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Geometries of a manifold with a symmetric-type quarter-symmetric non-metric connection

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Abstract. We introduce a symmetric-type quarter-symmetric non-metric connection family in a Riemannian manifold and study its geometrical properties. We also study the Schur's theorem of the symmetric-type quarter-symmetric projective non-metric connection family and the symmetric-type quarter-symmetric conformal non-metric connection family.

1. Introduction

The study of geometries of a manifold with certain connection has been an active field over the seven decades. The research in the area goes back to the concept of metric connection with torsion introduced by A. Hayden in the early 1932 in [18]. K. Yano in [28] introduced a semi-symmetric metric connection and studied the properties of the curvature tensor *w.r.t.* this connection.

K. yano in [29] posed a quarter-symmetric metric connection and interpreted some examples of these connections, and S. Golab in [14] introduced and investigated a quarter-symmetric connection. Further, Agache et al and De et al in [1, 7] respectively defined and studied the topologies of a manifold with the semi-symmetric non-metric connection. On the one hand, some geometries of a manifold with a semi-symmetric no-metric connection and by the Schur's theorem of a semi-symmetric non-metric connection are well known([8, 10, 11, 19–23]) based only on the second Bianchy identity. A semi-symmetric non-metric connection that is a geometrical model for scalar-tensor theories of gravitation was studied ([9]). Afterwards, several types of semi-symmetric non-metric connections were studied ([2–6, 30, 33–35]). Afterwards, several types of a quarter-symmetric non-metric connection were studied ([15–17, 31, 32]). And then Tang et al in [24–27] a generalized quarter-symmetric metric recurrent connection and a projective conformal quarter-symmetric metric connection were studied. In fact, there were few results about quarter-symmetric non-metric connections because of its formal complexity and computational difficulty. In Fu et al in [12, 13], the geometrical and physical properties of conformal and projective semi-symmetric metric recurrent connections were studied.

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Motivated by the previous researches we proposed newly in this note a symmetric-type quarter-symmetric non-metric connection family and its mutual connection family and study its projective invariant. And we also derive the conformal invariant of the symmetric-type quarter-symmetric non-metric connection which has the same geodesic as the Levi-Civita connection.

This paper is organized as follows. Section 1 considers the previous study results. Section 2 studies the symmetric-type quarter-symmetric non-metric connection family and its mutual connection family. Section 3 studies a symmetric-type quarter-symmetric projective non-metric connection family and its mutual connection family. Section 4 studies an invariant of the symmetric-type quarter-symmetric conformal non-metric connection which has the same geodesic as the Levi-Civita connection and its mutual connection.

2. A symmetric-type quarter-symmetric non-metric connection and its mutual connection

Let $(\mathcal{M}, g)$ be a Riemannian manifold ($dim\mathcal{M} \geq 3$), g be Riemannian metric on $\mathcal{M}$ and $\overset{\circ}{\nabla}$ be the Levi-Civita connection with respect to g. Let $T(\mathcal{M})$ denote the collection of all vector fields on $\mathcal{M}$.

Definition 2.1. *A connection $\overset{t}{\nabla}$ is called a symmetry-type quarter-symmetric non-metric connection if it satisfies the relations*

$$(\overset{t}{\nabla}_Z g)(X, Y) = -2(t-1)\pi(Z)U(X, Y) - t\pi(X)U(Z, Y) - t\pi(Y)U(Z, X),\ T(X, Y) = \pi(Y)UX - \pi(X)UY \quad (2.1)$$

where π is a 1-form and U is a $(1, 1)$-type symmetric tensor field in $(\mathcal{M}, g)$ and $t \in \mathbb{R}$ is a parameter of the connection family.

This connection family is expressed as follows

$$\overset{t}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + (t-1)\pi(X)UY + t\pi(Y)UX \quad (2.2)$$

Let (x^i) be a local coordinate system in $\mathcal{M}$ such that $g, \overset{\circ}{\nabla}, \overset{t}{\nabla}, \pi, T$ with the local expressions respectively, by $g_{ij}, \{^{k}_{ij}\}, \overset{t}{\Gamma}{}^{k}_{ij}, \pi_i, T^k_i$. At the same time, the formulas (2.1) and (2.2) can be rewritten as

$$\overset{t}{\nabla}_k g_{ij} = -2(t-1)\pi_k U_{ij} - t\pi_i U_{jk} - t\pi_j U_{ik}, T^k_{ij} = \pi_j U^k_i - \pi_i U^k_j. \quad (2.3)$$

and

$$\overset{t}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + (t-1)\pi_i U^k_j + t\pi_j U^k_i. \quad (2.4)$$

From the expression (2.4), the curvature tensor of $\overset{t}{\nabla}$ is given as

$$\overset{t}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + U^l_j \overset{t}{a}_{ik} - U^l_i \overset{t}{a}_{jk} + \overset{t}{b}{}^{l}_{jk}\pi_i + \overset{t}{b}{}^{l}_{ik}\pi_j + t(U^l_{ij} - U^l_{ji})\pi_k + (t-1)U^l_k \pi_{ij} \quad (2.5)$$

where $K^{\,k}_{ij}$ is the curvature tensor of $\overset{\circ}{\nabla}$ with respect to g_{ij} and the other notations are given as below

$$\begin{cases} \overset{t}{a}_{ik} &= t(\overset{\circ}{\nabla}_i \pi_k - (t-1)U^p_k \pi_i - tU^p_i \pi_p \pi_k), \\ \overset{t}{b}{}^{l}_{ij} &= (t-1)(\overset{\circ}{\nabla}_i U^l_j - tU^p_i U^l_p \pi_j), \\ U^{\,l}_{ij} &= \overset{\circ}{\nabla}_i U^l_j, \\ \pi_{ij} &= \overset{\circ}{\nabla}_i \pi_j - \overset{\circ}{\nabla}_j \pi_i. \end{cases} \quad (2.6)$$

Remark 2.1. *If $U_{ij} = g_{ij}$, then this connection family is a special type of the semi-symmetric metric recurrent connection family that was studied in [30, 35] and its physical model was studied in [9].*

Remark 2.2. *When U is the Ricci tensor, then this connection family studied in [25, 26].*

By (2.2) and (2.3) it is obvious that there holds the following.

