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Boundedness of derivatives and φ-normal functions

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Abstract. The aim of this paper is to study the families of normal and φ-normal functions on the unit disk $\mathbb{D}$, and to generalize some normal function criteria of Xu and Qiu [*An avoidance criterion for normal functions, C. R. Math 349(2011), 1159-1160*] and Yang [*A note on the avoidance criterion for normal functions, Anal. Math. Phys. 10, 35(2020)*] to the case where derivatives are bounded from above on zero sets.

1. Introduction and Main Results

For the sake of convenience we shall denote by $\mathcal{M}(D)$ the family of all functions meromorphic in a domain D in $\mathbb{C}$. A subfamily $\mathcal{F}$ of $\mathcal{M}(D)$ is said to be normal in D, in the sense of Montel, if every sequence of elements of $\mathcal{F}$ contains a subsequence which converges locally uniformly in D with respect to the spherical metric to a meromorphic function or ∞. One of the key results of normal families is the Marty's theorem which says that a subfamily of $\mathcal{F}$ of $\mathcal{M}(D)$ is normal in D if and only if the family $\left\{f^{\#} : f \in \mathcal{F}\right\}$ of the corresponding spherical derivatives $f^{\#} := \frac{|f'|}{1+|f|^2}$ is locally bounded in D.

A meromorphic function f on unit disc $\mathbb{D}$ is said to be *normal* in $\mathbb{D}$ if and only if the family $\mathcal{F} := \{f o \tau : \tau \in \mathcal{T}\}$ is normal in $\mathbb{D}$, where $\mathcal{T}$ denotes the set of all conformal self maps of $\mathbb{D}$.

The starting point for this paper is the following two results due to Y. Xu and H. Qiu [9]:

Theorem A *Let f be a meromorphic function in the unit disc $\mathbb{D}$, ψ_1, ψ_2 and ψ_3 be three functions meromorphic in $\mathbb{D}$ and continuous on closure of $\mathbb{D}$ such that $\psi_i \neq \psi_j$ $(1 \leq i < j \leq 3)$ on the unit circle $|z| = 1$. If $f(z) \neq \psi_i(z)$ $(i = 1, 2, 3)$ in $\mathbb{D}$, then f is normal.*

Theorem B *Let f be a meromorphic function in the unit disc $\mathbb{D}$, ψ_1, ψ_2 and ψ_3 be three functions meromorphic in $\mathbb{D}$ and continuous on closure of $\mathbb{D}$ such that $\psi_i \neq \psi_j$ $(1 \leq i < j \leq 3)$ on the unit circle $|z| = 1$, and let l_1, l_2, l_3 (possibly $+\infty$) be three positive integers with $1/l_1 + 1/l_2 + 1/l_3 < 1$. If all the zeros of $f(z) - \psi_i(z)$ have multiplicity at least l_i for $i = 1, 2, 3$ in $\mathbb{D}$, then f is normal.*

Often a theorem which assuming that a function does not vanish or vanishes to sufficiently large multiplicities can be strengthened by assuming that whenever it vanishes, their derivatives are bounded

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from above on zero sets. Here, in this paper we prove the following criterion for a normal function under a boundedness condition of the derivatives and hence obtain a generalization of above-mentioned results of Xu and Qiu:

Theorem 1.1. *Let f be a meromorphic function in the unit disc $\mathbb{D}$ and let $M > 0$ be a constant. Assume that there are*

1. *positive integer $l_1, l_2, \cdots, l_q$ (possibly $+\infty$) satisfying $\sum_{j=1}^{q} 1/l_j < q-2$,*
2. *meromorphic functions $\psi_1, \psi_2, \cdots, \psi_q$ in $\mathbb{D}$, continuous on closure of $\mathbb{D}$ such that $\psi_i(z) \neq \psi_j(z)$ $(1 \le i < j \le q)$ on the unit circle $|z| = 1$, and*
3. *sets $A_f = \cup_{j=1}^{q}\{z \in \mathbb{D} : f(z) = \psi_j(z) \neq \infty\}$ and $B_f = \cup_{j=1}^{q}\{z \in \mathbb{D} : f(z) = \psi_j(z) = \infty\}$ such that*

$$|f^{(k)}(z)| \le M \ on \ A_f \quad , \quad \left|\left(\frac{1}{f}\right)^{(k)}(z)\right| \le M \ on \ B_f$$

for all $k = 1, 2, \cdots, l_j - 1$.

