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On int-soft quasi-Γ-ideals of an ordered Γ-semigroup

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Abstract. In this paper, we introduce the concept of int-soft Γ-semigroup, int-soft quasi-Γ-semigroup and int-soft left (resp., right) Γ-semigroup of ordered Γ-semigroup over an initial universal set U. We investigate some properties of int-soft quasi-Γ-ideals and left (resp., right) Γ-ideals of ordered Γ-semigroup. Moreover, we define critical soft point of ordered Γ-semigroup. By using the notion of critical soft point, we define semiprime int-soft quasi-Γ-ideals of ordered Γ-semigroups. Characterizations of completely regular ordered Γ-semigroups in terms of their int-soft quasi-Γ-ideals and semiprime int-soft quasi-Γ-ideals are provided. Furthermore, we define the semilattices of left and right simple sub-Γ-semigroups of ordered Γ-semigroups and characterize them in terms of their int-soft quasi-Γ-ideals.

1. Introduction

In 1999, Molodtsov [32] introduced the concept of a soft set as a new mathematical tool to deal with uncertainities that appeared in many fields of research such as social science, enviromental science, medical, engineering etc. For example, we have a statement "numbers closer to 20", the classical set theory which is introduced by Cantor is no longer useful since there is uncertainty whether 19 is closer to 20 or not. These type of variables are non-Boolean. A few more examples of non-boolean variables are young people, tall people. To deal with such uncertainity, Zadeh [45] introduced the concept of fuzzy set in 1965. A fuzzy set of a classical set X is an object of the form $\{(x, f(x)) : x \in X \text{ and } f : X \longrightarrow [0, 1]\}$. Many theories like fuzzy set theory have been developed including probability theory [44], intuitionistic fuzzy set theory [5], rough set theory [33]. Molodtsov pointed out that all the theories lake parameterization tool and hence introduced the concept of a soft set. Maji et al. [31] defined operations for soft sets. Later on Ali et al. [3] defined new operations for soft sets. The concept of soft set has proven useful in many different fields such as optimization [28], data analysis [46], simulation [19]. Many researchers had applied the concept of soft set on different algebraic structures such as groups (see [2]), rings (see [6]) and semirings (see [10]). In 2009, Ali and Shabir [43] introduced the notion of soft semigroups. The notion of soft ordered semigroups is defined

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by Jun et al (see [16]). Soft ideal, soft bi-ideal and soft quasi-ideal of a semigroup are defined by Ali et al. [4].

There are two extensions of soft set called intersectional soft set (briefly int-soft set) and union soft set (briefly uni-soft set). In 2015, Sezer et al. [42] introduced the concept of a int-soft semigroup, a int-soft ideal and a int-soft bi-ideal of a semigroup. The notion of int-soft (generalized) bi-ideal of a semigroup is introduced by Jun and Song [17].

In 1981, M.K. Sen [37] introduced the notion of a Γ-semigroup as a generalization of semigroup. Later in 1986, Sen with Saha [39] changed the definition that was given by Sen in 1981 and gave more general definition. Ordered Γ-semigroup is defined by Sen and Seth [40] in 1993 as a generalization of ordered semigroup. Many classical notions and results of the theory of semigroups have been extended to Γ-semigroups. Green's relations in Γ-semigroup has been defined in [8, 9, 12, 36, 41]. For other results one can see [38]. Moreover, the concept of fuzzy set (see [21]) has been applied to ordered Γ-semigroup. Changphas and Thongkam ([7]) used the idea of soft set to Γ-semigroups and defined soft Γ-semigroup, soft Γ-subsemigroup, soft *l*-idealistic (*r*-idealistic). Also Γ-restricted product is defined in ([7]). Khan [26] applied the concept of fuzzy set to ordered Γ-semigroup and defined the notion of fuzzy interior Γ-ideal of ordered Γ-semigroup. Moreover, Khan [27] also defined generalized fuzzy bi-Γ-ideal of type (θ, λ) of ordered Γ-semigroup.

A lot of work has been done on soft set theory and Γ-structure theory by many researchers. However, the very fundamental results of int-soft Γ-semigroups in ordered Γ-semigroups remained untouched. The main motive of this paper is the study of some structural properties of ordered Γ-semigroups applying the int-soft set theory.

In this paper, we introduce the concept of int-soft Γ-semigroup, int-soft quasi−Γ-ideal and int-soft left (resp., right) Γ-ideal of ordered Γ-semigroup and investigate their properties. We define critical soft point and by using the concept of critical soft point, we introduce semiprime int-soft quasi-Γ-ideal of ordered Γ-semigroup. Furthermore, characterizations of completely regular by using int-soft quasi−Γ−ideal and semiprime int-soft quasi-Γ-ideal are discussed in detail. Next we define semilattice of left and right simple sub-Γ-semigroups of ordered Γ-semigroup and characterize semilattice of left and right simple sub-Γ-semigroups in terms of their int-soft quasi-Γ-ideal.

2. Preliminaries

In this section, we give some basic definitions and results, which are necessary for the following sections.

The definition of Γ-semigroup [24, 39] as a generalization of semigroup and ternary semigroup is given as follows:

Let S and Γ be two non-empty sets. Denote by the letters of the English alphabet the elements of M and with the letters of the Greek alphabet the elements of Γ. Then S is called a Γ-semigroup if

1. $a\gamma b \in S$, for all $\gamma \in \Gamma$.
2. $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
3. If $s_1, s_2, s_3, s_4 \in S, \gamma_1, \gamma_2 \in \Gamma$ such that $s_1 = s_3, \gamma_1 = \gamma_2$ and $s_2 = s_4$, then $s_1\gamma_1 s_2 = s_3\gamma_2 s_4$.

By an ordered Γ-semigroup S [40], we mean an algebraic structure $(S, \Gamma, \leq)$ that satisfies the given conditions:

(1) (S, Γ) is a Γ-semigroup,

(2) $(S, \leq)$ is a poset, and

(3) If $p \leq q \Longrightarrow p\lambda r \leq q\lambda r$ and $r\lambda p \leq r\lambda q$ $(\forall\, p, q, r \in S)$ $(\forall\, \lambda \in \Gamma)$.

For $A, B \subseteq S$, we define $A\Gamma B = \{a\gamma b \mid a \in A, \gamma \in \Gamma \text{ and } b \in B\}$. If $A = \{a\}$ or $B = \{b\}$, then we denote $\{a\}\Gamma B$, $A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively, by means of $a\Gamma B$, $A\Gamma b$ and $a\Gamma b$. Let a be any element of S.

For $\emptyset \neq A \subseteq S$, we define $(A] := \{s \in S \mid s \leq a \text{ for some } a \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$ (see [1]). For $x \in S$, we write $A_x = \{(p, q) \in S \mid x \leq p\lambda q \text{ for some } \lambda \in \Gamma\}$ (see [27]).

For other terminologies and definitions relevent to ordered Γ-semigroup, the reader is refereed to [13, 14, 24, 34, 35].

Example 2.1. *Let $S = \{u, v, w\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be two non-empty sets. Define binary operations on S in the tabels given below:*

α	u	v	w
u	u	v	w
v	v	w	u
w	w	u	v

β	u	v	w
u	v	w	u
v	w	u	v
w	u	v	w

γ	u	v	w
u	w	u	v
v	u	v	w
w	v	w	u

The order relation "$\leq$" is defined by: $\leq := \{(u, u), (v, v), (w, w)\}$. Then S is an ordered Γ-semigroup.

A sub-Γ-semigroup H of an ordered Γ-semigroup S is a non-empty subset of S such that $H\Gamma H \subseteq H$. A left (resp., right) Γ-ideal I of an ordered Γ-semigroup S is a non-empty subset of S which satisfies the following conditions:

1. $S\Gamma I \subseteq I$ (resp., $I\Gamma S \subseteq I$),
2. $(\forall p \in I$ and $\forall s \in S)$ $(s \leq p \Longrightarrow s \in I)$. In other words, $(I] = I$.

I is said to be a two sided Γ-ideal (or simply a Γ-ideal) of S, if I is a left Γ-ideal as well as a right Γ-ideal of S.

A quasi-Γ-ideal Q of an ordered Γ-semigroup S, is a non-empty subset of S which satisfies the following conditions:

1. $(S\Gamma Q] \cap (Q\Gamma S] \subseteq Q$,
2. $(\forall p \in Q$ and $\forall s \in S)$ $(s \leq p \Longrightarrow s \in Q)$. In otherwords, $(Q] = Q$.

A bi-Γ-ideal B of an ordered Γ-semigroup S, is a non-empty subset of S which satisfies the following conditions:

1. B is a sub-Γ-semigroup of S,
2. $B\Gamma S\Gamma B \subseteq B$, and
3. $(\forall p \in B$ and $\forall s \in S)$ $(s \leq p \Longrightarrow s \in B)$. In other words, $(B] = B$.

Lemma 2.2. *(see [30]) For any non-empty subsets K, L of an ordered Γ-semigroup S, the followings hold:*

1. *$K \subseteq (K]$ for any $K \subseteq S$.*
2. *If $K \subseteq L$, then $(K] \subseteq (L]$.*
3. *$(K] = ((K]]$.*
4. *$(K]\,\Gamma\,(L] \subseteq (K\Gamma L]$.*
5. *$((K]\,\Gamma\,(L]] = (K\Gamma L]$.*
6. *$(K \cup L] = (K] \cup (L]$.*
7. *$(K \cap L] \subseteq (K] \cap (L]$, (if $K \cap L \neq \emptyset$).*
8. *For every $p \in S$, the sets $(p\Gamma S]$, $(S\Gamma p]$ and $(S\Gamma p\Gamma S]$ are a right-Γ-ideal, a left-Γ-ideal and a Γ-ideal of S, respectively.*

Let p be any element of an ordered Γ-semigroup S. The quasi-Γ-ideal of S generated by an element p is denoted by $Q(p)$, is defined as, $Q(p) = (p \cup ((p\Gamma S] \cap (S\Gamma p])]$ (see [30]).

3. Basic operations of soft sets

Let U be a non-empty set called universal set and E be a non-empty set of all possible parameters with respect to U and $A, B, C \subseteq E$. By $P(U)$ we mean the power set of U and $\xi \subseteq U$.

Definition 3.1. *[32]. A pair (f_A, E) is called a soft set over U, where f_A is a mapping given by $f_A : E \longrightarrow P(U)$, is defined as:*

$$f_A(a) := \begin{cases} \xi, & \text{if } a \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In other words, a soft set is a parameterized family of subsets of the universal set U. For $a \in E$, $f_A(a)$ may be considered as a set of $a-$approxiamate elements of the soft set (f_A, E).

Examples of a soft set can be found in [32].