When $t = 0$, we denote $\overset{t}{\nabla}$ as $\overset{\circ}{D}$. Then the connection $\overset{\circ}{D}$ satisfies the relation

$$\overset{\circ}{D}_k g_{ij} = 2\pi_k U_{ij}, T^k_{ij} = \pi_j U^k_i - \pi_i U^k_j. \tag{2.7}$$

and the coefficient of $\overset{\circ}{D}$ is

$$\overset{\circ}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - \pi_i U^k_j.$$

When $t = 1$, we denote $\overset{t}{\nabla}$ as $\overset{l}{D}$. Then the connection $\overset{l}{D}$ satisfies the relation

$$\overset{l}{D}_k g_{ij} = -\pi_i U_{jk} - \pi_j U_{ik}, T^k_{ij} = \pi_j U^k_i - \pi_i U^k_j. \tag{2.8}$$

and the coefficient of $\overset{l}{D}$ is

$$\overset{l}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \pi_j U^k_i.$$

When $t = \frac{1}{2}$, we denote $\overset{t}{\nabla}$ as D. Then the connection D satisfies the relation

$$D_k g_{ij} = \pi_k U_{ij} - \frac{1}{2}\pi_i U_{jk} - \frac{1}{2}\pi_j U_{ik}, T^k_{ij} = \pi_j U^k_i - \pi_i U^k_j. \tag{2.9}$$

and the coefficient of D is

$$\tilde{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \frac{1}{2}\pi_j U^k_i - \frac{1}{2}\pi_i U^k_j. \tag{2.10}$$

From this expression, the curvature tensor of D is as below

$$\tilde{R}_{ijk}{}^l = K_{ijk}{}^l + U^l_j \tilde{a}_{ik} - U^l_i \tilde{a}_{jk} - \tilde{b}{}^l_{jk}\pi_i + \tilde{b}{}^l_{ik}\pi_j + \frac{1}{2}(U^l_{ij} - U^l_{ji})\pi_k - \frac{1}{2}U^l_k \pi_{ij}, \tag{2.11}$$

where

$$\begin{cases} \tilde{a}_{ik} & = \frac{1}{2}(\overset{\circ}{\nabla}_i \pi_k + \frac{1}{2}U^p_k \pi_p \pi_i - \frac{1}{2}U^p_i \pi_p \pi_k), \\ \tilde{b}{}^l_{ij} & = -\frac{1}{2}(\overset{\circ}{\nabla}_i U^l_j - \frac{1}{2}U^p_i U^l_p \pi_j). \end{cases} \tag{2.12}$$

The symmetric-type quarter-symmetric non-metric connection D is a connection which has the same geodesic as the Levi-Civita connection in a Riemannian manifold.

Let

$$\overset{t}{A}{}^l_{ijk} = U^l_j \overset{t}{a}_{ik} - \pi_i \overset{t}{b}{}^l_{jk} + tU^l_{ij}\pi_k + (t-1)U^l_k \overset{\circ}{\nabla}_i \pi_j.$$

Then from the expression (2.5), we get

$$\overset{t}{R}{}^l_{ijk} = K^l_{ijk} + \overset{t}{A}{}^l_{ijk} - \overset{t}{A}{}^l_{jik}.$$

So there exists the following

Theorem 2.1. *When $\overset{t}{A}{}^{l}_{ijk} = \overset{t}{A}{}^{l}_{jik}$, then the curvature tensor of $\overset{t}{\nabla}$ will keep unchanged under the connection transformation $\overset{\circ}{\nabla} \to \overset{t}{\nabla}$.*

Let α and β be two 1-form with the components

$$\alpha_i = U_i^k \pi_k, \beta_i = U_k^k \pi_i, \tag{2.13}$$

Theorem 2.2. *In a Riemannian manifold $(\mathcal{M}, g)$, if the 1-form α and β are of closed, then the volume curvature tensor of $\overset{t}{\nabla}$ is zero, namely*

$$\overset{t}{P}_{ij} = 0, \tag{2.14}$$

where $\overset{t}{P}_{ij} \hat{=} \overset{t}{R}_{ijkl} g^{kl}$ is a volume curvature tensor of $\overset{t}{\nabla}$.

Proof. Contracting the indices k and l of (2.5), then we have

$$\overset{t}{P}_{ij} = \overset{\circ}{P}_{ij} + U_j^k \overset{t}{a}_{ik} - U_i^k \overset{t}{a}_{jk} - \overset{t}{b}{}^{k}_{jk} \pi_i - \overset{t}{b}{}^{k}_{ik} \pi_j + t(U_{ij}^k - U_{ji}^k)\pi_k + (t-1)U_k^k \pi_{ij},$$

where $\overset{\circ}{P}_{ij}$ is the volume curvature tensor of $\overset{\circ}{\nabla}$ to g_{ij}.
On the other hand, there holds

$$\begin{aligned}
U_j^k \overset{t}{a}_{jk} - U_i^k \overset{t}{a}_{jk} &= t(U_j^k \overset{\circ}{\nabla}_i \pi_k - U_i^k \overset{t}{\nabla}_j \psi_k) - t(t-1)(\pi_i U_j^k - \pi_j U_i^k) U_k^p \pi_p - \pi_i \overset{t}{b}{}^{k}_{jk} + -\pi_j \overset{t}{b}{}^{k}_{ik}, \\
&= -(t-1)(\pi_i \overset{\circ}{\nabla}_j U_k^k - \pi_j \overset{\circ}{\nabla}_i U_k^k) + t(t-1)(\pi_i U_j^p - \pi_j U_i^p) U_p^k \pi_k, \\
t(U_{ij}^k - U_{ji}^k)\pi_k &= t(\overset{\circ}{\nabla}_i U_j^k - \overset{\circ}{\nabla}_j U_i^k)\pi_k, \\
(t-1)U_k^k \pi_{ij} &= (t-1)(U_k^k \overset{\circ}{\nabla}_i \pi_j - U_k^k \overset{\circ}{\nabla}_j \pi_i), \\
\overset{\circ}{P}_{ij} &= 0.
\end{aligned}$$

Hence using these expressions and the expression (2.13), we obtain

$$\overset{t}{P}_{ij} = t(\nabla_i \alpha_j - \overset{\circ}{\nabla}_j \alpha_i) - (t-1)(\overset{\circ}{\nabla}_i \beta_j - \overset{\circ}{\nabla}_j \beta_i). \tag{2.15}$$

If 1-form α and 1-form β are closed, then $\overset{\circ}{\nabla}_i \alpha_j - \overset{\circ}{\nabla}_j \alpha_i = 0$ and $\overset{\circ}{\nabla}_i \beta_j - \overset{\circ}{\nabla}_j \beta_i = 0$. Hence from the expression (2.15), we know that the formula (2.14) is tenable. □

It is well known that if the sectional curvature $k(E)$ at a point p is independent of E (a 2-dimensional subspace of $T_p(\mathcal{M})$), then the curvature tensor is

$$R^{\,l}_{ijk} = k(p)(\delta_i^l g_{jk} - \delta_j^l g_{ik}). \tag{2.16}$$

In this case, if $k(p) = const$, then the Riemannian manifold $(\mathcal{M}, g)$ is a constant curvature manifold.

Theorem 2.3. *Suppose that $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic symmetric-type quarter-symmetric non-metric connection family $\overset{t}{\nabla}$, If there holds*

$$s_h = 0, \tag{2.17}$$

where $s_h \hat{=} \frac{1}{n-1} T_{hp}^p$ (The Schur's theorem of the symmetric-type quarter- symmetric non-metric connection family), then the Riemannian manifold $(\mathcal{M}, g, \overset{t}{\nabla})$ is a constant curvature manifold.

