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Weak and strong convergence of inertial-type iterative algorithms for solving split general system of generalized equilibrium problem

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Abstract. In this article, we generalize several equilibrium and variational inequality problems by introducing new general system of split generalized equilibrium problem. We introduce an inertial-type iterative algorithm and prove its weak convergence to the common solution of general system of split generalized equilibrium problem and fixed point problem of demicontractive mapping. We also prove strong convergence of the proposed algorithm by using shrinking projection method. Finally, we give numerical experiments to validate the performance of our algorithm and compare it with other existing method.

1. Introduction

Equilibrium problem was firstly introduced by Fan [1] in 1972, but the most significant contribution to this problem was made by Blum and Oettli [2] and Noor and Oettli [3] in 1994. Equilibrium problem (EP) is a generalization of many mathematical models such as variational inequality problems, fixed point problems, optimization problems, nash EPs, minimization problems, saddle point problems, etc and having applications in physics, enginnering, economics, game theory, image reconstruction, transportation, network and elasticity [2, 4–8]. Therefore, this problem has been extended to more general problems in several ways.

In 2010, Ceng and Yao [9] introduced and studied a system of generalized equilibrium problem (SGEP). Several iterative methods have been proposed by many authors to solve SGEP. In 2011, Moudafi [10] introduced split equilibrium problem which is a natural extension of various optimization problems such as split feasibility problems, split inclusion problem, split variational iequality problem and split common fixed point problem, see [11–16]. In 2016, system of split equilibrium problem was introduced by Ugwunnadi and Ali [17] and it was further extended by Karahan et al. [18] to solve a new problem called system of split mixed equilibrium problem. For important results in this direction or in similar subjects, see [19–22].

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Inspired and motivated by the above mentioned results and the ongoing research in the direction of split equilibrium problem, we introduced a new problem called split general system of generalized equlibrium problem. This problem is general in the sense that it includes split equilibrium problems, split variational inequality problems, split feasibility problems, equilibrium problems, variational inequality problems and many other problems as its special cases.

Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $F_1, F_2 : C \times C \to \mathbb{R}$ and $G_1, G_2 : Q \times Q \to \mathbb{R}$ be bifunctions. Assume that $f_1, f_2 : C \to H_1$ and $h_1, h_2 : Q \to H_2$ are nonlinear mappings and $B : H_1 \to H_2$ is a bounded linear operator. Consider the following split general system of generalized equilibrium problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} F_1(x^*, x) + \langle f_1(y^*), x - x^* \rangle + \frac{1}{r_1} \langle x^* - y^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ F_2(y^*, y) + \langle f_2(x^*), y - y^* \rangle + \frac{1}{r_2} \langle y^* - x^*, y - y^* \rangle \geq 0 \text{ for all } y \in C \end{cases} \tag{1}$$

and $(u^*, v^*) = (Bx^*, By^*) \in Q \times Q$ such that

$$\begin{cases} G_1(u^*, u) + \langle h_1(v^*), u - u^* \rangle + \frac{1}{s_1} \langle u^* - v^*, u - u^* \rangle \geq 0 \text{ for all } u \in Q, \\ G_2(v^*, v) + \langle h_2(u^*), v - v^* \rangle + \frac{1}{s_2} \langle v^* - u^*, v - v^* \rangle \geq 0 \text{ for all } v \in Q, \end{cases} \tag{2}$$

where $r_1, r_2, s_1, s_2 > 0$.
Let the set of all solutions of (1) and (2) be denoted by Ψ_1 and Ψ_2 respectively, then set of all solutions of split general system of generalized equilibrium problem can be denoted by Ψ, where

$$\Psi = \{(x^*, y^*) \in \Psi_1 : (Bx^*, By^*) \in \Psi_2\}. \tag{3}$$

Next, we present some special cases of problem (3) which are given as follows:

1. If $F_1 = F_2 = F, G_1 = G_2 = G, f_1 = f_2 = f, h_1 = h_2 = h, r_1 = r_2, s_1 = s_2$ and $x^* = y^*$, then problem (3) reduces to the following modified split generalized equilibrium problem of finding $x^* \in C$ such that

   $$F(x^*, x) + \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C$$

   and $u^* = Bx^* \in Q$ such that

   $$G(u^*, u) + \langle h(u^*), u - u^* \rangle \geq 0 \text{ for all } u \in Q,$$

   which was introduced by Cheawchan and Kangtunyakarn [23].

2. If $f_1 = f_2 = h_1 = h_2 = 0$, then problem (3) reduces to following problem of finding $(x^*, y^*) \in C \times C$ such that

   $$\begin{cases} F_1(x^*, x) + \frac{1}{r_1} \langle x^* - y^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ F_2(y^*, y) + \frac{1}{r_2} \langle y^* - x^*, y - y^* \rangle \geq 0 \text{ for all } y \in C \end{cases}$$

   and $(u^*, v^*) = (Bx^*, By^*) \in Q \times Q$ such that

   $$\begin{cases} G_1(u^*, u) + \frac{1}{s_1} \langle u^* - v^*, u - u^* \rangle \geq 0 \text{ for all } u \in Q, \\ G_2(v^*, v) + \frac{1}{s_2} \langle v^* - u^*, v - v^* \rangle \geq 0 \text{ for all } v \in Q. \end{cases}$$

   which is a new split mixed equilibrium problem.

3. If $f_1 = f_2 = 0, h_1 = h_2 = 0, x^* = y^*$, then problem (3) reduces to following system of split equilibrium problem of finding $x^* \in C$ such that

   $$F_i(x^*, x) \geq 0 \text{ for all } x \in C, i = 1, 2$$

and $u^* = Bx^* \in Q$ such that

$$G_i(u^*, u) \geq 0 \quad \text{for all} \quad u \in Q, i = 1, 2.$$

4. If $F_1 = F_2 = F, G_1 = G_2 = G$ in problem (3), then problem (3) reduces to following split equilibrium problem of finding $x^* \in C$ such that

$$F(x^*, x) \geq 0 \quad \text{for all} \quad x \in C$$

and $u^* = Bx^* \in Q$ such that

$$G(u^*, u) \geq 0 \quad \text{for all} \quad u \in Q,$$

which was considered by He [24].

5. If $F_1 = F_2 = 0, G_1 = G_2 = 0$, then problem (3) reduces to following split general system of variational inequality problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle r_1 f_1(y^*) + x^* - y^*, x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in C, \\ \langle r_2 f_2(x^*) + y^* - x^*, y - y^* \rangle \geq 0 \quad \text{for all} \quad y \in C \end{cases}$$

and $(u^*, v^*) = (Bx^*, By^*) \in Q \times Q$ such that

$$\begin{cases} \left\langle s_1 h_1(y_2^*) + y_1^* - y_2^*, y_1 - y_1^* \right\rangle \geq 0 \quad \text{for all} \quad y_1 \in Q, \\ \left\langle s_2 h_2(y_1^*) + y_2^* - y_1^*, y_2 - y_2^* \right\rangle \geq 0 \quad \text{for all} \quad y_2 \in Q. \end{cases}$$

This problem was introduced by Siriyan and Kangtunyakarn [25].

6. If $f_1 = f_2 = f, h_1 = h_2 = h, r_1 = r_2, s_1 = s_2$ and $x^* = y^*$ in problem (5), then problem (3) reduces to following split variational inequality problem of finding $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in C$$

and $u^* = Bx^* \in Q$ such that

$$\langle h(u^*), u - u^* \rangle \geq 0 \quad \text{for all} \quad u \in Q,$$

which was introduced and discussed by Censor et al. [26].

7. If $f = h = 0$ in problem (6), then problem (3) reduces to following split feasibility problem of finding

$$x^* \in C \quad \text{such that} \quad u^* = Bx^* \in Q,$$

which was introduced by Censor and Elfving [27].

8. If $F_1 = G_1, F_2 = G_2, f_1 = h_1, f_2 = h_2, C = Q, H_1 = H_2, r_1 = s_1, r_2 = s_2$ and $B = I$, identity operator, then problem (3) reduces to following general system of generalized equilibrium problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} F_1(x^*, x) + \langle f_1(y^*), x - x^* \rangle + \frac{1}{r_1} \langle x^* - y^*, x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in C, \\ F_2(y^*, y) + \langle f_2(x^*), y - y^* \rangle + \frac{1}{r_2} \langle y^* - x^*, y - y^* \rangle \geq 0 \quad \text{for all} \quad y \in C, \end{cases} \tag{4}$$

which was considered by Ceng and Yao [9].

9. If $F_1 = F_2 = F, f_1 = f_2 = f$ in problem (8), then problem (3) reduces to following system of generalized equilibrium problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} F(x^*, x) + \langle f(y^*), x - x^* \rangle + \frac{1}{r_1} \langle x^* - y^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ F(y^*, y) + \langle f(x^*), y - y^* \rangle + \frac{1}{r_2} \langle y^* - x^*, y - y^* \rangle \geq 0 \text{ for all } y \in C, \end{cases}$$

which was discussed by Ceng and Yao [9].

10. If $x^* = y^*$ in problem (9), then problem (3) reduces to following generalized equilibrium problem of finding $x^* \in C$ such that

$$F(x^*, x) + \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C,$$

which was introduced by Takahashi and Takahashi [28].

11. If $f_1 = f_2 = 0, x^* = y^*$ in problem (8), then problem (3) reduces to following system of equilibrium problem of finding

$$\begin{cases} F_1(x^*, x) \geq 0 \text{ for all } x \in C, \\ F_2(x^*, x) \geq 0 \text{ for all } x \in C, \end{cases}$$

which was considered by Combettes and Hirstoaga [29].

12. If $F_1 = F_2 = F$ in problem (11), then problem (3) reduces to following equilibrium problem of finding $x^* \in C$ such that

$$F(x^*, x) \geq 0 \text{ for all } x \in C,$$

which was considered by Fan [1].