Then f is normal in $\mathbb{D}$.

One of the most important results characterizing normal functions in terms of their spherical derivatives was due to Lehto and Virtanen [4]. They modified Marty's criterion for a normal family to give a criterion for a function to be normal: *A necessary and sufficient condition for a meromorphic function f on unit disc $\mathbb{D}$ to be normal is*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)f^{\#}(z) < \infty.$$

Clearly, if f is a normal function on $\mathbb{D}$, then there exist a constant C_f (depending on f) such that $(1-|z|^2)f^{\#}(z) < C_f$ for each $z \in \mathbb{D}$. Recently, L. Yang [10] improved Theorem A for the families of meromorphic functions and obtained a constant C that depends only on the three fixed omitted meromorphic functions. Precisely, he proved:

Theorem C *Let ψ_1, ψ_2 and ψ_3 be three functions meromorphic in the unit disc $\mathbb{D}$ and continuous on closure of $\mathbb{D}$ such that $\psi_i \neq \psi_j$ $(1 \le i < j \le 3)$ on the unit circle $|z| = 1$. Let $\mathcal{F}$ be a subfamily of $\mathcal{M}(\mathbb{D})$ such that $f(z) \neq \psi_i(z)$ $(i = 1, 2, 3)$ in $\mathbb{D}$, for all $f \in \mathcal{F}$. Then there exists a constant C such that*

$$(1-|z|^2)f^{\#}(z) \le C$$

for each $z \in \mathbb{D}$ and $f \in \mathcal{F}$.

Remark. It is worthwhile to mention the conclusion of Theorem C is nothing but the definition of uniformly normal family (see [6]). That is, if there is a constant C such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)f^{\#}(z) < C$$

for each $z \in \mathbb{D}$ and $f \in \mathcal{F}$, then $\mathcal{F}$ is uniformly normal family in $\mathbb{D}$.

We prove the following improvement of Theorem C to the case where omitted functions are allowed to vary with the functions in family $\mathcal{F}$ and satisfy a condition on the spherical distance:

Theorem 1.2. *Let $\mathcal{F}$ be a subfamily of $\mathcal{M}(\mathbb{D})$ and $\epsilon > 0$. Assume that for each $f \in \mathcal{F}$ there exists meromorphic functions a_f, b_f, c_f (possibly $+\infty$) such that $f \neq a_f, b_f, c_f$ in $\mathbb{D}$ and*

$$min\{\sigma(a_f(z), b_f(z)), \sigma(b_f(z), c_f(z)), \sigma(c_f(z), a_f(z))\} \ge \epsilon$$

in $\mathbb{D}$, where σ denotes the spherical metric on extended complex plane $\overline{\mathbb{C}}$. Then $\mathcal{F}$ is uniformly normal family in $\mathbb{D}$.

Finally, we extend Theorem C for $\varphi-$normal functions, a concept that was an extension of normal function introduced by Aulaskari and Rattya [1] with the help of smoothly increasing function $\varphi : [0,1) \to (0,\infty)$ satisfying $\varphi(r)(1-r) \to \infty$ as $r \to 1^-$, and

$$\mathcal{R}_a(z) := \frac{\varphi(|a + z/\varphi(|a|))}{\varphi(|a|)} \to 1, \quad |a| \to 1^-,$$

uniformly on compact subsets of $\mathbb{C}$. For such function φ, a meromorphic function f on unit disc $\mathbb{D}$ is said to be *φ-normal* if

$$\sup_{z\in\mathbb{D}} \frac{f^{\#}(z)}{\varphi(|z|)} < \infty.$$

Clearly, if $\mathcal{F}$ is a subfamily of $\mathcal{M}(\mathbb{D})$ such that each function in $\mathcal{F}$ is a φ-normal function, then for each $f \in \mathcal{F}$, there exist a constant C_f such that $\sup_{z\in\mathbb{D}} \frac{f^{\#}(z)}{\varphi(|z|)} < C_f$ for each $z \in \mathbb{D}$. Now, we give the following definition:

Definition 1.3. *Let $\mathcal{F}$ be a subfamily of $\mathcal{M}(\mathbb{D})$. If there exist a constant C such that*

$$\sup_{z\in\mathbb{D}} \frac{f^{\#}(z)}{\varphi(|z|)} < C$$

for each $z \in \mathbb{D}$ and $f \in \mathcal{F}$, then $\mathcal{F}$ is uniformly φ-normal family in $\mathbb{D}$.