Proof. Substituting the expression (2.16) into the second Bianchi identity of the curvature tensor of the symmetric-type quarter-symmetric non-metric connection family, we get

$$\overset{t}{\nabla}_h \overset{t}{R}_{ijk}{}^l + \overset{t}{\nabla}_i \overset{t}{R}_{jhk}{}^l + \overset{t}{\nabla}_j \overset{t}{R}_{hik}{}^l = T^p_{hi} \overset{t}{R}_{jpk}{}^l + T^p_{ij} \overset{t}{R}_{hpk}{}^l + T^p_{jh} \overset{t}{R}_{ipk}{}^l,$$

and using the expression (2.3), then we arrive at

$$\begin{aligned}
&\overset{t}{\nabla}_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{t}{\nabla}_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{t}{\nabla}_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk})\\
&-(t-2)k(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})]\\
&= k(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})\\
&+\pi_h(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \pi_i(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \pi_j(\delta^l_h g_{ik} - \delta^l_i g_{hk})].
\end{aligned}$$

Contracting the indices i, l of this expression, then we have

$$\begin{aligned}
&(n-2)[g_{jk}\overset{t}{\nabla}_h k(p) - g_{hk}\overset{t}{\nabla}_j k(p) - (t-2)k(p)(\pi_h U_{jk} - \pi_j U_{hk})]\\
&= k(p)[(n-3)(\pi_h U_{jk} - \pi_j U_{hk}) + g_{hk}(U^i_j \pi_i - U^i_i \pi_j) + g_{jk}(U^i_i \pi_h - U^i_h \pi_i)].
\end{aligned}$$

Multiplying both sides of this expression by g^{jk}, then we obtain

$$(n-1)\overset{t}{\nabla}_h k(p) - (t-2)k(p)(\pi_h U^i_i - \pi_i U^i_h) = 2k(p)(\pi_h U^i_i - \pi_i U^i_h).$$

From this expression above we obtain

$$\overset{t}{\nabla}_h k(p) + t s_h k(p) = 0. \tag{2.18}$$

Consequently, if $s_h = 0$, then $k(p) = const$. □

Remark 2.3. *Theorem 2.3 is independent of t. From the expression (2.18), if $t = 0$, then $k(p) = const$.*

Thus we have the following

Theorem 2.4. *(**Schur's theorem**) Suppose $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic symmetric-type quarter-symmetric non-metric connection $\overset{\circ}{D}$. Then the Riemannian manifold $(\mathcal{M}, g, \overset{\circ}{D})$ is a constant curvature manifold.*

Remark 2.4. *The connection $\overset{\circ}{D}$ satisfies the relation (2.7). From Theorem 2.4, this connection is a symmetric-type quarter-symmetric non-metric connection with a constant curvature.*

From the expression (2.4), the coefficient of the mutual connection family $\overset{tm}{\nabla}$ of the symmetric-type quarter-symmetric non-metric connection family $\overset{t}{\nabla}$ is

$$\overset{tm}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + (t-1)\pi_j U^k_i, \tag{2.19}$$

and the mutual connection family $\overset{tm}{\nabla}$ satisfies the relation

$$\overset{tm}{\nabla}_k g_{ij} = -2t\pi_k U_{ij} - (t-1)\pi_i U_{jk} - (t-1)\pi_j U_{ik}, \quad \overset{tm}{T}_{ij}{}^k = \pi_i U^k_j - \pi_j U^k_i, \tag{2.20}$$

Using the expression (2.19), the curvature tensor of $\overset{tm}{\nabla}$ is

$$\overset{tm}{R}_{ijk}{}^l = K_{ijk}{}^l + \overset{tm}{a}_{ik} U^l_j - \overset{tm}{a}_{jk} U^l_i - \pi_i \overset{tm}{b}{}^l_{jk} + \pi_j \overset{tm}{b}{}^l_{ik} + (t-1)(U^l_{ij} - U^l_{ji})\pi_k + tU^l_k \pi_{ij}, \tag{2.21}$$

where

$$\begin{cases} \overset{tm}{a}_{ik} &= (t-1)[(\overset{\circ}{\nabla}_i\pi_k - tU_k^p\pi_p\pi_i) - (t-1)U_i^p\pi_i\pi_k], \\ \overset{tm}{b}{}^l_{ij} &= t(\overset{\circ}{\nabla}_iU_j^l - (t-1)U_i^pU_p^l\pi_j). \end{cases} \tag{2.22}$$

If $t = \frac{1}{2}$, then from the above two expressions we find the curvature tensor of the mutual connection $\overset{m}{D}$ of the connection D.

$$\overset{m}{\tilde{R}}_{ijk}{}^l = K_{ijk}{}^l + \overset{m}{\tilde{a}}_{ik}U_j^l - \overset{m}{\tilde{a}}_{jk}U_i^l - \pi_i\overset{m}{\tilde{b}}{}^l_{jk} + \pi_j\overset{m}{\tilde{b}}{}^l_{ik} - \frac{1}{2}(U_{ij}^l - U_{ji}^l)\pi_k + \frac{1}{2}U_k^l\pi_{ij}, \tag{2.23}$$

where

$$\begin{cases} \overset{m}{\tilde{a}}_{ik} &= -\frac{1}{2}[(\overset{\circ}{\nabla}_i\pi_k - \frac{1}{2}U_k^p\pi_p\pi_i) + \frac{1}{2}U_p^p\pi_i\pi_k], \\ \overset{m}{\tilde{b}}{}^l_{ij} &= \frac{1}{2}(\overset{\circ}{\nabla}_iU_j^l + \frac{1}{2}U_i^pU_p^l\pi_j). \end{cases} \tag{2.24}$$

The expressions (2.23) and (2.22) are not different in their forms from the expressions (2.5) and (2.6). Hence the volume flat and the constant curvature condition of the mutual connection family coincide with those of the symmetric-type quarter-symmetric connection family.

Thus, by using Theorem 2.2 and Theorem 2.3, we have the following

Theorem 2.5. *In a Riemannian manifold $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$, if 1-form α and β are closed, then the volume curvature tensor of $\overset{tm}{\nabla}$ is zero, namely*

$$\overset{tm}{P}_{ij} = 0,$$

where $\overset{tm}{P}_{ij} = \overset{tm}{R}_{ijkl}g^{kl}$ is a volume curvature tensor of $\overset{tm}{\nabla}$.

Theorem 2.6. *Suppose that $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic mutual connection family $\overset{tm}{\nabla}$ of $\overset{t}{\nabla}$. If*

$$s_h = 0,$$

then the Riemannian manifold $(\mathcal{M}, g, \overset{tm}{\nabla})$ is a constant curvature manifold.

Remark 2.5. *In this case, the expression (2.18) becomes*

$$\overset{tm}{\nabla}_h k(p) + (t-1)s_h k(p) = 0,$$

Hence if $t = 1$, then $k = const$.

