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Interior relative ideals in quasi-ordered semigroups

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Abstract. The concept of relative ideal in a semigroup was introduced in 1962 by A. D. Wallace. This concept was further developed by R. Hrmová (1967), and later by M. F. Ali, N. M. Khan and A. Mahboob (2020). In this paper, the notions of interior relative T-ideals and interior relative (H_1, H_2)-ideals in quasi-ordered semigroups are introduced and some fundamental properties of these ideals are studied. Also, among other things, the paper discusses about the minimality of interior relative ideals in such semigroups.

1. Introduction

In 1962, Wallace [19, 20], introduced the notion of relative ideals (T-ideals) on semigroup S. In 1967, Hrmová [5] generalized the notion of T-ideal by introducing the notion of an (H_1, H_2)-ideal of a semigroup S (where $T, H_1, H_2 \subseteq S$). T. K. Dutta studied relative ideals in groups ([4]). Then this class of generalized ideals in a quasi-ordered semigroup was the subject of interest of several researchers (see, for example [1–4, 10, 15]). Thus, relative bi-ideals and relative quasi ideals in ordered semigroups were introduced and analyzed in [10]. In some of the mentioned articles, relative ideals in hyper-semigroups are discussed ([2, 3]).

The concept of interior ideals in semigroups was introduced 1976 in article [12] by S. Lajos: A non-empty sub-semigroup J of a semigroup S is an interior ideal of S if $SJS \subseteq J$ holds. This concept in semigroups and in ordered semigroups has been considered by several authors (for example, [7–9, 11, 14, 16–18]).

In this paper, the concepts of interior T-ideals and interior (H_1, H_2)-ideals in a quasi-ordered semigroup are introduced as generalizations of the concept of interior ideals in a quasi-ordered semigroup. Also, some connections were found between (left, right) ideals, interior ideals, interior T-ideals and interior (H_1, H_2)-ideal. Thus, for example, some sufficient conditions have been found such that T-ideals and interior T-ideals in quasi-ordered semigroup coincide. Also, we have found some sufficient conditions for the (H_1, H_2)-ideals and the interior (H_1, H_2)-ideals in the quasi-ordered semigroup to coincide. It has been shown, in addition, that the family of all interior relative ideals (both types) forms a complete lattice, which allowed us to say something about the minimality of interior relative ideals in this family.

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2. Preliminaries

In this section, the notions and notations necessary for understanding the material presented in Section 3, which is the main part of this report, are presented. The examples that represent these notions are taken from the literature.

Let S be a quasi-ordered semigroup. For two subsets of A and B of S, we put $AB =: \{ab : a \in A \wedge b \in B\}$. If $A = \emptyset$, then $AB = B$. If $B = \emptyset$, then $AB = A$. We will write aB instead of $\{a\}B$ and Ab instead of $A\{b\}$. A non-empty subset A of a quasi-ordered semigroup S is a sub-semigroup of S if $AA \subseteq A$ holds. A non-empty subset J of a quasi-ordered semigroup S is a left (right) ideal of S if $SJ \subseteq J$ ($JS \subseteq J$, res.). A non-empty subset J of S is an ideal of S if J is a left ideal and a right ideal of S. It is easy to conclude that a (left, right) ideal in a semigroup S is a sub-semigroup of S. Indeed, if $SJ \subseteq J$, then $JJ \subseteq SJ \subseteq J$.

A relation $\preceq$ on a set S is a quasi-order if the following holds

(1) $(\forall x \in S)(x \preceq x)$ and

(2) $(\forall x, y, z \in S)((x \preceq y \wedge y \preceq z) \implies x \preceq z)$.

The quasi-order $\leqslant$ on a set S is an order relation on S if

(3) $(\forall x, y \in S)((x \leqslant y \wedge y \leqslant x) \implies x = y)$.

The quasi-order (res. the order) relation on a semigroup S must be compatible with the operation in S in the following sense

(4) $(\forall x, y, x \in S)(x \preceq y \implies (zx \preceq zy \wedge xz \preceq yz))$.

If a semigroup S is provided with a quasi-order relation (an order relation, res.) that satisfies the condition (4), then S is said to be a quasi-ordered semigroup (an ordered semigroup, res.). For any subset A of a quasi-ordered semigroup S, we put $(A] =: \{x \in S : (\exists a \in A)(x \preceq a)\}$. A (left, right) ideal J of a quasi-ordered semigroup S must satisfy the following condition

(5) $(\forall x, y \in S)((y \in J \wedge x \preceq y) \implies x \in J)$.

This condition can also be written in the form $(J] \subseteq J$. It can be concluded without any difficulties that $J \subseteq (J]$ and $((J]] = (J]$ is valid.