13. If $F_1 = F_2 = 0$ in problem (8), then problem (3) reduces to following general system of variational inequality problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle f_1(y^*), x - x^* \rangle + \frac{1}{r_1} \langle x^* - y^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ \langle f_2(x^*), y - y^* \rangle + \frac{1}{r_2} \langle y^* - x^*, y - y^* \rangle \geq 0 \text{ for all } y \in C, \end{cases}$$

which was introduced and considered by Ceng et al. [30].

14. If $f_1 = f_2 = f$ in problem (13), then problem (3) reduces to following system of variational inequality problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle f(y^*), x - x^* \rangle + \frac{1}{r_1} \langle x^* - y^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ \langle f(x^*), y - y^* \rangle + \frac{1}{r_2} \langle y^* - x^*, y - y^* \rangle \geq 0 \text{ for all } y \in C, \end{cases}$$

which was introduced and studied by Verma [31].

15. If $x^* = y^*$ in problem (14), then problem (3) reduces to following classical variational inequality problem of finding $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C.$$

In 1964, Polyak [32] employed the inertial extrapolation technique, based on heavy ball methods of two-order time dynamical system to equip the iterative algorithm with fast convergence characteristics. Inertial algorithm is a two step iteration where the next iterate is defined by making use of previous two iterates. Several researchers have used inertial extrapolation for constructing some fast iterative algorithms [33–37].

In 2008, Takahashi et al. [38] introduced shrinking projection method for finding fixed point of a nonexpansive single-valued mapping in Hilbert spaces. This method plays an important role in proving strong convergence for finding fixed points of nonlinear mapping. Many authors developed the shrinking projection method for solving equilibrium problems, fixed point problems, variational inequality problems etc.

In this paper, we are interested in studying the problem of finding a common solution for split general system of generalized equilibrium problem and fixed point problem for demicontractive mapping. The motivation for studying such problems is in its potential application to various mathematical models. We present a new inertial type algorithm and prove weak convergence of sequence generated by proposed algorithm under some mild conditions. The strong convergence theorem is also obtained by employing shrinking projection method. Our problem can be viewed as a generalization and improvement of various existing nonlinear analysis problems in the current literature. In particular, we have also applied our result in solving image restoration problem with the help of numerical example.

2. Preliminaries

Throughout the paper, we assume that H is a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and C is a nonempty closed convex subset of H. We denote weak and strong convergence by $'' \rightharpoonup ''$ and $'' \to ''$, respectively. Also, we use $\omega_w(x_n) = \{x : \text{there exists } x_{n_k} \rightharpoonup x\}$ to represent weak ω-limit set of $\{x_n\}$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C,$$

where P_C is called metric projection of H onto C and satisfies

$$\|P_C(x) - P_C(y)\| \leq \langle x - y, P_C(x) - P_C(y) \rangle . \tag{5}$$

Recall that an operator $S : H \to H$ is called

1. nonexpansive if
   $\|Sv - Sw\| \leq \|v - w\|$ for all $v, w \in H$.
2. quasi-nonexpansive if
   $\|Sv - w\| \leq \|v - w\|$ for all $v \in H$ and $w \in Fix(S) \neq \phi$.
3. firmly nonexpansive if
   $\langle Sv - Sw, v - w \rangle \geq \|Sv - Sw\|^2$ for all $v, w \in H$.
4. $\theta-$averaged if there exists a constant $\theta \in (0, 1)$ and a nonexpansive mapping $T : H \to H$ such that $S = (1 - \theta)I + \theta T$.
5. α-inverse strongly monotone (ism) if there exists $\alpha > 0$ such that
   $\langle Sv - Sw, v - w \rangle \geq \alpha \|Sv - Sw\|^2$ for all $v, w \in H$.
6. τ-demicontractive if $Fix(S) \neq \phi$ and there exists $\tau \in [0, 1)$ such that
   $\|Sv - w\|^2 \leq \|v - w\|^2 + \tau \|v - Sv\|^2$ for all $v \in H$ and $w \in Fix(S)$.

It is well known that every firmly nonexpansive mapping is nonexpansive, quasi-nonexpansive as well as 1/2-averaged. Every quasi-nonexpansive mapping satisfies the following property (see, [39])

$$2 \langle w - Sv, v - Sv \rangle \leq \|v - Sv\|^2 \text{ for all } v \in H, w \in Fix(S) \neq \phi. \tag{6}$$

Definition 2.1. *Let $\{x_n\}$ be a sequence in H and $S : H \to H$ be an operator, then $I - S$ is said to be demiclosed at zero if $x_n \rightharpoonup z^*$ and $(I - S)x_n \to 0$ implies $Sz^* = z^*$ i.e. $z^* \in Fix(S)$.*

Note that for every nonexpansive mapping $S : C \to C$, $I - S$ is demiclosed at zero, where C is nonempty closed, convex subset of H.

Next, we collect several lemmas, which we use in our results.

Lemma 2.2. *[40] Let $f : H \to H$ be β-ism operator, then $I - 2\beta f$ is nonexpansive.*

Lemma 2.3. *[41] Let H be a real Hilbert space, then*

1. $2\langle v, w\rangle = \|v\|^2 + \|w\|^2 - \|v - w\|^2 = \|v + w\|^2 - \|v\|^2 - \|w\|^2$ *for all $v, w \in H$.*
2. $\|v + w\|^2 \leq \|v\|^2 + 2\langle w, v + w\rangle$ *for all $v, w \in H$.*
3. $\|\alpha v + (1 - \alpha)w\|^2 = \alpha\|v\|^2 + (1 - \alpha)\|w\|^2 - \alpha(1 - \alpha)\|v - w\|^2$ *for all $v, w \in H$.*

Lemma 2.4. *[42, 43] Let C be a nonempty closed and convex subset of H, then following properties hold:*

1. $\|v - P_C u\|^2 + \|u - P_C u\|^2 \leq \|u - v\|^2$ *for all $u \in H, v \in C$.*
2. *For given $u \in H$ and $w \in C$, $w = P_C u$ iff $\langle u - w, v - w\rangle \leq 0$ for all $v \in C$.*

Lemma 2.5. *[44] Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative numbers such that $\alpha_{n+1} \leq \alpha_n + \beta_n$, $n \geq 0$. If $\sum_{n=0}^{\infty} \beta_n$ converges, then $\lim_{n\to\infty} \alpha_n$ exists.*

Lemma 2.6. *[45] Let $\{x_n\}$ be a sequence in H satisfying the properties:*

1. $\lim_{n\to\infty} \|x_n - x\|$ *exists for each $x \in C$;*
2. $\omega_w(x_n) \subseteq C$.

Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.7. *[46] Let $\{x_n\}$ be a sequence in H and $q = P_C u$ for $u \in H$. If $\{x_n\}$ satisfies the condition*

1. $\omega_w(x_n) \subset C$;
2. $\|x_n - u\| \leq \|q - u\|$ *for all n.*

Then $x_n \to q$.

Lemma 2.8. *[46] Let C be a nonempty closed convex subset of a real Hilbert space H. For every $x, y, z \in H$ and $\gamma \in \mathbb{R}$, the set $D = \{v \in C; \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + \gamma\}$ is closed and convex.*

Lemma 2.9. *[47] Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to H_2$ be a bounded linear operator with adjoint B^* such that L is spectral radius of operator B^*B. Consider $T : H_1 \to H_1$ a nonexpansive mapping, then $I - \gamma B^*(I - T)B$ is γL averaged.*

To solve problem (3), we need the following assumptions for a bifunction $F : C \times C \to \mathbb{R}$

(i) $F(u, u) = 0$ for all $u \in C$.

(ii) F is monotone i.e. $F(u, v) + F(v, u) \leq 0$ for all $u, v \in C$.

(iii) F is upper hemicontinuous i.e. for each $u, v, w \in C$,

$$\limsup_{t\to 0} F(tw + (1 - t)u, v) \leq F(u, v).$$

(iv) For each $u \in C$, the function $v \to F(u, v)$ is convex and lower semicontinuous.

Lemma 2.10. *[29] Assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies assumptions (i)-(iv). For $r > 0$ and $u \in H$, define a mapping $\mathrm{T}_r^F(u) = \{w \in C : F(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0 \text{ for all } v \in C\}$. Then the following statements hold:*

1. *$\mathrm{T}_r^F(x) \neq \phi$ for each $x \in H$ and T_r^F is single-valued.*
2. *T_r^F is firmly nonexpansive i.e.*

$$\|\mathrm{T}_r^F(u) - \mathrm{T}_r^F(v)\|^2 \leq \left\langle \mathrm{T}_r^F(u) - \mathrm{T}_r^F(v), u - v \right\rangle \text{ for all } u, v \in H.$$

3. *$Fix(\mathrm{T}_r^F) = EP(F, C)$.*
4. *Solution set $EP(F, C)$ is closed and convex.*

Lemma 2.11. *[9] Let C be a nonempty closed convex subset of Hilbert space H. Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two bifunctions satisfying assumptions (i)-(iv) and the mappings $f_1, f_2 : C \to H$ be β_1-ism and β_2-ism respectively. Let $r_1 \in (0, 2\beta_1)$ and $r_2 \in (0, 2\beta_2)$. Then $(x^*, y^*) \in C \times C$ is a solution of general system of generalized equilibrium problem (4) iff x^* is fixed point of the mapping $\Gamma : C \to C$ defined by*

$$\Gamma(x) = \mathrm{T}_{r_1}^{F_1}(I - r_1 f_1)\mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)x \text{ for all } x \in C,$$

where $y^* = \mathrm{T}_{r_2}^{F_2}(x^* - r_2 f_2 x^*)$.

Lemma 2.12. *Let C be a nonempty closed convex subset of Hilbert space H. Let $F_1, F_2 : C \times C \to \mathbb{R}$ be two bifunctions satisfying assumptions (i)-(iv) and the mappings $f_1, f_2 : C \to H$ be β_1-ism and β_2-ism respectively. Let $r_1 \in (0, 2\beta_1)$ and $r_2 \in (0, 2\beta_2)$. Consider a mapping $\Gamma : C \to C$ defined by*

$$\Gamma(x) = \mathrm{T}_{r_1}^{F_1}(I - r_1 f_1)\mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)x \text{ for all } x \in C,$$

then Γ is averaged mapping.