Definition 3.2. *(see [4, 7]). Let (f_C, E) and (g_D, E) be any two soft sets over a common universe U. Then (f_C, E) is said to be a soft subset of (g_D, E), denoted by $(f_C, E)\widetilde{\subseteq}(g_D, E)$, if $C \subseteq D$ and $f_C(p) \subseteq g_D(p)$, for all $p \in C$.*

Definition 3.3. *(see [18]). Two soft sets (f_C, E) and (g_D, E) are said to be equal if (f_C, E) is a soft subset of (g_D, E) and (g_D, E) is a soft subset of (f_C, E).*

Definition 3.4. *(see [18]). Let (f_C, E) and (g_D, E) be two soft sets over a common universe U. Then, the soft union of f_C and g_D, denoted by $f_C\widetilde{\cup}g_D$, is defined to be the soft set $\left(f_C\widetilde{\cup}g_D, E\right)$ over U in which $f_C\widetilde{\cup}g_D$ is defined by $\left(f_C\widetilde{\cup}g_D\right)(a) = f_C(a) \cup g_D(a)$, for all $a \in E$.*

Moreover, the set of all soft sets (f_S, S) over U is denoted by $S(U)$.

Definition 3.5. *(see [18]). Let (f_C, E) and (g_D, E) be two soft sets over a common universe U. Then, the soft intersection of f_C and g_D, denoted by $f_C\widetilde{\cap}g_D$, is defined to be the soft set $\left(f_C\widetilde{\cap}g_D, E\right)$ over U in which $f_C\widetilde{\cap}g_D$ is defined by $\left(f_C\widetilde{\cap}g_D\right)(a) = f_C(a) \cap g_D(a)$ for all $a \in E$.*

From now on S represents an ordered Γ-semigroup unless otherwise stated.

Definition 3.6. *For any two soft sets (f_S, S) and (g_S, S) over a common universe U, we denote the soft union-intersection product by $f_S \Gamma g_S$, and is defined by*

$$(f_S \Gamma g_S)(p) = \begin{cases} \bigcup\limits_{(s,t)\in A_p} \{f_S(s) \cap g_S(t)\}, & \text{if } A_p \neq \emptyset, \\ \emptyset, & \text{if } A_p = \emptyset, \end{cases} \quad \text{for all } p \in S.$$

Without any difficulty, the reader can prove that "Γ" on $S(U)$ is well defined. Moreover, we have the following lemma.

Lemma 3.7. *The set $\left(S(U), \Gamma, \widetilde{\subseteq}\right)$ forms an ordered Γ-semigroup.*

Proof. To show $S(U)$ is an ordered Γ-semigroup, first we show that $(S(U), \Gamma)$ is a Γ-semigroup. Let $f_S, g_S, h_S \in S(U)$. If $A_a = \emptyset$, then $((f_S \Gamma h_S) \Gamma g_S)(a) = \emptyset = (f_S \Gamma (h_S \Gamma g_S))(a)$. Let $A_a \neq \emptyset$. We have

$$\begin{aligned}
((f_S \Gamma h_S) \Gamma g_S)(a) &= \bigcup_{(s,t)\in A_a} \{(f_S \Gamma h_S)(s) \cap g_S(t)\} \\
&= \bigcup_{(s,t)\in A_a} \left\{ \bigcup_{(k,l)\in A_s} \{f_S(k) \cap h_S(l)\} \cap g_S(t) \right\} \\
&= \bigcup_{(k\lambda l,t)\in A_a} \{(f_S(k) \cap h_S(l)) \cap g_S(t)\} \\
&= \bigcup_{(k,l\mu t)\in A_a} \{f_S(k) \cap (h_S(l) \cap g_S(t))\} \\
&\subseteq \bigcup_{(k,l\mu t)\in A_a} \left\{ f_S(k) \cap \left(\bigcup_{(l,t)\in A_{l\mu t}} \{h_S(l) \cap g_S(t)\} \right) \right\} \\
&= \bigcup_{(k,l\mu t)\in A_a} \{f_S(k) \cap (h_S \Gamma g_S)(l\mu t)\} \\
&= (f_S \Gamma (h_S \Gamma g_S))(a).
\end{aligned}$$

This implies that $((f_S\Gamma h_S)\Gamma g_S)(a) \subseteq (f_S\Gamma(h_S\Gamma g_S))(a)$. Similarly we can show that $(f_S\Gamma(h_S\Gamma g_S))(a) \subseteq ((f_S\Gamma h_S)\Gamma g_S)(a)$. Hence $((f_S\Gamma h_S)\Gamma g_S)(a) = (f_S\Gamma(h_S\Gamma g_S))(a)$. Now we show that the order relation "$\widetilde{\subseteq}$" on $S(U)$ is compatible with "Γ". Let $f_S, h_S, g_S \in S(U)$ and $a \in S$ be such that $f_S\widetilde{\subseteq}h_S$. If $A_a = \emptyset$, then $(f_S\Gamma g_S)(a) = \emptyset = (h_S\Gamma g_S)(a)$. Let $A_a \neq \emptyset$. Then

$$\begin{aligned}(f_S\Gamma g_S)(a) &= \bigcup_{(s,t)\in A_a}\{f_S(s)\cap g_S(t)\}\\ &\subseteq \bigcup_{(s,t)\in A_a}\{h_S(s)\cap g_S(t)\}\ \ (\text{since } f_S(s)\subseteq h_S(s)\ \forall\, s\in S)\\ &= (h_S\Gamma g_S)(a).\end{aligned}$$

Thus $(f_S\Gamma g_S)(a) \subseteq (h_S\Gamma g_S)(a)$. Similary we can prove that $(g_S\Gamma f_S)(a) \subseteq (g_S\Gamma h_S)(a)$. Therefore, $\left(S(U), \Gamma, \widetilde{\subseteq}\right)$ is an ordered Γ-semigroup. □

Lemma 3.8. *For any soft sets $(f_S, S), (h_S, S)$ and (g_S, S) over U, we have the following:*

1. $\left(f_S\Gamma\left(g_S\widetilde{\cap}h_S\right), S\right) = (f_S\Gamma g_S, S)\,\widetilde{\cap}\,(f_S\Gamma h_S, S)$,
2. $\left(f_S\Gamma\left(g_S\widetilde{\cup}h_S\right), S\right) = (f_S\Gamma g_S, S)\,\widetilde{\cup}\,(f_S\Gamma h_S, S)$,
3. $\left(f_S\widetilde{\cap}\left(g_S\widetilde{\cup}h_S\right), S\right) = \left(f_S\widetilde{\cap}g_S, S\right)\widetilde{\cup}\left(f_S\widetilde{\cap}h_S, S\right)$,
4. $\left(f_S\widetilde{\cup}\left(g_S\widetilde{\cap}h_S\right), S\right) = \left(f_S\widetilde{\cup}g_S, S\right)\widetilde{\cap}\left(f_S\widetilde{\cup}h_S, S\right)$.

Proof. (1) Let $a \in S$. If $A_a = \emptyset$, then $\left(f_S\Gamma\left(g_S\widetilde{\cap}h_S\right)\right)(a) = \emptyset = \left((f_S\Gamma g_S)\widetilde{\cap}(f_S\Gamma h_S)\right)(a)$. Let $A_a \neq \emptyset$. Then

$$\begin{aligned}\left(f_S\Gamma\left(g_S\widetilde{\cap}h_S\right)\right)(a) &= \bigcup_{(s,t)\in A_a}\left\{f_S(s)\cap\left(g_S\widetilde{\cap}h_S\right)(t)\right\}\\ &= \bigcup_{(s,t)\in A_a}\{f_S(s)\cap(g_S(t)\cap h_S(t))\}\\ &= \bigcup_{(s,t)\in A_a}\{(f_S(s)\cap g_S(t))\cap(f_S(s)\cap h_S(t))\}\\ &= \left(\bigcup_{(s,t)\in A_a}\{f_S(s)\cap g_S(t)\}\right)\cap\left(\bigcup_{(s,t)\in A_a}\{f_S(s)\cap h_S(t)\}\right)\\ &= (f_S\Gamma g_S)(a)\cap(f_S\Gamma h_S)(a)\\ &= \left((f_S\Gamma g_S)\widetilde{\cap}(f_S\Gamma h_S)\right)(a),\end{aligned}$$

hence, $\left(f_S\Gamma\left(g_S\widetilde{\cap}h_S\right), S\right) = (f_S\Gamma g_S, S)\,\widetilde{\cap}\,(f_S\Gamma h_S, S)$.

(2) Let $a \in S$. If $A_a = \emptyset$, then $\left(f_S\Gamma\left(g_S\widetilde{\cup}h_S\right)\right)(a) = \emptyset = \left((f_S\Gamma g_S)\widetilde{\cup}(f_S\Gamma h_S)\right)(a)$. Let $A_a \neq \emptyset$. Then

$$\begin{aligned}\left(f_S\Gamma\left(g_S\widetilde{\cup}h_S\right)\right)(a) &= \bigcup_{(s,t)\in A_a}\left\{f_S(s)\cap\left(g_S\widetilde{\cup}h_S\right)(t)\right\}\\ &= \bigcup_{(s,t)\in A_a}\{f_S(s)\cap(g_S(t)\cup h_S(t))\}\\ &= \bigcup_{(s,t)\in A_a}\{(f_S(s)\cap g_S(t))\cup(f_S(s)\cap h_S(t))\}\\ &= \left(\bigcup_{(s,t)\in A_a}\{f_S(s)\cap g_S(t)\}\right)\cup\left(\bigcup_{(s,t)\in A_a}\{f_S(s)\cap h_S(t)\}\right)\\ &= (f_S\Gamma g_S)(a)\cup(f_S\Gamma h_S)(a)\\ &= \left((f_S\Gamma g_S)\widetilde{\cup}(f_S\Gamma h_S)\right)(a).\end{aligned}$$

Hence, $\left(f_S\Gamma\left(g_S\widetilde{\cup}h_S\right), S\right) = (f_S\Gamma g_S, S)\,\widetilde{\cup}\,(f_S\Gamma h_S, S)$.

The proof of (3) and (4) are straightforward. □

Definition 3.9. *In an ordered Γ-semigroup $(S, \Gamma, \leq)$, the soft set "$\emptyset_S$" of S over U defined as:*

$$\emptyset_S : S \longrightarrow P(U), \ s \longmapsto \emptyset_S(s) = \emptyset, \forall\, s \in S,$$

to be the "least element" of $S(U)$, is said to be a null soft set over U.

The soft set "T_S"of S over U defined as:

$$T_S : S \longrightarrow P(U), \ s \longmapsto T_S(s) = U, \forall\, s \in S,$$

to be the "greatest element" of $S(U)$, is said to be a whole soft set over U.

Definition 3.10. *Let $\emptyset \neq C \subseteq S$, we denote, the characterictic soft set over U by "χ_C", and is defined by:*

$$\chi_C(p) = \begin{cases} U, & \text{if } p \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Lemma 3.11. *Let C, D be any two non-empty subsets of an ordered Γ-semigroup S. Then the following statements are true:*

1. $C \subseteq D$ *iff* $\chi_C \widetilde{\subseteq} \chi_D$,
2. $\chi_C \widetilde{\cap} \chi_D = \chi_{C \cap D}$,
3. $\chi_C \widetilde{\cup} \chi_D = \chi_{C \cup D}$,
4. $\chi_C \Gamma \chi_D = \chi_{(C \Gamma D]}$.

Proof. The proof of (1), (2) and (3) is straightforward.