In the following definition, the concept of (left, right) T-ideals in quasi-ordered semigroups is presented.

Definition 2.1 ([19]). *Let S be a quasi-ordered semigroup and let J and T be any non-empty subsets of S. Then J is said to be a left T-ideal of S if the following holds*

(6) $TJ \subseteq J$ and

(7) $(\forall x, y \in S)((x \in T \wedge x \preceq y \wedge y \in J) \implies x \in J)$.

Dually we can define a right T-ideal of S if (7) is valid and the following holds

(8) $JT \subseteq J$.

Further on, J is said to be a T-ideal of S if it is both a left T-ideal and a right T-ideal of S.

Remark 2.2. *It should be noted here that in the determination of the (left, right) T-ideal J of a quasi-ordered semigroup S there is no requirement that the ideal J must be a sub-semigroup of S.*

Example 2.3 ([4], Example 1). *Let $M_2(\mathbb{Q})$ be the set of all 2×2 nonsingular matrices over the field $\mathbb{Q}$ of rational numbers ordered by the standard order relation. Then M_2 is an ordered semi-group. Let $J := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in 2\mathbb{Z}\}$ and $T := \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the ring of integers. Then J is a left T-ideal as well as a right T-ideal of M_2.*

If $T = S$, then the notion of a left T-ideal (resp. a right T-ideal) of a quasi-ordered semigroup S coincides with the notion of a (left, right) ideal S. An ideal J of a quasi-ordered semigroup S is a T-ideal of S for any subset T of S, but the converse is not true in general because it does not have to be a sub-semigroup of S. The family of all (left, right) T-ideals in a quasi-ordered semigroup S is denoted by $\mathfrak{J}_S(T)$ ($\mathfrak{l}\mathfrak{J}_S(T)$, $\mathfrak{r}\mathfrak{J}_S(T)$, res.). Especially $\mathfrak{J}(S) =: \mathfrak{J}_S(S)$. Therefore, $\mathfrak{J}(S) \subseteq \mathfrak{J}_S(T)$. One may easily, by direct verification, check that each of the families $\mathfrak{J}_S(T)$, $\mathfrak{l}\mathfrak{J}_S(T)$ and $\mathfrak{r}\mathfrak{J}_S(T)$ is a complete lattice.

Let A and T be any non-empty subsets of a quasi-ordered semigroup S. We define

$$(A]_T = \{t \in T | (\exists a \in A)(t \preceq a)\}$$

and, at that, we will write $(A]_S = (A]$. It is clear that $(A]_T \subseteq (A]$ is valid. On other properties of this set, the reader can look at Lemma 1.5 in [10]. The condition (7) can be written in the form $(J]_T \subseteq J$.

Example 2.4 ([10], Example 1.3). *Let $S = \{a, b, c, d\}$. Define the operation on S as shown in the following table*

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	a	a	b	a

Define an order on S as $\leqslant =: \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (a, c)\}$. Clearly S is an ordered semigroup. Let $A = \{a, b\}$, $B = \{a, d\}$. It is easy to check that A is a B-ideal of S, but not an ideal of S.

Example 2.5 ([5], Proposition 1). *Let $(G, \cdot, 1, \leqslant)$ be an ordered group and A be a non-empty subset of G. Then there exists a sub-semigroup $T =: \{t \in G : tA \subseteq A\}$ of G such that $(A]_T$ is left T-ideal of G.*

Example 2.6 ([10], Lemma 1.5(7)). *If T is sub-semigroup of a quasi-ordered semigroup S, then $(TaT]_T$ is T-ideal of S for any $a \in S$.*

A generalization of this class of ideals in a quasi-ordered semigroup was designed in [5] by R. Hrmová.

Definition 2.7 ([5]). *Let S be a quasi-ordered semigroup and let H_1 and H_2 be any non-empty subsets of S. A non-empty subset J of S is said to be an (H_1, H_2)-ideal, or a relative ideal of S, if the following holds*
(9) $H_1 J \subseteq J$, $JH_2 \subseteq J$ *and*
(10) $(\forall x, y \in S)((x \in H_1 \cup H_2 \wedge x \preceq y \wedge y \in J) \Longrightarrow x \in J)$.
Condition (10) can be written in the form $(J]_{H_1} \cup (J]_{H_2} = (J]_{H_1 \cup H_2} \subseteq J$. If $H_1 = \emptyset$ or $H_2 = \emptyset$, then the (H_1, H_2)-ideal becomes one sided relative ideal of S.