Proof. Firstly, we prove that $I - r_i f_i$ is averaged for $i = 1, 2$. Note that

$$I - r_i f_i = (I - \frac{r_i}{2\beta_i})I + \frac{r_i}{2\beta_i}(I - 2\beta_i f_i), i = 1, 2$$

and $\frac{r_i}{2\beta_i} \in (0, 1)$. By applying Lemma 2.2, $I - 2\beta_i f_i$ is nonexpansive and therefore $I - r_i f_i$ is averaged mapping for $i = 1, 2$. Also, from Lemma 2.10, $\mathrm{T}_{r_i}^{F_i}$ is firmly nonexpansive i.e. 1/2-averaged. As composition of averaged mappings is averaged, hence Γ is averaged. □

3. Main Results

In this section, we present two new inertial type iterative algorithm to find the common solution of general system of split generalized equilibrium problem and fixed point problem of demicontractive mapping. Strong convergence of the suggested algorithm is proved using shrinking projection method. Firstly, we prove following lemma.

Lemma 3.1. *Let C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively. Let $F_1, F_2 : C \times C \to \mathbb{R}$ and $G_1, G_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying assumptions (i)-(iv). Let $f_1, f_2 : H_1 \to H_1$ be β_1, β_2-ism and $h_1, h_2 : H_2 \to H_2$ be ρ_1, ρ_2-ism mappings respectively. Assume that $B : H_1 \to H_2$ is a bounded linear operator with adjoint B^* such that $\gamma \in (0, 1/L)$ where L is spectral radius of operator B^*B. Let $r_1 \in (0, 2\beta_1)$, $r_2 \in (0, 2\beta_2)$,*

$s_1 \in (0, 2\rho_1)$ and $s_2 \in (0, 2\rho_2)$. Define $K_C : H_1 \to C$ by $K_C(x) = T_{r_1}^{F_1}(I - r_1 f_1)T_{r_2}^{F_2}(I - r_2 f_2)x$ for all $x \in H_1$ and $K_Q : H_2 \to Q$ by $K_Q(x) = T_{s_1}^{G_1}(I - s_1 h_1)T_{s_2}^{G_2}(I - s_2 h_2)x$ for all $x \in H_2$. Assume that

$$\Psi = \{(x^*, y^*) \in \Psi_1 : (Bx^*, By^*) \in \Psi_2\}. \tag{7}$$

Then $(x^*, y^*) \in \Psi$ iff $x^* = K_C(x^* - \gamma B^*(I - K_Q)Bx^*)$, where $y^* = T_{r_2}^{F_2}(I - r_2 f_2)x^*$ and $v^* = T_{s_2}^{G_2}(I - s_2 h_2)u^*$ with $u^* = Bx^*$ and $v^* = By^*$.

Proof. Firstly, we consider $(x^*, y^*) \in \Psi$, then $(x^*, y^*) \in \Psi_1$ and $(u^*, v^*) \in \Psi_2$ where $u^* = Bx^*$ and $v^* = By^*$. Since $(x^*, y^*) \in \Psi_1$, we have

$$\begin{cases} F_1(x^*, x) + \langle f_1(y^*), x - x^*\rangle + \frac{1}{r_1}\langle x^* - y^*, x - x^*\rangle \geq 0 \text{ for all } x \in C, \\ F_2(y^*, y) + \langle f_2(x^*), y - y^*\rangle + \frac{1}{r_2}\langle y^* - x^*, y - y^*\rangle \geq 0 \text{ for all } y \in C. \end{cases}$$

Then from Lemma 2.11, we have $x^* = T_{r_1}^{F_1}(I - r_1 f_1)y^*$ where $y^* = T_{r_2}^{F_2}(I - r_2 f_2)x^*$ i.e.

$$x^* = T_{r_1}^{F_1}(I - r_1 f_1)T_{r_2}^{F_2}(I - r_2 f_2)x^* = K_C(x^*). \tag{8}$$

Since $(u^*, v^*) = (Bx^*, By^*) \in \Psi_2$, we have

$$\begin{cases} G_1(u^*, u) + \langle h_1(v^*), u - u^*\rangle + \frac{1}{s_1}\langle u^* - v^*, u - u^*\rangle \geq 0 \text{ for all } u \in Q, \\ G_2(v^*, v) + \langle h_2(u^*), v - v^*\rangle + \frac{1}{s_2}\langle v^* - u^*, v - v^*\rangle \geq 0 \text{ for all } v \in Q. \end{cases}$$

Again, from Lemma 2.11, we have $u^* = T_{s_1}^{G_1}(I - s_1 h_1)v^*$ where $v^* = T_{s_2}^{G_2}(I - s_2 h_2)u^*$ i.e.

$$u^* = T_{s_1}^{G_1}(I - s_1 h_1)T_{s_2}^{G_2}(I - s_2 h_2)u^* = K_Q(u^*) = K_Q(Bx^*). \tag{9}$$

From equations (8) and (9), we conclude $x^* = K_C(x^* - \gamma B^*(I - K_Q)Bx^*)$.

Conversely, let $x^* = K_C(x^* - \gamma B^*(I - K_Q)Bx^*)$ and $(w_1, w_2) \in \Psi$ i.e. $(w_1, w_2) \in \Psi_1$ and $(w_1^*, w_2^*) \in \Psi_2$ where $w_1^* = Bw_1$ and $w_2^* = Bw_2$. Since f_1, f_2 are β_1, β_2-ism respectively and h_1, h_2 are ρ_1, ρ_2-ism respectively, then from Lemma 2.12, K_C and K_Q are averaged mappings and hence nonexpansive. From $(w_1, w_2) \in \Psi$, we obtain $w_1 = K_C(w_1 - \gamma B^*(I - K_Q)Bw_1)$. Further from equation (9), $Bw_1 = w_1^* = K_Q(w_1^*)$. Now, using equation (6), we obtain

$$\begin{aligned} ||x^* - w_1||^2 &= ||K_C(x^* - \gamma B^*(I - K_Q)Bx^*) - K_C(w_1 - \gamma B^*(I - K_Q)Bw_1)||^2 \\ &\leq ||(x^* - \gamma B^*(I - K_Q)Bx^*) - (w_1 - \gamma B^*(I - K_Q)Bw_1)||^2 \\ &= ||x^* - \gamma B^*(I - K_Q)Bx^* - w_1||^2 \\ &= ||x^* - w_1 - \gamma B^*(I - K_Q)Bx^*||^2 \\ &= ||x^* - w_1||^2 + \gamma^2||B^*(I - K_Q)Bx^*||^2 - 2\gamma\left\langle x^* - w_1, B^*(I - K_Q)Bx^*\right\rangle \\ &= ||x^* - w_1||^2 + \gamma^2||B^*(I - K_Q)Bx^*||^2 - 2\gamma\left\langle Bx^* - Bw_1, (I - K_Q)Bx^*\right\rangle \\ &= ||x^* - w_1||^2 + \gamma^2||B^*(I - K_Q)Bx^*||^2 + 2\gamma\Big[\left\langle Bw_1 - K_QBx^*, Bx^* - K_QBx^*\right\rangle \\ &\quad - \left\langle Bx^* - K_QBx^*, Bx^* - K_QBx^*\right\rangle\Big] \\ &= ||x^* - w_1||^2 + \gamma^2||B^*(I - K_Q)Bx^*||^2 - 2\gamma||(I - K_Q)Bx^*||^2 + 2\gamma\left\langle Bw_1 - K_QBx^*, Bx^* - K_QBx^*\right\rangle \\ &\leq ||x^* - w_1||^2 + \gamma^2L||(I - K_Q)Bx^*||^2 - 2\gamma||(I - K_Q)Bx^*||^2 + \gamma||Bx^* - K_QBx^*||^2 \\ &= ||x^* - w_1||^2 + \gamma^2L||(I - K_Q)Bx^*||^2 - \gamma||(I - K_Q)Bx^*||^2 \\ &= ||x^* - w_1||^2 - \gamma(1 - \gamma L)||(I - K_Q)Bx^*||^2, \end{aligned} \tag{10}$$

which implies $\|(I-K_Q)Bx^*\|=0$ i.e. $Bx^*\in Fix(K_Q)$. Hence $u^*=Bx^*=K_Q(u^*)=\mathrm{T}_{s_1}^{G_1}(I-s_1h_1)\mathrm{T}_{s_2}^{G_2}(I-s_2h_2)u^*$ which implies $u^*=\mathrm{T}_{s_1}^{G_1}(I-s_1h_1)v^*$ where $v^*=\mathrm{T}_{s_2}^{G_2}(I-s_2h_2)u^*$. From Lemma 2.11, we obtain $(u^*,v^*)\in\Psi_2$. Also, $x^*=K_C(x^*-\gamma B^*(I-K_Q)Bx^*)=K_C(x^*)$ i.e. $x^*\in Fix(K_C)$ which implies $x^*=\mathrm{T}_{r_1}^{F_1}(I-r_1f_1)\mathrm{T}_{r_2}^{F_2}(I-r_2f_2)x^*$ where $y^*=\mathrm{T}_{r_2}^{F_2}(I-r_2f_2)x^*$ and $x^*=\mathrm{T}_{r_1}^{F_1}(I-r_1f_1)y^*$. Again from Lemma 2.11, we obtain $(x^*,y^*)\in\Psi_1$. Hence $(x^*,y^*)\in\Psi$. □

Now, we provide our algorithm and its converge analysis.