(4) Let $m \in S$. Then $\chi_C \Gamma \chi_D = \chi_{(C\Gamma D]}$. Indeed: let $m \in (C\Gamma D]$. Then $\chi_{(C\Gamma D]}(m) = U$.
Since $m \leq c\lambda d$ for some $c \in C$, $d \in D$ and $\lambda \in \Gamma$, then $(c, d) \in A_m$. Thus we have

$$(\chi_C \Gamma \chi_D)(m) = \bigcup_{(s,t) \in A_m} \chi_C(s) \cap \chi_D(t) \supseteq \chi_C(c) \cap \chi_D(d) = U \cap U = U.$$

Thus $(\chi_C \Gamma \chi_D)(m) \supseteq U$. But $\chi_C \Gamma \chi_D$ is a soft subset of S, so $(\chi_C \Gamma \chi_D)(m) \subseteq U$. Therefore, $(\chi_C \Gamma \chi_D)(m) = U = \chi_{(C\Gamma D]}(m)$.

Suppose $m \notin (C\Gamma D]$. If $A_m = \emptyset$, then $\chi_{(C\Gamma D]}(m) = \emptyset = (\chi_C \Gamma \chi_D)(m)$. Let $A_m \neq \emptyset$. Since $m \notin (C\Gamma D]$, then this implies that there does not exist any $c \in C$ or $d \in D$ such that $m \leq c\lambda d$ for some $\lambda \in \Gamma$.

Let $m \leq a\mu b$ for some $a \in C^c$ or $b \in D^c$ and $\mu \in \Gamma$. If $a \in C^c$, then $\chi_C(a) = \emptyset$. Since $(a, b) \in A_m$, then we have $(\chi_C \Gamma \chi_D)(m) = \bigcup_{(a,b) \in A_m} \chi_C(a) \cap \chi_D(b) = \emptyset = \chi_{(C\Gamma D]}(m)$.

If $b \in D^c$, then $\chi_D(b) = \emptyset$. Thus, $(\chi_C \Gamma \chi_D)(m) = \bigcup_{(a,b) \in A_m} \chi_C(a) \cap \chi_D(b) = \emptyset = \chi_{(C\Gamma D]}(m)$.

Hence in any case, $\chi_C \Gamma \chi_D = \chi_{(C\Gamma D]}$. □

Definition 3.12. *A soft set (f_S, S) of an ordered Γ-semigroup S over U is called int-soft Γ-semigroup over U if:*

$$(f_S(p\gamma q) \supseteq f_S(p) \cap f_S(q)) \ \ (\forall\, p, q \in S \text{ and } \gamma \in \Gamma).$$

Example 3.13. *Let $S = \{p, q, r\}$ and $\Gamma = \{\lambda, \mu\}$ be two non-empty sets. Let $U = \{e, x, x^2, y, xy, x^2y\}$ is a universal set. Then S is an ordered Γ-semigroup with respect to the binary operations and order relation defined below:*

λ	p	q	r
p	p	q	r
q	q	p	r
r	r	r	r

μ	p	q	r
p	q	p	r
q	p	q	r
r	r	r	r

The order relation is defined by:

$$\leq := \{(p, p), (q, q)\, (r, r), (p, r), (q, r)\}$$

Let f_S be a soft set defined by:

$f_S(p) = \{e, x, x^2\}$, $f_S(q) = \{y, xy\}$, $f_S(r) = \{e, x^2y\}$. *Clearly f_S is an int-soft Γ-semigroup of an ordered Γ-semigroup S.*

4. Int-soft quasi-Γ-ideals

Definition 4.1. *A soft set (f_S, S) of an ordered Γ-semigroup S over U is called int-soft left (resp., right) Γ-ideal if:*

1. $(\forall\, p, q \in S)\ (\forall\, \gamma \in \Gamma)\ (f_S(p\gamma q) \supseteq f_S(q))$ *(resp.,* $(f_S(q\gamma p) \supseteq f_S(q)))$,
2. $(\forall\, p, q \in S)$ *(If* $p \leq q \Longrightarrow f_S(p) \supseteq f_S(q))$.

Example 4.2. *Let* $S = [0, 1]$ *and* $\Gamma = \{1/n : n \in \mathbb{N}\}$. *Then S is an ordered Γ-semigroup under usual multiplication and usual partial order. Let* $U = \mathbb{N}$ *be a universal set. We define a soft set* f_S *over U given below:*

$$f_S(p) = \begin{cases} \mathbb{N} \setminus \{p\}, & \text{if } p \in [0, 1/2], \\ \emptyset, & \text{otherwise.} \end{cases}$$

Clearly it is int-soft left Γ-ideal as well as int-soft right Γ-ideal of S over U.

Proposition 4.3. *Let* f_S *be any soft set of an ordered Γ-semigroup S over U. The following statements are equivalent:*

1. *A soft set* f_S *is an int-soft right (resp., int-soft left) Γ-ideal of S.*
2. *(i)* $f_S \Gamma \chi_S \widetilde{\subseteq} f_S$ $\left(\text{resp., } \chi_S \Gamma f_S \widetilde{\subseteq} f_S\right)$,
   (ii) $(\forall\, p, q \in S)$ *(If* $p \leq q \Longrightarrow f_S(p) \supseteq f_S(q))$.

Proof. (1) ⟹ (2). Let $p \in S$ and f_S be an int-soft right Γ-ideal of S. If $A_p = \emptyset$, then we have, $(f_S \Gamma \chi_S)(p) = \emptyset \subseteq f_S(p)$. Let $A_p \neq \emptyset$. We have $(f_S \Gamma \chi_S)(p) = \underset{(s,t) \in A_p}{\cup} f_S(s) \cap \chi_S(t) = \underset{(s,t) \in A_p}{\cup} f_S(s)$

On the other hand, $f_S(s) \subseteq f_S(p)$ for every $(s, t) \in A_p$. Indeed: Since $(s, t) \in A_p$, then we have $p \leq s\lambda t$ for some $\lambda \in \Gamma$. Moreover, since (f_S, S) is an int-soft right Γ-ideal over U, then we obtain $f_S(p) \supseteq f_S(s\lambda t) \supseteq f_S(s)$ and so $f_S(p) \supseteq f_S(s)$. Hence $(f_S \Gamma \chi_S)(p) \subseteq f_S(p)$. Since f_S is an int-soft right Γ-ideal of S, then the property (ii) holds.

(2) ⟹ (1). By hypothesis for every $s, t \in S$ and $\lambda \in \Gamma$, we have

$$f_S(s\lambda t) \supseteq (f_S \Gamma \chi_S)(s\lambda t) = \underset{(u,v) \in A_{s\lambda t}}{\cup} f_S(u) \cap \chi_S(v) \supseteq f_S(s) \cap \chi_S(t) = f_S(s).$$

Therefore, $f_S(s\lambda t) \supseteq f_S(s)$ and so f_S is an int-soft right Γ-ideal of S over U. □

Proposition 4.4. *Let* $(S, \Gamma, \leq)$ *be an ordered Γ-semigroup and* $\emptyset \neq C \subseteq S$. *Then the following statements are equivalent:*

1. *C is a right Γ-ideal (resp., left Γ-ideal) of S.*
2. *The characteristic soft set* (χ_C, S) *is an int-soft right (resp., left) Γ-ideal over U.*

Proof. (1) ⟹ (2). Let C is a right Γ-ideal of S, so $C\Gamma S \subseteq C$. For point (4) of Lemma 3.11, we have, $\chi_C \Gamma \chi_S = \chi_{(C\Gamma S]} \subseteq \chi_C$. Thus $\chi_C \Gamma \chi_S \subseteq \chi_C$. Let $p, q \in S$ be such that $p \leq q$. If $q \in C$, then $\chi_C(q) = U$. Since $p \leq q$ and C is a right-Γ-ideal of S, then we have $p \in C$. Hence$\chi_C(p) = U = \chi_C(q)$. If $q \notin C$, then $\chi_C(q) = \emptyset \subseteq \chi_C(p)$ and so $\chi_C(p) \supseteq \chi_C(q)$. By Proposition 4.3, (χ_C, S) is an int-soft right Γ-ideal over U.

(2) ⟹ (1). Let $c \in C\Gamma S$ implies $c \in (C\Gamma S]$. Thus $\chi_{(C\Gamma S]} = U$. Since $\chi_{(C\Gamma S]}(c) = \chi_C \Gamma \chi_S(c)$, then we have $U = \chi_C \Gamma \chi_S(c) \subseteq \chi_C(c)$, (by Proposition 4.3). Therefore $U \subseteq \chi_C(c)$ and we have $\chi_C(c) \subseteq U$. Thus $\chi_C(c) = U$ implies $c \in C$, that is, $C\Gamma S \subseteq C$. Let $p, q \in S$ be such that $p \in C$ and $q \leq p$. Then $q \in C$. Indeed: Since $q \leq p$, then $\chi_C(q) \supseteq \chi_C(p) = U$ (by hypothesis), but $\chi_C(q) \subseteq U$, and so $\chi_C(q) = U$. Thus $q \in C$. Therefore, C is a right Γ-ideal of S. □

Definition 4.5. *An element s of an ordered Γ-semigroup* $(S, \Gamma, \leq)$ *is said to be left (resp., right) regular if there exist some* $r \in S$ *and* $\lambda, \mu \in \Gamma$ *such that* $s \leq r\lambda s\mu s$ *(resp.,* $s \leq s\lambda s\mu r$*).*

S is said to be left (resp., right) regular, if all the elements of S are left (resp., right) regular. Equivalent definitions are:

1. *For every $K \subseteq S$, $K \subseteq (S\Gamma K\Gamma K]$ (resp., $K \subseteq (K\Gamma K\Gamma S]$),*
2. *For every element $s \in S$, $s \in (S\Gamma s\Gamma s]$ (resp., $s \in (s\Gamma s\Gamma S]$).*

Definition 4.6. *An element s of an ordered Γ-semigroup $(S, \Gamma, \leq)$ is said to be regular if there exist some element $r \in S$ and $\lambda, \mu \in \Gamma$ such that $s \leq s\lambda r\mu s$.*

If every element of an ordered Γ-semigroup S is regular, then S is called regular ordered Γ-semigroup. Equivalent definitions are:

1. *For every $K \subseteq S$, $K \subseteq (K\Gamma S\Gamma K]$,*
2. *For every $s \in S$, $s \in (s\Gamma S\Gamma s]$.*

Proposition 4.7. *In an ordered Γ-semigroup, we have*

$$f_S \widetilde{\cap} g_S \widetilde{\supseteq} f_S \Gamma g_S.$$

for every int-soft right Γ-ideal f_S and for every int-soft left Γ-ideal g_S over U.

