Filomat 38:14 (2024), 4955–4972
https://doi.org/10.2298/FIL2414955A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the characterization of the set of reflexive generalized inverses of finite potent endomorphisms

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Abstract. The aim of this work is to characterize the set of reflexive generalized inverses of a finite potent endomorphism. As an application we determine the structure of the set of reflexive generalized inverses of an square matrix with entries in an arbitrary field and we offer an algorithm for the explicit computation of these matrices. Several examples of the given characterization are also provided.

1. Introduction

If $A \in \mathrm{Mat}_{n\times m}(k)$ is a matrix with entries in an arbitrary field k, a matrix $A^- \in \mathrm{Mat}_{m\times n}(k)$ is $\{1\}$-inverse of A when $AA^-A = A$ and A^- is a $\{2\}$-inverse of A when $A^-AA^- = A^-$. Matrices that are simultaneously $\{1\}$-inverses and $\{2\}$-inverses are called "reflexive generalized inverses".

A classic example of a reflexive generalized inverse of a matrix A is the Moore-Penrose inverse $A^\dagger$ described by E. H. Moore in [2] and R. Penrose in [9].

Moreover, if we define the index of an square matrix A, that is denoted by $i(A)$, as the smallest integer such that $\mathrm{rk}(A^{i(A)}) = \mathrm{rk}(A^{i(A)+1})$, given $\tilde{A} \in \mathrm{Mat}_{n\times n}(k)$ with $i(\tilde{A}) \leq 1$, it is well-known that there exists the "group inverse" of $\tilde{A}$ as the unique matrix $\tilde{A}^\# \in \mathrm{Mat}_{n\times n}(k)$ satisfying that:

- $\tilde{A} \cdot \tilde{A}^\# \cdot \tilde{A} = \tilde{A}$;
- $\tilde{A}^\# \cdot \tilde{A} \cdot \tilde{A}^\# = \tilde{A}^\#$;
- $\tilde{A}^\# \cdot \tilde{A} = \tilde{A} \cdot \tilde{A}^\#$.

Accordingly, $\tilde{A}^\#$ is a reflexive generalized inverse of $\tilde{A}$. The set of reflexive generalized inverses of A is usually denoted by $A(1,2)$.

During the last years, the second-named author of this work has extended different notions of the theory of matrices to finite potent endomorphisms on arbitrary vector spaces. Thus, the existence and uniqueness of the Drazin inverse, the CMP inverse and the DMPs inverses of finite potent endomorphisms has been

2020 *Mathematics Subject Classification*. 2020 Mathematics Subject Classification: 15A09, 15A04, 65F05.
Keywords. Reflexive Generalized Inverse, Algorithm, Finite Potent Endomorphism, Square Matrices
Received: 09 May 2023; Accepted: 24 December 2023
Communicated by Dijana Mosić
The second-named author of this work is partially supported by the Spanish Government research project no. PGC2018-099599-B-I00 and the Regional Government of Castile and Leon research project no. J416/463AC03.
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proved in [7], [4] and [6] respectively. From these extensions, the main properties of these generalized inverses of matrices have been recovered and new relations between them have been given.

In this context, the authors of this work have recently characterized the set of {1}-inverses of a finite potent endomorphism and have offered an algorithm for the explicit computation of all {1}-inverses of an arbitrary square matrix in [8].

This paper deals with the answer to the following questions:

- which is the structure of the set of reflexive generalized inverses of a finite potent endomorphism?
- how many reflexive generalized inverses does an arbitrary square matrix with entries in an arbitrary field admit?

As far as we know the answer to the first question is not stated explicitly in the literature and for the second question, according to the statements of [10, Section III.1.1], it is known the structure of the set $A(1,2)$ when A is a matrix with entries in $\mathbb{R}$ or $\mathbb{C}$.

In this work, for a square matrix A with entries in an arbitrary field k, according to Theorem 5.1, we show that

$$A(1,2) \simeq [\prod_{i=1}^{\operatorname{rk} A_1} N(A)] \times [\prod_{i=1}^{\operatorname{rk} A_2} N(A)] \times [\prod_{i=1}^{\dim N(A)} k^{\operatorname{rk} A}] \simeq k^{2[\dim N(A)]\cdot(\operatorname{rk} A)},$$

where $A = A_1 + A_2$ is the core-nilpotent decomposition of A.

The paper is organized as follows. In Section 2 we recall the basic definitions of the theory of finite potent endomorphisms and a summary of statements of the articles [1], [3], [4] and [8].

Section 3 deals with the characterization of the set of reflexive generalized inverses of a finite potent endomorphism on an arbitrary vector space and Section 4 is devoted to study the set $A(1,2)$ of a square matrix with entries in an arbitrary field as a particular case of the statements proved for finite potent endomorphisms previously.

Finally, Section 5 contains the algorithm for computing the set of reflexive generalized inverses of a square matrix and several examples with the explicit application of this algorithm.

2. Preliminaries

2.1. Finite Potent Endomorphisms

Let k be an arbitrary field, let V be a k-vector space and let us consider an endomorphism φ of V. We say that φ is "finite potent" if $\varphi^n V$ is finite dimensional for some n, where the power means the composition $\varphi \circ \overset{n)}{\dots} \circ \varphi$. This definition was introduced by J. Tate in [11] as a basic tool for his elegant definition of Abstract Residues.

In 2007, M. Argerami, F. Szechtman and R. Tifenbach showed in [1] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|_{U_\varphi}}$ is nilpotent, W_φ is finite dimensional and $\varphi_{|_{W_\varphi}} \colon W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via φ, and the explicit definition of the above φ-invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } \varphi^m(v) = 0 \text{ for some } m \in \mathbb{N}\}$;
- $W_\varphi = \left\{\begin{array}{l} v \in V \text{ such that } p(\varphi)(v) = 0 \text{ for some } p(x) \in k[x] \\ \text{relatively prime to } x \end{array}\right\}$.

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \operatorname{Ker} \varphi^m$ and $W_\varphi = \operatorname{Ker} p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST-decomposition of V.