So there exists the following

Theorem 2.7. *(**Schur's theorem**) Suppose $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic mutual connection $\overset{lm}{D}$ of the symmetric-type quarter-symmetric non-metric connection $\overset{l}{D}$. Then the Riemannian manifold $(\mathcal{M}, g, \overset{lm}{D})$ is a constant curvature manifold.*

3. Geometries of manifolds associated with a symmetric-type quarter-symmetric projective non-metric connection

Definition 3.1. *Definition. A connection family $\overset{p}{\nabla}$ is called a symmetric-type quarter-symmetric projective non-metric connection family if $\overset{p}{\nabla}$ is a projective equivalent to $\overset{t}{\nabla}$.*

From the expression (2.4), the coefficient of $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \delta^k_j\psi_i + \delta^k_i\psi_j + (t-1)\pi_i U^k_j + t\pi_j U^k_i, \tag{3.1}$$

where ψ_i is a projective component of $\overset{p}{\nabla}$. And from the expression (3.1), the connection family $\overset{p}{\nabla}$ satisfies the relation

$$\overset{p}{\nabla}_k g_{ij} = -2\psi_k g_{ij} - \psi_i g_{jk} - 2(t-1)\pi_k U_{ij} - t\pi_i U_{jk} - t\pi_j U_{ik}, \;\; T^{\ k}_{ij} = \pi_j U^k_i - \pi_i U^k_j. \tag{3.2}$$

From (3.1), the curvature tensor of $\overset{p}{\nabla}$ is

$$\begin{aligned}\overset{p}{R}_{ijk}{}^{l} \;&=\; K_{ijk}{}^{l} + U^l_j\overset{t}{a}_{ik} - U^l_i\overset{t}{a}_{jk} + \delta^l_j c_{ik} - \delta^l_i c_{jk} - \pi_i\overset{t}{b}{}^{l}_{jk} + \pi_j\overset{t}{b}{}^{l}_{ik} + t(U^l_{ij} - U^l_{ji})\pi_k + (t-1)U^l_k\pi_{ij}\\ &+\; T^l_{ij}\psi_k + \delta^l_k\psi_{ij},\end{aligned} \tag{3.3}$$

where

$$\begin{cases} c_{ik} &= \overset{\circ}{\nabla}_i\psi_k - \psi_i\psi_k - (t-1)\pi_i U^p_k\psi_p - t\pi_k U^p_i\psi_p,\\ \psi_{ij} &= \overset{\circ}{\nabla}_i\psi_j - \overset{\circ}{\nabla}_j\psi_i.\end{cases} \tag{3.4}$$

Using (2.5), the expression (3.3) becomes

$$\overset{p}{R}_{ijk}{}^{l} = \overset{t}{R}_{ijk}{}^{l} + \delta^l_j c_{ik} - \delta^l_i c_{jk} + T^l_{ij}\psi_k + \delta^l_k\psi_{ij}, \tag{3.5}$$

Theorem 3.1. *In a Riemannian manifold $(\mathcal{M}, g)$ if 1-form ψ, α and β are closed, then the volume curvature tensor of $\overset{p}{\nabla}$ is zero, namely*

$$\overset{p}{P}_{ij} = 0, \tag{3.6}$$

where $\overset{p}{P}_{ij} = \overset{p}{R}_{ijkl}g^{kl}$ is the volume curvature tensor of $\overset{p}{\nabla}$.

Proof. Contracting the indices k and l of the expression (3.5), then we have

$$\overset{p}{P}_{ij} = \overset{t}{P}_{ij} + c_{ij} - c_{ji} + T^k_{ij}\psi_k + n\psi_{ij}.$$

On the other hand, from (3.4), we obtain

$$c_{ij} - c_{ji} = \psi_{ij} - T^p_{ij}\psi_p.$$

Substituting this expression into the above expression, we obtain

$$\overset{p}{P}_{ij} = \overset{t}{P}_{ij} + (n+1)\psi_{ij}. \tag{3.7}$$

If 1-form ψ, α and β are closed, then $\overset{t}{P}_{ij} = 0$ by Theorem 2.1 and $\psi_{ij} = 0$. Hence from the expression (3.7), we obtain the expression (3.6) is tenable. □

Theorem 3.2. *Suppose that $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic symmetric-type quarter-symmetric projective non-metric connection family $\overset{p}{\nabla}$. If there holds*

$$\psi_h - ts_h = 0, \tag{3.8}$$

then the Riemannian manifold $(\mathcal{M}, g, \overset{p}{\nabla})$ is a constant curvature manifold.

Proof. Proof. Substituting (2.16) into the second Bianchi identity of the curvature tensor of the symmetric-type quarter-symmetric projective non-metric connection family $\overset{p}{\nabla}$, we get

$$\overset{p}{\nabla}_h R_{ijk}{}^l + \overset{p}{\nabla}_i R_{jhk}{}^l + \overset{p}{\nabla}_j R_{hik}{}^l = \overset{m}{T}_{hi}\overset{p}{R}_{jmk}{}^l + \overset{m}{T}_{ij}\overset{p}{R}_{hmk}{}^l + \overset{m}{T}_{jh}\overset{p}{R}_{imk}{}^l,$$

and using the expression (3.2), then we have

$$\begin{aligned}
&\overset{p}{\nabla}_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{p}{\nabla}_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{p}{\nabla}_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk})\\
&-k(p)[\psi_h(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \psi_i(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \psi_j(\delta^l_h g_{ik} - \delta^l_i g_{hk})]\\
&-(t-2)k(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})]\\
&= k(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})\\
&+\pi_h(U^l_i g_{jk} - U^l_j g_{ik}) + \pi_i(U^l_j g_{hk} - U^l_h g_{jk}) + \pi_j(U^l_h g_{ik} - U^l_i g_{hk})].
\end{aligned}$$

Contracting the indices i, l of this expression, then we have

$$\begin{aligned}
&(n-2)(g_{jk}\overset{p}{\nabla}_h k(p) - g_{hk}\overset{p}{\nabla}_j k(p)) - (n-2)k(p)(\psi_h U_{jk} - \psi_j U_{hk}) - (n-2)(t-2)k(p)(\pi_h U_{jk} - \pi_j U_{hk})\\
&= k(p)[(n-3)(\pi_h U_{jk} - \pi_j U_{hk}) + g_{hk}(U^i_j \pi_i - U^i_i \pi_j) + g_{jk}(U^i_i \pi_h - U^i_h \pi_i)].
\end{aligned}$$

Multiplying both sides of this expression by g^{jk}, then we obtain

$$\begin{aligned}
&(n-1)(n-2)\overset{p}{\nabla}_h k(p) - (n-1)(n-2)k(p)\psi_h - (n-2)(t-2)k(p)(\pi_h U^i_i - \pi_i U^i_h)\\
&= 2(n-2)k(p)k(p)(U^i_i \pi_h - U^i_h \pi_i).
\end{aligned}$$

From this expression above we obtain

$$\overset{p}{\nabla}_h k(p) - k(p)(\psi_h - ts_h) = 0.$$

Consequently, if $\psi_h - ts_h = 0$, then $k = const$. □

From the expression (3.1), the coefficient of the mutual connection family $\overset{pm}{\nabla}$ of $\overset{p}{\nabla}$ is

$$\overset{pm}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \delta^k_j \psi_i + \delta^k_i \psi_j + t\pi_i U^k_j + (t-1)\pi_j U^k_i,$$

and from this expression, the curvature tensor of the mutual connection family $\overset{pm}{\nabla}$ is

$$\begin{aligned}
\overset{pm}{R}_{ijk}{}^l \;=\;& K_{ijk}{}^l + \delta^l_j \overset{m}{c}_{ik} - \delta^l_i \overset{m}{c}_{jk} + U^l_j \overset{m}{a}_{ik} - U^l_i \overset{m}{a}_{jk} - \pi_i \overset{tm}{b}{}^l_{jk} + \pi_j \overset{tm}{b}{}^l_{ik} + (t-1)(U^l_{ij} - U^l_{ji})\pi_k + tU^l_k \pi_{ij}\\
+\;& T^l_{ij}\psi_k + \delta^l_k \psi_{ij},
\end{aligned} \tag{3.9}$$

where

$$\overset{m}{c}_{ij} = \overset{\circ}{\nabla}_i \psi_j - \psi_i \psi_j - t\pi_i U^p_j \pi_p - (t-1)U^p_i \psi_p \pi_j. \tag{3.10}$$

By (2.23) and (3.9), we get

$$\overset{pm}{R}_{ijk}{}^{l} = \overset{tm}{R}_{ijk}{}^{l} + \delta^l_j \overset{m}{c}_{ik} - \delta^l_i \overset{m}{c}_{jk} - T^l_{ij}\psi_k + \delta^l_k \psi_{ij}. \tag{3.11}$$

The expression (3.10) and (3.11) are not different in their forms from the expressions (3.4) and (3.5). Hence the volume flat the constant curvature condition of the mutual connection family coincide with those of the symmetric-type quarter-symmetric projective non-metric connection family.