Proof. Let $p \in S$. If $A_p = \emptyset$, then $(f_S \Gamma g_S)(p) = \emptyset \subseteq \left(f_S \widetilde{\cap} g_S\right)(p)$. Let $A_p \neq \emptyset$. We have

$$(f_S \Gamma g_S)(p) = \underset{(s,t)\in A_p}{\cup} \{f_S(s) \cap g_S(t)\} \cdot$$

Since $(s, t) \in A_p$, then we have $p \leq s\lambda t$ for some $\lambda \in \Gamma$. Since f_S is an int-soft right $\Gamma-$ideal, then we deduce $f_S(p) \supseteq f_S(s\lambda t) \supseteq f_S(s)$ and so $f_S(p) \supseteq f_S(s)$. Also g_S is an int-soft left Γ-ideal, so we have $g_S(p) \supseteq g_S(s\lambda t) \supseteq g_S(t)$ implies $g_S(p) \supseteq g_S(t)$. Thus we have,

$$(f_S \Gamma g_S)(p) = \underset{(s,t)\in A_p}{\cup} \{f_S(s) \cap g_S(t)\} \subseteq f_S(p) \cap g_S(p) = \left(f_S \widetilde{\cap} g_S\right)(p).$$

Therefore, $f_S \Gamma g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$ or $f_S \widetilde{\cap} g_S \widetilde{\supseteq} f_S \Gamma g_S$. □

Proposition 4.8. *Let $(S, \Gamma, \leq)$ be a regular ordered Γ-semigroup. Then for each int-soft right (resp., left) Γ-ideal (f_S, S) and for each soft subset (g_S, S), we have*

$$f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \Gamma g_S \text{ (resp., } g_S \widetilde{\cap} f_S \widetilde{\subseteq} g_S \Gamma f_S).$$

Proof. Let (f_S, S) be an int-soft right Γ-ideal and (g_S, S) be any soft set over U. Let $p \in S$. As S is regular, then $p \leq p\lambda q\mu p$ for some $q \in S$ and $\lambda, \mu \in \Gamma$. Thus $(p\lambda q, p) \in A_p$. Since $A_p \neq \emptyset$, then we have

$$(f_S \Gamma g_S)(p) = \underset{(s,t)\in A_p}{\cup} \{f_S(s) \cap g_S(t)\} \supseteq f_S(p\lambda q) \cap g_S(p)$$

Since f_S is an int-soft right Γ-ideal of S, then $f_S(p\lambda q) \supseteq f_S(p)$. We have, $(f_S \Gamma g_S)(p) \supseteq f_S(p\lambda q) \cap g_S(p) \supseteq f_S(p) \cap g_S(p)$. Therefore, $f_S \Gamma g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$ or $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \Gamma g_S$. □

Corollary 4.9. *In a regular ordered Γ-semigroup S, we have*

$$f_S \widetilde{\cap} g_S = f_S \Gamma g_S.$$

for every int-soft right Γ-ideal f_S and for every int-soft left Γ-ideal g_S over U.

Definition 4.10. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then a soft set (f_S, S) of S over U is said to be int-soft quasi-Γ-ideal of S if:*

1. $(f_S \Gamma \chi_S; S) \widetilde{\cap} (\chi_S \Gamma f_S; S) \widetilde{\subseteq} f_S$,
2. $(\forall\, p, q \in S)$ (*If* $p \leq q \Longrightarrow f_S(p) \supseteq f_S(q)$).

Example 4.11. *Let $S = P$ (set of prime numbers) and $1 \in S$. Let Γ be any non-empty set. Let $U = E$ (set of even numbers) be a universal set. Define $S \times \Gamma \times S \longrightarrow S$ by*

$$p\lambda q = \begin{cases} p, & \text{if } p = q, \\ 1, & \text{otherwise,} \end{cases}$$

for all $p, q \in S$ and $\lambda \in \Gamma$.

Define an order relation by $p \leq q$ iff p/q. Then S is an ordered Γ-semigroup. Let f_S be a soft set is defined by:

$$f_S(t) = \begin{cases} 2E, & \text{if } t = 1, \\ 4E, & \text{if } t \in [2, 50] \cap P, \\ 8E, & \text{if } t \in [51, \infty] \cap P. \end{cases}$$

The soft set f_S is an int-soft quasi-Γ-ideal over U.

Theorem 4.12. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and $\emptyset \neq C \subseteq S$, then the following statements are equivalent:*

1. *C is a quasi-Γ-ideal of S.*
2. *The characteristic soft set (χ_C, S) is an int-soft quasi-Γ-ideal of S.*

Proof. (1) $\Longrightarrow$ (2). Let a be any element of S. Then we have

$$\begin{aligned}\left((\chi_C\Gamma\chi_S) \widetilde{\cap} (\chi_S\Gamma\chi_C)\right)(a) &= (\chi_C\Gamma\chi_S)(a) \cap (\chi_S\Gamma\chi_C)(a) \\ &= \chi_{(C\Gamma S]}(a) \cap \chi_{(S\Gamma C]}(a) = \chi_{(C\Gamma S]\cap(S\Gamma C]}(a).\end{aligned}$$

Since C is a quasi-Γ-ideal of S, then $(C\Gamma S]\cap(S\Gamma C] \subseteq C$, and so $\chi_{(C\Gamma S]\cap(S\Gamma C]} \subseteq \chi_C$. Therefore, $(\chi_C\Gamma\chi_S) \widetilde{\cap} (\chi_S\Gamma\chi_C)(a) \subseteq \chi_C(a)$.

Let $p, q \in S$ be such that $p \leq q$. If $q \in C$, then $p \in C$. Indeed: if $q \in C$, then $\chi_C(q) = U$. Since $p \leq q$ and C is a quasi-Γ-ideal of S, then we have $p \in C$. Hence $\chi_C(p) = U = \chi_C(q)$. If $q \notin C$, then $\chi_C(q) = \emptyset \subseteq \chi_C(p)$. Hence in any case, $\chi_C(p) \supseteq \chi_C(q)$. Therefore χ_C is an int-soft quasi-Γ-ideal over U.

(2) $\Longrightarrow$ (1). Let χ_C be an int-soft quasi-Γ-ideal over U and let $a \in (C\Gamma S] \cap (S\Gamma C]$. Cleraly, we have $a \leq b\lambda p$ and $a \leq q\mu c$ for some $b, c \in C$, $\lambda, \mu \in \Gamma$ and $p, q \in S$. Hence $(b, p), (q, c) \in A_a$. Since χ_C is an int-soft quasi-Γ-ideal over U, then we have

$$\begin{aligned}\chi_C(a) &\supseteq \left((\chi_C\Gamma\chi_S) \widetilde{\cap} (\chi_S\Gamma\chi_C)\right)(a) \\ &= (\chi_C\Gamma\chi_S)(a) \cap (\chi_S\Gamma\chi_C)(a) \\ &= \Big(\bigcup_{(s,t)\in A_a} \{\chi_C(s) \cap \chi_S(t)\}\Big) \cap \Big(\bigcup_{(s,t)\in A_a} \{\chi_S(s) \cap \chi_C(t)\}\Big) \\ &\supseteq \{\chi_C(b) \cap \chi_S(p)\} \cap \{\chi_S(q) \cap \chi_C(c)\} \\ &= \chi_C(b) \cap \chi_C(c) = U.\end{aligned}$$

Thus $\chi_C(a) \supseteq U$, but $\chi_C(a) \subseteq U$. Hence $\chi_C(a) = U$. This implies $a \in C$. Hence we have $(C\Gamma S] \cap (S\Gamma C] \subseteq C$.

Now let $S \ni s \leq r \in C$. This means that $s \in C$. Indeed: Since $s \leq r$, then by hypothesis, $\chi_C(s) \supseteq \chi_C(r)$. Also $r \in C$, implies $\chi_C(r) = U$. Thus we have $\chi_C(s) \supseteq \chi_C(r) = U$ and so $\chi_C(s) \supseteq U$. Also $U \supseteq \chi_C(s)$, and so $\chi_C(s) = U$ which means $s \in C$. Therefore C is a quasi-Γ-ideal of S. □

Theorem 4.13. *Every one sided int-soft Γ-ideal (f_S, S) of an ordered Γ-semigroup S is an int-soft quasi-Γ-ideal over U.*

Proof. Assume that the soft set f_S is an int-soft right Γ-ideal over U. Let $a \in S$, then

$$((f_S\Gamma\chi_S) \widetilde{\cap} (\chi_S\Gamma f_S))(a) = (f_S\Gamma\chi_S)(a) \cap (\chi_S\Gamma f_S)(a) \subseteq (f_S\Gamma\chi_S)(a).$$

Since f_S is a int-soft right Γ-ideal over U, then we have $(f_S\Gamma\chi_S)(a) \subseteq f_S(a)$ (by Proposition 4.3). Therefore $((f_S\Gamma\chi_S)\widetilde{\cap}(\chi_S\Gamma f_S))(a) \subseteq f_S(a)$. Also for all $p,q \in S$ such that $p \le q \Longrightarrow f_S(p) \supseteq f_S(q)$. Thus (f_S, S) is an int-soft quasi-Γ-ideal of S. Similarly, we can prove it for int-soft left Γ-ideal of S over U.

The converse is not true. This is supported by the next example. □

Example 4.14. *Let $S = \{0, p, q, r\}$ and $\Gamma = \{\lambda\}$. Let $U = \{1, 2, 3\}$ is a universal set. Define a binary operation on S in the table given below:*

λ	0	p	q	r
0	0	0	0	0
p	0	p	q	0
q	0	0	0	0
r	0	r	0	0

The order relation "$\le$"is defined by:

$$\le := \{(0,0), (p,p), (q,q), (r,r)\}$$

This is an ordered Γ-semigroup. Define a soft set f_S over U such that $f_S(0) = \{1,2,3\}$, $f_S(p) = \{1,2\}$, $f_S(q) = \{1\}$ and $f_S(r) = \{2,3\}$. Clearly it is an int-soft quasi-Γ-ideal of S, but it is not int-soft left Γ-ideal nor int-soft right Γ-ideal, since $f_S(p\lambda q) = f_S(q) = \{1\} \not\supseteq f_S(p)$ and $f_S(r\lambda p) = f_S(r) = \{2,3\} \not\supseteq f_S(p)$.

Definition 4.15. *Let C be a non-empty subset of an ordered Γ-semigroup S and let (f_C, S) be a soft set over U. For all $\delta \subseteq U$, the set*

$$\mathcal{U}(f_S;\delta) = \{w \in C \mid f_C(w) \supseteq \delta\}.$$

is said an upper δ-inclusion of (f_C, S).

Theorem 4.16. *Let $(S, \Gamma, \le)$ is an ordered Γ-semigroup. Then for any soft set f_S over U, the given statements are equivalent:*

1. $(\forall\delta \subseteq U)\ \mathcal{U}(f_S;\delta) \neq \emptyset$, *$\mathcal{U}(f_S;\delta)$ is a quasi-Γ-ideal of S.*
2. *$(f_S; S)$ is an int-soft quasi-Γ-ideal of S over U.*

Proof. (1) $\Longrightarrow$ (2). Assume that for every $\delta \subseteq U$, the set $\emptyset \neq \mathcal{U}(f_S;\delta)$ is a quasi-Γ-ideal of S. If $A_p = \emptyset$, then $(f_S\Gamma\chi_S)(p) = \emptyset = (\chi_S\Gamma f_S)(p)$ and so $(f_S\Gamma\chi_S)(p) \cap (\chi_S\Gamma f_S)(p) = \emptyset \subseteq f_S(p)$.