Moreover, we shall call "index of φ", $i(\varphi)$, to the nilpotent order of $\varphi_{|U_\varphi}$. One has that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism.

Basic examples of finite potent endomorphisms are all endomorphisms of a finite-dimensional vector space and finite rank or nilpotent endomorphisms of infinite-dimensional vector space.

2.2. CN Decomposition of a Finite Potent Endomorphism

Let V be again an arbitrary k-vector space. Given a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \operatorname{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

According to [4, Theorem 3.2], if φ^D is the Drazin inverse of φ, one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$ is the core part of φ. Also, φ_2 is named the nilpotent part of φ and one has that

$$\varphi = \varphi_1 \Longleftrightarrow U_\varphi = \operatorname{Ker} \varphi \Longleftrightarrow W_\varphi = \operatorname{Im} \varphi \Longleftrightarrow (\varphi^D)^D = \varphi \Longleftrightarrow i(\varphi) \leq 1 \,. \tag{1}$$

Moreover, if $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of V induced by φ, then φ_1 and φ_2 are the unique linear maps such that:

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ \varphi(v) & \text{if } v \in U_\varphi \end{cases} \,. \tag{2}$$

2.3. Jordan Bases of a nilpotent endomorphism

Let V be a vector space over an arbitrary field k and let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism. If n is the nilpotency index of g, according to the statements of [3], setting $U_i^g = \operatorname{Ker} g^i/[\operatorname{Ker} g^{i-1} + g(\operatorname{Ker} g^{i+1})]$ with $i \in \{1, 2, \dots, n\}$, $\mu_i(V, g) = \dim_k U_i^g$ and $S_{\mu_i(V,g)}$ a set such that $\# S_{\mu_i(V,g)} = \mu_i(V, g)$ with $S_{\mu_i(V,g)} \cap S_{\mu_j(V,g)} = \emptyset$ for all $i \neq j$, one has that there exists a family of vectors $\{v_{s_i}\}$ that determines a Jordan basis of g:

$$B = \bigcup_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq n}} \{v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\} \,. \tag{3}$$

Moreover, if we write $H_{s_i}^g = \langle v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i}) \rangle$, the basis B induces a decomposition

$$V = \bigoplus_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq n}} H_{s_i}^g \,. \tag{4}$$

2.4. Bases of a Finite Potent endomorphism

Let us now consider a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ with CN-decomposition $\varphi = \varphi_1 + \varphi_2$ and that induces the AST-decomposition $V = U_\varphi \oplus W_\varphi$. Keeping the above notation, if n is the nilpotency order of φ_2, we can construct a basis $B_V = B_{W_\varphi} \cup B_{U_\varphi}$ of V where

$$B_{W_\varphi} = \{w_1, \dots, w_r\}$$

is a basis of W_φ ($r = \dim_k W_\varphi$) and

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\}$$

is a Jordan basis of U_φ determined by $\varphi_{|U_\varphi}$.

If $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition of φ, it is clear that

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \{v_{s_i}, \varphi_2(v_{s_i}), \dots, \varphi_2^{i-1}(v_{s_i})\}$$

and

$$\operatorname{Ker} \varphi = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \langle \varphi^{i-1}(v_{s_i}) \rangle = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \langle \varphi_2^{i-1}(v_{s_i}) \rangle .$$

2.5. {1}-Inverses of a Finite Potent Endomorphism

Let V be a k-vector space and let $f \in \operatorname{End}_k(V)$ be an arbitrary endomorphism. Recall that $f^- \in \operatorname{End}_k(V)$ is a {1}-inverse of $f \in \operatorname{End}_k(V)$ when

$$f \circ f^- \circ f = f .$$

It is known from [8, Lemma 3.2] that $f^- \in \operatorname{End}_k(V)$ is a {1}-inverse of f if and only if for every $v \in V$ we have that

$$f^-(f(v)) = v + u$$

with $u \in \operatorname{Ker} f$.

With the above notation, according to [8, Proposition 3.3], one has that:

Proposition 2.1. *If $\varphi \in \operatorname{End}_k(V)$ is a finite potent endomorphism, then an endomorphism $\hat{\varphi} \in \operatorname{End}_k(V)$ is a {1}-inverse of φ if an only if $\hat{\varphi}$ satisfies that*

- $\hat{\varphi}(w_h) = (\varphi_{|W_\varphi})^{-1}(w_h) + u_h$ *for each* $h \in \{1, \dots, r\}$;
- $\hat{\varphi}(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$;
- $\hat{\varphi}(v_{s_i}) = \tilde{v}_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$;

where $\tilde{v}_{s_i} \in V$ *and* $u_h, u^j_{s_i} \in \operatorname{Ker} \varphi$ *for each* $h \in \{1, \dots, r\}$ *and for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$.

3. Reflexive Generalized Inverses of Finite Potent Endomorphisms

Let V be again an arbitrary k-vector space.

Definition 3.1. *Given an endomorphism $f \in \mathrm{End}_k(V)$, we say that $\hat{f} \in \mathrm{End}_k(V)$ is a "reflexive generalized inverse" of f when it satisfies that:*

- $\hat{f} \circ f \circ \hat{f} = \hat{f}$;
- $f \circ \hat{f} \circ f = f$.

It is clear that $\hat{f}$ is a reflexive generalized inverse of f when $\hat{f}$ is a $\{1\}$-inverse of f and f is a $\{1\}$-inverse of $\hat{f}$.

Our purpose is now to characterize the set of all reflexive generalized inverses of a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$. We shall denote this set by $X_\varphi(1,2)$.

With the above notation, let us fix a basis $B_V = B_{W_\varphi} \cup B_{U_\varphi}$ of V with

$$B_{W_\varphi} = \{w_1, \dots, w_r\}$$

and

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\}\,.$$

If $C = (c_{ij})$ is the matrix associated to $\varphi_{|_{W_\varphi}}$ in the basis B_{W_φ}, we have that

$$\varphi(w_j) = \sum_{i=1}^{r} c_{ij} w_i$$

for every $j \in \{1, \dots, r\}$.