Theorem 1.4. *Let $\varphi : [0,1) \to (0,\infty)$ be a smoothly increasing function, $\mathcal{F}$ be a subfamily of $\mathcal{M}(\mathbb{D})$ and let $M > 0$ be a constant. Assume that there are*

1. *positive integer $l_1, l_2, \cdots, l_q$ (possibly $+\infty$) satisfying $\sum_{j=1}^{q} 1/l_j < q-2$,*
2. *meromorphic functions $\psi_{1f}, \psi_{2f}, \cdots, \psi_{qf}$ ($f \in \mathcal{F}$) in $\mathbb{D}$, positive constant ϵ such that $\sigma(\psi_{if}(z), \psi_{jf}(z)) \geq \epsilon$ ($1 \leq i < j \leq q$) for all $z \in \mathbb{D}$, where σ denotes the spherical metric on extended complex plane $\overline{\mathbb{C}}$, and*
3. *sets $A_f = \cup_{j=1}^{q}\{z \in \mathbb{D} : f(z) = \psi_{jf}(z) \neq \infty\}$ and $B_f = \cup_{j=1}^{q}\{z \in \mathbb{D} : f(z) = \psi_{jf}(z) = \infty\}$ such that*

$$|f^{(k)}(z)| \leq M\varphi^k(|z|) \ \text{on}\ A_f \quad , \quad \left|\left(\frac{1}{f}\right)^{(k)}(z)\right| \leq M\varphi^k(|z|) \ \text{on}\ B_f$$

for all $k = 1, 2, \cdots, l_j - 1$.

Then $\mathcal{F}$ is uniformly φ-normal family in $\mathbb{D}$.

2. Proof of the Main Results

We assume that the reader is familiar with the basic notions used in Value distribution theory of meromorphic functions such as $T(r,f), m(r,f), N(r,f), o(T(r,f))$ etc. For deeper insight one can refer to [3]. Further for the proof of our main results we require the following lemmas.

Lemma 2.1. *(Zalcman's Lemma) [11] Let $\mathcal{F}$ be a subfamily of $\mathcal{M}(\mathbb{D})$. Then $\mathcal{F}$ is not normal in $\mathbb{D}$ if and only if there exist*

- *a real number r: $r < 1$,*
- *points z_n: $|z_n| < r$,*
- *positive numbers ρ_n: $\rho_n \to 0$,*

- *functions* $f_n \in \mathcal{F}$

such that

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta)$$

converges locally uniformly with respect to the spherical metric to $g(\zeta)$*, where* $g(\zeta)$ *is a non-constant meromorphic function on* $\mathbb{C}$.

Lemma 2.2. *(Lohwater-Pommerenke Theorem)[5] A function* f *meromorphic in* $\mathbb{D}$ *is a normal function if and only if there do not exist sequences* $\{z_n\}$ *and* $\{\rho_n\}$ *with* $z_n \in \mathbb{D}$*, and* $\rho_n > 0$*,* $\rho_n \to 0$ *such that* $g_n(\zeta) = f(z_n + \rho_n \zeta)$ *converges uniformly on each compact subset of* $\mathbb{C}$ *to a function* $g(\zeta)$*, where* $g(\zeta)$ *is a non-constant meromorphic function.*