Theorem 3.2. *Let C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively. Let $F_1,F_2:C\times C\to\mathbb{R}$ and $G_1,G_2:Q\times Q\to\mathbb{R}$ be bifunctions satisfying assumptions (i)-(iv). Let $f_1,f_2:H_1\to H_1$ be β_1,β_2-ism and $h_1,h_2:H_2\to H_2$ be ρ_1,ρ_2-ism mappings respectively. Assume that $T:H_1\to H_1$ is η demicontractive mapping such that $I-T$ is demiclosed at zero. Let $B:H_1\to H_2$ be a bounded linear operator with adjoint B^* such that $\gamma\in(0,1/L)$ where L is spectral radius of operator B^*B. Let $r_1\in(0,2\beta_1)$, $r_2\in(0,2\beta_2)$, $s_1\in(0,2\rho_1)$ and $s_2\in(0,2\rho_2)$. Define $K_C:H_1\to C$ by $K_C(x)=\mathrm{T}_{r_1}^{F_1}(I-r_1f_1)\mathrm{T}_{r_2}^{F_2}(I-r_2f_2)x$ for all $x\in H_1$ and $K_Q:H_2\to Q$ by $K_Q(x)=\mathrm{T}_{s_1}^{G_1}(I-s_1h_1)\mathrm{T}_{s_2}^{G_2}(I-s_2h_2)x$ for all $x\in H_2$. Define $K:H_1\to C$ by $K(x)=K_C(x-\gamma B^*(I-K_Q)Bx)$ for all $x\in H_1$.*

For given $x_0,x_1\in C$, let iterative sequence $\{x_n\}$ be generated as

$$\begin{cases} y_n=x_n+\theta_n(x_n-x_{n-1}),\\ u_n=K_C(y_n-\gamma B^*(I-K_Q)By_n),\\ x_{n+1}=(1-\alpha_n)u_n+\alpha_nT(u_n),\end{cases}\tag{11}$$

where $\theta_n\in[0,\theta]$ for some $\theta\in[0,1)$ and $\alpha_n\in(\delta,1-\eta-\delta)$ for some $\delta>0$. Assume that $\Omega=Fix(K)\cap Fix(T)\neq\phi$ and $\sum_{n=0}^{\infty}\theta_n\|x_n-x_{n-1}\|<\infty$, then the generated sequence $\{x_n\}$ converges weakly to a point $p\in\Omega$ where $(p,q)\in\Psi$ such that $q=\mathrm{T}_{r_2}^{F_2}(I-r_2f_2)p$ and $v=\mathrm{T}_{s_2}^{G_2}(I-s_2h_2)u$ with $u=Bp$ and $v=Bq$.

Proof. Firstly, we show that $\{x_n\}$ is bounded.
Let $x^*\in\Omega$, then we have, $(x^*,y^*)\in\Psi$ such that $y^*=\mathrm{T}_{r_2}^{F_2}(I-r_2f_2)x^*$ and $v^*=\mathrm{T}_{s_2}^{G_2}(I-s_2h_2)u^*$ with $u^*=Bx^*$ and $v^*=By^*$. From definition of $\{y_n\}$, we have

$$\begin{aligned}\|y_n-x^*\| &=\|x_n+\theta_n(x_n-x_{n-1})-x^*\|\\ &\le\|x_n-x^*\|+\theta_n\|x_n-x_{n-1}\|.\end{aligned}\tag{12}$$

Consider $z_n=y_n-\gamma B^*(I-K_Q)By_n$, then using $\gamma\in(0,1/L)$ and equation (6), we have

$$\begin{aligned}\|z_n-x^*\|^2&=\|y_n-\gamma B^*(I-K_Q)By_n-x^*\|^2\\ &=\|y_n-x^*\|^2+\gamma^2\|B^*(I-K_Q)By_n\|^2-2\gamma\left\langle y_n-x^*,B^*(I-K_Q)By_n\right\rangle\\ &=\|y_n-x^*\|^2+\gamma^2\left\langle B^*(I-K_Q)By_n,B^*(I-K_Q)By_n\right\rangle-2\gamma\left\langle By_n-Bx^*,(I-K_Q)By_n\right\rangle\\ &=\|y_n-x^*\|^2+\gamma^2\left\langle (I-K_Q)By_n,BB^*(I-K_Q)By_n\right\rangle-2\gamma\Big[\left\langle By_n-K_QBy_n,By_n-K_QBy_n\right\rangle\\ &\quad+\left\langle K_QBy_n-Bx^*,By_n-K_QBy_n\right\rangle\Big]\\ &\le\|y_n-x^*\|^2+\gamma^2L\|(I-K_Q)By_n\|^2-2\gamma\Big[\|By_n-K_QBy_n\|^2-\frac{1}{2}\|By_n-K_QBy_n\|^2\Big]\\ &=\|y_n-x^*\|^2+(\gamma^2L-\gamma)\|(I-K_Q)By_n\|^2\\ &=\|y_n-x^*\|^2-\gamma(1-\gamma L)\|(I-K_Q)By_n\|^2 \qquad (13)\\ &\le\|y_n-x^*\|^2. \qquad (14)\end{aligned}$$

Since K_C is averaged mapping from Lemma 2.12 and hence nonexpansive. Hence

$$\begin{aligned}\|u_n-x^*\|&=\|K_C(z_n)-K_C(x^*)\|\\ &\le\|z_n-x^*\|.\end{aligned}\tag{15}$$

From definition of $\{x_n\}$, $\alpha_n \in (\delta, 1-\eta-\delta)$ for some $\delta > 0$ and Lemma 2.3, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)u_n + \alpha_n T(u_n) - x^*\|^2 \\
&\le \alpha_n \|T(u_n) - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|u_n - T(u_n)\|^2 \\
&\le \alpha_n \Big[\|u_n - x^*\|^2 + \eta\|u_n - T(u_n)\|^2\Big] + (1-\alpha_n)\|u_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|u_n - T(u_n)\|^2 \\
&= \|u_n - x^*\|^2 - \alpha_n(1-\eta-\alpha_n)\|u_n - T(u_n)\|^2 \qquad (16)\\
&\le \|u_n - x^*\|^2. \qquad (17)
\end{aligned}$$

Using equations (12), (14) and (15) in equation (17), we obtain

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|. \tag{18}$$

Using Lemma 2.5 and $\sum_{n=0}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$, we conclude $\lim_{n\to\infty} \|x_n - x^*\|$ exists. In particular, $\{x_n\}, \{y_n\}, \{u_n\}$ all are bounded.

Using equations (14), (15), (16) and Lemma 2.3, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 - \alpha_n(1-\eta-\alpha_n)\|u_n - T(u_n)\|^2 \\
&\le \|y_n - x^*\|^2 - \alpha_n(1-\eta-\alpha_n)\|u_n - T(u_n)\|^2 \\
&\le \|x_n - x^*\|^2 + 2\theta_n \langle x_n - x_{n-1}, y_n - x^* \rangle - \alpha_n(1-\eta-\alpha_n)\|u_n - T(u_n)\|^2, \qquad (19)
\end{aligned}$$

i.e.

$$\begin{aligned}
0 &\le \alpha_n(1-\eta-\alpha_n)\|u_n - T(u_n)\|^2 \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\theta_n \langle x_n - x_{n-1}, y_n - x^* \rangle \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\theta_n \|x_n - x_{n-1}\|.\|y_n - x^*\|.
\end{aligned}$$

As $\lim_{n\to\infty} \|x_n - x^*\|$ exists, therefore utilizing $\sum_{n=0}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$ and $\alpha_n \in (\delta, 1-\eta-\delta)$ for some $\delta > 0$, we have

$$\lim_{n\to\infty} \|u_n - T(u_n)\| = 0. \tag{20}$$

Now,

$$\begin{aligned}
\|x_{n+1} - u_n\| &= \|(1-\alpha_n)u_n + \alpha_n T(u_n) - u_n\| \\
&= \|\alpha_n(u_n - T(u_n))\| \\
&= \alpha_n \|u_n - T(u_n)\|. \qquad (21)
\end{aligned}$$

Hence using equation (20), we get

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{22}$$

Further

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0. \tag{23}$$

From equations (13), (15) (17) and Lemma 2.3, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 \\
&\le \|y_n - x^*\|^2 - \gamma(1-\gamma L)\|(I - K_Q)By_n\|^2 \\
&\le \|x_n - x^*\|^2 + 2\theta_n \langle x_n - x_{n-1}, y_n - x^* \rangle - \gamma(1-\gamma L)\|(I - K_Q)By_n\|^2, \qquad (24)
\end{aligned}$$

which implies

$$0 \le \gamma(1-\gamma L)\|(I-K_Q)By_n\|^2 \\ \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\|.$$

Since $\lim_{n\to\infty} \|x_n - x^*\|$ exists, therefore utilizing $\sum_{n=0}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$ and $\gamma \in (0, 1/L)$, we have

$$\lim_{n\to\infty} \|(I-K_Q)By_n\| = 0. \tag{25}$$

As averaged mappings are nonexpansive, then from definition of $\{u_n\}$ and Lemma 2.12, we get

$$\|u_n - K_C(u_n)\| = \|K_C(z_n) - K_C(u_n)\| \\ \le \|z_n - u_n\|. \tag{26}$$

Consider $v_n = \mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)z_n$. We know that every firmly nonexpansive mapping is nonexpansive. Since f_1 and f_2 are β_1 and β_2-ism mappings respectively, using $y^* = \mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)x^*$ and firmly nonexpansiveness of $\mathrm{T}_{r_1}^{F_1}$, we obtain