If $H_1 = H_2 = S$, then the (H_1, H_2)-ideal of a quasi-ordered semigroup S is an ideal of S. We denote the family of all (H_1, H_2)-ideal of S by $\mathfrak{J}_S(H_1, H_2)$. The notion of (H_1, H_2)-ideals of quasi-ordered semigroup S is, evidently, not only a generalization of the concept (left, right) ideals of S but also a generalization of the notion of (left, right) T-ideals of S. So, $\mathfrak{J}(S) \subseteq \mathfrak{J}_S(H_1, H_2)$ and $\mathfrak{J}_S(T) = \mathfrak{J}_S(T, T)$.

Example 2.8 ([5], Example 1.1). *Let H_1 and H_2 be sub-semigroups of a semigroup S. Then for every $a \in S$, the set $J =: \{a\} \cup H_1 a \cup aH_2 \cup H_1 aH_2$ is a (H_1, H_2)-ideal of S.*

Example 2.9 ([5], Example 1.4). *Let J be a bi-ideal of a semigroup S, i.e. a sub-semigroup of S such that $JSJ \subseteq J$. Then J is a (JS, SJ)-ideal of S.*

About relative bi-ideals in ordered semigroups, any interested reader can look at the article [10].

Example 2.10 ([10], Example 2.4). *Let $S =: \{a, b, c, d\}$ and let the internal operation on S shown in the following table*

$\cdot$	a	b	c	d
a	a	a	a	a
b	b	b	b	d
c	a	b	b	d
d	a	d	d	d

Define an order on S as $\preceq =: \{(a,a),(b,b),(c,c),(d,d),(a,d)\}$. *Clearly S is an ordered semigroup. Let* $J = \{a,b\}$, $H_1 = \{a,c\}$ *and* $H_2 = \{b,c\}$. *Then* $H_1 \cup H_2 = \{a,b,c\}$. *It is easy to check that J is an* (H_1,H_2)*-ideal of S, but not an ideal of S.*

According to [5], the concept of relative ideals of a quasi-ordered semigroup S in both of these meanings can be further developed in two directions:

- Given a subset J of a quasi-ordered semigroup S to find subsets H_1, H_2 of S such that $J \in \mathfrak{J}_S(H_1,H_2)$;
- Study of the elements of the lattice $\mathfrak{J}_S(H_1,H_2)$, which have some recognizable properties for a given subsets H_1, H_2 of S.

Thus, for example, minimal relative ideals in semigroups were analyzed in [5]. Relative bi-ideals and relative quasi ideals in ordered semigroups have been the subject of study in [10] by N. M. Khan and Md. F. Ali. This report presents the results obtained on interior relative ideals (both types) in a quasi-ordered semigroup.

3. The main results: Interior relative ideals

In this section, as the main part of this report, the concepts of interior T-ideals and interior (H_1,H_2)-ideals in quasi-ordered semigroups are introduced and their properties are discussed. It is shown that any T-ideal ((H_1,H_2)-ideal) is an interior T-ideal (interior (H_1,H_2)-ideal, res.). Since, in the general case, the converse does not have to hold, some additional conditions imposed on the quasi-ordered semigroup were found which ensure that all T-ideals ((H_1,H_2)-ideals) in such a semigroup are interior T-ideals (interior (H_1,H_2)-ideals, res.).

3.1. *Interior T-ideals*

The concept of interior ideals of a semigroup S has been introduced by S. Lajos in [12] as a sub-semigroup J of S such that $SJS \subseteq J$. The interior ideals of semigroups have been also studied by G. Szász in [16, 17]. In [7, 8] N. Kehayopulu and M. Tsingelis introduced the concepts of interior ideals in ordered semigroups. In this section, we introduce the notion of relative interior ideals in quasi-ordered semigroups.

The following definition introduces the concept of interior T-ideals in a quasi-ordered semigroup.

Definition 3.1. *Let S be a quasi-ordered semigroup and let T and J be any non-empty subsets of S. The subset J is said to be an interior T-ideal of S if (7) is valid and the following holds*
(11) $TJT \subseteq J$.

It is obvious that any interior ideal of a quasi-ordered semigroup S is an interior T-ideal of S for any non-empty subset T. Indeed, if $SJS \subseteq J$ and $(J] \subseteq J$, then $TJT \subseteq SJS \subseteq J$ and $(J]_T \subseteq (J] \subseteq J$. The family of all interior T-ideals of S determined by the subset T shall be denoted by $\mathfrak{Int}_S(T)$. Additionally for simplicity, in particular, $\mathfrak{Int}(S) =: \mathfrak{Int}_S(S)$. So, $\mathfrak{Int}(S) \subseteq \mathfrak{Int}_S(T)$.

Remark 3.2. *It should be noted here, also, that in the determination of the interior T-ideal J of a quasi-ordered semigroup S does not require that the set J must be a sub-semigroup of S.*

Example 3.3. *Let* $S = \{0,1,2,3,4\}$ *and operation '·' defined on S as follows:*

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	3	2
3	0	3	3	3	3
4	0	4	2	3	4

Then S forms a semigroup. The quasi-order relation on S is given by

$\preceq = \{(0,0), (0,1), (0,2), (0,3), (0,4), (1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,4)\}$.