Let $A_p \neq \emptyset$. Assume that for every $(s,t) \in A_p$, there exists some $\beta_i \subseteq U$ such that $f_S(s) \cap f_S(t) = \beta_i$ and so $f_S(s) \supseteq \beta_i$ and $f_S(t) \supseteq \beta_i$ for some i. Thus we have $s, t \in \mathcal{U}(f_S;\beta_i)$. Since $(s,t) \in A_p$, then $p \le s\lambda t$, for some $\lambda \in \Gamma$. We have $p \in (\mathcal{U}(f_S;\beta_i)\Gamma S]$ and $p \in (S\Gamma\mathcal{U}(f_S;\beta_i)]$, for each i. Therefore $p \in (\mathcal{U}(f_S;\beta_i)\Gamma S] \cap (S\Gamma\mathcal{U}(f_S;\beta_i)] \subseteq \mathcal{U}(f_S;\beta_i)$ and so $p \in \mathcal{U}(f_S;\beta_i)$ for each i. Consequently, $f_S(p) \supseteq \beta_i$ for each i. We have

$$\begin{aligned}((f_S\Gamma\chi_S)\widetilde{\cap}(\chi_S\Gamma f_S))(p) &= (f_S\Gamma\chi_S)(p) \cap (\chi_S\Gamma f_S)(p)\\ &= \left(\underset{(s,t)\in A_p}{\cup}\{f_S(s) \cap \chi_S(t)\}\right) \cap \left(\underset{(s,t)\in A_p}{\cup}\{\chi_S(s) \cap f_S(t)\}\right)\\ &= \underset{(s,t)\in A_p}{\cup}\{f_S(s) \cap f_S(t)\} = \cup\beta_i \subseteq f_S(p).\end{aligned}$$

Thus $((f_S\Gamma\chi_S)\widetilde{\cap}(\chi_S\Gamma f_S))(p) \subseteq f_S(p)$. Let $p, q \in S$ be such that $p \le q$. Then $f_S(p) \supseteq f_S(q)$. Indeed: Let $f_S(p) \subset f_S(q)$. Then there exists $\alpha \subseteq U$ such that $f_S(p) \subseteq \alpha \subseteq f_S(q) \Longrightarrow q \in \mathcal{U}(f_S;\alpha)$ and $p \notin \mathcal{U}(f_S;\alpha)$, which is a contradiction because $\mathcal{U}(f_S;\alpha)$ is a quasi-Γ-ideal of S. Hence $f_S(p) \supseteq f_S(q)$. Therefore, $(f_S; S)$ is an int-soft quasi-Γ-ideal over U.

(2) $\Longrightarrow$ (1) Let $a \in (\mathcal{U}(f_S;\delta)\Gamma S] \cap (S\Gamma\mathcal{U}(f_S;\delta)] \Longrightarrow a \in (\mathcal{U}(f_S;\delta)\Gamma S]$ and $a \in (S\Gamma\mathcal{U}(f_S;\delta)]$. Then there exist some $m,n \in \mathcal{U}(f_S;\delta)$, $p,q \in S$ and $\lambda,\mu \in \Gamma$ such that $a \leq m\lambda p$ and $a \leq q\mu n$. Then $(m,p),(q,n) \in A_a$. Since $m,n \in \mathcal{U}(f_S;\delta)$, then $f_S(m) \supseteq \delta$ and $f_S(n) \supseteq \delta$. Then we have

$$(f_S\Gamma\chi_S)(a) = \bigcup_{(s,t)\in A_a} f_S(s) \cap \chi_S(t) \supseteq f_S(m) \cap \chi_S(p) \supseteq \delta \cap U = \delta.$$

Hence $(f_S\Gamma\chi_S)(a) \supseteq \delta$. Furthermore, we have

$$(\chi_S\Gamma f_S)(a) = \bigcup_{(s,t)\in A_a} \chi_S(t) \cap f_S(s) \supseteq \chi_S(q) \cap f_S(n) \supseteq U \cap \delta = \delta.$$

Hence $(\chi_S\Gamma f_S)(a) \supseteq \delta$ and $(f_S\Gamma\chi_S)(a) \cap (\chi_S\Gamma f_S)(a) \supseteq \delta$. But f_S is an int-soft quasi-Γ-ideal over U, so $f_S(a) \supseteq (f_S\Gamma\chi_S)(a) \cap (\chi_S\Gamma f_S)(a) \supseteq \delta \Longrightarrow f_S(a) \supseteq \delta$ and so $a \in \mathcal{U}(f_S;\delta)$. Thus $(\mathcal{U}(f_S;\delta)\Gamma S] \cap (S\Gamma\mathcal{U}(f_S;\delta)] \subseteq \mathcal{U}(f_S;\delta)$.

Let $a,b \in S$ be such that $a \leq b$ and $b \in \mathcal{U}(f_S;\delta)$, then $a \in \mathcal{U}(f_S;\delta)$. Indeed: As $b \in \mathcal{U}(f_S;\delta)$, then $f_S(b) \supseteq \delta$. Since f_S is an int-soft quasi-Γ-ideal over U, then $f_S(a) \supseteq f_S(b) \supseteq \delta \Longrightarrow f_S(a) \supseteq \delta$. Hence $a \in \mathcal{U}(f_S;\delta)$. Therefore, $\mathcal{U}(f_S;\delta)$ is a quasi-Γ-ideal of S. □

Theorem 4.17. *Let $(S,\Gamma,\leq)$ be an ordered Γ-semigroup, then the soft intersection of any int-soft right Γ-ideal (f_S,S) and any int-soft left Γ-ideal (g_S,S) over U is an int-soft quasi-Γ-ideal of S over U.*

Proof. Let (f_S,S) be any int-soft right Γ-ideal and (g_S,S) be any int-soft left Γ-ideal over U. By Proposition 4.3, if $a \in S$, then we have

$$\left(\left(f_S\widetilde{\cap}g_S\right)\Gamma\chi_S\right)(a) \subseteq (f_S\Gamma\chi_S)(a) \subseteq f_S(a)$$

Thus $\left(\left(f_S\widetilde{\cap}g_S\right)\Gamma\chi_S\right)(a) \subseteq f_S(a)$. Moreover, we have

$$\left(\chi_S\Gamma\left(f_S\widetilde{\cap}g_S\right)\right)(a) \subseteq (\chi_S\Gamma g_S)(a) \subseteq g_S(a)$$

Thus $\left(\chi_S\Gamma\left(f_S\widetilde{\cap}g_S\right)\right)(a) \subseteq g_S(a)$. Then $\left(\left(f_S\widetilde{\cap}g_S\right)\Gamma\chi_S\right)(a) \cap \left(\chi_S\Gamma\left(f_S\widetilde{\cap}g_S\right)\right)(a) \subseteq f_S(a) \cap g_S(a) = (f_S\widetilde{\cap}g_S)(a)$. Thus $\left(\left(f_S\widetilde{\cap}g_S\right)\Gamma\chi_S\right)\widetilde{\cap}\left(\chi_S\Gamma\left(f_S\widetilde{\cap}g_S\right)\right) \subseteq f_S\widetilde{\cap}g_S$.

Let $s,t \in S$ be such that $s \leq t$, then $\left(f_S\widetilde{\cap}g_S\right)(s) \supseteq \left(f_S\widetilde{\cap}g_S\right)(t)$. Indeed: Since f_S and g_S are int-soft right Γ-ideal and int-soft left Γ-ideal over U, then $f_S(s) \supseteq f_S(t)$ and $g_S(s) \supseteq g_S(t)$, so $f_S(s) \cap g_S(s) \supseteq f_S(t) \cap g_S(t)$. Therefore, $\left(f_S\widetilde{\cap}g_S\right)(s) = f_S(s) \cap g_S(s) \supseteq f_S(t) \cap g_S(t) = \left(f_S\widetilde{\cap}g_S\right)(t)$. Hence $\left(f_S\widetilde{\cap}g_S;S\right)$ is an int-soft quasi-Γ-ideal over U. □

Theorem 4.18. *(see [1]) In an ordered Γ-semigroup S, the following statements are equivalent:*

1. *S is regular.*
2. *The set $(R\Gamma L]$ is a quasi-Γ-ideal of S for each right Γ-ideal R and for each left Γ-ideal L of S.*

Theorem 4.19. *In an ordered Γ-semigroup S, the given statements are equivalent:*

1. *S is regular.*
2. *for any int-soft right Γ-ideal (f_S,S) and for any int-soft left Γ-ideal (g_S,S) of S, the product $(f_S\Gamma g_S,S)$ is an int-soft quasi-Γ-ideal over U.*

Proof. (1) $\Longrightarrow$ (2) Since S is regular, then by Corollary 4.9, $\left(f_S\widetilde{\cap}g_S,S\right) = (f_S\Gamma g_S,S)$. By Theorem 4.17, $\left(f_S\widetilde{\cap}g_S,S\right)$ is an int-soft quasi-Γ-ideal over U and so $(f_S\Gamma g_S,S)$ is an int-soft quasi-Γ-ideal of S over U.

(2) $\Longrightarrow$ (1) Let R be any right Γ-ideal and L be any left Γ-ideal of S. By Proposition 4.4 and assumption, $(\chi_R\Gamma\chi_L,S)$ is an int-soft quasi-Γ-ideal over U. Moreover, $(\chi_R\Gamma\chi_L,S) = \left(\chi_{(R\Gamma L]},S\right)$, so $\chi_{(R\Gamma L]}$ is an int-soft quasi-Γ-ideal over U. Thus $(R\Gamma L]$ is a quasi-Γ-ideal of S (by Theorem 4.12). Hence by Theorem 4.18, S is regular. □

5. Characterizations of completely regular ordered Γ-semigroup by using its int-soft quasi-Γ-ideals

In this section, characterizations of completely regular ordered Γ-semigroups by using int-soft quasi-Γ-ideals and semiprime int-soft quasi-Γ-ideals are provided.

Definition 5.1. *[22] An ordered* Γ*-semigroup* $(S, \Gamma, \leq)$ *is called completely regular if* S *is left regular, right regular and regular.*

In other words, an element $s \in S$ *is called completely regular if there exist some element* $r \in S$ *and* $\lambda, \mu, \rho, \sigma \in S$ *such that* $s \leq s\lambda s\mu r\rho s\sigma s$*.*

If every element of an ordered Γ-semigroup S is completely regular, then S is called completely regular ordered Γ-semigroup. Equivalent definitions are:

1. For every $K \subseteq S$, $K \subseteq (K\Gamma K\Gamma S\Gamma K\Gamma K]$.
2.) For every element $s \in S$, $s \in (s\Gamma s\Gamma S\Gamma s\Gamma s]$.