If $\hat{\varphi} \in \mathrm{End}_k(V)$ is a reflexive generalized inverse of φ, since $\hat{\varphi}$ is a $\{1\}$-inverse of φ, then it follows from Proposition 2.1 that

$$\hat{\varphi}(w_h) = (\varphi_{|_{W_\varphi}})^{-1}(w_h) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \leq i' \leq n}} \lambda^h_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}})\,, \tag{5}$$

with $\lambda^h_{s_{i'}} \in k$ for each $s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)}$ and each $h \in \{1, \dots, r\}$ and where only a finite number of the scalars $\{\lambda^h_{s_{i'}}\}$ are different from zero.

Moreover, one has that

$$\hat{\varphi}(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \leq i' \leq n}} \beta^{s_i,j}_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}}) \tag{6}$$

with $\beta^{s_i,j}_{s_{i'}} = 0$ for almost all $s_{i'} \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$.

Finally, we have that $\hat{\varphi}(v_{s_i}) = \tilde{v}_{s_i}$ where $\tilde{v}_{s_i} \in V$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and for all $i \in \{1, \dots, n\}$.

The explicit characterization of the reflexive generalized inverses of a finite potent endomorphism is proved in the following:

Proposition 3.2. *Let V be an arbitrary k-vector space and let us consider a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$. With the previous notation, an endomorphism $\hat{\varphi} \in \operatorname{End}_k(V)$ is a reflexive generalized inverse if and only if it satisfies (5), (6) and*

$$\hat{\varphi}(v_{s_i}) = \sum_{j=1}^{r} \gamma_j^{s_i} \cdot w_j + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \leq i' \leq n}} [\sum_{l=0}^{i'-1} \xi_{s_{i'}}^{s_i,l} \varphi^l(v_{s_{i'}})] \tag{7}$$

with

$$\xi_{s_{i'}}^{s_i,i'-1} = \sum_{j,h=1}^{r} (\lambda_{s_{i'}}^h \cdot c_{hj} \cdot \gamma_j^{s_i}) + \sum_{\substack{s_{i''} \in S_{\mu_{i''}(U_\varphi,\varphi)} \\ 1 \leq i'' \leq n}} (\sum_{l=0}^{i''-2} [\xi_{s_{i''}}^{s_i,l} \cdot \beta_{s_{i'}}^{s_{i''},l+1}]), \tag{8}$$

with $\xi_{s_{i'}}^{s_i,l} = 0$ for almost all $s_{i'} \in S_{\mu_i(U_\varphi,\varphi)}$ and $l \in \{1, \dots, i'-1\}$.

Proof. Let $\hat{\varphi}$ be a reflexive generalized inverse of φ. Since $\hat{\varphi}$ is a $\{1\}$-inverse, one can easily check that the three conditions of this proposition hold.

Conversely, let us consider an endomorphism $\hat{\varphi} \in \operatorname{End}_k(V)$ satisfying (5) and (6) and the required condition for $\hat{\varphi}(v_{s_i})$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and for all $i \in \{1, \dots, n\}$. In this case, it follows from Proposition 2.1 that $\hat{\varphi}$ is a $\{1\}$-inverse of φ and a computation shows that

$$(\hat{\varphi} \circ \varphi \circ \hat{\varphi})(w_h) = \hat{\varphi}(w_h)$$

for each $h \in \{1, \dots, r\}$ and

$$(\hat{\varphi} \circ \varphi \circ \hat{\varphi})(\varphi^j(v_{s_i})) = \hat{\varphi}(\varphi^j(v_{s_i}))$$

for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$.

Hence, to prove the claim we only have to check that

$$(\hat{\varphi} \circ \varphi \circ \hat{\varphi})(v_{s_i}) = \hat{\varphi}(v_{s_i})$$

for each $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and for all $i \in \{1, \dots, n\}$.

Thus, since

$$\hat{\varphi}(\varphi^j(v_{s_{i''}})) = \varphi^{j-1}(v_{s_{i''}}) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \leq i' \leq n}} \beta_{s_{i'}}^{s_{i''},j} \cdot \varphi^{i'-1}(v_{s_{i'}}),$$