Then S is a quasi-ordered semigroup. By direct verification one can establish that the set $\{0,4\}$ *is an interior* $\{0,1\}$*-ideal and an interior* $\{0,1,2\}$*-ideal in S but they are not interior ideals of S because, for example, we have* $4 \in J$ *but* $3 \cdot 4 \cdot 2 = 3 \notin J$*. However,* $\{0,4\}$ *is not a* $\{0,1,2\}$*-ideal of S because, for example, we have* $2 \cdot 4 = 2 \notin \{0,4\}$*.*

Example 3.4. *Let S be a quasi-ordered semigroup,* $a \in S$ *and a subset T of S be a sub-semigroup of S. The T-ideal* $(TaT]_T$*, generated in Example 2.6, is an interior T-ideal.*

First, let us show that the concept of interior T-ideals is a generalization of the concept of T-ideals in a quasi-ordered semigroup.

Theorem 3.5. *Let T be a non-empty subset of a quasi-ordered semigroup S. Then any T-ideal of S is an interior T-ideal of S, i.e., the following holds* $\mathfrak{J}_S(T) \subseteq \mathfrak{Int}_S(T)$*.*

Proof. Let J be a T-ideal of a quasi-ordered semigroup S. This means that the subset J satisfies the conditions (6), (7) and (8). Let us prove (11): $TJT = T(JT) \subseteq TJ \subseteq J$. □

The reversal of the previous theorem cannot be valid. The inverse of the previous theorem is valid if the subset T of S satisfies the following condition
(A) $(\forall x \in S)(\exists y \in T)(x \preceq xy)$.

Theorem 3.6. *Let S be a quasi-ordered semigroup and T be a subset of S satisfying the condition (A). Then any interior T-ideal of S is a left T-ideal of S, i.e., the following inclusion* $\mathfrak{Int}_S(T) \subseteq \mathfrak{l}\mathfrak{J}_S(T)$ *is valid.*

Proof. Let J be an interior T-ideal of S and let $x, u \in S$ be arbitrary elements such that $x \in J$ and $u \in T$. Then there exists an element $y_x \in T$ such that $x \preceq xy_x$. Thus $ux \preceq uxy_x$ by (4). By assumption (11), we have $uxy_x \in J$ because $x \in J$. Now, from $ux \preceq uxy_x$ and $uxy_x \in J$ it follows $ux \in J$ according to (7). This means that J is a left T -ideal of S. □

By analogy, a dual theorem can be proved:

Theorem 3.7. *Let S be a quasi-ordered semigroup and T be a subset of S satisfying the condition*
(B) $(\forall x \in S)(\exists y \in T)(x \preceq yx)$.
Then any interior T-ideal of S is a right T-ideal of S.

Let us now analyze the following option:

Theorem 3.8. *Suppose that a subset T of a quasi-ordered semigroup S is a sub-semigroup of S and let S satisfy the following condition*
(C) $(\forall x \in S)(\exists u, v \in T)(x \preceq uxv)$.
Then any interior T-ideal of S is a T-ideal of S, that is, $\mathfrak{Int}_S(T) \subseteq \mathfrak{J}_S(T)$ *is valid.*

Proof. Let J be an interior T-ideal of S. This means that J satisfies the conditions (7) and (11). Let us prove (6) and (8).

Let $z, x \in S$ be arbitrary element such that $x \in J$ and $z \in T$. Then there are elements $u, v \in T$ such that $x \preceq uxv$ by (C). Thus $zx \preceq (zu)xv$ by (4). Hence $(zu)xv \in J$ by (11) since $zu \in T$. Hence $zx \in J$ according to (7). This means that (6) is valid.

(8) can be proved by analogy with the previous demonstration. □

Let us recall the term intra-regularity in a quasi-ordered semigroup S ([6, 13]): A quasi-ordered semigroup S is intra-regular if holds

$$(\forall x \in S)(\exists u, v \in S)(x \preceq ux^2v).$$

The following definition introduces the concept of T-intra-regular quasi-ordered semigroups:

Definition 3.9. *Let T be a subset of a quasi-ordered semigroup S. S is said to be a T-intra-regular quasi-ordered semigroup if holds*

(D) $(\forall x \in S)(\exists u, v \in T)(x \preceq ux^2v)$.

Theorem 3.10. *Let S be a T-intra-regular quasi-ordered semigroup for the sub-semigroup $T \subseteq S$. If an interior T-ideal J of S is a sub-semigroup of S, then J is a T-ideal of S.*

Proof. Let J be an interior T-ideal of a T-intra-regular quasi-ordered semigroup S such that $JJ \subseteq J$. This means that (D) is a valid formula. Let us prove (6) and (8).