Lemma 2.3. *(Simultaneous rescaling version of Zalcman's Lemma) [2] Let* p *be a natural number and* $\mathcal{F} \subseteq (\mathcal{M}(\mathbb{D}))^p$*. Assume that there exist* $j_0 \in \{1, \cdots, p\}$ *such that the family* $\pi_{j_0}(\mathcal{F})$ *of projections is not normal at* $z_0 \in \mathbb{D}$*. Then there exist sequences* $\{f_n\} = \{(f_{1,n}, \cdots, f_{p,n})\} \subseteq \mathcal{F}$*,* $\{z_n\} \subset \mathbb{D}$ *with* $z_n \to z_0$ *and positive numbers* $\{\rho_n\}$ *with* $\rho_n \to 0$ *and such that for all* $j = 1, \cdots, p$ *the sequences* $\{g_{j,n}\}$ *defined by*

$$g_{j,n} := f_{j,n}(z_n + \rho_n \zeta)$$

converge to functions $g_j \in \mathcal{M}(\mathbb{C}) \cup \{\infty\}$ *locally uniformly in* $\mathbb{C}$ *(with respect to the spherical metric) where at least one of the functions* $g_1, \cdots, g_p$ *is not constant.*

Lemma 2.4. *[2] Let* $\mathcal{F} \subseteq (\mathcal{M}(\mathbb{D}))^2$ *be a family of pairs of meromorphic functions in* $\mathbb{D}$ *and* $\epsilon > 0$*. Assume that*

$$\sigma(a(z), b(z)) \geq \epsilon, \text{ for all } (a, b) \in \mathcal{F} \text{ and all } z \in \mathbb{D}.$$

Then the families $\{a : (a, b) \in \mathcal{F}\}$ *and* $\{b : (a, b) \in \mathcal{F}\}$ *are normal in* $\mathbb{D}$*.*

Proof. **[Proof of Theorem 1.2.]** Suppose that $\mathcal{F}$ is not uniformly normal in $\mathbb{D}$. Then we can find sequences $\{f_\nu\} \subset \mathcal{F}$, $\{a_{f_\nu}\}, \{b_{f_\nu}\}, \{c_{f_\nu}\} \subseteq \mathcal{M}(\mathbb{D}) \cup \{\infty\}$ and $\{z_\nu\} \subset \mathbb{D}$ such that

$$\min\{\sigma(a_{f_\nu}(z), b_{f_\nu}(z)), \sigma(b_{f_\nu}(z), c_{f_\nu}(z)), \sigma(c_{f_\nu}(z), a_{f_\nu}(z))\} \geq \epsilon$$

and

$$(1 - |z_\nu|^2) f_\nu^\#(z_\nu) \to \infty \text{ as } \nu \to \infty.$$

Since

$$(1 - |z_\nu|) f_\nu^\#(z_\nu) = \frac{1 - |z_\nu|^2}{1 + |z_\nu|} f_\nu^\#(z_\nu) \geq \frac{1}{2}(1 - |z_\nu|^2) f_\nu^\#(z_\nu),$$

it follows that

$$(1 - |z_\nu|) f_\nu^\#(z_\nu) \to \infty \text{ as } \nu \to \infty. \tag{1}$$

We define

$$g_\nu(z) := f_\nu(z_\nu + (1 - |z_\nu|) z)$$

and

$$a_{g_\nu}(z) := a_{f_\nu}(z_\nu + (1 - |z_\nu|) z), b_{g_\nu}(z) := b_{f_\nu}(z_\nu + (1 - |z_\nu|) z), c_{g_\nu}(z) := c_{f_\nu}(z_\nu + (1 - |z_\nu|) z)$$

for $z \in \mathbb{D}$. Then by (1), we have

$$g_\nu^\#(0) = (1 - |z_\nu|) f_\nu^\#(z_\nu) \to \infty \text{ as } \nu \to \infty.$$

Thus by Marty's theorem, it follows that $\{g_\nu\}$ is not normal at the origin.

