2}$, $x=\frac{b+t}{2}$ and $y=\frac{c+s}{2}$, $x=\frac{b+t}{2}$ and $y=\frac{d+s}{2}$ in the other integrals of right side of (19), respectively, we have

$$\begin{aligned}
&(IR)\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d F(x,y)\,dydx \qquad (21)\\
=\;& \frac{1}{4}(IR)\int_a^b\int_c^d F\left(\frac{a+t}{2},\frac{d+s}{2}\right)dsdt\\
\supseteq\;& \frac{1}{4}(IR)\int_a^b\int_c^d\left(\frac{1}{(t-a)(d-s)}(IR)\int_a^t\int_s^d F(x,y)\,dydx\right)dsdt\\
=\;& \frac{1}{4}(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{d-y}\right)F(x,y)\,dydx,
\end{aligned}$$

$$\begin{aligned}
&(IR)\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}} F(x,y)\,dydx \qquad (22)\\
=\;& \frac{1}{4}(IR)\int_a^b\int_c^d F\left(\frac{b+t}{2},\frac{c+s}{2}\right)dsdt\\
\supseteq\;& \frac{1}{4}(IR)\int_a^b\int_c^d\left(\frac{1}{(b-t)(s-c)}(IR)\int_t^b\int_c^s F(x,y)\,dydx\right)dsdt\\
=\;& \frac{1}{4}(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{y-c}\right)F(x,y)\,dydx,
\end{aligned}$$

$$\begin{aligned}
&(IR)\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d F(x,y)\,dydx \qquad (23)\\
=\;& \frac{1}{4}(IR)\int_a^b\int_c^d F\left(\frac{b+t}{2},\frac{d+s}{2}\right)dsdt\\
\supseteq\;& \frac{1}{4}(IR)\int_a^b\int_c^d\left(\frac{1}{(b-t)(d-s)}(IR)\int_t^b\int_s^d F(x,y)\,dydx\right)dsdt\\
=\;& \frac{1}{4}(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{d-y}\right)F(x,y)\,dydx.
\end{aligned}$$

By substituting the inclusions (20)-(23) in (19) and using the last inclusion of (17) for $\alpha=\beta=2$, we have

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\supseteq\;& \frac{1}{(b-a)(d-c)}(IR)\int_a^b\int_c^d F(x,y)\,dydx\\
\supseteq\;& \frac{1}{4(b-a)(d-c)}\left\{(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{y-c}\right)F(x,y)\,dydx\right.
\end{aligned}$$

$$
\begin{aligned}
&+(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{x-a}\right)\left(\ln\frac{d-c}{d-y}\right)F(x,y)\,dydx\\
&+(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{y-c}\right)F(x,y)\,dydx\\
&+(IR)\int_a^b\int_c^d\left(\ln\frac{b-a}{b-x}\right)\left(\ln\frac{d-c}{d-y}\right)F(x,y)\,dydx\Bigg\}\\
\supseteq\ &\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}
\end{aligned}
$$

which is proved the inclusion (18). □

Theorem 5.6. *Let $F:\Delta\subset\mathbb{R}^2\to\mathbb{R}_I^+$ be interval-valued co-ordinated convex on Δ in $\mathbb{R}^2$ and $F\in\mathcal{ID}_{(\Delta)}$. Let $\upsilon_1,\upsilon_2:[0,1]\to\mathbb{R}_I^+$ be two functions such that $\upsilon_1,\upsilon_2\in\mathcal{IR}_{([0,1])}$. Then, one has the inclusions:*

$$
\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right) \qquad (24)\\
\supseteq\ &\frac{1}{4\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b\Upsilon_1(x)F\left(x,\frac{c+d}{2}\right)dx+\frac{1}{4\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d\Upsilon_2(y)F\left(\frac{a+b}{2},y\right)dy\\
\supseteq\ &\frac{1}{4\Psi_{\upsilon_1}\Psi_{\upsilon_2}(b-a)(d-c)}(IR)\int_a^b\int_c^d\Upsilon_1(x)\Upsilon_2(y)F(x,y)\,dydx\\
\supseteq\ &\frac{1}{8\Psi_{\upsilon_1}(b-a)}\left[(IR)\int_a^b\Upsilon_1(x)F(x,c)\,dx+(IR)\int_a^b\Upsilon_1(x)F(x,d)\,dx\right]\\
&+\frac{1}{8\Psi_{\upsilon_2}(d-c)}\left[(IR)\int_c^d\Upsilon_2(y)F(a,y)\,dy+(IR)\int_c^d\Upsilon_2(y)F(b,y)\,dy\right]\\
\supseteq\ &\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.
\end{aligned}
$$