Let $x \in J$ and $z \in T$ be arbitrary elements. Then $x^2 \in J$ and there exist elements $u, v \in T$ such that $x \preceq ux^2v$. Thus $zx \preceq (zu)x^2v$ by (D). Since J is an interior T-ideal, we have that $x^2 \in J$ implies $(zu)x^2v \in J$. Hence $zx \in J$ by (7). This means that (6) is valid.

(8) can be proved by analogy with the previous demonstration. □

The family $\mathfrak{Int}_S(T)$ of all interior T-ideals of a quasi-ordered semigroup S is not empty because $(TaT]_T \in \mathfrak{Int}_S(T)$ for any $a \in S$ according to Example 3.4. The validity of the following theorem can be established by direct verification:

Theorem 3.11. *The family $\mathfrak{Int}_T(S)$ is a complete lattice.*

Proof. Let $\{J_i\}_{i\in I}$ be a family of interior T-ideals of a quasi-ordered semigroup S. Let us prove (11) and (7) for subsets $\bigcup_{i\in I} J_i$ and $\bigcap_{i\in I} J_i$.

$T(\bigcup_{i\in I} J_i)T \subseteq \bigcup_{i\in I} TJ_iT \subseteq \bigcup_{i\in I} J_i$ and $T(\bigcap_{i\in I} J_i)T \subseteq \bigcap_{i\in I} TJ_iT \subseteq \bigcap_{i\in I} J_i$.

$(\bigcup_{i\in I} J_i]_T \subseteq \bigcup_{i\in I}(J_i]_T \subseteq \bigcup_{i\in I} J_i$ and $(\bigcap_{i\in I} J_i]_T \subseteq \bigcap_{i\in I}(J_i]_T \subseteq \bigcap_{i\in I} J_i$. □

Let T be a subset of a quasi-ordered semigroup S. Before embarking on further conclusions, let us recall the terms 'minimum T-ideal' and 'maximum T-ideal' in a quasi-ordered semigroup: We shall say that an interior T-ideal A is a minimal interior T-ideal of S if there is no interior T-ideal B of S such that $B \subset A$ ([5]). Also, dually, we shall say that an interior T-ideal A is a maximal interior T-ideal of S if there is no interior T-ideal B of S such that $A \subset B$. It is easy to conclude that if A and B are two minimum interiors T-ideals of a quasi-ordered semigroup S, then $A \cap B = \emptyset$, because, otherwise, according to the previous theorem, $A \cap B$ would be an interior T-ideal of S contained in A and contained in B, which is impossible.

Corollary 3.12. *For any subset X of S there exists the maximal interior T-ideal contained in X and the minimal interior T-ideal containing X.*

Proof. Let $\mathfrak{X}$ be the family of all interior T-ideals of S contained in X. Then $F_{max}(X) =: \cup\mathfrak{X}$ is the maximal interior T-ideal contained in X. If B is an interior T-ideal of S contained in X, then $B \in \mathfrak{X}$. So, $B \subseteq \cup\mathfrak{X}$. Therefore, $\cup\mathfrak{X}$ is the maximal interior T-ideal of S contained in X.

Let $\mathfrak{Y}$ be the family of all interior T-ideals of S containing X. Then $F^{min}(X) =: \cap\mathfrak{Y}$ is the minimal interior T-ideal of S containing X. If B is an interior T-ideal of S containing X, then $B \in \mathfrak{Y}$. So, $B \supseteq \cap\mathfrak{Y}$. Therefore, $\cap\mathfrak{Y}$ is the minimal interior T-ideal containing X. □

Corollary 3.13. *For any $a \in S$ there exists the maximal interior T-ideal J_a of S such that $a \notin J_a$ and the minimal interior T-ideal of S containing a.*

Proof. The proof of the first part of this Corollary can be obtained from the previous Corollary by taking $X = S \setminus \{a\}$. The proof of the second part of this Corollary can be obtained from the previous Corollary by taking $X = \{a\}$. □

A nontrivial result on the minimum interior T-ideal in a quasi-ordered semigroup S gives the following theorem.