Consider the family of quadruples $\widehat{\mathcal{F}} = \left\{(g_\nu, a_{g_\nu}, b_{g_\nu}, c_{g_\nu}) : \nu \in \mathbb{N}\right\}$. Since $\pi_1(\widehat{\mathcal{F}}) = \{g_\nu\}$ fails to normal at the origin, Lemma 2.3 guarantees the existence of subsequences $\left\{\{g_\nu\}, \left\{a_{g_\nu}\right\}, \left\{b_{g_\nu}\right\}, \left\{c_{g_\nu}\right\}\right\} \subset \widehat{\mathcal{F}}$ (for the sake of simplicity, we take the same sequences), points $w_\nu \to 0$, positive numbers $\rho_\nu \to 0$ such that g_ν omits $a_{g_\nu}, b_{g_\nu}, c_{g_\nu}$ and

$$\min\{\sigma(a_{g_\nu}(z), b_{g_\nu}(z)), \sigma(b_{g_\nu}(z), c_{g_\nu}(z)), \sigma(c_{g_\nu}(z), a_{g_\nu}(z))\} \geq \epsilon \tag{2}$$

for all $z \in \mathbb{D}$ and all ν and such that the sequences $\{G_\nu\}, \{A_\nu\}, \{B_\nu\}$ and $\{C_\nu\}$ defined by

$$G_\nu(\xi) := g_\nu(w_\nu + \rho_\nu \xi),$$

$$A_\nu(\xi) := a_{g_\nu}(w_\nu + \rho_\nu \xi),\ B_\nu(\xi) := b_{g_\nu}(w_\nu + \rho_\nu \xi),\ C_\nu(\xi) := c_{g_\nu}(w_\nu + \rho_\nu \xi)$$

converge locally uniformly in $\mathbb{C}$ to functions $G, A, B, C \in \mathcal{M}(\mathbb{C}) \cup \{\infty\}$, respectively, not all of which are constant. Further, Lemma 2.4 ensures that the families $\{a_{g_\nu}\}, \{b_{g_\nu}\}$ and $\{c_{g_\nu}\}$ are normal, and so by Lemma 2.1 we find that A, B and C are constant and consequently G is non-constant. On the other hand, g_ν omits $a_{g_\nu}, b_{g_\nu}, c_{g_\nu}$, we have by Hurwitz's theorem, G omits three distinct constant A, B and C. Hence, by Picard's theorem, G is constant. This is a contradiction. □

Proof. **[Proof of Theorem 1.4.]** Suppose that $\mathcal{F}$ is not uniformly φ-normal family on $\mathbb{D}$. Then there exist sequences $\{f_\nu\} \subset \mathcal{F}$, $\{z_\nu\} \subset \mathbb{D}$ and

- positive integer $l_1, l_2, \cdots, l_q$ (posibbly $+\infty$) satisfying $\sum_{j=1}^{q} 1/l_j < q - 2$,
- meromorphic functions $\psi_{1f_\nu}, \psi_{2f_\nu}, \cdots, \psi_{qf_\nu}$ $(f \in \mathcal{F})$ in $\mathbb{D}$, positive constant ϵ such that $\sigma(\psi_{if_\nu}(z), \psi_{jf_\nu}(z)) \geq \epsilon$ $(1 \leq i < j \leq q)$ for all $z \in \mathbb{D}$, and
- sets $A_{f_\nu} = \cup_{j=1}^{q}\{z \in \mathbb{D} : f_\nu(z) = \psi_{jf_\nu}(z) \neq \infty\}$ and $B_{f_\nu} = \cup_{j=1}^{q}\{z \in \mathbb{D} : f_\nu(z) = \psi_{jf_\nu}(z) = \infty\}$ with $|f_\nu^{(k)}(z)| \leq M\varphi^k(|z|)\ z \in A_{f_\nu}$, $\left|\left(\frac{1}{f_\nu}\right)^{(k)}(z)\right| \leq M\varphi^k(|z|)\ z \in B_{f_\nu}$ $(1 \leq k \leq l_j - 1)$

such that

$$\frac{f_\nu^{\#}(z_\nu)}{\varphi(|z_\nu|)} \to \infty \text{ as } \nu \to \infty. \tag{3}$$

Passing to a subsequence (if necessary), we may assume that $z_\nu \to z_0 \in \overline{\mathbb{D}}$. We consider the following cases:

Case 1. $|z_0| = 1$. Consider the family

$$\left\{g_\nu(z) := f_\nu\left(z_\nu + \frac{z}{\varphi(|z_\nu|)}\right),\ z \in \mathbb{D}\right\}.$$