bearing in mind that

$$
\begin{aligned}
(\hat{\varphi}\circ\varphi\circ\hat{\varphi})(v_{s_i}) &= (\hat{\varphi}\circ\varphi)\Big(\sum_{j=1}^{r}\gamma_j^{s_i}\cdot w_j + \sum_{\substack{s_{i''}\,\in\, S_{\mu_{i''}(U_\varphi,\varphi)}\\ 1\leq i''\leq n}} [\sum_{l=0}^{i''-1}\xi_{s_{i''}}^{s_i,l}\varphi^l(v_{s_{i''}})]\Big)\\
&= \hat{\varphi}\Big(\sum_{j=1}^{r}\gamma_j^{s_i}\cdot\varphi(w_j) + \sum_{\substack{s_{i''}\,\in\, S_{\mu_{i''}(U_\varphi,\varphi)}\\ 1\leq i''\leq n}} [\sum_{l=0}^{i''-1}\xi_{s_{i''}}^{s_i,l}\varphi^l(v_{s_{i''}})]\Big)\\
&= \hat{\varphi}\Big(\sum_{j,h=1}^{r}[\gamma_j^{s_i}\cdot c_{hj}\cdot w_h] + \sum_{\substack{s_{i''}\,\in\, S_{\mu_{i''}(U_\varphi,\varphi)}\\ 1\leq i''\leq n}} [\sum_{l=0}^{i''-1}\xi_{s_{i''}}^{s_i,l}\varphi^l(v_{s_{i''}})]\Big)\\
&= \sum_{j=1}^{r}\gamma_j^{s_i}\cdot[\sum_{h=1}^{r}c_{hj}\cdot(\varphi_{|W_\varphi})^{-1}(w_h)] + \sum_{\substack{s_{i'}\,\in\, S_{\mu_{i'}(U_\varphi,\varphi)}\\ 1\leq i'\leq n}} [\sum_{j,h=1}^{r}(\gamma_j^{s_i}\cdot c_{hj}\cdot\lambda_{s_{i'}}^{h})]\cdot\varphi^{i'-1}(v_{s_{i'}})\\
&+ \sum_{\substack{s_{i''}\,\in\, S_{\mu_{i''}(U_\varphi,\varphi)}\\ 1\leq i''\leq n}} \Big([\sum_{l=0}^{i''-2}\xi_{s_{i''}}^{s_i,l}\varphi^l(v_{s_{i''}})] + [\sum_{l=0}^{i''-2}\xi_{s_{i''}}^{s_i,l}\cdot\Big[\sum_{\substack{s_{i'}\,\in\, S_{\mu_{i'}(U_\varphi,\varphi)}\\ 1\leq i'\leq n}}\beta_{s_{i'}}^{s_{i''},l+1}\cdot\varphi^{i'-1}(v_{s_{i'}})\Big]\Big)\\
&= \sum_{j=1}^{r}\gamma_j^{s_i}\cdot w_j + \sum_{\substack{s_{i'}\,\in\, S_{\mu_{i'}(U_\varphi,\varphi)}\\ 1\leq i'\leq n}} \Big([\sum_{l=0}^{i'-2}\xi_{s_{i'}}^{s_i,l}\varphi^l(v_{s_{i'}})]\\
&+ \sum_{\substack{s_{i'}\,\in\, S_{\mu_{i'}(U_\varphi,\varphi)}\\ 1\leq i'\leq n}} \Big(\sum_{j,h=1}^{r}(\lambda_{s_{i'}}^{h}\cdot c_{hj}\cdot\gamma_j^{s_i}) + \sum_{\substack{s_{i''}\,\in\, S_{\mu_{i''}(U_\varphi,\varphi)}\\ 1\leq i''\leq n}} (\sum_{l=0}^{i''-2}[\xi_{s_{i''}}^{s_i,l}\cdot\beta_{s_{i'}}^{s_{i''},l+1}])\Big)\cdot\varphi^{i'-1}(v_{s_{i'}})\,,
\end{aligned}
$$

the assertion is deduced. □

Remark 3.3. *We wish to point out that, in general, a reflexive generalized inverse of a finite potent endomorphism is not a finite potent operator.*

Let us now denote by $X_\varphi(1,2)$ the set of all reflexive generalized inverses of a finite potent endomorphism $\varphi\in\mathrm{End}_k(V)$.

Theorem 3.4 (Structure of $X_\varphi(1,2)$). *If $\varphi\in\mathrm{End}_k(V)$ is a finite potent endomorphism, then there exists a bijection*

$$
X_\varphi(1,2)\simeq[\prod_{h=1}^{r}\mathrm{Ker}\,\varphi]\times\Big[\prod_{\substack{s_i\,\in\, S_{\mu_i(U_\varphi,\varphi)}\\ 1\leq i\leq n}}[(V/\mathrm{Ker}\,\varphi)\times\prod_{j=1}^{i-1}\mathrm{Ker}\,\varphi]\Big]. \tag{9}
$$

Proof. If we denote:

- $u_h = \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \le i' \le n}} \lambda^h_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}}) \in \operatorname{Ker}\varphi;$

- $u^j_{s_i} = \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \le i' \le n}} \beta^{s_i,j}_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}}) \in \operatorname{Ker}\varphi;$

- $[\tilde{v}_{s_i}] = \Big[\sum_{j=1}^r \gamma^{s_i}_j \cdot w_j + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \le i' \le n}} \Big([\sum_{l=0}^{i'-2} \xi^{s_i,l}_{s_{i'}} \varphi^l(v_{s_{i'}})] \Big] \in V/\operatorname{Ker}\varphi,$

it follows from Proposition 3.2 that the map

$$X_\varphi(1,2) \longrightarrow [\prod_{h=1}^r \operatorname{Ker}\varphi] \times \Big[\prod_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le n}} [(V/\operatorname{Ker}\varphi) \times \prod_{j=1}^{i-1} \operatorname{Ker}\varphi] \Big]$$
$$\hat{\varphi} \longmapsto (u_h, ([\tilde{v}_{s_i}], u^j_{s_i})_{j \in \{1,\dots,i-1\}})_{h \in \{1,\dots,r\}; s_i \in S_{\mu_i(U_\varphi,\varphi)}}$$

is a bijection.

□

To conclude this section we shall compute the set $X_\varphi(1,2)$ for a certain finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$.

Example 3.5. *Let V be a vector space of countable dimension over an arbitrary ground field k. Let $\{v_1, v_2, v_3, \dots\}$ be a basis of V indexed by the natural numbers and let us consider the finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ defined as:*

$$\varphi(v_i) = \begin{cases} v_1 - v_2 + v_4 & \text{if} \quad i = 1 \\ -2v_1 - v_5 & \text{if} \quad i = 2 \\ v_2 + v_4 + v_5 & \text{if} \quad i = 3 \\ v_{i+1} & \text{if} \quad i = 2r \\ 0 & \text{if} \quad i = 2r+1 \end{cases}$$

for all $r \ge 2$. We shall characterize the set $X_\varphi(1,2)$.

A computation shows that the AST-decomposition of φ is $V = W_\varphi \oplus U_\varphi$ with

$$W_\varphi = \langle v_1 + v_2 - v_4 + 2v_5, -2v_1 + v_2 - v_4 - v_5 \rangle \text{ and } U_\varphi = \langle 2v_1 + v_2 + 2v_3, v_4, v_5, \dots \rangle ,$$

that the matrix associated with $\varphi_{|W_f}$ in this basis is

$$\varphi_{|W_\varphi} \equiv \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

and that $\operatorname{Ker}\varphi = \langle v_{2r+1} \rangle_{r \ge 2}$.