Theorem 3.14. *Let T be a sub-semigroup of a quasi-ordered semigroup S, A be a minimal interior T-ideal of S, $L\,(\subseteq T)$ be a left T-ideal of S and $R\,(\subseteq T)$ be a right T-ideal of S. Then $A = LaR$ for any $a \in A$.*

Proof. First, we have that LaR is an interior T-ideal of S. Indeed, the following $T(LaR)T = (TL)a(RT) \subseteq LaR$ is valid. On the other hand, we have $LaR \subseteq LAR \subseteq TAT \subseteq A$ since A is an interior T-ideal of S. Therefore, $LaR = A$ because A is minimal interior T-ideal of S. □

We conclude this subsection with the following two theorems:

Theorem 3.15. *If J is an interior T-ideal and A is a sub-semigroup of a quasi-ordered semigroup S, then $J \cap A$ is an interior $T \cap A$-ideal of A, provided that the sets $A \cap T$ and $A \cap J$ are not-empty.*

Proof. First, the following holds

$$(T \cap A)(A \cap J)(T \cap A) \subseteq (T \cap A)A(T \cap A) \cap (T \cap A)J(T \cap A) \subseteq AAA \cap TJT \subseteq A \cap J.$$

Second, for $x \in A \cap T$ and $y \in J$ such that $x \preceq y$, we have $x \in J \cap A$ by (7). This means $(A \cap J]_{A \cap T} \subseteq J \cap A$. □

In what follows we need the notion of T-bi-ideal ([10]): Let S be a quasi-ordered semigroup and let T, B be any non-empty subsets of S. Then B is said to be an T-bi-ideal of S if (7) is valid and the following $BTB \subseteq B$ holds.

Theorem 3.16. *Let S be a quasi-ordered semigroups, $T \subseteq S$, $J\,(\subseteq T)$ be an interior T-ideal of S and $B\,(\subseteq T)$ be a T-bi-ideal of S. If S is T-regular, then $J \cap B = BJB$.*

Proof. On the one hand, we have $BJB \subseteq BTB \subseteq B$ and $BJB \subseteq TJT \subseteq J$. Hence $BJB \subseteq B \cap J$. On the other hand, let $x \in B \cap J$. Then $x \in J$, $x \in B$ and there is $y \in T$ such that $x \preceq xyx$. Since $xyx \in xTx \subseteq JTJ$ and $xyx \in xTx \subseteq BTB$, we have $xyx \in (xTx)T(xTx) = (xTx)(TxT)x \subseteq (BTB)(TJT)B \subseteq BJB$. Thus $x \in BJB$ by (7) and $B \cap J = BJB$. □

3.2. *Interior* (H_1, H_2)*-ideals*

The following definition introduces the concept of interior relative ideals of the second type in a quasi-ordered semigroup.

Definition 3.17. *Let S be a quasi-ordered semigroup and J, H_1 and H_2 be any non-empty subsets of S. Then the set J is said to be an interior (H_1, H_2)-ideal or a relative interior ideal of S with respect to subsets H_1 and H_2 if the following holds*

(10) $(\forall x, y \in S)((x \in H_1 \cup H_2 \wedge x \preceq y \wedge y \in J) \Longrightarrow x \in J)$ and

(12) $H_1 J H_2 \subseteq J$.

It is obvious that any interior ideal of a quasi-ordered semigroup S is an interior (H_1, H_2)-ideal of S for any non-empty subsets H_1 and H_2 of S. The family of all interior relative ideals of S determined by the subsets H_1 and H_2 shall be denoted by $\mathfrak{Int}_S(H_1, H_2)$. Additionally, in particular, $\mathfrak{Int}(S) = \mathfrak{Int}_S(S, S)$ and $\mathfrak{Int}_S(T, T) = \mathfrak{Int}_S(T)$. So, $\mathfrak{Int}(S) \subseteq \mathfrak{Int}_S(H_1, H_2)$.

Example 3.18. *Let S be a quasi-ordered semigroup and $H_1, H_2 \subseteq S$. If $L \in \mathfrak{I}_S(H_1, \emptyset)$ and $R \in \mathfrak{I}_S(\emptyset, H_2)$, then $(LR]_{H_1 \cup H_2} \in \mathfrak{Int}_S(H_1, H_2)$.*

Example 3.19. *Let S be as in Example 3.3. Direct verification can prove that the set $\{0, 4\}$ is an interior $(\{0, 1\}, \{0, 1, 4\})$-ideal of S.*

The first, let us state that the concept of interior (H_1, H_2)-ideals is a generalization of the concept (H_1, H_2)-ideals, ie. the following inclusion $\mathfrak{I}_S(H_1, H_2) \subseteq \mathfrak{Int}_S(H_1, H_2)$ is valid:

Theorem 3.20. *Let S be a quasi-ordered semigroup and $H_1, H_2 \subseteq S$. Any (H_1, H_2)-ideal of S is an interior (H_1, H_2)-ideal of S.*

The reverse of the previous theorem is valid in some special cases.