Since $\varphi : [0, 1) \to (0, \infty)$ is a smoothly increasing function satisfying $\varphi(r)(1 - r) \to \infty$ as $r \to 1^-$, we can assume that $\varphi(r)(1 - r) \geq 1$ for all $r \in [0, 1)$. Using this and $|z_\nu| \to 1^-$, we conclude that for ν sufficiently large

$$\varphi(|z_\nu|)(1 - |z_\nu|) \geq 1.$$

Therefore,

$$\left|z_\nu + \frac{z}{\varphi(|z_\nu|)}\right| \leq |z_\nu| + \frac{|z|}{\varphi(|z_\nu|)} < |z_\nu| + \frac{1}{\varphi(|z_\nu|)} \leq |z_\nu| + (1 - |z_\nu|) = 1$$

for each $z \in \mathbb{D}$, so that the function g_ν is well-defined in $\mathbb{D}$, for all ν.
Now, by using (3), we get

$$g_\nu^{\#}(0) = \frac{f_\nu^{\#}(z_\nu)}{\varphi(|z_\nu|)} \to \infty \text{ as } \nu \to \infty,$$

Thus Marty's theorem implies that $\{g_\nu\}$ is not normal at 0. By Lemma 2.1, we can find a subsequence of $\{g_\nu\}$, one may take $\{g_\nu\}$ itself, $\{v_\nu\} \subset \mathbb{D}$ with $v_\nu \to 0$, positive numbers σ_ν with $\sigma_\nu \to 0$ such that

$$G_\nu(\xi) := g_\nu(v_\nu + \sigma_\nu \xi) = f_\nu\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi}{\varphi(|z_\nu|)}\right)$$

locally uniformly with respect to the spherical metric to $G(\xi)$, where $G(\xi)$ is a non-constant meromorphic function on $\mathbb{C}$. Therefore, on every compact subset of $\mathbb{C}$ that contains no poles of G, $G_\nu^{(j)}$ converges uniformly to $G^{(j)}$ for all $j \in \mathbb{N}$. Similarly, on every compact subset of $\mathbb{C}$ containing no zeros of G, $(1/G_\nu)^{(j)}$ converges uniformly to $(1/G)^{(j)}$ for all $j \in \mathbb{N}$. Also, Lemma 2.4 ensures that $\{\psi_{jf_\nu}\}_{\nu \geq 1}$ is a normal family on $\mathbb{D}$ and so passing to a subsequence, we assume that $\{\psi_{jf_\nu}\}_{\nu\geq 1}$ converges spherically uniformly on $\mathbb{C}$ to a meromorphic function ψ_j (or ∞) for all $j = 1, 2, \ldots, q$. Thus

$$\psi_{jf_\nu}\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi}{\varphi(|z_\nu|)}\right) \to \psi_j(z_0)$$

spherically locally uniformly on $\mathbb{C}$. Now, we claim:

1. For any $j \in \{1, 2, \cdots, q\}$, if $\psi_j(z_0) \neq \infty$, then all the zeros of $G - \psi_j(0)$ have multiplicity at least l_j.

2. For some $j \in \{1, 2, \ldots, q\}$, if $\psi_j(z_0) = \infty$, then all the poles of G have multiplicity at least l_j.

Suppose that $\psi_j(0) \neq \infty$ and for any fixed j, let ξ_0 be a zero $G(\xi) - \psi_{jf_\nu}(0)$. Then G is holomorphic at ξ_0. By Hurwitz's theorem there exist a sequence of points $\xi_\nu \to \xi_0$ such that for all ν sufficiently large, $\psi_{jf_\nu}\left(z_\nu + \frac{v_\nu+\sigma_\nu\xi_\nu}{\varphi(|z_\nu|)}\right) \neq \infty$ and

$$0 = G_\nu(\xi_\nu) - \psi_{jf_\nu}\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right) = f_\nu\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right) - \psi_{jf_\nu}\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right)$$

$$\Rightarrow f_\nu\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right) = \psi_{jf_\nu}\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right), \text{ for sufficiently large } \nu.$$