Proof. Since $F:\Delta\to\mathbb{R}_I^+$ is interval-valued convex on the co-ordinates, it follows that the mapping $g_x:[c,d]\to\mathbb{R}$, $g_x(y)=F(x,y)$, is interval-valued convex on $[c,d]$ for all $x\in[a,b]$. Then by using inclusions (7), we can write

$$
g_x\left(\frac{c+d}{2}\right)\supseteq\frac{1}{2\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d\Upsilon_2(y)g_x(y)dy\supseteq\frac{g_x(c)+g_x(d)}{2},\quad x\in[a,b].
$$

That is,

$$
\begin{aligned}
F\left(x,\frac{c+d}{2}\right)\ &\supseteq\ \frac{1}{2\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d\Upsilon_2(y)F(x,y)\,dy\\
&\supseteq\ \frac{F(x,c)+F(x,d)}{2}, \qquad (25)
\end{aligned}
$$

for all $x\in[a,b]$. Then, multiplying both sides of (25) by $\frac{1}{2\Psi_{\upsilon_1}(b-a)}\Upsilon_1(x)$, and integrating with respect to x over $[a,b]$, we have

$$
\frac{1}{2\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b\Upsilon_1(x)F\left(x,\frac{c+d}{2}\right)dx \qquad (26)
$$

$$\supseteq \frac{1}{4\Psi_{\upsilon_1}\Psi_{\upsilon_2}(b-a)(d-c)}(IR)\int_a^b\int_c^d \Upsilon_1(x)\Upsilon_2(y)F(x,y)\,dydx$$
$$\supseteq \frac{1}{4\Psi_{\upsilon_1}(b-a)}\left[(IR)\int_a^b \Upsilon_1(x)F(x,c)\,dx+(IR)\int_a^b \Upsilon_1(x)F(x,d)\,dx\right].$$

By similar argument applied for the mapping $g_y:[a,b]\to\mathbb{R}$, $g_y(x)=F(x,y)$, we have

$$\frac{1}{2\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \Upsilon_2(y)F\left(\frac{a+b}{2},y\right)dy \tag{27}$$
$$\supseteq \frac{1}{4\Psi_{\upsilon_1}\Psi_{\upsilon_2}(b-a)(d-c)}(IR)\int_a^b\int_c^d \Upsilon_1(x)\Upsilon_2(y)F(x,y)\,dydx$$
$$\supseteq \frac{1}{4\Psi_{\upsilon_2}(d-c)}\left[(IR)\int_c^d \Upsilon_2(y)F(a,y)\,dy+(IR)\int_c^d \Upsilon_2(y)F(b,y)\,dy\right].$$

Adding the inclusions (26) and (27), we have

$$\frac{1}{2\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b \Upsilon_1(x)F\left(x,\frac{c+d}{2}\right)dx+\frac{1}{2\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \Upsilon_2(y)F\left(\frac{a+b}{2},y\right)dy$$
$$\supseteq \frac{1}{2\Psi_{\upsilon_1}\Psi_{\upsilon_2}(b-a)(d-c)}(IR)\int_a^b\int_c^d \Upsilon_1(x)\Upsilon_2(y)F(x,y)\,dydx$$
$$\supseteq \frac{1}{4\Psi_{\upsilon_1}(b-a)}\left[(IR)\int_a^b \Upsilon_1(x)F(x,c)\,dx+(IR)\int_a^b \Upsilon_1(x)F(x,d)\,dx\right]$$
$$+\frac{1}{4\Psi_{\upsilon_2}(d-c)}\left[(IR)\int_c^d \Upsilon_2(y)F(a,y)\,dy+(IR)\int_c^d \Upsilon_2(y)F(b,y)\,dy\right],$$

which give the second and the third inclusions in (24).

Now, by using the first inclusion in (7), we also have

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\supseteq \frac{1}{2\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b \Upsilon_1(x)F\left(x,\frac{c+d}{2}\right)dx,$$

and

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\supseteq \frac{1}{2\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \Upsilon_2(y)F\left(\frac{a+b}{2},y\right)dy$$

by addition,

$$F\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\supseteq \frac{1}{4\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b \Upsilon_1(x)F\left(x,\frac{c+d}{2}\right)dx+\frac{1}{4\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \Upsilon_2(y)F\left(\frac{a+b}{2},y\right)dy,$$

which give the first inclusion in (24).

Finaly, by using the second incluson in (7) we can also state,

$$\frac{1}{2\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b \Upsilon_1(x)F(x,c)\,dx\supseteq \frac{F(a,c)+F(b,c)}{2},$$

$$\frac{1}{2\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b \Upsilon_1(x)F(x,d)\,dx \supseteq \frac{F(a,d)+F(b,d)}{2},$$

$$\frac{1}{2\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \Upsilon_2(y)F(a,y)\,dy \supseteq \frac{F(a,c)+F(a,d)}{2},$$

and

$$\frac{1}{2\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \Upsilon_2(y)F(b,y)\,dy \supseteq \frac{F(b,c)+F(b,d)}{2},$$

which give, by addition, the last inclusion in (24). □

Remark 5.7. *If in Theorem 5.6 with* $\upsilon_1(t)=[t,t]$, $\upsilon_2(s)=[s,s]$ *on* $[0,1]$, *then the inclusions (24) become the inclusions*