Theorem 3.21. *Let S be a quasi-ordered semigroup and H_1, H_2 be subsets of S satisfying the condition* (A) *(assuming in that formula that $T = H_1$). Then any interior (H_1, H_2)-ideal of S is a left H_1-ideal of S.*

Theorem 3.22. *Let S be a quasi-ordered semigroup and H_1, H_2 be subsets of S satisfying the condition* (B) *(assuming in that formula that $T = H_2$). Then any interior (H_1, H_2)-ideal of S is a right H_2-ideal of S.*

Also, it can be proven:

Theorem 3.23. *Suppose that subsets H_1, H_2 of a quasi-ordered semigroup S are sub-semigroups of S and let S satisfy the following condition*

(C2) $(\forall x \in S)(\exists u \in H_1)(\exists v \in H_2)(x \preceq uxv)$.

Then any interior (H_1, H_2)-ideal of S is a (H_1, H_2)-ideal of S,

Proof. Let J be an interior (H_1, H_2)-ideal of a quasi-ordered semigroup. This means that J satisfies the conditions (10) and (12). Let us prove (9).

Let $u, x \in S$ be such that $u \in H_1$ and $x \in J$. Since S satisfies the condition (C2) we can find $h_1 \in H_1$ and $h_2 \in H_2$ such that $x \preceq h_1xh_2$. Then $ux \preceq (uh_1)xh_2$ by (4). On the other hand, we have $uh_1 \in H_1$ because of $H_1H_1 \subseteq H_1$ and $(uh_1)xh_2 \in J$ because J is an interior (H_1, H_2)-ideal of S. Hence $ux \in J$ by (10). The implication $(x \in J \wedge v \in H_2) \Longrightarrow xv \in J$ can be proved analogously to the previous demonstration. □

The following definition introduces the concept of (H_1, H_2)-intra-regular quasi-ordered semigroups:

Definition 3.24. *Let H_1, H_2 be subsets of a quasi-ordered semigroup S. S is said to be a (H_1, H_2)-intra-regular quasi-ordered semigroup if holds*

(D2) $(\forall x \in S)(\exists u \in H_1)(\exists v \in H_2)(x \preceq ux^2v)$.

Theorem 3.25. *Let H_1 H_2 be sub-semigroups of a (H_1, H_2)-intra-regular quasi-ordered semigroup S. If an interior (H_1, H_2)-ideal J of S is a sub-semigroup of S, then it is a (H_1, H_2)-ideal of S.*

Proof. Let S be a (H_1, H_2)-intra-regular quasi-ordered semigroup and J be an interior a (H_1, H_2)-ideal of S is a sub-semigroup of S. This means that J satisfies the conditions (10), (12) and $JJ \subseteq J$. Let us prove (9).

Let $u, x \in S$ such that $u \in H_1$ and $x \in J$. Then $x^2 \in J$ since J is a sub-semigroup of S. Since S is (H_1, H_2)-intra-regular, we can find the elements $h_1 \in H_1$ and $h_2 \in H_2$ such that $x \preceq h_1x^2h_2$ where $uh_1 \in H_1$. Thus $ux \preceq (uh_1)x^2h_2$ by (4). On the other hand, we have $(uh_1)x^2h_2 \in J$ because J is an interior (H_1, H_2)-ideal of S. Hence $ux \in J$ by (10). The conclusion $xv \in J$ for any $v \in H_2$ can be obtained analogously to the previous one. □

Theorem 3.26. *The family $\mathfrak{Int}_S(H_1, H_2)$ of all interior (H_1, H_2)-ideals of a quasi-ordered semigroup S forms a complete lattice.*

Proof. Let $\{J_i\}_{i \in I}$ be a family of interior (H_1, H_2)-ideals of a quasi-ordered semigroup S. Let us prove (11) and (7) for subsets $\bigcup_{i \in I} J_i$ and $\bigcap_{i \in I} J_i$:

$H_1(\bigcup_{i \in I} J_i)H_2 \subseteq \bigcup_{i \in I} H_1J_iH_2 \subseteq \bigcup_{i \in I} J_i$ and

$H_1(\bigcap_{i \in I} J_i)H_2 \subseteq \bigcap_{i \in I} H_1J_iH_2 \subseteq \bigcap_{i \in I} J_i$.

$(\bigcup_{i \in I} J_i]_{H_1 \cup H_2} \subseteq \bigcup_{i \in I}(J_i]_{H_1 \cup H_2} \subseteq \bigcup_{i \in I} J_i$ and

$(\bigcap_{i \in I} J_i]_{H_1 \cup H_2} \subseteq \bigcap_{i \in I}(J_i]_{H_1 \cup H_2} \subseteq \bigcap_{i \in I} J_i$. □

Let H_1, H_2 be subsets of a semigroup S. We shall say that an interior (H_1, H_2)-ideal A of S is a minimal interior (H_1, H_2)-ideal of S if there is no interior (H_1, H_2)-ideal B of S such that $B \subset A$ ([5]). Also, dually, we shall say that an interior (H_1, H_2)-ideal A is a maximal interior (H_1, H_2)-ideal of S if there is no interior (H_1, H_2)-ideal B of S such that $A \subset B$. It is easy to conclude that if A and B are two minimum interior (H_1, H_2)-ideals of a quasi-ordered semigroup S, then $A \cap B = \emptyset$, because, otherwise, according to the previous theorem, $A \cap B$ would be an interior (H_1, H_2)-ideal of S contained in A and contained in B, which is impossible.