By hypothesis, we have

$$\left|f_\nu^{(k)}\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right)\right| \leq M\varphi^k\left(\left|z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right|\right) \tag{4}$$

for ν sufficiently large and $k = 1, 2, \cdots, l_j - 1$.
Now,

$$\begin{aligned} G_\nu^{(k)}(\xi_\nu) &= \left(\frac{\sigma_\nu}{\varphi(|z_\nu|)}\right)^k \left|f_\nu^{(k)}\left(z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right)\right| \\ &\leq \left(\frac{\sigma_\nu}{\varphi(|z_\nu|)}\right)^k M\varphi^k\left(\left|z_\nu + \frac{v_\nu + \sigma_\nu \xi_\nu}{\varphi(|z_\nu|)}\right|\right) \\ &= \sigma_\nu^k M \left[\frac{\varphi\left(\left|z_\nu + \frac{v_\nu+\sigma_\nu\xi_\nu}{\varphi(|z_\nu|)}\right|\right)}{\varphi(|z_\nu|)}\right]^k. \end{aligned}$$

Since

$$\frac{\varphi\left(\left|z_\nu + \frac{v_\nu+\sigma_\nu\xi_\nu}{\varphi(|z_\nu|)}\right|\right)}{\varphi(|z_\nu|)} \to 1 \text{ as } \nu \to \infty,$$

we have

$$G^{(k)}(\xi_0) = \lim_{\nu\to\infty} G_\nu^{(k)}(\xi_\nu) = \lim_{\nu\to\infty} \sigma_\nu^k M \left[\frac{\varphi\left(\left|z_\nu + \frac{v_\nu+\sigma_\nu\xi_\nu}{\varphi(|z_\nu|)}\right|\right)}{\varphi(|z_\nu|)} \right]^k = 0$$

for all $k = 1, 2, \cdots, l_j - 1$. Thus ξ_0 is a zero of $G - \psi_{if_\nu}(0)$ of multiplicity at least l_j. This proves claim (1).

Suppose that $\psi_j(z_0) = \infty$. Then, on every compact subset of $\mathbb{D}$ that contains no zeros of $\psi_{jf_\nu}(z)$, $1/\psi_{jf_\nu}\left(z_\nu + \frac{v_\nu+\sigma_\nu\xi}{\varphi(|z_\nu|)}\right)$ converges locally uniformly to 0 with respect to Euclidean metric. Let η_0 be a zero of $1/G$. Then by Hurwitz's theorem there exist a sequence of points $\eta_\nu \to \eta_0$ such that for all ν sufficiently large, $\psi_{jf_\nu}\left(z_\nu + \frac{v_\nu+\sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right) = \infty$ and

$$0 = \frac{1}{G_\nu(\eta_\nu)} - \frac{1}{\psi_{jf_\nu}\left(z_\nu + \frac{v_\nu+\sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right)} = \frac{1}{f_\nu\left(z_\nu + \frac{v_\nu+\sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right)} - \frac{1}{\psi_{jf_\nu}\left(z_\nu + \frac{v_\nu+\sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right)}$$

$$\Rightarrow \frac{1}{f_\nu\left(z_\nu + \frac{v_\nu+\sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right)} = \frac{1}{\psi_{jf_\nu}\left(z_\nu + \frac{v_\nu+\sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right)}$$

$$\Rightarrow f_n\left(z_\nu + \frac{v_\nu + \sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right) = \psi_{jf_n}\left(z_\nu + \frac{v_\nu + \sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right).$$

By hypothesis, we have

$$\left|\left(\frac{1}{f_\nu}\right)^{(k)}\left(z_\nu + \frac{v_\nu + \sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right)\right| \le M\varphi^k\left(\left|z_\nu + \frac{v_\nu + \sigma_\nu\eta_\nu}{\varphi(|z_\nu|)}\right|\right)$$

for all ν sufficiently large and $k = 1, 2, \cdots, l_j - 1$. Thus by applying the same argument as in claim (1), we find that η_0 is a zero of $1/G$ of multiplicity at least l_j. Hence η_0 is a pole of G of multiplicity at least l_j, This proves claim (2).