So using Theorem 3.1 and Theorem 3.2, we have the following

Theorem 3.3. *In a Riemannian manifold* $(\mathcal{M}, g)$, *if 1-form* ψ, α *and* β *are closed, then the volume curvature tensor of* $\overset{pm}{\nabla}$ *is zero, namely*

$$\overset{pm}{P}_{ij} = 0, \tag{3.12}$$

where $\overset{pm}{P}_{ij} = \overset{pm}{R}_{ijkl} g^{kl}$ *is the volume curvature tensor of* $\overset{pm}{\nabla}$.

Theorem 3.4. *Suppose that* $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ *is a connected Riemannian manifold associated with an isotropic mutual connection family* $\overset{tm}{\nabla}$ *of* $\overset{t}{\nabla}$. *If*

$$\psi_h - (t-1)s_h = 0, \tag{3.13}$$

then the Riemannian manifold $(\mathcal{M}, g, \overset{tm}{\nabla})$ *is a constant curvature manifold.*

Theorem 3.5. *In a Riemannian manifold* $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$, *if a 1-form* ψ *is a closed form, then the tensor below*

$$\overset{t}{W}{}^{l}_{ijk} + \overset{tm}{W}{}^{l}_{ijk}, \tag{3.14}$$

is an invariant under the projective connection transformation $\overset{t}{\nabla} \to \overset{p}{\nabla}, \overset{tm}{\nabla} \to \overset{pm}{\nabla}$, *where*

$$\begin{cases} \overset{t}{W}{}^{l}_{ijk} = \overset{t}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{t}{R}_{jk} - \delta^l_j \overset{t}{R}_{ik}), \\ \overset{tm}{W}{}^{l}_{ijk} = \overset{tm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^l_i \overset{tm}{R}_{jk} - \delta^l_j \overset{tm}{R}_{ik}). \end{cases} \tag{3.15}$$

where $\overset{t}{W}{}^{l}_{ijk}, \overset{tm}{W}{}^{l}_{ijk}$ *are Weyl projective curvature tensor of* $\overset{t}{\nabla}$ *and* $\overset{tm}{\nabla}$ *respectively.*

Proof. Adding the expressions (3.5) and (3.11), we obtain

$$\overset{p}{R}{}^{l}_{ijk} + \overset{pm}{R}{}^{l}_{ijk} = \overset{t}{R}{}^{l}_{ijk} + \overset{tm}{R}{}^{l}_{ijk} + \delta^l_j - \alpha_{ik} - \delta^l_i \alpha_{jk} + 2\delta^l_k \psi_{ij}, \tag{3.16}$$

where $\alpha_{ij} = c_{ij} + \overset{m}{c}_{ij}$, If ψ is closed, then $\psi_{ij} = \overset{\circ}{\nabla}_i \psi_j - \overset{\circ}{\nabla}_j \psi_i = 0$. From this fact, the expression (3.16) is

$$\overset{p}{R}{}^{l}_{ijk} + \overset{pm}{R}{}^{l}_{ijk} = \overset{t}{R}{}^{l}_{ijk} + \overset{tm}{R}{}^{l}_{ijk} + \delta^l_j - \alpha_{ik} - \delta^l_i \alpha_{jk}. \tag{3.17}$$

Contracting the indices i, l of (3.17), we get

$$\overset{p}{R}_{jk} + \overset{pm}{R}_{jk} = \overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - (n-1)\alpha_{jk}. \tag{3.18}$$

From this expression above we find

$$\alpha_{jk} = \frac{1}{n-1}(\overset{t}{R}_{jk} + \overset{tm}{R}_{jk} - \overset{p}{R}_{jk} - \overset{pm}{R}_{jk})$$

Substituting this expression above into (3.17) and putting

$$\begin{cases} \overset{p}{W}{}^{l}_{ijk} = \overset{p}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^{l}_{i}\overset{p}{R}_{jk} - \delta^{l}_{j}\overset{p}{R}_{ik}), \\ \overset{pm}{W}{}^{l}_{ijk} = \overset{pm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^{l}_{i}\overset{pm}{R}_{jk} - \delta^{l}_{j}\overset{pm}{R}_{ik}). \end{cases} \tag{3.19}$$

Then by a direct computation, we obtain

$$\overset{p}{W}{}^{l}_{ijk} + \overset{pm}{W}{}^{l}_{ijk} = \overset{t}{W}{}^{l}_{ijk} + \overset{tm}{W}{}^{l}_{ijk}. \tag{3.20}$$

This ends the proof of Theorem 3.5. □

Remark 3.1. *The expression (3.19) is a Weyl projective curvature tensor of* $\overset{p}{\nabla}$ *and* $\overset{pm}{\nabla}$ *respectively. And the expression (3.20) is independent of the parameter* t.

Theorem 3.6. *In a Riemannian manifold* $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$*, the tensor below*

$$\overset{t}{\overline{W}}{}^{l}_{ijk} + \overset{tm}{\overline{W}}{}^{l}_{ijk}, \tag{3.21}$$

is an invariant under the projective connection transformation $\overset{t}{\nabla} \to \overset{p}{\nabla}$ *and* $\overset{tm}{\nabla} \to \overset{pm}{\nabla}$*, where*

$$\begin{cases} \overset{t}{\overline{W}}{}^{l}_{ijk} = \overset{t}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^{l}_{i}\overset{t}{R}_{jk} - \delta^{l}_{j}\overset{t}{R}_{ik}) + \frac{1}{n^2-1}\Big[\delta^{l}_{i}(\overset{t}{R}_{jk} - \overset{t}{R}_{kj}) - \delta^{l}_{j}(\overset{t}{R}_{ik} - \overset{t}{R}_{ki}) + (n-1)\delta^{l}_{k}(\overset{t}{R}_{ij} - \overset{t}{R}_{ji})\Big], \\ \overset{tm}{\overline{W}}{}^{l}_{ijk} = \overset{tm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^{l}_{i}\overset{tm}{R}_{jk} - \delta^{l}_{j}\overset{tm}{R}_{ik}) + \frac{1}{n^2-1}\Big[\delta^{l}_{i}(\overset{tm}{R}_{jk} - \overset{tm}{R}_{kj}) - \delta^{l}_{j}(\overset{tm}{R}_{ik} - \overset{tm}{R}_{ki}) + (n-1)\delta^{l}_{k}(\overset{tm}{R}_{ij} - \overset{tm}{R}_{ji})\Big]. \end{cases} \tag{3.22}$$

where $\overset{t}{\overline{W}}{}^{l}_{ijk}, \overset{tm}{\overline{W}}{}^{l}_{ijk}$ *are the generalized Weyl projective curvature tensor of* $\overset{t}{\nabla}$ *and* $\overset{tm}{\nabla}$*, respectively.*