Let us write

$$\begin{aligned} v'_3 &= \lambda_1^3(v_1 + v_2 - v_4 + 2v_5) + \lambda_2^3(-2v_1 + v_2 - v_4 - v_5)+ \\ &+ \lambda_3^3(2v_1 + v_2 + 2v_3) + \lambda_4^3(4v_4 + v_5) + \sum_{s\geq 3} \lambda_{2s}^3 v_{2s} = \\ &= (\lambda_1^3 - 2\lambda_2^3 + 2\lambda_3^3)v_1 + (\lambda_1^3 + \lambda_2^3 + \lambda_3^3)v_2 + 2\lambda_3^3 v_3 + \\ &+ (-\lambda_1^3 - \lambda_2^3 + 4\lambda_4^3)v_4 + (2\lambda_1^3 - \lambda_2^3 + \lambda_4^3)v_5 + \sum_{s\geq 3} \lambda_{2s}^3 v_{2s} \end{aligned} \tag{10}$$

and

$$\begin{aligned} v'_{2r} &= (\lambda_1^{2r} - 2\lambda_2^{2r} + 2\lambda_3^{2r})v_1 + (\lambda_1^{2r} + \lambda_2^{2r} + \lambda_3^{2r})v_2 + 2\lambda_3^{2r} v_3 + \\ &+ (-\lambda_1^{2r} - \lambda_2^{2r} + 4\lambda_4^{2r})v_4 + (2\lambda_1^{2r} - \lambda_2^{2r} + \lambda_4^{2r})v_5 + \sum_{s\geq 3} \lambda_{2s}^{2r} v_{2s} \end{aligned} \tag{11}$$

for all $r \geq 3$ *and with* $\lambda_1^3, \lambda_2^3, \lambda_3^3, \lambda_4^3, \lambda_{2s}^3, \lambda_1^{2r}, \lambda_2^{2r}, \lambda_3^{2r}, \lambda_4^{2r}, \lambda_{2s}^{2r} \in k$ *for every* $s \geq 3$.

Given an endomorphism $\hat{\varphi} \in X_\varphi(1)$*, since* $\varphi(2v_1 + v_2 + 2v_3) = 4v_4 + v_5$*, if we consider the basis* $B_V = B_{W_\varphi} \cup B_{U_\varphi}$ *with*

$$B_{W_\varphi} = \{v_1 + v_2 - v_4 + 2v_5, -2v_1 + v_2 - v_4 - v_5\}$$

and

$$B_{U_\varphi} = \{2v_1 + v_2 + 2v_3, 4v_4 + v_5, 4v_5\} \cup \Big[\bigcup_{r\geq 3} \{v_{2r}, v_{2r+1}\}\Big],$$

it follows from Proposition 3.2 that:

- $\hat{\varphi}(v_1 + v_2 - v_4 + 2v_5) = -v_1 - v_2 + v_4 - 2v_5 + u_1$;
- $\hat{\varphi}(-2v_1 + v_2 - v_4 - v_5) = -v_1 + \frac{1}{2}v_2 - \frac{1}{2}v_4 - \frac{1}{2}v_5 + u_2$;
- $\hat{\varphi}(2v_1 + v_2 + 2v_3) = v'_3$;
- $\hat{\varphi}(4v_4 + v_5) = 2v_1 + v_2 + 2v_3 + u_4$;
- $\hat{\varphi}(4v_5) = 4v_4 + v_5 + u_5$;
- $\hat{\varphi}(v_{2r}) = v'_{2r}$;
- $\hat{\varphi}(v_{2r+1}) = v_{2r} + u_{2r+1}$;

with $u_1, u_2, u_4, u_5, u_{2r+1} \in \operatorname{Ker}\varphi$ *and* v'_3 *and* v'_{2r} *as above.*

Accordingly, from a non-difficult computation we obtain that the above equalities are equivalent to the following:

- $\hat{\varphi}(v_1) = -\frac{1}{2}v_2 - \frac{1}{2}v_4 - \frac{3}{4}v_5 + \frac{1}{3}u_1 - \frac{1}{3}u_2 - \frac{1}{4}u_5$;
- $\hat{\varphi}(v_2) = -\frac{1}{2}v_1 - \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{3}{4}v_4 - \frac{29}{16}v_5 - \frac{2}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{4}u_4 - \frac{5}{16}u_5$;
- $\hat{\varphi}(v_3) = \frac{1}{2}v_1 + \frac{5}{4}v_2 - \frac{1}{2}v_3 + \frac{5}{4}v_4 + \frac{53}{16}v_5 + \frac{1}{3}u_2 - \frac{1}{4}u_4 + \frac{13}{16}u_5 + v'_3$;
- $\hat{\varphi}(v_4) = \frac{1}{2}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{1}{4}v_4 - \frac{1}{16}v_5 + \frac{1}{4}u_4 - \frac{1}{16}u_5$;
- $\hat{\varphi}(v_5) = v_4 + \frac{1}{4}v_5 + \frac{1}{4}u_5$;
- $\hat{\varphi}(v_{2r}) = v'_{2r}$;

- $\hat{\varphi}(v_{2r+1}) = v_{2r} + u_{2r+1}$;

and, therefore, one has that an endomorphism $\hat{\varphi} \in \mathrm{End}_k(V)$ *is a reflexive generalized inverse of* φ *if and only if*

$$\hat{\varphi}(v_i) = \begin{cases} -\frac{1}{2}v_2 - \frac{1}{2}v_4 - \frac{3}{4}v_5 + \frac{1}{3}u_1 - \frac{1}{3}u_2 - \frac{1}{4}u_5 & \text{if} \quad i = 1 \\ -\frac{1}{2}v_1 - \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{3}{4}v_4 - \frac{29}{16}v_5 - \frac{2}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{4}u_4 - \frac{5}{16}u_5 & \text{if} \quad i = 2 \\ \frac{1}{2}v_1 + \frac{5}{4}v_2 - \frac{1}{2}v_3 + \frac{5}{4}v_4 + \frac{53}{16}v_5 + \frac{1}{3}u_2 - \frac{1}{4}u_4 + \frac{13}{16}u_5 + v'_3 & \text{if} \quad i = 3 \\ \frac{1}{2}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 - \frac{1}{4}v_4 - \frac{1}{16}v_5 + \frac{1}{4}u_4 - \frac{1}{16}u_5 & \text{if} \quad i = 4 \\ v_4 + \frac{1}{4}v_5 + \frac{1}{4}u_5 & \text{if} \quad i = 5 \\ v'_i & \text{if} \quad i = 2r \\ v_{i-1} + u_i & \text{if} \quad i = 2r+1 \end{cases}$$

with $u_1, u_2, u_4, u_5, u_{2r+1} \in \mathrm{Ker}\,\varphi$ *and* v'_3 *and* v'_{2r} *satisfying (10) and (11) for every* $r \geq 3$ *respectively.*