Lemma 5.2. *Every int-soft quasi-*Γ*-ideal* (f_S, S) *of an ordered* Γ*-semigroup* S*, is an int-soft* Γ*-semigroup over* U*.*

Proof. Let (f_S, S) be an int-soft quasi-Γ-ideal of an ordered Γ-semigroup S over U. Let $p, q \in S$ and $\lambda \in S$. Since $(p, q) \in A_{p\lambda q}$ and (f_S, S) is an int-soft quasi-Γ-ideal of S, then we have

$$\begin{aligned} f_S(p\lambda q) &\supseteq \left((f_S\Gamma\chi_S)\,\widetilde{\cap}\,(\chi_S\Gamma f_S)\right)(p\lambda q) \\ &= (f_S\Gamma\chi_S)(p\lambda q) \cap (\chi_S\Gamma f_S)(p\lambda q) \\ &= \Big(\bigcup_{(s,t)\in A_{p\lambda q}}\{f_S(s) \cap \chi_S(t)\}\Big) \cap \Big(\bigcup_{(s,t)\in A_{p\lambda q}}\{\chi_S(s) \cap f_S(t)\}\Big) \\ &\supseteq \{f_S(p) \cap \chi_S(q)\} \cap \{\chi_S(p) \cap f_S(q)\} \\ &= f_S(p) \cap f_S(q). \end{aligned}$$

Thus $f_S(p\lambda q) \supseteq f_S(p) \cap f_S(q)$ for all $p, q \in S$ and $\lambda \in \Gamma$. Therefore, (f_S, S) is a int-soft Γ-semigroup of S. □

Lemma 5.3. *([22]) Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semigroup, then the following statements are equivalent:*

1. *S is completely regular.*
2. *Every bi-*Γ*-ideal B of S is semiprime.*

Lemma 5.4. *([23]). Every quasi-*Γ*-ideal Q of an ordered* Γ*-semigroup S is a bi-*Γ*-ideal of S.*

Lemma 5.5. *(see [1]) In a regular ordered* Γ*-semigroup S, every bi-*Γ*-ideal B of S is a quasi-*Γ*-ideal of S.*

Theorem 5.6. *In an ordered* Γ*-semigroup S, the follwoing statements are equivalent:*

1. *S is completely regular.*
2. *For all int-soft quasi-*Γ*-ideal* (f_S, S) *of S, we have,* $(f_S(p), S) = \left(\bigcap_{\theta\in\Gamma} f_S(p\theta p), S\right)$ *for all* $p \in S$ *and S is regular.*

Proof. (1) ⟹ (2). Let (f_S, S) be an int-soft quasi-Γ-ideal over U and $p \in S$. Since S is left regular as well as right regular, then $p \leq u\theta p\mu p$ and $p \leq p\rho p\sigma v$ for some $u, v \in S$ and $\theta, \mu, \rho, \sigma \in \Gamma$. Then $(u, p\mu p), (p\rho p, v) \in A_p$ and we have

$$\begin{aligned} f_S(p) &\supseteq \left((f_S\Gamma\chi_S)\,\widetilde{\cap}\,(\chi_S\Gamma f_S)\right)(p) \\ &= (f_S\Gamma\chi_S)(p) \cap (\chi_S\Gamma f_S)(p) \\ &= \Big(\bigcup_{(s,t)\in A_p}\{f_S(s) \cap \chi_S(t)\}\Big) \cap \Big(\bigcup_{(s,t)\in A_p}\{\chi_S(s) \cap f_S(t)\}\Big) \\ &\supseteq \{f_S(p\rho p) \cap \chi_S(v)\} \cap \{\chi_S(u) \cap f_S(p\mu p)\} \\ &= f_S(p\rho p) \cap f_S(p\mu p) \\ &\supseteq \bigcap_{\theta\in\Gamma} f_S(p\theta p). \end{aligned}$$

Thus $f_S(p) \supseteq \underset{\theta\in\Gamma}{\cap} f_S(p\theta p)$. Since f_S is an int-soft quasi-Γ-ideal over U, then by Lemma 5.2, (f_S, S) is an int-soft Γ-semigroup of S. Hence $\underset{\theta\in\Gamma}{\cap} f_S(p\theta p) \supseteq f_S(p)$. Thus $f_S(p) = \underset{\theta\in\Gamma}{\cap} f_S(p\theta p)$ for all $p \in S$. Since S is completely regular, then S is regular.

(2) $\Longrightarrow$ (1). Let $p \in S$ and A be any quasi-Γ-ideal of S. By Lemma 5.4, A is a bi-Γ-ideal of S. Let $p\theta p \in A$, for all $\theta \in \Gamma$. By Theorem 4.12 and (2), $\chi_A(p) = \underset{\theta\in\Gamma}{\cap} \chi_A(p\theta p)$. Since $p\theta p \in A$, then $\chi_A(p\theta p) = U$. This implies $\underset{\theta\in\Gamma}{\cap} \chi_A(p\theta p) = U = \chi_A(p)$. Thus $p \in A$ and hence A is semiprime. By Lemma 5.5, since S is regular, A is a bi-Γ-ideal of S and it is semiprime. By Theorem 5.3, S is completely regular. □

Definition 5.7. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup and $\emptyset \neq C \subseteq S$. Then C is called semiprime if for all $p \in S$ and $\lambda \in \Gamma$, $p\lambda p \in C$ implies $p \in C$. Equivalently, for every $D \subseteq S$, $D\Gamma D \subseteq C$ implies $D \subseteq C$.*

Definition 5.8. *Let a be any element of an ordered Γ-semigroup $(S, \Gamma, \leq)$ and let $\emptyset \neq \lambda \subseteq U$. The soft set $((a]_\lambda, S)$ such that*

$$(a]_\lambda : S \longrightarrow P(U),\ a_\lambda(p) = \begin{cases} \lambda, & \text{if } p \in (a], \\ \emptyset, & \text{otherwise.} \end{cases}$$

is called critical soft point of ordered Γ-semigroup $(S, \Gamma, \leq)$.

Definition 5.9. *Let (f_S, S) be an int-soft quasi-Γ-ideal of an ordered Γ-semigroup S over U. (f_S, S) is called semiprime if for every critical soft point $(a]_\lambda$ of S*

$$(a]_\lambda \Gamma (a]_\lambda \subseteq f_S \text{ implies } (a]_\lambda \subseteq f_S.$$

For any soft set (f_S, S) of S over U, we define, $((f_S]](x) = \underset{x\leq y}{\cup} f_S(y)$ for any $x \in S$.

If for any soft set (f_S, S) of S, then we have $(((f_S]], S) = (f_S, S)$, then f_S is called strongly convex.

Lemma 5.10. *Let (f_S, S) be a soft set of an ordered Γ-semigroup S. Then, (f_S, S) is a strongly convex soft set of S if and only if $x \leq y$ implies $f_S(y) \subseteq f_S(x)$, for all $x, y \in S$.*

Proof. Straightforward. □

Lemma 5.11. *Let $((a]_\lambda, S)$ and $((a]_\mu, S)$ be critical soft points of an ordered Γ-semigroup S and let (f_S, S) be a soft set of S over U. Then the following statements are true:*

1. *If (f_S, S) is strongly convex soft set of S, then $(a]_\lambda \subseteq f_S$ iff $f_S(a) \supseteq \lambda$.*
2. $(a]_\lambda \Gamma (b]_\mu = \underset{c\in a\Gamma b}{\cup} (c]_{\lambda\cap\mu}$.

Proof. (1) Let (f_S, S) be a strongly convex soft subset of S.

If $(a]_\lambda \subseteq f_S$, we have $\lambda = (a]_\lambda(a) \subseteq f_S(a)$ and so $f_S(a) \supseteq \lambda$.

If $f_S(a) \supseteq \lambda$, then $(a]_\lambda \subseteq f_S$. Indeed, if $p \notin (a]_\lambda$, we obtain $(a]_\lambda(p) = \emptyset \subseteq f_S(p)$.

Moreover, if $p \in (a]_\lambda$, we have $p \leq a$. Hence, by Lemma 5.10 and hypothesis of strong convexity, we deduce $f_S(p) \supseteq f_S(a) \supseteq (a]_\lambda(p) = \lambda$. Thus $(a]_\lambda \subseteq f_S$.

(2) Let $p \in (c]$, for some $c \in a\Gamma b$ and so $p \leq c \in a\Gamma b$. Thus $p \leq a\alpha b$, for some $\alpha \in \Gamma$ and so $(a, b) \in A_p$. We have ,

$$\left((a]_\lambda \Gamma (b]_\mu\right)(p) = \underset{(s,t)\in A_p}{\cup} (a]_\lambda(s) \cap (b]_\mu(t) \supseteq (a]_\lambda(a) \cap (b]_\mu(b) = \lambda \cap \mu.$$

Hence $\left((a]_\lambda \Gamma b_\mu\right)(p) \supseteq \lambda\cap\mu$. Since $(a]_\lambda(s) \cap (b]_\mu(t) \subseteq \lambda\cap\mu$ for any $s, t \in S$, then $\left((a]_\lambda \Gamma (b]_\mu\right)(p) \subseteq \lambda\cap\mu$ and so $\left((a]_\lambda \Gamma (b]_\mu\right)(p) = \lambda\cap\mu = \underset{c\in a\Gamma b}{\cup} (c]_{\lambda\cap\mu}(p)$. If $p \notin (c]$, for all $c \in a\Gamma b$, then we have $\underset{c\in a\Gamma b}{\cup} (c]_{\lambda\cap\mu}(p) = \emptyset$. On the other hand, $\left((a]_\lambda \Gamma (b]_\mu\right)(p) = \emptyset$. In fact, if $\left((a]_\lambda \Gamma (b]_\mu\right)(p) \neq \emptyset$, then we have $\left((a]_\lambda \Gamma b_\mu\right)(p) = \underset{(s,t)\in A_p}{\cup} (a]_\lambda(s) \cap (b]_\mu(t) \neq \emptyset$. Therefore, there exist $(u, v) \in A_p$ such that $(a]_\lambda(u) = \lambda$ and $(b]_\mu(v) = \mu$. Thus $u \in (a]$ and $v \in (b]$. Since $(u, v) \in A_p$, then $p \leq u\lambda v$ for some $\lambda \in \Gamma$ and so $p \in ((a]\Gamma(b]] = (a\Gamma b]$. Then $p \leq a\alpha b$, for some $\alpha \in \Gamma$ which is impossible. In this case, $\left((a]_\lambda \Gamma (b]_\mu\right)(p) = \emptyset = \underset{c\in a\Gamma b}{\cup} (c]_{\lambda\cap\mu}$. □

Theorem 5.12. *Let f_S be an int-soft quasi-Γ-ideal of an ordered Γ-semigroup S. Then the following statements are equivalent:*

1. *f_S is semiprime.*
2. *$f_S(p) \supseteq \bigcap_{q \in p\Gamma p} f_S(q)$, for any $p \in S$.*
3. *$f_S(p) = \bigcap_{q \in p\Gamma p} f_S(q)$, for any $p \in S$.*

Proof. (1) $\Longrightarrow$ (2). Let $p \in S$. We claim $f_S(p) \supseteq \bigcap_{q \in p\Gamma p} f_S(q)$. Indeed: If $f_S(p) \subset \bigcap_{q \in p\Gamma p} f_S(q)$, then there exists $\lambda \subseteq U$ such that $f_S(p) \subset \lambda \subseteq \bigcap_{q \in p\Gamma p} f_S(q)$. Then $f_S(q) \supseteq \lambda$, for all $q \in p\Gamma p$. By Lemma 5.11 (1), $(q]_\lambda \subseteq f_S$, for all $q \in p\Gamma p$. By Lemma 5.11 (2), $(p]_\lambda \Gamma (p]_\lambda = \bigcup_{q \in p\Gamma p} (q]_\lambda \subseteq f_S$. Since f_S is semiprime int-soft quasi-Γ-ideal of S, then $(p]_\lambda \subseteq f_S$. By Lemma 5.11 (1), $f_S(p) \supseteq \lambda$ which is a contradiction. Therefore, $f_S(p) \supseteq \bigcap_{q \in p\Gamma p} f_S(q)$, for any $p \in S$.