Proof. Contracting the indices i, l of (3.16) and using $\psi_{ij} = -\psi_{ji}$, we get

$$\overset{p}{R}_{jk} + \overset{pm}{R}_{jk} = \overset{t}{R}_{jk} + \overset{tp}{R}_{jk} - (n-1)\alpha_{jk} - 2\psi_{jk}. \tag{3.23}$$

Alternating the indices j and k of this expression and using $\alpha_{jk} - \alpha_{kj} = 2\psi_{jk}$, we obtain

$$\overset{p}{R}_{jk} - \overset{p}{R}_{kj} + \overset{pm}{R}_{jk} - \overset{pm}{R}_{kj} = \overset{t}{R}_{jk} - \overset{t}{R}_{kj} + \overset{tm}{R}_{jk} - \overset{tm}{R}_{kj} - 2(n+1)\psi_{jk}.$$

By a similar computation just as that in Theorem 3.5, we then have

$$\overset{t}{\overline{W}}{}^{l}_{ijk} + \overset{tm}{\overline{W}}{}^{l}_{ijk} = \overset{p}{\overline{W}}{}^{l}_{ijk} + \overset{pm}{\overline{W}}{}^{l}_{ijk}. \tag{3.24}$$

where

$$\begin{cases} \overset{p}{\overline{W}}{}^{l}_{ijk} = \overset{p}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^{l}_{i}\overset{p}{R}_{jk} - \delta^{l}_{j}\overset{p}{R}_{ik}) + \frac{1}{n^2-1}\Big[\delta^{l}_{i}(\overset{p}{R}_{jk} - \overset{p}{R}_{kj}) - \delta^{l}_{j}(\overset{p}{R}_{ik} - \overset{p}{R}_{ki}) + (n-1)\delta^{l}_{k}(\overset{p}{R}_{ij} - \overset{p}{R}_{ji})\Big], \\ \overset{pm}{\overline{W}}{}^{l}_{ijk} = \overset{pm}{R}{}^{l}_{ijk} - \frac{1}{n-1}(\delta^{l}_{i}\overset{pm}{R}_{jk} - \delta^{l}_{j}\overset{pm}{R}_{ik}) + \frac{1}{n^2-1}\Big[\delta^{l}_{i}(\overset{pm}{R}_{jk} - \overset{pm}{R}_{kj}) - \delta^{l}_{j}(\overset{pm}{R}_{ik} - \overset{pm}{R}_{ki}) + (n-1)\delta^{l}_{k}(\overset{pm}{R}_{ij} - \overset{pm}{R}_{ji})\Big]. \end{cases} \tag{3.25}$$

This completes the proof Theorem 3.6. □

Remark 3.2. *Theorem 3.5 and Theorem 3.6 show that the proposed invariant is independent of the parameter* t.

4. Geometries of manifolds with a symmetric-type quarter-symmetric conformal non-metric connection

Definition 4.1. *A connection family $\overset{c}{\nabla}$ is called a symmetric-type quarter-symmetric conformal non-metric connection family, if the connection $\overset{c}{\nabla}$ is a conformal equivalent to $\overset{t}{\nabla}$.*

From (2.4), the coefficient of is

$$\overset{c}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \delta^{k}_{j}\sigma_i + \delta^{k}_{i}\sigma_j - g_{ij}\sigma^{k} + (t-1)\pi_i U^{k}_{j} + t\pi_j U^{k}_{i}, \tag{4.1}$$

where σ_i is a conformal component of the connection family $\overset{c}{\nabla}$ with respect to the conformal transformation of g_{ij}, namely $\bar{g}_{ij} = e^{2\sigma} g_{ij}(\sigma_i = \partial_i\sigma)$. From (4.1), the connection family $\overset{c}{\nabla}$ satisfies the relation

$$\overset{c}{\nabla}_k g_{ij} = -2\sigma_k g_{ij} - 2(t-1)\pi_k U_{ij} - t\pi_i U_{jk} - t\pi_j U_{ik}, \;\; T^{\;k}_{ij} = \pi_j U^{k}_{i} - \pi_i U^{k}_{j}. \tag{4.2}$$

The curvature tensor of $\overset{c}{\nabla}$ is

$$\begin{aligned}\overset{c}{R}_{ijk}{}^{l} \;&=\; K_{ijk}{}^{l} + \delta^{l}_{j}e_{ik} - \delta^{l}_{i}e_{jk} + g_{ik}f^{l}_{j} - g_{jk}f^{l}_{i} + U^{l}_{j}\overset{t}{a}_{ik} - U^{l}_{i}\overset{t}{a}_{jk} - \pi_i \overset{t}{b}{}^{l}_{jk} + \pi_j \overset{t}{b}{}^{l}_{ik} + t(U^{l}_{ij} - U^{l}_{ji})\pi_k \\ &+\; (t-1)U^{l}_{k}\pi_{ij} - T^{l}_{ij}\sigma_k - tT_{ijk}\sigma^{l},\end{aligned} \tag{4.3}$$

where

$$\begin{cases} e_{ik} &= \overset{\circ}{\nabla}_i\sigma_k - \sigma_i\sigma_k - (t-1)\pi_i U^{p}_{k}\sigma_p - t\pi_k U^{p}_{i}\sigma_p + \frac{1}{2}g_{ik}\sigma^{p}\sigma_p, \\ f_{ij} &= \overset{\circ}{\nabla}_i\sigma_k - \sigma_i\sigma_k + (t-1)\pi_i U^{p}_{k}\sigma_p + t\pi_k U^{p}_{i}\sigma_p + \frac{1}{2}g_{ik}\sigma^{p}\sigma_p. \end{cases} \tag{4.4}$$

Using (2.5), the expression (4.3) becomes

$$\overset{c}{R}_{ijk}{}^{l} = \overset{t}{R}_{ijk}{}^{l} + \delta^{l}_{j}e_{ik} - \delta^{l}_{i}e_{jk} + g_{ik}f^{l}_{j} - g_{jk}f^{l}_{i} + T^{l}_{ij}\sigma_k - (t-1)T_{ijk}\sigma^{l}. \tag{4.5}$$

Theorem 4.1. *In a Riemannian manifold $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ if 1-form α and β are closed form, then the volume curvature tensor of $\overset{c}{\nabla}$ is zero, namely*

$$\overset{c}{P}_{ij} = 0. \tag{4.6}$$

where $\overset{c}{P}_{ij} = \overset{c}{R}_{ijkl}g^{kl}$ is the volume curvature tensor of $\overset{c}{\nabla}$.