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\supseteq\;& \frac{1}{2}\left[\frac{1}{b-a}(IR)\int_a^b F\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}(IR)\int_c^d F\left(\frac{a+b}{2},y\right)dy\right]\\
\supseteq\;& \frac{1}{(b-a)(d-c)}(IR)\int_a^b\int_c^d F(x,y)\,dydx\\
\supseteq\;& \frac{1}{4}\left[\frac{1}{b-a}(IR)\int_a^b F(x,c)\,dx+\frac{1}{b-a}(IR)\int_a^b F(x,d)\,dx\right.\\
&\left.+\frac{1}{d-c}(IR)\int_c^d F(a,y)\,dy+\frac{1}{d-c}(IR)\int_c^d F(b,y)\,dy\right]\\
\supseteq\;& \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}
\end{aligned}$$

which are proved by Zhao et al. in [30].

Remark 5.8. *If in Theorem 5.6 with* $\upsilon_1(t)=\left[t^{\alpha-1},t^{\alpha-1}\right]$ $(\alpha>0)$, $\upsilon_2(s)=\left[s^{\beta-1},s^{\beta-1}\right]$ $(\beta>0)$ *on* $[0,1]$, *then the inclusions (24) become the fractional integral inclusions*

$$\begin{aligned}
&F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\supseteq\;& \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{a+}^{\alpha}F\left(b,\frac{c+d}{2}\right)+J_{b-}^{\alpha}F\left(a,\frac{c+d}{2}\right)\right]\\
&+\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{c+}^{\beta}F\left(\frac{a+b}{2},d\right)+J_{d-}^{\beta}F\left(\frac{a+b}{2},c\right)\right]\\
\supseteq\;& \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J_{a+,c+}^{\alpha,\beta}F(b,d)+J_{a+,d-}^{\alpha,\beta}F(b,c)+J_{b-,c+}^{\alpha,\beta}F(a,d)+J_{b-,d-}^{\alpha,\beta}F(a,c)\right]\\
\supseteq\;& \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J_{a+}^{\alpha}F(b,c)+J_{a+}^{\alpha}F(b,d)+J_{b-}^{\alpha}F(a,c)+J_{b-}^{\alpha}F(a,d)\right]\\
&+\frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J_{c+}^{\beta}F(a,d)+J_{c+}^{\beta}F(b,d)+J_{d-}^{\beta}F(a,c)+J_{d-}^{\beta}F(b,c)\right]
\end{aligned}$$

$$\supseteq \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}$$

which are proved by Budak et al. in [1].

Corollary 5.9. *Under assumption of Theorem 5.6 with* $\upsilon_1(t) = \left[(-\ln t)^{\alpha-1}, (-\ln t)^{\alpha-1}\right]$ $(\alpha > 0)$, $\upsilon_2(s) = \left[(-\ln s)^{\beta-1}, (-\ln s)^{\beta-1}\right]$ $(\beta$ *on* $[0,1]$, *we get the following inclusions*

$$\begin{aligned}
& F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\supseteq\ & \frac{1}{4\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1} + \left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right] F\left(x, \frac{c+d}{2}\right)dx \\
& + \frac{1}{4\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1} + \left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right] F\left(\frac{a+b}{2}, y\right)dy \\
\supseteq\ & \frac{1}{4\Psi_{\upsilon_1}\Psi_{\upsilon_2}(b-a)(d-c)}(IR)\int_a^b \int_c^d \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1} + \left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right] \\
& \times \left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1} + \left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right] F(x,y)\,dydx \\
\supseteq\ & \frac{1}{8\Psi_{\upsilon_1}(b-a)}(IR)\int_a^b \left[\left[\ln\left(\frac{b-a}{b-x}\right)\right]^{\alpha-1} + \left[\ln\left(\frac{b-a}{x-a}\right)\right]^{\alpha-1}\right] [F(x,c)+F(x,d)]\,dx \\
& + \frac{1}{8\Psi_{\upsilon_2}(d-c)}(IR)\int_c^d \left[\left[\ln\left(\frac{d-c}{d-y}\right)\right]^{\beta-1} + \left[\ln\left(\frac{d-c}{y-c}\right)\right]^{\beta-1}\right] [F(a,y)+F(b,y)]\,dy \\
\supseteq\ & \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{4}.
\end{aligned}$$

6. Conclusion

In this paper, by using interval-valued convex and interval-valued co-ordinated convex mappings, we obtain some interval-valued weighted Hermite–Hadamard inequalities for differentiable convex mappings. In addition, we discussed the special cases of the main results and we showed that our results generalize several well-known Riemann Liouville fractional type integrals and Logarithmic integrals. In future works, authors can obtain some new results by applying inclusions for other kinds of convex functions to our newly proved identities.

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