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|K_C(z_n) - K_C(x^*)\|^2 \\
&= \|\mathrm{T}_{r_1}^{F_1}(I - r_1 f_1)\mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)z_n - \mathrm{T}_{r_1}^{F_1}(I - r_1 f_1)\mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)x^*\|^2 \\
&= \|\mathrm{T}_{r_1}^{F_1}(I - r_1 f_1)v_n - \mathrm{T}_{r_1}^{F_1}(I - r_1 f_1)y^*\|^2 \\
&= \|\mathrm{T}_{r_1}^{F_1}[v_n - r_1 f_1(v_n)] - \mathrm{T}_{r_1}^{F_1}[y^* - r_1 f_1(y^*)]\|^2 \\
&\le \|v_n - r_1 f_1(v_n) - [y^* - r_1 f_1(y^*)]\|^2 \\
&\le \|v_n - y^* - r_1[f_1(v_n) - f_1(y^*)]\|^2 \\
&\le \|v_n - y^*\|^2 + r_1^2\|f_1(v_n) - f_1(y^*)\|^2 - 2r_1\langle v_n - y^*, f_1(v_n) - f_1(y^*)\rangle \\
&\le \|v_n - y^*\|^2 + r_1^2\|f_1(v_n) - f_1(y^*)\|^2 - 2r_1\beta_1\|f_1(v_n) - f_1(y^*)\|^2 \\
&= \|v_n - y^*\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\
&= \|\mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)z_n - \mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)x^*\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\
&= \|(I - r_2 f_2)z_n - (I - r_2 f_2)x^*\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\
&= \|(z_n - x^*) - r_2(f_2 z_n - f_2 x^*)\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\
&\le \|z_n - x^*\|^2 - r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2.
\end{aligned} \tag{27}$$

Using equation (14), (27) in equation (17) and from Lemma 2.3, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 \\
&\le \|z_n - x^*\|^2 - r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\
&\le \|y_n - x^*\|^2 - r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\
&\le \|x_n - x^*\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - x^*\rangle - r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 \\
&\quad - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2,
\end{aligned} \tag{28}$$

which implies

$$0 \le r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\ \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\|$$

and

$$\begin{aligned}0 &\le r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\|.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x^*\|$ exists, therefore utilizing $\sum_{n=0}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$, $r_1 \in (0, 2\beta_1)$ and $r_2 \in (0, 2\beta_2)$, we get

$$\lim_{n\to\infty} \|f_1(v_n) - f_1(y^*)\| = \lim_{n\to\infty} \|f_2(z_n) - f_2(x^*)\| = 0. \tag{29}$$

Since f_2 is β_2-ism, using firmly nonexpansiveness of $T_{r_2}^{F_2}$ and Lemma 2.3, we get

$$\begin{aligned}\|v_n - y^*\|^2 &= \|T_{r_2}^{F_2}(I - r_2f_2)z_n - T_{r_2}^{F_2}(I - r_2f_2)x^*\|^2 \\ &\le \langle v_n - y^*, (z_n - r_2f_2(z_n)) - (x^* - r_2f_2(x^*))\rangle \\ &= \frac{1}{2}\Big[\|v_n - y^*\|^2 + \|(z_n - x^*) - r_2(f_2(z_n) - f_2(x^*))\|^2 \\ &\quad - \|z_n - x^* - r_2(f_2(z_n) - f_2(x^*)) - (v_n - y^*)\|^2\Big] \\ &\le \frac{1}{2}\Big[\|v_n - y^*\|^2 + \|z_n - x^*\|^2 - r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 \\ &\quad - \|z_n - x^* - r_2(f_2(z_n) - f_2(x^*)) - (v_n - y^*)\|^2\Big] \\ &= \frac{1}{2}\Big[\|v_n - y^*\|^2 + \|z_n - x^*\|^2 - r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 \\ &\quad - \|(z_n - v_n) - r_2(f_2(z_n) - f_2(x^*)) - (x^* - y^*)\|^2\Big] \\ &= \frac{1}{2}\Big[\|v_n - y^*\|^2 + \|z_n - x^*\|^2 - r_2(2\beta_2 - r_2)\|f_2(z_n) - f_2(x^*)\|^2 \\ &\quad - \|(z_n - v_n - (x^* - y^*)\|^2 - r_2^2\|f_2(z_n) - f_2(x^*)\|^2 \\ &\quad + 2r_2\langle z_n - v_n - (x^* - y^*), f_2(z_n) - f_2(x^*)\rangle\Big],\end{aligned}$$

which implies

$$\|v_n - y^*\|^2 \le \|z_n - x^*\|^2 - \|(z_n - v_n - (x^* - y^*)\|^2 + 2r_2\|z_n - v_n - (x^* - y^*)\|.\|f_2(z_n) - f_2(x^*)\|. \tag{30}$$

Since f_1 is β_1-ism, using firmly nonexpansiveness of $T_{r_1}^{F_1}$ and Lemma 2.3, we get

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|K_C(z_n) - K_C(x^*)\|^2 \\
&= \|T_{r_1}^{F_1}(I - r_1 f_1)T_{r_2}^{F_2}(I - r_2 f_2)z_n - T_{r_1}^{F_1}(I - r_1 f_1)T_{r_2}^{F_2}(I - r_2 f_2)x^*\|^2 \\
&= \|T_{r_1}^{F_1}(I - r_1 f_1)v_n - T_{r_1}^{F_1}(I - r_1 f_1)y^*\|^2 \\
&= \|T_{r_1}^{F_1}[v_n - r_1 f_1(v_n)] - T_{r_1}^{F_1}[y^* - r_1 f_1(y^*)]\|^2 \\
&\leq \langle u_n - x^*, (v_n - r_1 f_1(v_n)) - (y^* - r_1 f_1(y^*))\rangle \\
&= \frac{1}{2}\Big[\|u_n - x^*\|^2 + \|v_n - y^* - r_1(f_1(v_n) - f_1(y^*))\|^2 \\
&\quad - \|(v_n - y^*) - r_1(f_1(v_n) - f_1(y^*)) - (u_n - x^*)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|u_n - x^*\|^2 + \|v_n - y^*\|^2 - r_1(2\beta_1 - r_1)\|f_1(v_n) - f_1(y^*)\|^2 \\
&\quad - \|(v_n - u_n) - r_1(f_1(v_n) - f_1(y^*)) + (x^* - y^*)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|u_n - x^*\|^2 + \|v_n - y^*\|^2 - \|v_n - u_n + (x^* - y^*)\|^2 - r_1^2\|f_1(v_n) - f_1(y^*)\|^2 \\
&\quad + 2r_1\langle v_n - u_n + (x^* - y^*), f_1(v_n) - f_1(y^*)\rangle\Big] \\
&\leq \frac{1}{2}\Big[\|u_n - x^*\|^2 + \|v_n - y^*\|^2 - \|v_n - u_n + (x^* - y^*)\|^2 \\
&\quad + 2r_1\|v_n - u_n + (x^* - y^*)\|.\|f_1(v_n) - f_1(y^*)\|\Big],
\end{aligned}$$

i.e.

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|v_n - y^*\|^2 - \|v_n - u_n + (x^* - y^*)\|^2 + 2r_1\|v_n - u_n + (x^* - y^*)\|.\|f_1(v_n) - f_1(y^*)\| \\
&\leq \|z_n - x^*\|^2 - \|(z_n - v_n - (x^* - y^*)\|^2 + 2r_2\|z_n - v_n - (x^* - y^*)\|.\|f_2(z_n) - f_2(x^*)\| \\
&\quad - \|v_n - u_n + (x^* - y^*)\|^2 + 2r_1\|v_n - u_n + (x^* - y^*)\|.\|f_1(v_n) - f_1(y^*)\|. \qquad (31)
\end{aligned}$$

Using equations (14), (17) and (31), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|u_n - x^*\|^2 \\
&\leq \|z_n - x^*\|^2 - \|(z_n - v_n - (x^* - y^*)\|^2 + 2r_2\|z_n - v_n - (x^* - y^*)\|.\|f_2(z_n) - f_2(x^*)\| \\
&\quad - \|v_n - u_n + (x^* - y^*)\|^2 + 2r_1\|v_n - u_n + (x^* - y^*)\|.\|f_1(v_n) - f_1(y^*)\| \\
&\leq \|y_n - x^*\|^2 - \|(z_n - v_n - (x^* - y^*)\|^2 + 2r_2\|z_n - v_n - (x^* - y^*)\|.\|f_2(z_n) - f_2(x^*)\| \\
&\quad - \|v_n - u_n + (x^* - y^*)\|^2 + 2r_1\|v_n - u_n + (x^* - y^*)\|.\|f_1(v_n) - f_1(y^*)\| \\
&\leq \|x_n - x^*\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - x^*\rangle - \|(z_n - v_n - (x^* - y^*)\|^2 \\
&\quad + 2r_2\|z_n - v_n - (x^* - y^*)\|.\|f_2(z_n) - f_2(x^*)\| - \|v_n - u_n + (x^* - y^*)\|^2 \\
&\quad + 2r_1\|v_n - u_n + (x^* - y^*)\|.\|f_1(v_n) - f_1(y^*)\|, \qquad (32)
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\leq \|(z_n - v_n) - (x^* - y^*)\|^2 + \|v_n - u_n + (x^* - y^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\| + 2r_2\|z_n - v_n - (x^* - y^*)\|.\|f_2(z_n) - f_2(x^*)\| \\
&\quad + 2r_1\|v_n - u_n + (x^* - y^*)\|.\|f_1(v_n) - f_1(y^*)\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x^*\|$ exists, therefore utilizing $\sum_{n=0}^{\infty}\theta_n\|x_n - x_{n-1}\| < \infty$ and equation (29), we conclude

$$\lim_{n\to\infty}\|z_n - v_n - (x^* - y^*)\| = \lim_{n\to\infty}\|v_n - u_n + (x^* - y^*)\| = 0. \qquad (33)$$

Note that

$$\begin{aligned}\|z_n - u_n\| &= \|z_n - v_n - (x^* - y^*) + v_n - u_n + (x^* - y^*)\| \\ &\leq \|z_n - v_n - (x^* - y^*)\| + \|v_n - u_n + (x^* - y^*)\|,\end{aligned} \tag{34}$$

then using (33), we get

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{35}$$

From equations (26) and (35), we have

$$\lim_{n\to\infty} \|u_n - K_C(u_n)\| = 0. \tag{36}$$

Also

$$\begin{aligned}\|z_n - y_n\|^2 &= \gamma^2\|B^*(I - K_Q)By_n\|^2 \\ &\leq \gamma^2 L\|(I - K_Q)By_n\|^2.\end{aligned} \tag{37}$$