Proof. Contracting the indices k and l of the expression (4.5), then we have

$$\overset{c}{P}_{ij} = \overset{t}{P}_{ij} + e_{ij} - e_{ji} + f_{ji} - f_{ij} + T^{k}_{ij}\sigma_k - (t-1)T_{ijk}\sigma^{k}.$$

On the other hand, by (4.4), we have

$$e_{ij} - e_{ji} = -T^{k}_{ij}\sigma_k; \;\; f_{ij} - f_{ji} = (t-1)T^{k}_{ijk}\sigma^{k}.$$

Substituting these expressions into the above expression, we obtain

$$\overset{c}{P}_{ij} = \overset{t}{P}_{ij}. \tag{4.7}$$

If 1-form α and β are closed, then $\overset{t}{P}_{ij} = 0$(Theorem 2.2). Hence from the expression (4.7), we obtain the expression (4.6). □

Theorem 4.2. *Suppose that $(\mathcal{M}, g)(dim\mathcal{M} \geq 3)$ is a connected Riemannian manifold associated with an isotropic symmetric-type quarter-symmetric conformal non-metric connection family $\overset{c}{\nabla}$. If there holds*

$$2\sigma_h - ts_h = 0, \tag{4.8}$$

then the Riemannian manifold $(\mathcal{M}, g, \overset{c}{\nabla})$ is a constant curvature manifold.

Proof. Substituting the expression (2.16) into the second Bianchi identity of the curvature tensor of the symmetric-type quarter-symmetric conformal non-metric connection family $\overset{c}{\nabla}$, we get

$$\overset{c}{\nabla}_h \overset{c}{R}_{ijk}{}^l + \overset{c}{\nabla}_i \overset{c}{R}_{jhk}{}^l + \overset{c}{\nabla}_j \overset{c}{R}_{hik}{}^l = T^m_{hi}\overset{c}{R}_{jmk}{}^l + T^m_{ij}\overset{c}{R}_{hmk}{}^l + T^m_{jh}\overset{c}{R}_{imk}{}^l,$$

and using the expression (4.2), then we have

$$\begin{aligned}
&\overset{c}{\nabla}_h k(p)(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \overset{c}{\nabla}_i k(p)(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \overset{c}{\nabla}_j k(p)(\delta^l_h g_{ik} - \delta^l_i g_{hk}) \\
&-2k(p)[\sigma_h(\delta^l_i g_{jk} - \delta^l_j g_{ik}) + \sigma_i(\delta^l_j g_{hk} - \delta^l_h g_{jk}) + \sigma_j(\delta^l_h g_{ik} - \delta^l_i g_{hk})] \\
&-(t-2)k(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})] \\
&= k(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk}) \\
&+\pi_h(U^l_i g_{jk} - U^l_j g_{ik}) + \pi_i(U^l_j g_{hk} - U^l_h g_{jk}) + \pi_j(U^l_h g_{ik} - U^l_i g_{hk})].
\end{aligned}$$

Contracting the indices i, l of this expression, then we arrive at

$$\begin{aligned}
&(n-2)(g_{jk}\overset{c}{\nabla}_h k(p) - g_{hk}\overset{c}{\nabla}_j k(p)) - 2(n-2)k(p)(\sigma_h g_{jk} - \sigma_j g_{hk}) - (n-2)(t-2)k(p)(\pi_h U_{jk} - \pi_j U_{hk}) \\
&= k(p)[(n-3)(\pi_h U_{jk} - \pi_j U_{hk}) + g_{hk}(U^i_j \pi_i - U^i_i \pi_j) + g_{jk}(U^i_i \pi_h - U^i_h \pi_i)].
\end{aligned}$$

Multiplying both sides of this expression by g^{jk}, then we obtain

$$\begin{aligned}
&(n-1)(n-2)\overset{c}{\nabla}_h k(p) - 2(n-1)(n-2)k(p)\sigma_h - (n-2)(t-2)k(p)(\pi_h U^i_i - \pi_i U^i_h) \\
&= 2(n-2)k(p)k(p)(U^i_i \pi_h - U^i_h \pi_i).
\end{aligned}$$

From this equation above we obtain

$$\overset{c}{\nabla}_h k(p) - k(p)(2\sigma_h - ts_h) = 0.$$

Consequently, if $2\sigma_h - ts_h = 0$, then $k(p) = const$. □

From the expression (4.1), the coefficient of the mutual connection $\overset{cm}{\nabla}$ of $\overset{c}{\nabla}$ as

$$\overset{cm}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \delta^k_j \sigma_i + \delta^k_i \sigma_j - g_{ij}\sigma^k + t\pi_i U^k_j + (t-1)\pi_j U^k_i,$$

and from this expression, the curvature tensor of the mutual connection family $\overset{pm}{\nabla}$ is

$$\begin{aligned}
\overset{cm}{R}_{ijk}{}^l &= K_{ijk}{}^l + \delta^l_j \overset{m}{e}_{ik} - \delta^l_i \overset{m}{e}_{jk} + g_{ik}\overset{m}{f}{}^l_j - g_{jk}\overset{m}{f}{}^l_i + U^l_j \overset{m}{a}_{ik} - U^l_i \overset{m}{a}_{jk} - \pi_i \overset{tm}{b}{}^l_{jk} + \pi_j \overset{tm}{b}{}^l_{ik} \\
&+ (t-1)(U^l_{ij} - U^l_{ji})\pi_k + tU^l_k \pi_{ij} - T^l_{ij}\sigma_k - tT_{ijk}\sigma^l,
\end{aligned} \tag{4.9}$$

where

$$\begin{cases} \overset{m}{c}_{ij} &= \overset{\circ}{\nabla}_i \sigma_j - \sigma_i \sigma_j - t\pi_i U^p_j \pi_p - (t-1)U^p_i \sigma_p \pi_j + \frac{1}{2} g_{ij}\sigma_p \sigma^p, \\ \overset{m}{f}_{ij} &= \overset{\circ}{\nabla}_i \sigma_j - \sigma_i \sigma_j + t\pi_i U_{jp}\sigma^p + (t-1)U_{ij}\sigma^p \pi_p + \frac{1}{2} g_{ij}\sigma_p \sigma^p \end{cases} \tag{4.10}$$

Using the expression (2.11) and (4.9), we have

$$\overset{cm}{R}_{ijk}{}^{l} = \overset{tm}{R}_{ijk}{}^{l} + \delta^l_j \overset{m}{e}_{ik} - \delta^l_i \overset{m}{e}_{jk} + g_{ik}\overset{m}{f}{}^{l}_{j} - g_{jk}\overset{m}{f}{}^{l}_{i} - T^l_{ij}\sigma_k - tT_{ijk}\sigma^l. \tag{4.11}$$

The expressions (4.10) and (4.11) are not different in the form from the expressions (4.4) and (4.5). Hence the volume flat and constant curvature condition of the mutual connection family coincide with those of the symmetric-type quarter-symmetric conformal connection family.

Thus, we know that by Theorem 4.1 and Theorem 4.2 there holds

Theorem 4.3. *In a Riemannian manifold $(\mathcal{M}, g)$ if 1-form α and β are closed, then the volume curvature tensor of $\overset{cm}{\nabla}$ is zero, namely*

$$\overset{cm}{P}_{ij} = 0 \tag{4.12}$$

where $\overset{cm}{P}_{ij} = \overset{cm}{R}_{ijkl}g^{kl}$ is the volume curvature tensor of $\overset{cm}{\nabla}$.

Theorem 4.4. *Suppose that $(\mathcal{M}, g)$ is a connected Riemannian manifold associated with an isotropic mutual connection family $\overset{cm}{\nabla}$ of $\overset{c}{\nabla}$. If*

$$\sigma_h - (t-1)s_h = 0, \tag{4.13}$$

then the Riemannian manifold $(\mathcal{M}, g, \overset{cm}{\nabla})$ is a constant curvature manifold.