(2) $\Longrightarrow$ (3). Let $p \in S$. Since f_S is an int-soft quasi-Γ-ideal, then by Lemma 5.2, f_S is an int-soft Γ-semigroup and so, $\bigcap_{q \in p\Gamma p} f_S(q) \supseteq f_S(p)$, for all $q \in p\Gamma p$. By (2), we have $f_S(p) = \bigcap_{q \in p\Gamma p} f_S(q)$, for any $p \in S$.

(3) $\Longrightarrow$ (1). Let $p \in S$ and $(p]_\lambda \Gamma (p]_\lambda \subseteq f_S$, $\lambda \subseteq U$. By Lemma 5.11 (2), $(p]_\lambda \Gamma (p]_\lambda = \bigcup_{q \in p\Gamma p} (q]_\lambda \subseteq f_S$. Then $(q]_\lambda \subseteq f_S$ for every $q \in p\Gamma p$ and so, by Lemma 5.11 (1), $f_S(q) \supseteq \lambda$, for every $q \in p\Gamma p$. Thus we have, $f_S(p) = \bigcap_{q \in p\Gamma p} f_S(q) \supseteq \lambda$ and so, $f_S(p) \supseteq \lambda$. Since f_S is strongly convex, then by Lemma 5.11 (1), we deduce $(p]_\lambda \subseteq f_S$. Therefore, f_S is semiprime. $\square$

Theorem 5.13. *In an ordered Γ-semigroup S, the following statements are equivalent:*

1. *S is completely regular.*
2. *Each int-soft quasi-Γ-ideal (f_S, S) of S over U is semiprime int-soft quasi-Γ-ideal and S is regular.*

Proof. (1) $\Longrightarrow$ (2). Let (f_S, S) be an int-soft quasi-Γ-ideal of an ordered Γ-semigroup S over U. Let $p \in S$. Since S is a left as well as right regular, then $p \leq u\lambda p\mu p$ and $p \leq p\rho p\sigma v$ for some $u, v \in S$ and $\lambda, \mu, \rho, \sigma \in \Gamma$. Then $(u, p\mu p), (p\rho p, v) \in A_p$ and we have

$$
\begin{aligned}
f_S(p) &\supseteq \left((f_S\Gamma\chi_S)\,\widetilde{\cap}\,(\chi_S\Gamma f_S)\right)(p) \\
&= (f_S\Gamma\chi_S)(p) \cap (\chi_S\Gamma f_S)(p) \\
&= \Big(\bigcup_{(s,t)\in A_p} \{f_S(s) \cap \chi_S(t)\}\Big) \cap \Big(\bigcup_{(s,t)\in A_p} \{\chi_S(s) \cap f_S(t)\}\Big) \\
&\supseteq \{f_S(p\rho p) \cap \chi_S(v)\} \cap \{\chi_S(u) \cap f_S(p\mu p)\} \\
&= f_S(p\rho p) \cap f_S(p\mu p) \\
&\supseteq \bigcap_{\lambda\in\Gamma} f_S(p\lambda p).
\end{aligned}
$$

Thus $f_S(p) \supseteq \bigcap_{\lambda\in\Gamma} f_S(p\lambda p)$. Then by Theorem 5.12, (f_S, S) is semiprime int-soft quasi-Γ-ideal over U. Moreover, S is regular since S is completely regular.

(2) $\Longrightarrow$ (1). Let $p \in S$. Let A be any quasi-Γ-ideal of S, then by lemma 5.4, A is a bi-Γ-ideal of S. Let $p\theta p \in A$ for all $\theta \in \Gamma$. Then by Theorem 4.12 and (2), we have, $\chi_A(p) = \bigcap_{\theta\in\Gamma} \chi_A(p\theta p)$. Since $p\theta p \in A$, then $\chi_A(p\theta p) = U$ imply $\bigcap_{\theta\in\Gamma} \chi_A(p\theta p) = U = \chi_A(p)$. Thus $p \in A$. Hence A is semiprime. Since S is regular, then by Lemma 5.5, A is any bi-Γ-ideal of S and is semiprime. Thus by Theorem 5.3, S is a completely regular. $\square$

6. Characterizations of semilattices of left as well as right simple ordered Γ-semigroups in terms of int-soft quasi-Γ-ideals

In this section, we define the semilattices of left and right simple semigroups of ordered Γ-semigroups and characterize them in terms of their int-soft quasi-Γ-ideals.

Definition 6.1. *([15]) A sub-Γ-semigroup F of an ordered Γ-semgroup S is said to be a filter of S if:*

1. $(\forall\ u, v \in S$ *and* $\forall\ \lambda \in \Gamma)$ $(u\lambda v \in F \Longrightarrow u \in F$ *and* $v \in F)$.
2. $(\forall\ u \in F$ *and* $\forall\ v \in S)$ $(v \leq u \Longrightarrow v \in F)$.

For $a \in S$, the least filter of S generated by a, is denoted by $N(a)$. We denote, the relation on S by $\mathcal{N}$, is defined by $\mathcal{N} := \{(u, v) \in S \times S \mid N(u) = N(v)\}$. The relation $\mathcal{N}$ on S is an equivalence relation on S.

Definition 6.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. Then*

1. *A congruence (see [29]) on S is an equivalence relation σ on S such that, if $(u, v) \in \sigma \Longrightarrow (u\lambda s, v\lambda s) \in \sigma$ and $(s\lambda u, s\lambda v) \in \sigma$ for every $s \in S$ and $\lambda \in \Gamma$.*
2. *A semilattice congruence (see [29]) on S is a congruence σ on S such that $(u, u\lambda u) \in \sigma$ and $(u\lambda v, v\lambda u) \in \sigma$ for every $u, v \in S$ and $\lambda \in \Gamma$.*

Note if σ is a semilattice congruence on S, then for every $s \in S$, the σ-class $(s)_\sigma$ of S is a sub-Γ-semigroup of S. Moreover, $\mathcal{N}$ is a semilattice congruence on S (see [29]).

Definition 6.3. *An oredered Γ-semigroup $(S, \Gamma, \leq)$ is called a semilattice of left as well as right simple sub-Γ-semigroups if there exists a semilattice X and a family $\left\{S_\beta\right\}_{\beta \in X}$ of left as well as right simple sub-Γ-semigroup of S such that:*

1. $(\forall \alpha, \beta \in X, \alpha \neq \beta)$ $(S_\alpha \cap S_\beta) = \emptyset$,
2. $S = \underset{\alpha \in X}{\cup} S_\alpha$,
3. $(\alpha, \beta \in X)\left(S_\alpha \Gamma S_\beta \subseteq S_{\alpha\beta}\right)$.

Equivalently, if there exists a semilattice congruence σ on S such that every $\sigma-$class $(s)_\sigma$ of S is a left simple as well as right simple sub-Γ-semigroup of S, for every $s \in S$.

Lemma 6.4. *([20]) For every $s \in S$, the $\mathcal{N}-$class $(s)_{\mathcal{N}}$ of an ordered Γ-semigroup S is a right (resp., left) simple sub-Γ-semigroup of S if and only if every right (resp., left) ideal is a left (resp., right) ideal of S and is semiprime.*

Theorem 6.5. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. The following statements are equivalent:*

1. *S is a semilattice of left as well as right simple sub-Γ-semigroups.*
2. *For every int-soft quasi-Γ-ideal (f_S, S) over U, we have,*
   (i) $(f_S(u), S) = \left(\underset{\lambda \in \Gamma}{\cap} f_S(u\lambda u), S\right)$,
   (ii) $(f_S(u\theta v), S) = (f_S(v\theta u), S)$ *for all $u, v \in S$ and $\theta \in \Gamma$.*

Proof. (1) $\Longrightarrow$ (2). Let S be a semilattice of left as well as right simple sub-Γ-semigroups and (f_S, S) be an int-soft quasi-Γ-ideal over U.

To proof (i), by Theorem 5.6, it is enough to prove that S is completely regular. Let $u \in S = \underset{\alpha \in X}{\cup} S_\alpha$, then $u \in S_\alpha$ for some $\alpha \in X$. Since each S_α is left as well as right simple, then $S_\alpha = (u\Gamma S_\alpha]$ and $S_\alpha = (S_\alpha \Gamma u]$. So we have,

$$
\begin{aligned}
(u\Gamma S_\alpha] &= (u\Gamma (S_\alpha \Gamma u]] \\
&= ((u]\,\Gamma\,((S_\alpha \Gamma a]]] \\
&= ((u]\,\Gamma\,(S_\alpha \Gamma a]] \\
&= (u\Gamma\,(S_\alpha \Gamma u)]
\end{aligned}
$$

Thus $u \in (u\Gamma S_\alpha] = (u\Gamma(S_\alpha\Gamma u)]$ implies $u \in (u\Gamma(S_\alpha\Gamma u)]$, then there exist $v \in S_\alpha$ and $\theta, \mu \in \Gamma$ such that $u \leq u\theta v\mu u$. Since $v \in S_\alpha = (u\Gamma S_\alpha\Gamma u]$ implies $v \in (u\Gamma S_\alpha\Gamma u]$, then there exist $s \in S_\alpha$ and $\rho, \lambda \in \Gamma$ such that $v \leq u\rho s\lambda u$. Thus we have, $u \leq u\theta v\mu u \leq u\theta(u\rho s\lambda u)\mu u \leq (u\theta u)\rho s\lambda(u\mu u) \in u\Gamma u\Gamma S\Gamma u\Gamma u$ implies $u \in (u\Gamma u\Gamma S\Gamma u\Gamma u]$. Therefore, S is completely regular.