From equation (25), we have

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{38}$$

Further

$$\|u_n - x_n\| \leq \|u_n - z_n\| + \|z_n - y_n\| + \|y_n - x_n\|.$$

Hence, using equations (23), (35) and (38), we get

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{39}$$

Now

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - x_n\|.$$

Hence using equations (22) and (39), we obtain $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. As $\{x_n\}$ is bounded, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in H_1$ and hence $p \in \omega_w(x_n)$. From boundedness of $\{u_n\}$ and $\{y_n\}$ and from equations (23) and (39), there exists subsequences $\{u_{n_i}\}$ and $\{y_{n_i}\}$ of $\{u_n\}$ and $\{y_n\}$ respectively such that $u_{n_i} \rightharpoonup p$ and $y_{n_i} \rightharpoonup p$. From equation (36) and Lemma 2.12, we have $p \in Fix(K_C)$ i.e. $K_C(p) = p$. Since B is a bounded linear operator. Hence $y_{n_i} \rightharpoonup p$ implies $By_{n_i} \rightharpoonup Bp$. From equation (25) and Lemma 2.12, $Bp \in Fix(K_Q)$ i.e. $K_Q(u) = u$ where $u = Bp$ which implies $p = K_C(p - \gamma B^*(I - K_Q)Bp)$ i.e. $p \in Fix(K)$. Further, from equation (20) and demiclosedness of $I - T$ at zero, we get $p \in Fix(T)$. Hence, we conclude $p \in Fix(K) \cap Fix(T)$ i.e. $p \in \Omega$ which implies $\omega_w(x_n) \subset \Omega$. Further, from Lemma 2.6, we obtain $x_n \rightharpoonup p$. From Lemma 3.1, we conclude that $(p, q) \in \Psi$ where $q = \mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)p$ and $v = \mathrm{T}_{s_2}^{G_2}(I - s_2 h_2)u$ with $u = Bp$ and $v = Bq$. This completes the proof. □

Theorem 3.3. *Let C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively. Let $F_1, F_2 : C \times C \to \mathbb{R}$ and $G_1, G_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying assumptions (i)-(iv). Let $f_1, f_2 : H_1 \to H_1$ be β_1, β_2-ism and $h_1, h_2 : H_2 \to H_2$ be ρ_1, ρ_2-ism mappings respectively. Assume that $T : H_1 \to H_1$ is η demicontractive mapping such that $I - T$ is demiclosed at zero. Let $B : H_1 \to H_2$ be a bounded linear operator with adjoint B^* such that $\gamma \in (0, 1/L)$ where L is spectral radius of operator B^*B. Let $r_1 \in (0, 2\beta_1)$, $r_2 \in (0, 2\beta_2)$, $s_1 \in (0, 2\rho_1)$ and $s_2 \in (0, 2\rho_2)$. Define $K_C : H_1 \to C$ by $K_C(x) = \mathrm{T}_{r_1}^{F_1}(I - r_1 f_1)\mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)x$ for all $x \in H_1$ and $K_Q : H_2 \to Q$ by $K_Q(x) = \mathrm{T}_{s_1}^{G_1}(I - s_1 h_1)\mathrm{T}_{s_2}^{G_2}(I - s_2 h_2)x$ for all $x \in H_2$. Define $K : H_1 \to C$ by $K(x) = K_C(x - \gamma B^*(I - K_Q)Bx)$ for all $x \in H_1$.*

For given $x_0, x_1 \in C$, let iterative sequence $\{x_n\}$ be generated as

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = K_C(y_n - \gamma B^*(I - K_Q)By_n), \\ t_n = (1 - \alpha_n)u_n + \alpha_n T(u_n), \\ C_{n+1} = \{z \in C_n; \|t_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\ -2\theta_n \langle x_n - z, x_{n-1} - x_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_1, n \geq 1, \end{cases} \tag{40}$$

where $\theta_n \in [0, \theta]$ for some $\theta \in [0, 1)$ and $\alpha_n \in (\delta, 1 - \eta - \delta)$ for some $\delta > 0$. Assume that $\Omega = Fix(K) \cap Fix(T) \neq \phi$ and $\sum_{n=0}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$, then the generated sequence $\{x_n\}$ converges strongly to a point $p \in \Omega$ where $(p, q) \in \Psi$ such that $q = \mathrm{T}_{r_2}^{F_2}(I - r_2 f_2)p$ and $v = \mathrm{T}_{s_2}^{G_2}(I - s_2 h_2)u$ with $u = Bp$ and $v = Bq$.

Proof. Firstly, we show that sequence $\{x_n\}$ is well defined. As fixed point set of demicontractive mapping is closed, convex set and hence solution set Ω is closed and convex. Moreover, from Lemma 2.8, C_{n+1} is closed and convex for each $n \geq 1$. Firstly, we show $\Omega \subset C_n$ for all $n \geq 1$. Obviously, $\Omega \subset C_1 = C$. By induction, assume that $\Omega \subset C_n$ for some $n \geq 1$. We have to show $\Omega \subset C_{n+1}$. For any $x^* \in \Omega$, using Lemma 2.3 and proceeding similarly as in Theorem 3.2, we have

$$\|t_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \alpha_n(1 - \eta - \alpha_n)\|u_n - T(u_n)\|^2 \tag{41}$$
$$\leq \|u_n - x^*\|^2. \tag{42}$$
$$\leq \|z_n - x^*\|^2$$
$$\leq \|y_n - x^*\|^2 - \gamma(1 - \gamma L)\|(I - K_Q)By_n\|^2 \tag{43}$$

Using equations (42), (43) and $\gamma \in (0, 1/L)$, we get

$$\begin{aligned} \|t_n - x^*\|^2 &\leq \|y_n - x^*\|^2 \\ &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &= \|x_n - x^*\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_n - x^*, x_{n-1} - x_n\rangle, \end{aligned} \tag{44}$$

which implies $x^* \in C_{n+1}$. Hence $\Omega \subset C_n$ for all $n \geq 1$. Summing up these facts, we conclude that C_{n+1} is nonempty, closed and convex for all $n \geq 1$ and hence $\{x_n\}$ is well defined.

Now, we will show that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Since Ω is nonempty closed and convex subset of H_1, there exists unique $z^* \in \Omega$ such that $z^* = P_\Omega x_1$. From $x_{n+1} = P_{C_{n+1}}x_1$, we have

$$\|x_{n+1} - x_1\| \leq \|x^* - x_1\| \quad \text{for all} \quad x^* \in \Omega \subset C_{n+1}. \tag{45}$$

In particular,

$$\|x_{n+1} - x_1\| \leq \|P_\Omega x_1 - x_1\|, \tag{46}$$

i.e. $\{x_n\}$ is bounded.

On the other hand, $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$ implies

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|. \tag{47}$$

Hence $\{x_n\}$ is nondecreasing and $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Now using Lemma 2.4, we get

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 + x_1 - x_n\|^2 \\ &= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_1\rangle \\ &= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n + x_n - x_1\rangle \\ &= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_n - x_1, x_{n+1} - x_n\rangle \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2. \end{aligned} \tag{48}$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{49}$$

Since $x_{n+1} \in C_{n+1}$, therefore

$$\|t_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_n - x_{n+1}, x_{n-1} - x_n\rangle. \tag{50}$$

Utilizing equation (49) and $\sum_{n=0}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$, we obtain

$$\lim_{n\to\infty} \|t_n - x_{n+1}\| = 0. \tag{51}$$

Using triangle inequality, we get

$$\|t_n - x_n\| \leq \|t_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \tag{52}$$

Using equations (49) and (51), we get

$$\lim_{n\to\infty} \|t_n - x_n\| = 0 \tag{53}$$

and

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0. \tag{54}$$

Now, using equation (41) and (43), we have

$$\begin{aligned} \|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \alpha_n(1 - \eta - \alpha_n)\|u_n - T(u_n)\|^2 \\ &\leq \|y_n - x^*\|^2 - \alpha_n(1 - \eta - \alpha_n)\|u_n - T(u_n)\|^2 \\ &\leq \|x_n - x^*\|^2 + 2\theta_n \langle x_n - x_{n-1}, y_n - x^*\rangle - \alpha_n(1 - \eta - \alpha_n)\|u_n - T(u_n)\|^2, \end{aligned} \tag{55}$$

which implies

$$\begin{aligned} \alpha_n(1 - \eta - \alpha_n)\|u_n - T(u_n)\|^2 &\leq \|x_n - x^*\|^2 - \|t_n - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\| \\ &\leq \Big(\|x_n - x^*\| + \|t_n - x^*\|\Big)\|x_n - t_n\| + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\|. \end{aligned}$$

Using equation (53) and $\sum_{n=0}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$, we get

$$\lim_{n\to\infty} \|u_n - T(u_n)\| = 0. \tag{56}$$

Also,

$$\lim_{n\to\infty} \|t_n - u_n\| = \lim_{n\to\infty} \alpha_n\|u_n - T(u_n)\| = 0. \tag{57}$$

Note that

$$\|x_n - u_n\| \leq \|x_n - t_n\| + \|t_n - u_n\|. \tag{58}$$

Using equations (53) and (57), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{59}$$

Consider $z_n = y_n - \gamma B^*(I - K_Q)By_n$. Proceeding similarly as in Theorem 3.2 and using equation (53), we can show