If $t = \frac{1}{2}$, and using Remark. 2.3, the expression (4.5) becomes

$$\overset{c}{\tilde{R}}_{ijk}{}^{l} = \tilde{R}_{ijk}{}^{l} + \delta^l_j \tilde{e}_{ik} - \delta^l_i \tilde{e}_{jk} + g_{ik}\tilde{f}{}^{l}_{j} - g_{jk}\tilde{f}{}^{l}_{i} + T^l_{ij}\sigma_k + \frac{1}{2}T_{ijk}\sigma^l. \tag{4.14}$$

where

$$\begin{cases} \tilde{e}_{ij} &= \overset{\circ}{\nabla}_i\sigma_j - \sigma_i\sigma_j + \frac{1}{2}\pi_i U^p_j \sigma_p - \frac{1}{2}\pi_j U^p_i \pi_p + \frac{1}{2}g_{ij}\sigma_p\sigma^p, \\ \tilde{f}_{ij} &= \overset{\circ}{\nabla}_i\sigma_j - \sigma_i\sigma_j + \frac{1}{2}\pi_i U_{jp}\sigma^p + \frac{1}{2}\sigma^p U_{ij}\pi_p + \frac{1}{2}g_{ij}\sigma_p\sigma^p. \end{cases} \tag{4.15}$$

From the expressions (4.10) and (4.11), we have

$$\overset{cm}{\tilde{R}}_{ijk}{}^{l} = \overset{m}{\tilde{R}}_{ijk}{}^{l} + \delta^l_j \overset{m}{\tilde{e}}_{ik} - \delta^l_i \overset{m}{\tilde{e}}_{jk} + g_{ik}\overset{m}{\tilde{f}}{}^{l}_{j} - g_{jk}\overset{m}{\tilde{f}}{}^{l}_{i} + T^l_{ij}\sigma_k - \frac{1}{2}T_{ijk}\sigma^l. \tag{4.16}$$

where

$$\begin{cases} \overset{m}{\tilde{e}}_{ij} &= \overset{\circ}{\nabla}_i\sigma_j - \sigma_i\sigma_j - \frac{1}{2}\pi_i U^p_j \sigma_p + \frac{1}{2}\pi_j U^p_i \pi_p + \frac{1}{2}g_{ij}\sigma_p\sigma^p, \\ \overset{m}{\tilde{f}}_{ij} &= \overset{\circ}{\nabla}_i\sigma_j - \sigma_i\sigma_j + \frac{1}{2}\pi_i U_{jp}\sigma^p - \frac{1}{2}\sigma^p U_{ij}\pi_p + \frac{1}{2}g_{ij}\sigma_p\sigma^p. \end{cases} \tag{4.17}$$

Theorem 4.5. *In a Riemannian manifold $(\mathcal{M}, g)$, the tensor below*

$$\tilde{C}_{ijk}{}^{l} + \overset{m}{\tilde{C}}_{ijk}{}^{l} \tag{4.18}$$

is an invariant under the conformal connection transformation $\tilde{D} \to \overset{c}{\tilde{D}}; \overset{c}{\tilde{D}} \to \overset{cm}{\tilde{D}}$, where

$$\begin{cases} \tilde{C}_{ijk}{}^{l} &= \tilde{R}_{ijk}{}^{l} - \frac{1}{n-2}(\delta^l_i \tilde{R}_{jk} - \delta^l_j \tilde{R}_{ik} + g_{jk}\tilde{R}{}^{l}_{i} - g_{ik}\tilde{R}{}^{l}_{j}) - \frac{\tilde{R}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}), \\ \overset{m}{\tilde{C}}_{ijk}{}^{l} &= \overset{m}{\tilde{R}}_{ijk}{}^{l} - \frac{1}{n-2}(\delta^l_i \overset{m}{\tilde{R}}_{jk} - \delta^l_j \overset{m}{\tilde{R}}_{ik} + g_{jk}\overset{m}{\tilde{R}}{}^{l}_{i} - g_{ik}\overset{m}{\tilde{R}}{}^{l}_{j}) - \frac{\overset{m}{\tilde{R}}}{(n-1)(n-2)}(\delta^l_j g_{ik} - \delta^l_i g_{jk}). \end{cases} \tag{4.19}$$

($\tilde{C}_{ijk}{}^{l}, \overset{m}{\tilde{C}}_{ijk}{}^{l}$ are called the so-called Weyl conformal curvature tensor of $D, \overset{m}{\tilde{D}}$, respectively)

Proof. Adding the expressions (4.14) and (4.16), we obtain

$$\overset{c}{\tilde{R}}_{ijk}{}^{l} + \overset{cm}{\tilde{R}}_{ijk}{}^{l} = \tilde{R}_{ijk}{}^{l} + \overset{m}{\tilde{R}}_{ijk}{}^{l} + \delta^{l}_{j}\beta_{ik} - \delta^{l}_{i}\beta_{jk} + g_{ik}\beta^{l}_{j} - g_{jk}\beta^{l}_{i}, \tag{4.20}$$

where β_{ij} from the expressions (4.15) and (4.17) satisties

$$\beta_{ij} = \tilde{e}_{ij} + \overset{m}{\tilde{e}}_{ij} = \tilde{f}_{ij} + \overset{m}{\tilde{f}}_{ij} = 2\overset{\circ}{\nabla}_{i}\sigma_{j} - 2\sigma_{i}\sigma_{j} + g_{ij}\sigma^{k}\sigma_{k}.$$

Contracting the indices i, l of the expression (4.20), we get

$$\overset{c}{\tilde{R}}_{jk} + \overset{cm}{\tilde{R}}_{jk} = \tilde{R}_{jk} + \overset{m}{\tilde{R}}_{jk} - (n-2)\beta_{jk} - g_{jk}\beta^{i}_{i}. \tag{4.21}$$

Multiplying both sides of expression (4.21) by g^{jk}, then we arrive at

$$\overset{c}{\tilde{R}} + \overset{cm}{\tilde{R}} = \tilde{R} + \overset{m}{\tilde{R}} - 2(n-1)\beta^{i}_{i}.$$

From this expression above we have

$$\beta^{i}_{i} = \frac{1}{2(n-1)}(\tilde{R} + \overset{m}{\tilde{R}} - \overset{c}{\tilde{R}} - \overset{cm}{\tilde{R}}).$$

By a direct computation similar to that in Theorem 3.4, we obtain

$$\overset{c}{\tilde{C}}{}^{l}_{ijk} + \overset{cm}{\tilde{C}}{}^{l}_{ijk} = \tilde{C}^{l}_{ijk} + \overset{m}{\tilde{C}}{}^{l}_{ijk}. \tag{4.22}$$

where $\overset{c}{\tilde{C}}{}^{l}_{ijk} + \overset{cm}{\tilde{C}}{}^{l}_{ijk}$ are Weyl conformal curvature tensors of symmetric-type quarter-symmetric conformal non-metric connection $\overset{c}{D}$ and its mutual connection $\overset{cm}{D}$, respectively. This ends the proof of Theorem 4.5. □

5. Ackonowedement

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