Remark 3.6. *In [5], the second-named author has recently introduced the notion of "core-nilpotent endomorphism" of a infinite dimensional vector space. We wish to point out that, using arguments similar to those made in this section, it is possible to study the set of reflexive generalized inverses of a core-nilpotent endomorphism of an arbitrary k-vector space.*

4. Reflexive Generalized Inverses of square matrices

Let E be a k-vector space of dimension n with k an arbitrary field.

Our purpose is now to characterize the set of all reflexive generalized inverses of an endomorphism $f \in \mathrm{End}_k(E)$.

With the above notation, let us fix a basis $B_E = B_{W_f} \cup B_{U_f}$ of E with

$$B_{W_f} = \{w_1, \dots, w_r\}$$

and

$$B_{U_f} = \bigcup_{1 \leq l \leq s} \{u_l, f(u_l), \dots, f^{n_l - 1}(u_l)\}.$$

If $C = (c_{ij})$ is the matrix associated to $f_{|W_f}$ in the basis B_{W_f}, we have that

$$f(w_j) = \sum_{i=1}^{r} c_{ij} w_i$$

for every $j \in \{1, \dots, r\}$.

Setting $m_h = \sum_{i=1}^{h} n_i$ for every $1 \leq h \leq s$, if $J = (d_{ij})$ is the matrix associated with f on the basis B_V one has that:

1. $d_{ij} = c_{ij}$ for all $1 \leq i, j \leq r$;
2. $d_{ij} = 0$ for each $i > r$ and $1 \leq j \leq r$;
3. $d_{ij} = 0$ for each $j = r + m_h$ with $h \in \{1, \dots, s\}$;
4. $d_{ij} = \delta_{i,j+1}$ for every $j \geq r + 1$ and $j \neq r + m_h$ with $h \in \{1, \dots, s\}$,

where $\delta_{i,j+1}$ is the Kronecker delta.

Note that $n = r + m_s = r + \sum_{i=1}^{s} n_i$.

Lemma 4.1. *With the previous notation, if $\Lambda = (\alpha_{ij}) \in Mat_{n\times n}(k)$, then one has that:*

$$\sum_{j=1}^{s}\Big[\sum_{z=0}^{n_j-2} \alpha_{(r+m_{j'})(r+m_{j-1}+z+2)} \cdot \alpha_{(r+m_{j-1}+z+1)(r+m_{l-1}+1)}\Big] = $$

$$= \sum_{i,h=r+1}^{n} \Big(\alpha_{(r+m_{j'})h} \cdot d_{hi} \cdot \alpha_{i(r+m_{l-1}+1)}\Big),$$

for every $j' \in \{1,\dots,s\}$.

Proof. Since, for every $i \in \mathbb{N}$ with $r+1 \leq i \leq n$ and $i \neq r+m_h$ for all $h \in \{1,\dots,s\}$ there exists a unique decomposition

$$i = r + m_{j-1} + z + 1\,,$$

where $j \in \{1,\dots,s\}$ and $z \in \{0,\dots,n_j-2\}$, the statement is immediately deduced from the expression of the matrix $J = (d_{ij})$. □

Corollary 4.2. *Keeping the notation, we have that:*

$$\sum_{j=1}^{s}\Big[\sum_{z=0}^{n_j-2} \alpha_{(r+m_{j'})(r+m_{j-1}+z+2)} \cdot \alpha_{(r+m_{j-1}+z+1)(r+m_{l-1}+1)}\Big] = $$

$$= \sum_{h=r+1}^{n} \Big(\sum_{i=1}^{n} \alpha_{(r+m_{j'})h} \cdot d_{hi} \cdot \alpha_{i(r+m_{l-1}+1)}\Big).$$

Proof. Bearing in mind that $d_{ij} = 0$ for each $j = r+m_h$ with $h \in \{1,\dots,s\}$, the claim is a direct consequence of Lemma 4.1. □

If $\hat{f} \in \mathrm{End}_k(E)$ is a reflexive generalized inverse of f, since $\hat{f}$ is a {1}-inverse of f, then it follows from Proposition 2.1 that:

$$\hat{f}(w_h) = (f_{|W_f})^{-1}(w_h) + \sum_{j=1}^{s} \alpha_{(r+m_j)h} \cdot f^{n_j-1}(u_j)\,, \tag{12}$$

with $\alpha_{(r+m_j)h} \in k$ for each $h \in \{1,\dots,r\}$. Moreover, denoting $m_0 = 0$, one has that:

$$\hat{f}(f^t(u_l)) = f^{t-1}(u_l) + \sum_{j=1}^{s} \alpha_{(r+m_j)(r+m_{l-1}+t+1)} \cdot f^{n_j-1}(u_j) \tag{13}$$

for $1 \leq t \leq n_l - 1$ and all $l \in \{1,\dots,s\}$.