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{60}$$

Now using equations (42) and (43), we get

$$\begin{aligned}\|t_n - x^*\|^2 &\leq \|u_n - x^*\|^2 \\ &\leq \|y_n - x^*\|^2 - \gamma(1-\gamma L)\|(I-K_Q)By_n\|^2 \\ &\leq \|x_n - x^*\|^2 + 2\theta_n \langle x_n - x_{n-1}, y_n - x^*\rangle - \gamma(1-\gamma L)\|(I-K_Q)By_n\|^2, \end{aligned} \tag{61}$$

which implies

$$\begin{aligned}0 \leq \gamma(1-\gamma L)\|(I-K_Q)By_n\|^2 &\leq \|x_n - x^*\|^2 - \|t_n - x^*\|^2 + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\| \\ &\leq \Big(\|x_n - x^*\| + \|t_n - x^*\|\Big)\|x_n - t_n\| + 2\theta_n\|x_n - x_{n-1}\|.\|y_n - x^*\|.\end{aligned}$$

Using equation (53), $\gamma \in (0, 1/L)$ and $\sum_{n=0}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$, we get

$$\lim_{n\to\infty} \|(I-K_Q)By_n\| = 0. \tag{62}$$

Now,

$$\begin{aligned}\|u_n - K_C(u_n)\| &= \|K_C(z_n) - K_C(u_n)\| \\ &\leq \|z_n - u_n\|.\end{aligned} \tag{63}$$

Hence

$$\lim_{n\to\infty} \|u_n - K_C(u_n)\| = 0. \tag{64}$$

Since $\{x_n\}$ is bounded, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in H_1$ and hence $p \in \omega_w(x_n)$. From equations (54) and (59), it follows that $\{y_n\}$ and $\{u_n\}$ are also bounded. Hence, there exists subsequences $\{y_{n_i}\}$ and $\{u_{n_i}\}$ of $\{y_n\}$ and $\{u_n\}$ respectively such that $y_{n_i} \rightharpoonup p$ and $u_{n_i} \rightharpoonup p$. Since B is a bounded linear operator, hence $y_{n_i} \rightharpoonup p$ implies $By_{n_i} \rightharpoonup Bp$.

Since K_C and K_Q are averaged mappings and hence nonexpansive mappings. It follows that $I - K_C$ and $I - K_Q$ are demiclosed at zero. It follows from equations (62) and (64) that $p \in Fix(K_C)$ and $Bp \in Fix(K_Q)$ which implies $p = K_C(p - \gamma B^*(I - K_Q)Bp)$ i.e. $p \in Fix(K)$. Further from equation (56) and demiclosedness of $I - T$ at zero, we have $p \in Fix(T)$ i.e. $p \in Fix(K) \cap Fix(T)$ which implies $p \in \Omega$. Hence $\omega_w(x_n) \subset \Omega$. Since $x_n = P_{C_n}x_1$ and $\Omega \subset C_n$, it follows from Lemma 2.4 that $\langle x_1 - x_n, x_n - x^*\rangle \geq 0$ for all $x^* \in \Omega$. Hence, we have $\langle x_1 - p, p - x^*\rangle \geq 0$ for all $x^* \in \Omega$ which implies $p = P_\Omega x_1$. From Lemma 2.7, we conclude that $x_n \to p$. Further, from Lemma 3.1, we conclude that $(p, q) \in \Psi$ where $q = T_{r_2}^{F_2}(I - r_2 f_2)p$ and $v = T_{s_2}^{G_2}(I - s_2 h_2)u$ with $u = Bp$ and $v = Bq$. This completes the proof. □

4. Numerical Example

In this section, we present numerical examples to illustrate the performance of the proposed method. The computations are carried out using MATLAB program on a Lenovo X250, Intel (R) Core i7 vPro with RAM 8.00GB. We show the numerical behaviour of the sequences generated by Algorithm 40 and also compare the performance with the non-inertial version (i.e., $\theta_n = 0$) and Algorithm 3.3 of [13] (namely, KJ algorithm).

Example 4.1. *Let $H_1 = H_2 = H_3 = \mathbb{R}$, be the set of real numbers with inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$. Let f_1, f_2 be mappings from $\mathbb{R}$ to $\mathbb{R}$ defined as $f_1(x) = \frac{x-3}{5}$ and $f_2(x) = \frac{x-3}{7}$ for all $x \in \mathbb{R}$ and let h_1, h_2 be mappings from $\mathbb{R}$ to $\mathbb{R}$ defined as $h_1(x) = \frac{x-6}{3}$ and $h_2(x) = \frac{x-6}{5}$ for all $x \in \mathbb{R}$. It is easy to see that f_1, f_2, h_1, h_2 are 1-inverse strongly monotone mappings. Then, we can choose $r_1, r_2, s_1, s_2 \in (0, 2)$. Assume that $B : \mathbb{R} \to \mathbb{R}$ is a bounded linear operator defined as $B(x) = 2x$ for all $x \in \mathbb{R}$, then the spectral radius of operator B^*B is $L = 4$. So, we can choose $\gamma \in (0, \frac{1}{4})$, say $\gamma = \frac{1}{8}$. Further, we take a demicontractive mapping $T : \mathbb{R} \to \mathbb{R}$ such that $T(x) = \frac{7x+6}{9}$. We define the bifunctions $F_1, F_2 : C \times C \to \mathbb{R}$ as*

$$F_1(x, y) = 2(x-3)(y-x) \text{ and } F_2(x, y) = (x-3)(y-x),$$

for all $x, y \in C$ *and* $G_1, G_2 : Q \times Q \to \mathbb{R}$ *as*

$$G_1(x, y) = 2(x-6)(y-x) \text{ and } G_2(x, y) = 3(x-6)(y-x),$$

for all $x, y \in Q$. *It can be seen that* F_1, F_2, G_1 *and* G_2 *satisfy assumptions* $(i) - (iv)$. *By definition of* $T_{r_1}^{F_1}$, $y = T_{r_1}^{F_1}(x)$ *implies*

$$\begin{aligned} F_1(y,z) + \frac{1}{r_1}\langle z-y, y-x\rangle &\geq 0 \text{ for all } z \in C, \\ 2(y-3)(z-y) + \frac{1}{r_1}\langle z-y, y-x\rangle &\geq 0 \text{ for all } z \in C, \\ 2r_1(y-3)(z-y) + (z-y)(y-x) &\geq 0 \text{ for all } z \in C, \\ (z-y)[2r_1(y-3) + (y-x)] &\geq 0 \text{ for all } z \in C. \end{aligned}$$

Since $T_{r_1}^{F_1}$ *is single valued, so* $2r_1(y-3) + (y-x) = 0$ *which implies* $y = T_{r_1}^{F_1}(x) = \frac{6r_1+x}{2r_1+1}$. *Similarly, we can find*

$$T_{r_2}^{F_2}(x) = \frac{3r_2 + x}{r_2 + 1}, T_{s_1}^{G_1}(x) = \frac{12s_1 + x}{2s_1 + 1}, T_{s_2}^{G_2}(x) = \frac{18s_2 + x}{3s_2 + 1}.$$

It can be seen that $Fix(T_{r_1}^{F_1}) = Fix(T_{r_2}^{F_2}) = \{3\}$ *and* $Fix(T_{s_1}^{G_1}) = Fix(T_{s_2}^{G_2}) = \{6\}$. *Let* $r_1 = s_1 = 0.5$ *and* $r_2 = s_2 = 1$. *Under above said assumptions, we obtain* $K_C(x) = \frac{219+67x}{140}$, $K_Q(x) = \frac{186+29x}{60}$ *and* $K(x) = \frac{5963x+32511}{16800}$ *for all* $x \in \mathbb{R}$. *We choose*

$$\theta_n = \begin{cases} \min\{\frac{1}{n^3\|x_n - x_{n-1}\|}, 0.8\}, & \text{if } x_n \neq x_{n-1} \\ 0.8, & \text{otherwise} \end{cases}$$

and $\alpha_n = \frac{1}{2n}$, *then Algorithm 40 reduces to the following:*
Algorithm M:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = \frac{5963y_n+32511}{16800}, \\ t_n = (1 - \frac{1}{2n})u_n + \frac{1}{2n}.\frac{7u_n+6}{9}, \\ C_{n+1} = \{z \in C_n; \|t_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\ -2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_1, n \geq 1, \end{cases}$$

It is easy to see that $3 \in Fix(K)$. *Further, from Lemma 3.1, we can say* $(3, 3) \in \Psi$. *Also, it can be seen that* $3 \in Fix(T)$. *Hence* $3 \in \Omega$ *and solution set* $\Omega \neq \phi$. *Hence from Theorem 3.3, we conclude that* $\{x_n\}$ *converges strongly to* $3 \in \Omega$, *where* $(3, 3) \in \Psi$. *We compare the performance of Algorithm M with Algorithm 3.3 of [13] (named, KJ Algorithm) choosing* $r_n = \frac{1}{2}$, $\alpha_n = \frac{1}{n+1}$, $\delta_n = \frac{5n}{7n+9}$ *and* $\beta_n = 1 - \alpha_n - \delta_n$. *We test the algorithms using the following starting points:*
Case I: $x_0 = \frac{1}{\sqrt{33}}$ *and* $x_1 = \frac{3}{\sqrt{17}}$;
Case II: $x_0 = \frac{1}{3}$ *and* $x_1 = \sqrt{\sqrt{7}}$;
Case III: $x_0 = 5.0$ *and* $x_1 = 2.5$;
Case IV: $x_0 = 1.0$ *and* $x_1 = 7.2$.
We use $\|x_{n+1} - x_n\| < 10^{-4}$ *as stopping criterion in each case. The numerical results are shown in Table 4.1 and Figure 1.*

		Algorithm M	Alg. M ($\theta_n = 0$)	KJ Algorithm
Case I	No of Iter.	13	23	25
	CPU time (sec)	0.0032	0.0136	0.0151
Case II	No of Iter.	12	21	27
	CPU time (sec)	0.0015	0.0064	0.0083
Case III	No of Iter.	9	16	24
	CPU time (sec)	0.0017	0.0134	0.0159
Case IV	No of Iter.	13	22	25
	CPU time (sec)	0.0021	0.0086	0.0135

Table 1: Computational result for Example 4.1.