Corollary 3.27. *For any subset X of S there exists the maximal interior (H_1, H_2)-ideal contained in X and the minimal interior (H_1, H_2)-ideal containing X.*

Proof. Let $\mathfrak{X}$ be the family of all interior (H_1, H_2)-ideals of S contained in X. Then $F_{max}(X) =: \cup\mathfrak{X}$ is the maximal interior (H_1, H_2)-ideal contained in X. If B is any interior (H_1, H_2)-ideal of S contained in X, then $B \in \mathfrak{X}$. So, $B \subseteq \cup\mathfrak{X}$. Therefore, $\cup\mathfrak{X}$ is the maximal interior (H_1, H_2)-ideal of S contained in X.

Let $\mathfrak{Y}$ be the family of all interior (H_1, H_2)-ideals of S containing X. Then $F^{min}(X) =: \cap\mathfrak{Y}$ is the minimal interior (H_1, H_2)-ideal of S containing X. If B is any interior (H_1, H_2)-ideal of S containing X, then $B \in \mathfrak{Y}$. So, $B \supseteq \cap\mathfrak{Y}$. Therefore, $\cap\mathfrak{Y}$ is the minimal interior (H_1, H_2)-ideal containing X. □

Corollary 3.28. *For any $a \in S$ there exists the maximal interior (H_1, H_2)-ideal J_a of S such that $a \notin J_a$ and the minimal interior (H_1, H_2)-ideal of S containing the element a.*

Proof. The proof of the first part of this Corollary can be obtained from the previous Corollary by taking $X = S \setminus \{a\}$. The proof of the second part of this Corollary can be obtained from the previous Corollary by taking $X = \{a\}$. □

Here, too, the analogous property of the minimality of the interior (H_1, H_2)-ideals in a quasi-ordered semigroup can be shown, as shown in the previous subsection.

Theorem 3.29. *Let H_1, H_2 be sub-semigroups of a quasi-ordered semigroup S, A be a minimal interior (H_1, H_2)-ideal of S, $L\,(\subseteq H_1)$ be a left H_1-ideal of S and $R\,(\subseteq H_2)$ be a right H_2-ideal of S. Then $A = LaR$ for any $a \in A$.*

Proof. First, we have that LaR is an interior (H_1, H_2)-ideal of S. Indeed, the following $H_1(LaR)H_2 = (H_1L)a(RH_2) \subseteq LaR$ is valid. On the other hand, we have $LaR \subseteq LAR \subseteq H_1AH_2 \subseteq A$ since A is an interior (H_1, H_2)-ideal of S. Therefore, $LaR = A$ because A is an minimal interior (H_1, H_2)-ideal of S. □

Corollary 3.30. *Let H_1, H_2 be sub-semigroups of a quasi-ordered semigroup S and A be a minimal interior (H_1, H_2)-ideal of S. Then $A = H_1aH_2$ for every $a \in A$.*

However, as shown in [5], Example 2.2, if L is a minimum interior H_1-ideal of S and R is a minimum interior H_2-ideal of S, then LaR need not be a minimal interior (H_1, H_2)-ideal of S.

In what follows we need the following term ([10]): The (H_1, H_2)-ideal J of a quasi-ordered semigroup S is said to be (H_1, H_2)-semiprime if holds

(13) $(\forall A \subseteq H_1 \cup H_2)(A^2 \subseteq J \Longrightarrow A \subseteq J)$.

We can now demonstrate the following theorem:

Theorem 3.31. *In an intra-regular quasi-ordered semigroup S any interior (H_1, H_2)-ideal in S is (H_1, H_2)-semiprime.*

Proof. Let J be an interior (H_1, H_2)-ideal of a quasi-ordered semigroup S and let A be any subset of $H_1 \cup H_2$ such that $A^2 \subseteq J$. Let us prove (13). For any $a \in A$ there are $u \in H_1$ and $v \in H_2$ such that $a \preceq ua^2v$. On the other hand, we have $ua^2v \in H_1A^2H_2 \subseteq H_1JH_2 \subseteq J$. Thus $a \in J$ by (10). Hence $A \subseteq J$. □

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