Since, by assumption on spherical distance, $\psi_1(z_0), \psi_2(z_0), \cdots, \psi_q(z_0)$ are q-distinct points in extended complex plane, applying Second fundamental theorem of Nevanlinna to G, we obtain

$$\begin{aligned}(q-2)T(r,G) &\le \sum_{i=1}^{q} \overline{N}\left(r, \frac{1}{G - \psi_j(z_0)}\right) + o(T(r,G)) \\ &\le \sum_{j=1}^{q} \frac{1}{l_j} N\left(r, \frac{1}{G - \psi_j(z_0)}\right) + o(T(r,G)) \\ &\le \sum_{j=1}^{q} \frac{1}{l_j} T(r,G) + o(T(r,G)).\end{aligned}$$

That is,

$$\left[(q-2) - \sum_{j=1}^{q} \frac{1}{l_j}\right] T(r,G) \le o(T(r,G))$$

which is a contradiction to the fact that $\sum_{j=1}^{q} \frac{1}{l_j} < q - 2$.

Case 2. $0 \le |z_0| < 1$. Since φ is increasing, the following inequality

$$\frac{f_\nu^{\#}(z_\nu)}{\varphi(0)} \ge \frac{f_\nu^{\#}(z_\nu)}{\varphi(|z_\nu|)} \tag{5}$$

must holds for every positive integer ν. From (3) and (5), we find that

$$f_\nu^\#(z_\nu) \to \infty \text{ as } \nu \to \infty,$$

and so by Marty's theorem, $\{f_\nu\}_{\nu=1}^{\infty}$ is not normal at z_0. By Lemma 2.1, there exist a subsequence of f_ν, one may take f_ν itself, $\{u_\nu\} \subset \mathbb{D}$ with $u_\nu \to z_0$ and positive numbers ρ_ν with $\rho_\nu \to 0$
such that

$$F_\nu(\eta) := f_\nu(u_\nu + \rho_\nu \eta)$$

locally uniformly with respect to the spherical metric to F, where $F(\eta)$ is a non-constant meromorphic function on $\mathbb{C}$. Therefore, on every compact subset of $\mathbb{C}$ that contains no poles of F, $F_\nu^{(j)}$ converges uniformly to $F^{(j)}$ for all $j \in \mathbb{N}$. Similarly, on every compact subset of $\mathbb{C}$ containing no zeros of F, $(1/F_\nu)^{(j)}$ converges uniformly to $(1/F)^{(j)}$ for all $j \in \mathbb{N}$. Now, by applying the same argument as in Case 1, we get

1. For any $j \in \{1, 2, \cdots, q\}$, if $\psi_j(z_0) \neq \infty$, then all the zeros of $F - \psi_j(z_0)$ have multiplicity at least l_j.
2. For some $j \in \{1, 2, \ldots, q\}$, if $\psi_j(z_0) = \infty$, then all the poles of F have multiplicity at least l_j.

Again, by applying Second fundamental theorem of Nevanlinna to F and $\psi_j(z_0)(1 \leq j \leq q)$ with assumption $\sum_{j=1}^{q} \frac{1}{l_j} < q - 2$, we get a contradiction.

This completes the proof of the Theorem 1.4. □

Proof. **[Proof of Theorem 1.1.]** Suppose that f is not normal in $\mathbb{D}$. By Lemma 2.2, there exists a sequence $\{z_n\} \subset \mathbb{D}$ and positive numbers $\{\rho_n\}$ with $\rho_n \to 0$ such that

$$g_n(\xi) := f(z_n + \rho_n \xi)$$

converges uniformly on each compact subset of $\mathbb{C}$ to a non-constant meromorphic function $g(\xi)$ on $\mathbb{C}$. Passing to a subsequence (if necessary) we assume that $z_n \to z_0 \in \overline{\mathbb{D}}$. If $z_0 \in \mathbb{D}$, then we have

$$g_n(\xi) = f(z_n + \rho_n \xi) \to f(z_0).$$

This implies that $g(\xi) \equiv f(z_0)$, which is not possible since g is non constant meromorphic function on $\mathbb{C}$. Thus we must have $|z_0| = 1$.

We omit the rest of the proof of Theorem 1.1 as it almost relies on the same argument used in Theorem 1.4. □

3. Compliance with Ethical Standards

- Conflicts of Interest: The authors declare that they have no conflict of interest.
- Data Availability Statement: Data Sharing is not applicable to this article.
- Authors received no specific grant/funding for the research, authorship or publication of this article
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