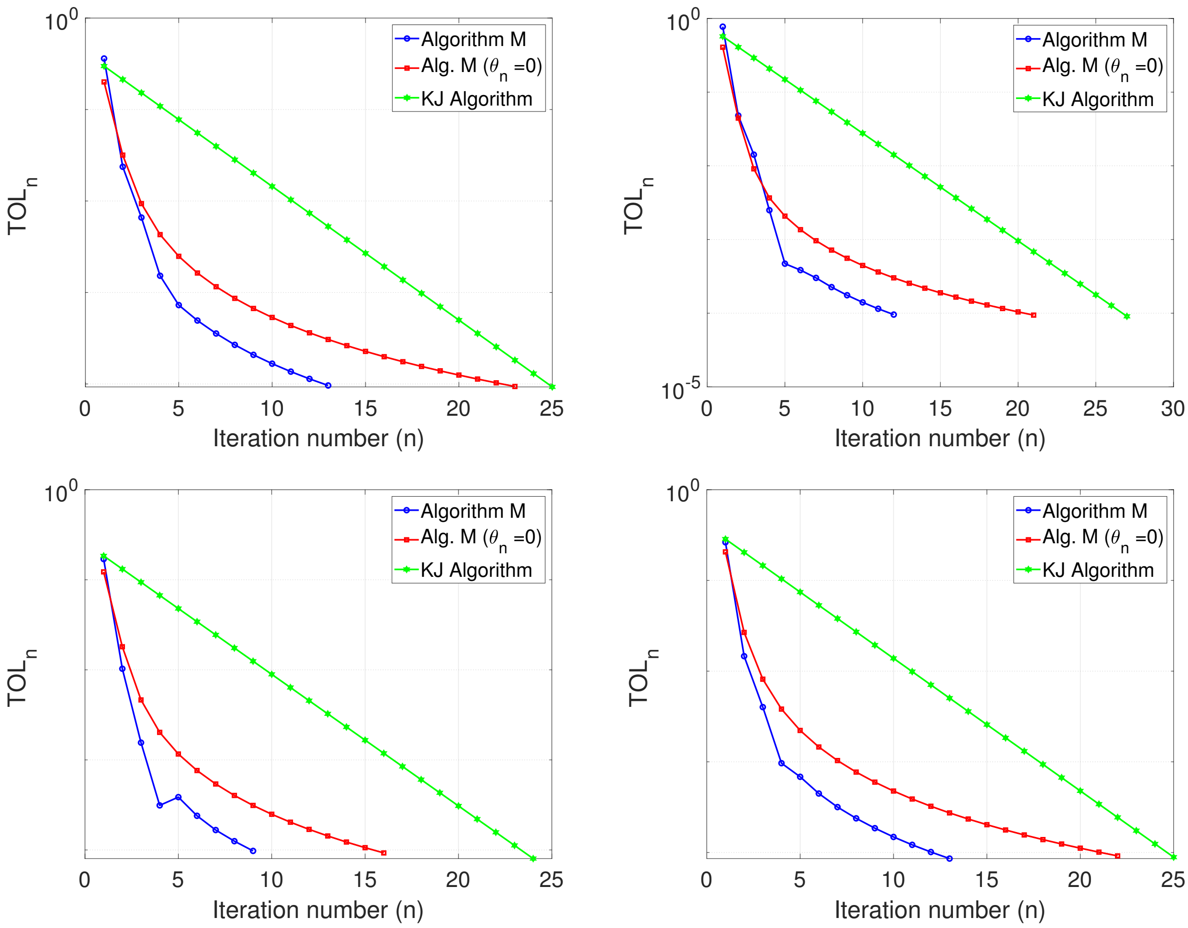

Figure 1: Example 4.1, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Example 4.2. *Next, we apply the proposed algorithm to image restoration problem which can be formulated by the inversion of the following observation model:*

$$b = Ax + c, \tag{65}$$

where $x \in \mathbb{R}^{n\times 1}$ is the original image, c is the additive noise, $b \in \mathbb{R}^{m\times 1}$ is the degraded image and $A \in \mathbb{R}^{m\times n}$ is the blurring matrix. In order to solve problem (65)*, we employed the regularization technique given by*

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \tag{66}$$

where $\lambda > 0$ is the regularization parameter, $\|x\|_2$ is the Euclidean norm of x and $\|x\|_1 = \sum_{i=1}^{N} |a_i|$ is the l_1-norm of x. This technique has been considered as a great tool in several branches of science and engineering due to the difficulty in computing the inverse A^{-1} from $A^{-1}(b - c)$ while solving (65)*. It is easy to see that the regularization problem* (65) *is equivalent to the least absolute shrinkage selection operator (LASSO) problem defined by*

$$\min_{x\in C} \frac{1}{2}\|b - Ax\|_2^2, \tag{67}$$

where $C = \{x \in \mathbb{R}^n \ : \ \|x\|_1 \leq \lambda\}$. Consequently, (67) *is a subclass of the split feasibility problem (SFP) defined by*

$$\textbf{\textit{find}} \quad x \in C \quad \textbf{\textit{such that}} \quad Ax \in Q, \tag{68}$$

where in this case $Q := \{b\}$. Note that if in problem (1)-(2)*, we set $F_1 = F_2 = F$, $f_1 = f_2 = 0$, $G_1 = G_2 = G$ and $h_1 = h_2 = 0$, we obtain the split equilibrium problem (SEQ) and if in addition, $F(x, y) = I_C(x) - I_C(y)$ and $G(u, v) = I_Q(u) - I_Q(v)$, where I_C and I_Q are the identity operator on C and Q respectively, then the SEQ becomes the SFP* (68)*. Hence, we can apply our algorithm to the SFP with the resolvent operator T_r^F being the projection operator on C and T_r^G is the projection operator onto Q. In order to implement our algorithm, we choose the following parameters: $\theta_n = \frac{1}{n^2}$, $\alpha_n = \frac{1}{n+1}$, $\delta = 10^{-5}$, $\eta = 10^{-2}$, and $T(x) = \frac{x}{2}$. Our aim here is to recover the original image x given the data of the blurred image b. We consider the grey scale image of m pixels wide and n pixel height, each value is known to be in the range* $[0, 255]$*. Let $D = m \times n$. The quality of the restored image is measured by the signal-to-noise ratio defined as*

$$SNR = 20 \times \log_{10}\left(\frac{\|x\|_2}{\|x - x^*\|_2}\right),$$

where x is the original image and x^* is the restored image. Typically, the larger the SNR, the better the quality of the restored image. In our experiments, we used the grey test image Cameraman (256×256*) in Image Processing Toolbox in MATLAB, while each test image is degraded by Gaussian* 7×7 *blur kernel with standard deviation* 4*. We also choose the initial values as $x_0 = \mathbf{0} \in \mathbb{R}^D$ and $x_1 = \mathbf{1} \in \mathbb{R}^D$. Figure 2 showed the original, blurred and restored images by the algoritgms. Figure 3 showed the graphs of SNR against number of iterations for each algorithm and in Figure 3, we show the graphs of time and SNR values against number of iterations for the algorithm. We compare the performance of the proposed algorithm (Algorithm M) with its non-inertial version (Algorithm M with $\theta_n = 0$). The experiment shows that both methods are effective in reconstructing the blurred image, however, the time taken by Algorithm M with $\theta_n = 0$ is more (Average time = 11.8354s) than the time taken by the proposed algorithm (Average time = 10. 2971s). More so the SNR value of Algorithm M is 33.8238 while that of Algorithm M is 33.8019.*

Figure 2: Example 4.2, Top shows original image of Cameraman (left) and blurred image of Cameraman (right); Bottom (Left) shows recovered image by Algorithm M and Bottom (Right) shows recovered image by Algorithm M with $\theta_n = 0$.

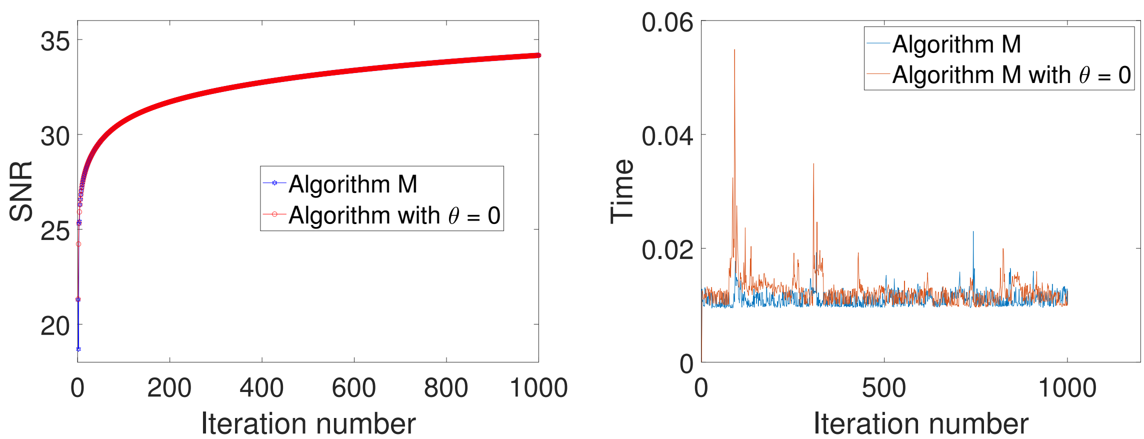

Figure 3: Example 4.2, graphs of SNR (Left) and Time (Right) against number of iterations .

5. Conclusion

In this paper, we introduce a new problem called split general system of generalized equlibrium problem. We consider new inertial type algorithm which converges weakly to a common solution of split general system of generalized equilibrium problem and fixed point of a demicontractive mapping in real Hilbert spaces. Strong convergence of new inertial type algorithm is also obtained by using shrinking projection method. Significance and applicability of our result lies in the fact that it generalizes several nonlinear analysis problems as its special cases. Numerical experiment is also presented to demonstrate the efficiency of our proposed method as well as comparing with other existing method in the literature.

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