(ii) Let $u, v \in S = \bigcup_{\alpha\in X} S_\alpha$. Then $u \in S_\alpha$ and $v \in S_\beta$ for some $\alpha, \beta \in X$. Thus for all $\theta \in \Gamma$, $u\theta v \in S_\alpha\Gamma S_\beta \subseteq S_{\alpha\beta}$ implies $u\theta v \in S_{\alpha\beta}$. Also $v\theta u \in S_\beta\Gamma S_\alpha = S_{\beta\alpha}$ implies $v\theta u \in S_{\beta\alpha} = S_{\alpha\beta}$ (because X is a semilattice). Thus $v\theta u \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is left as well as right simple sub-Γ-semigroup of S, then $S_{\alpha\beta} = \left(S_{\alpha\beta}\Gamma c\right]$ and $S_{\alpha\beta} = \left(c\Gamma S_{\alpha\beta}\right]$ for all $c \in S_{\alpha\beta}$.Since $u\theta v \in S_{\alpha\beta}$, then $u\theta v \in \left(v\theta u\Gamma S_{\alpha\beta}\right]$ and $u\theta v \in \left(S_{\alpha\beta}\Gamma v\theta u\right]$ implies $u\theta v \in \left(v\theta u\Gamma S_{\alpha\beta}\right] \cap \left(S_{\alpha\beta}\Gamma v\theta u\right] \subseteq (v\theta u\Gamma S] \cap (S\Gamma v\theta u]$. Thus $u\theta v \in (v\theta u\Gamma S] \cap (S\Gamma v\theta u]$ implies $u\theta v \in (v\theta u\Gamma S]$ and $u\theta v \in (S\Gamma v\theta u]$, then there exist $\rho, \lambda \in \Gamma$ and $x, y \in S$ such that $u\theta v \leq (v\theta u)\rho x$ and $u\theta v \leq y\lambda(v\theta u)$. So $(y, v\theta u), (v\theta u, x) \in A_{u\theta v}$. Since f_S is an int-soft quasi-Γ-ideal over U, then we have

$$\begin{aligned} f_S(u\theta v) &\supseteq \left((f_S\Gamma\chi_S)\,\widetilde{\cap}\,(\chi_S\Gamma f_S)\right)(u\theta v) \\ &= (f_S\Gamma\chi_S)(u\theta v) \cap (\chi_S\Gamma f_S)(u\theta v) \\ &= \Big(\bigcup_{(s,t)\in A_{u\theta v}} \{f_S(s) \cap \chi_S(t)\}\Big) \cap \Big(\bigcup_{(s,t)\in A_{u\theta v}} \{\chi_S(s) \cap f_S(t)\}\Big) \\ &\supseteq \{f_S(v\theta u) \cap \chi_S(x)\} \cap \{\chi_S(y) \cap f_S(v\theta u)\} \\ &= f_S(v\theta u) \cap f_S(v\theta u) = f_S(v\theta u). \end{aligned}$$

Hence $f_S(u\theta v) \supseteq f_S(v\theta u)$. Similary we can proof that $f_S(v\theta u) \supseteq f_S(u\theta v)$. Hence $f_S(u\theta v) = f_S(v\theta u)$ for all $u, v \in S$ and $\theta \in \Gamma$.

(2) $\Longrightarrow$ (1) Let for every int-soft quasi-Γ-ideals (f_S, S) of S, (i) and (ii) hold. By using Lemma 6.4, we prove (1). Let L is a left Γ-ideal of S and $u \in L$, $v \in S$, so by Proposition 4.4 and Theorem 4.13, we have χ_L is a int-soft quasi-Γ-ideal of S. Then by (2), $\chi_L(u\theta v) = \chi_L(v\theta u)$. Since $v\theta u \in S\Gamma L \subseteq L$ implies $v\theta u \in L$, then $\chi_L(v\theta u) = U = \chi_L(u\theta v)$. Thus $u\theta v \in L$. Hence $L\Gamma S \subseteq L$. Also for all $u \in L$ and for all $v \in S$,if $v \leq u \Longrightarrow v \in L$. Therefore L is right Γ-ideal of S. Let $u \in S$ such that $u\theta u \in L$ for all $\theta \in \Gamma$. Since L is a quasi-Γ-ideal of S, then by using Theorem 4.12 and (2), we have $\chi_L(u) = \bigcap_{\theta\in\Gamma} \chi_L(u\theta u)$. Since $u\theta u \in L$ for all $\theta \in \Gamma$, then we have $\bigcap_{\theta\in\Gamma} \chi_L(u\theta u) = U = \chi_L(u)$. Thus $u \in L$. Therefore L is semiprime. Similarly we can proof this for every right Γ-ideal of S. □

Lemma 6.6. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. If $u \leq u\theta u$, $\forall\ u \in S$ and $\theta \in \Gamma$, then for every int-soft quasi-Γ-ideal f_S over U, we have,*

$$(f_S(u), S) = \Big(\bigcap_{\theta\in\Gamma} f_S(u\theta u), S\Big) \text{ for all } u \in S.$$

Proof. Reader can easily prove this by using Lemma 5.2. □

Theorem 6.7. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semigroup. If $u \leq u\theta u$, $\forall\ u \in S$ and $\theta \in \Gamma$, then the following statements are equivalent:*

1. *$u\lambda v \in (v\lambda u\Gamma S] \cap (S\Gamma v\lambda u]$ for each $u, v \in S$ and $\lambda \in \Gamma$.*
2. *For every int-soft quasi-Γ-ideal (f_S, S) of S, we have, $(f_S(u\lambda v), S) = (f_S(v\lambda u), S)$ for every $u, v \in S$ and $\lambda \in \Gamma$.*

Proof. (1) $\Longrightarrow$ (2). As $u\lambda v \in (v\lambda u\Gamma S] \cap (S\Gamma v\lambda u]$, then $u\lambda v \in (v\lambda u\Gamma S]$ and $u\lambda v \in (S\Gamma v\lambda u]$. If $u\lambda v \in (v\lambda u\Gamma S]$, then there exist $\rho \in \Gamma$ and $p \in S$ such that $u\lambda v \leq (v\lambda u)\rho p$. By (1), we have $(v\lambda u)\rho p \in (p\rho(v\lambda u)\Gamma S] \cap (S\Gamma p\rho(v\lambda u)]$ implies $(v\lambda u)\rho p \in (S\Gamma p\rho(v\lambda u)]$, then there exist $\theta \in \Gamma$ and $q \in S$ such that $(v\lambda u)\rho p \leq (q\theta p)\rho(v\lambda u)$. Thus $u\lambda v \leq (v\lambda u)\rho p \leq (q\theta p)\rho(v\lambda u) \Longrightarrow u\lambda v \leq (q\theta p)\rho(v\lambda u)$. So $(q\theta p, v\lambda u) \in A_{u\lambda v}$. If $u\lambda v \in (S\Gamma v\lambda u]$, then there exist $m \in S$ and $\alpha \in \Gamma$ such that $u\lambda v \leq m\alpha(v\lambda u)$. By (1), we have $m\alpha(v\lambda u) \in ((v\lambda u)\alpha m\Gamma S] \cap (S\Gamma(v\lambda u)\alpha m]$ implies $m\alpha(v\lambda u) \in ((v\lambda u)\alpha m\Gamma S]$, then there exist $n \in S$ and $\beta \in \Gamma$ such that $m\alpha(v\lambda u) \leq (v\lambda u)\alpha(m\beta n)$ and so $u\lambda v \leq m\alpha(v\lambda u) \leq (v\lambda u)\alpha(m\beta n) \Longrightarrow u\lambda v \leq (v\lambda u)\alpha(m\beta n)$. So $(v\lambda u, m\beta n) \in A_{u\lambda v}$.

Now let (f_S, S) is an int-soft quasi-ideal over U and $A_{u\lambda v} \neq \emptyset$, Then,

$$\begin{aligned} f_S(u\lambda v) &\supseteq \left((f_S\Gamma\chi_S)\,\widetilde{\cap}\,(\chi_S\Gamma f_S)\right)(u\lambda v) \\ &= (f_S\Gamma\chi_S)(u\lambda v) \cap (\chi_S\Gamma f_S)(u\lambda v) \\ &= \bigcup_{(s,t)\in A_{u\lambda v}} \{f_S(s) \cap \chi_S(t)\} \cup \bigcup_{(s,t)\in A_{u\lambda v}} \{\chi_S(s) \cap f_S(t)\} \\ &\supseteq \{f_S(v\lambda u) \cap \chi_S(m\beta n)\} \cup \{\chi_S(q\theta p) \cap f_S(v\lambda u)\} \\ &= f_S(v\lambda u) \cup f_S(v\lambda u) = f_S(v\lambda u). \end{aligned}$$

Thus $f_S(u\lambda v) \supseteq f_S(v\lambda u)$. Similarly we can prove that $(f_S(v\lambda u), S) \supseteq (f_S(u\lambda v), S)$. Therefore, $(f_S(v\lambda u), S) = (f_S(u\lambda v), S)$ for all $u, v \in S$ and $\lambda \in \Gamma$.

(2) $\Longrightarrow$ (1). Let (f_S, S) be an int-soft quasi-Γ-ideal over U. Since $u \leq u\lambda u$ for all $u \in S$ and $\lambda \in \Gamma$, then by Lemma 6.6, we have $f_S(u) = \bigcap_{\lambda\in\Gamma} f_S(u\lambda u)$ for all $\lambda \in \Gamma$. By (2), we have $(f_S(v\lambda u), S) = (f_S(u\lambda v), S)$ for every $u, v \in S$ and $\lambda \in \Gamma$. By Theorem 6.5, S is a semilattice of left simple as well as right simple sub-Γ-semigroups of S.

Let $u, v \in S = \bigcup_{\alpha\in X} S_\alpha$, then $u \in S_\alpha$ and $v \in S_\beta$ for some $\alpha, \beta \in X$. Then for all $\lambda \in \Gamma$, $u\lambda v \in S_\alpha\Gamma S_\beta \subseteq S_{\alpha\beta}$ implies $u\lambda v \in S_{\alpha\beta}$. Also $v\lambda u \in S_\beta\Gamma S_\alpha = S_{\beta\alpha}$ implies $v\lambda u \in S_{\beta\alpha} = S_{\alpha\beta}$ (because X is a semilattice). Hence $v\lambda u \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is left as well as right simple sub-Γ-semigroup of S, then $S_{\alpha\beta} = \left(S_{\alpha\beta}\Gamma w\right]$ and $S_{\alpha\beta} = \left(w\Gamma S_{\alpha\beta}\right]$ for all $w \in S_{\alpha\beta}$. Since $u\lambda v \in S_{\alpha\beta}$, then $u\lambda v \in \left(v\lambda u\Gamma S_{\alpha\beta}\right]$ and $u\lambda v \in \left(S_{\alpha\beta}\Gamma v\lambda u\right]$ implies $u\lambda v \in \left(v\lambda u\Gamma S_{\alpha\beta}\right] \cap \left(S_{\alpha\beta}\Gamma v\lambda u\right] \subseteq (v\lambda u\Gamma S] \cap (S\Gamma v\lambda u]$. Thus $u\lambda v \in (v\lambda u\Gamma S] \cap (S\Gamma v\lambda u]$. This completes the proof. □

7. Conclusion

In this paper, we have extended int-soft ideal theory of ordered semigroups to ordered Γ-semigroups. The results that have been proved in *po*-semigroups by using their fuzzy ideals, fuzzy quasi-ideals and uni-soft quasi ideals, we can prove them for ordered Γ-semigroups by using their int-soft quasi-Γ-ideals. The results that relate int-soft quasi-Γ-ideals and int-soft one sided Γ-ideals have been proved. Using the notion of int-soft quasi-Γ-ideals in ordered Γ-semigroups, we characterized completely regular ordered Γ-semigroups by using their int-soft quasi-Γ-ideals. The notion of critical soft point in *po*-Γ-semigroups have been defined. By using the definition of critical soft point, we have defined the notion of semiprime int-soft quasi-Γ-ideal and characterized completely regular ordered Γ-semigroups by using their semiprime int-soft quasi-Γ-ideals. Moreover, semilattices of left as well as right simple sub-Γ-semigroups of ordered Γ-semigroups have been defined and some characterizations of semilattices of left as well as right simple sub-Γ-semigroups by using their int-soft quasi-Γ-ideals have been provided. From above results, we can conclude that the theory of int-soft ideals can be extended to other structures and Γ-structures.

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