Finally, we have that $\hat{f}(u_i) = \tilde{u}_i$ where $\tilde{u}_i \in E$ for all $i \in \{1,\dots,s\}$. The explicit characterization of the reflexive generalized inverses of an endomorphism on a finite-dimensional k-vector space is proved in the following:

Proposition 4.3. *Let E be an arbitrary finite dimensional k-vector space and let us consider an endomorphism $f \in \mathrm{End}_k(E)$. With the previous notation, one has that $\hat{f} \in \mathrm{End}_k(E)$ is a reflexive generalized inverse of f if and only if it satisfies (12), (13) and*

$$\hat{f}(u_l) = \sum_{i=1}^{r} \alpha_{i(r+m_{l-1}+1)} \cdot w_i + \sum_{j=1}^{s}[\sum_{z=0}^{n_j-1} \alpha_{(r+m_{j-1}+z+1)(r+m_{l-1}+1)} f^z(u_j)] \tag{14}$$

with

$$\alpha_{(r+m_j)(r+m_{l-1}+1)} = \sum_{h,i=1}^{n} \alpha_{(r+m_j)h} \cdot d_{hi} \cdot \alpha_{i(r+m_{l-1}+1)} \,, \tag{15}$$

where $\alpha_{(r+m_{j-1}+z+1)(r+m_{l-1}+1)} \in k$ *for every* $l \in \{1, \dots, s\}$ *and* $z \in \{0, \dots, n_j - 1\}$.

Proof. Relating the above notation with the notation of Section 3, if we write $v_{s_i} = u_l$, we have that the vector v_{s_i} takes the place $r + m_{l-1} + 1$ in the basis B_E for every $l \in \{1, \dots, s\}$. Moreover, the vector $f^t(v_{s_i})$ takes the place $r + m_{l-1} + t + 1$ in the basis B_E for every $l \in \{1, \dots, s\}$ and $t \in \{1, \dots, n_l - 1\}$.

Thus, writing $v_{s_{i'}} = v_j$, it makes sense to denote:

- $\lambda^h_{s_{i'}} = \alpha_{(r+m_j)h}$;
- $\gamma^{s_i}_i = \alpha_{i(r+m_{l-1}+1)}$;
- $\beta^{s_i,t}_{s_{i'}} = \alpha_{(r+m_j)(r+m_{l-1}+t+1)}$;
- $\xi^{s_i,z}_{s_{i'}} = \alpha_{(r+m_{j-1}+z+1)(r+m_{l-1}+1)}$.

Accordingly, bearing in mind that with this notation $i = n_l$ and $i' = n_j$ and writing $v_{s_{i''}} = u_{j'}$ with $j' \in \{1, \dots, s\}$, one has that:

- $\xi^{s_i,i'-1}_{s_{i'}} = \alpha_{(r+m_j)(r+m_{l-1}+1)}$;
- $\xi^{s_i,l}_{s_{i''}} = \alpha_{(r+m_{j'-1}+l+1)(r+m_{l-1}+1)}$;
- $\beta^{s_{i''},l+1}_{s_{i'}} = \alpha_{(r+m_j)(r+m_{j'-1}+l+2)}$;

and, therefore, applying Lemma 4.1 and Corollary 4.2 we obtain that:

$$\begin{aligned}
&\sum_{j'=1}^{s} [\sum_{l=0}^{n_{j'}-2} \alpha_{(r+m_j)(r+m_{j'-1}+l+2)} \cdot \alpha_{(r+m_{j'-1}+l+1)(r+m_{l-1}+1)}] = \\
&= \sum_{i,h=r+1}^{n} \left(\alpha_{(r+m_j)h} \cdot d_{hi} \cdot \alpha_{i(r+m_{l-1}+1)}\right) = \\
&= \sum_{h=r+1}^{n} \Big(\sum_{i=1}^{n} \alpha_{(r+m_j)h} \cdot d_{hi} \cdot \alpha_{i(r+m_{l-1}+1)}\Big),
\end{aligned}$$

for every $j \in \{1, \dots, s\}$.

Thus, replacing φ with f, the statement is deduced from Proposition 3.2 because (5) implies (12), (13) is deduced from (6), from (7) we obtain (14) and (15) is a particular case of (8). □

Corollary 4.4. *With the notation of Proposition 4.3, one has that:*

$$\alpha_{(r+m_j)(r+m_{l-1}+1)} = \sum_{\substack{h,i=1 \\ h \neq r+m_t+1 \\ t \in \{0,\dots,s-1\}}}^{n} \alpha_{(r+m_j)h} \cdot d_{hi} \cdot \alpha_{i(r+m_{l-1}+1)}$$

for all $j, l \in \{1, \dots, s\}$.

Proof. Bearing in mind that $d_{(r+m_t+1)i} = 0$ for every $i \in \{1, \dots, n\}$ and each $t \in \{0, \dots, s-1\}$, the statement is immediately deduced from Proposition 4.3. □

5. Algorithm for the computation of reflexive generalized inverses of a square matrix

This final section is devoted to the application of the previous results to characterize the set $A(1,2)$ of reflexive generalized inverses of a finite square matrix A with entries in an arbitrary field k and to offer an algorithm for the explicit computation of $A(1,2)$.

Theorem 5.1 (Structure of $A(1,2)$). *Let $A \in Mat_{m\times m}(k)$ be a square matrix with entries in an arbitrary field k and let $A = A_1 + A_2$ be its core-nilpotent decomposition. Then, the structure of $A(1,2)$ is determined from the following bijection:*

$$A(1,2) \simeq [\prod_{i=1}^{rk\, A_1} N(A)] \times [\prod_{i=1}^{rk\, A_2} N(A)] \times [\prod_{i=1}^{dim\, N(A)} k^{rk\, A}] \simeq k^{2[dim\, N(A)]\cdot(rk\, A)} . \tag{16}$$

Proof. Bearing in mind the well-known relationship between finite square matrices and endomorphisms of finite-dimensional vector spaces, the statement is immediately deduced from Proposition 4.3. □

Accordingly, an algorithm for computing the set $A(1,2)$ is the following:

1. Write A in its Jordan canonical form: $A = PJP^{-1}$.
2. If $C \in \text{Mat}_{r\times r}(k)$, let $\{m_1, m_2, \dots, m_s\}$ be the set of natural numbers such that $m_i \leq m_{i+1}$ and each $(r + m_i)$-column of J are zero.
3. If $J = \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}$ with C invertible and N nilpotent, compute the inverse C^{-1} .
4. Calculate the nullspace $N(J)$.
5. Put $J' = \begin{pmatrix} C^{-1} & 0 \\ 0 & N^t \end{pmatrix}$.
6. Add a general element of $N(J)$ in the non-zero columns of J' to get a matrix J''.
7. Obtain a matrix $\tilde{J}$ by completing the zero columns of J'' with arbitrary parameters except the i- rows of this columns with
$$i \in \{r + m_1, r + m_2, \dots, r + m_s\} .$$
Note that the zero columns of J'' are the j-columns with
$$j \in \{r + 1, r + m_1 + 1, \dots, r + m_{s-1} + 1\} .$$
8. Setting $C = (c_{ij})$ and $\tilde{J} = (\alpha'_{ij})$, and applying the expression (15), we write $\hat{J} = (\alpha_{ij})$ with
$$\alpha_{ij} = \begin{pmatrix} \alpha'_{i1} & \alpha'_{i2} & \dots & \alpha'_{in} \end{pmatrix} \cdot J \cdot \begin{pmatrix} \alpha'_{1j} \\ \alpha'_{2j} \\ \vdots \\ \alpha'_{nj} \end{pmatrix} ,$$
for each $i \in \{r + m_1, \dots, r + m_s\}$ and $j \in \{r + 1, r + m_1 + 1, \dots, r + m_{s-1} + 1\}$ and being $\alpha_{hs} = \alpha'_{hs}$ otherwise.
9. Compute $\hat{A} \in A(1,2)$ depending on $2[\dim N(A)] \cdot (\text{rk}\, A)$ parameters as $\hat{A} = P\hat{J}P^{-1}$.

Remark 5.2. *We wish to point out that (8) of this algorithm makes sense from Corollary 4.4.*

5.1. Illustrative Examples

To finish this work we shall offer several examples of the explicit application of the above algorithm for the calculation of the set of reflexive generalized inverses of a square matrix.

Example 5.3. *We shall now characterize the set of reflexive generalized inverses of the matrix*

$$A = \begin{pmatrix} 1+3i & 3+6i & 5+9i \\ -2-i & -6-2i & -10-3i \\ 1 & 3 & 5 \end{pmatrix} \in Mat_{3\times 3}(\mathbb{C})\,.$$

Since the Jordan canonical form of A is

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix},$$

for the sake of clarity, let us fix the notation exactly as it is stated in the algorithm previously presented. Therefore,

$$J = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice that in this case, $C = (i) \in Mat_{1\times 1}(\mathbb{C})$ *and we have* $\{m_1\}$ *with* $m_1 = 2$ *such that the* $(1+2) = (3)-$*column of* J *is zero. It is obvious that* $C^{-1} = (-i)$ *and that* $N(J) = \{(0,0,\lambda)\}_{\lambda\in\mathbb{C}}$. *Bearing this in mind, let us set*

$$J' = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Adding a general element of $N(J)$ *in the non-zero columns of* J', *we obtain*

$$J'' = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ \alpha'_{31} & 0 & \alpha'_{33} \end{pmatrix}.$$

Now, we shall complete the zero columns of J'' *with arbitrary parameters except for the* $i-$*rows of these colums with*

$$i \in \{r + m_1\} = \{1+2\} = \{3\}.$$

It is clear that the zero columns of J'' *are the* $j-$*columns with*

$$j \in \{r+1\} = \{2\}.$$

Hence,

$$\tilde{J} = \begin{pmatrix} -i & \alpha'_{12} & 0 \\ 0 & \alpha'_{22} & 1 \\ \alpha'_{31} & 0 & \alpha'_{33} \end{pmatrix}.$$

Let us denote $\hat{J} = (\alpha_{ij})$ *and* $\tilde{J} = (\alpha'_{ij})$. *The only entry of* $\hat{J}$ *left to compute is*

$$\alpha_{32} = \begin{pmatrix} \alpha'_{31} & 0 & \alpha'_{33} \end{pmatrix} \cdot \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{12} \\ \alpha'_{22} \\ 0 \end{pmatrix} = i\alpha'_{31} \cdot \alpha'_{12} + \alpha'_{33} \cdot \alpha'_{22}.$$

Therefore, we get

$$\hat{J} = \begin{pmatrix} -i & \alpha'_{12} & 0 \\ 0 & \alpha'_{22} & 1 \\ \alpha'_{31} & (i\alpha'_{31}\cdot\alpha'_{12} + \alpha'_{33}\cdot\alpha'_{22}) & \alpha'_{33} \end{pmatrix}.$$

To conclude, we get that $\hat{A}$ is a reflexive generalized inverse of A if and only if :

$$\hat{A} = \begin{pmatrix} 3 & -2 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} -i & \alpha'_{12} & 0 \\ 0 & \alpha'_{22} & 1 \\ \alpha'_{31} & (i\alpha'_{31}\cdot\alpha'_{12} + \alpha'_{33}\cdot\alpha'_{22}) & \alpha'_{33} \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\alpha'_{31}, \alpha'_{12}, \alpha'_{22}, \alpha'_{33} \in \mathbb{C}$. Realize that dim $N(A) = 1$ and $rk(A) = 2$ that gives us $A(1,2) \simeq \mathbb{C}^4$, which is coherent with the computations made in this example.

Example 5.4. *Let us now compute the set of reflexive generalized inverses of the matrix*

$$A = \begin{pmatrix} 16 & 33 & 50 & 0 \\ -7 & -16 & -25 & 0 \\ 1 & 3 & 5 & 0 \\ 6 & 23 & 40 & 0 \end{pmatrix} \in Mat_{4\times4}(\mathbb{R}).$$

Since the Jordan canonical form of A is

$$A = \begin{pmatrix} 3 & -2 & 1 & 0 \\ -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 11 & 1 \end{pmatrix}\cdot\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 5 & -3 & 1 \end{pmatrix},$$

similarly to that above, let us make clear the notation in every step of the algorithm. Hence,

$$J = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, we have that $C = (5) \in Mat_{1\times1}(\mathbb{R})$ and $\{m_1, m_2\}$ with $m_1 = 2$ and $m_2 = 3$, such that the $(1+2) = (3)$−column of J is zero and so the $(1+3) = (4)$−column of J is zero too. It is obvious that $C^{-1} = (\frac{1}{5})$ and that $N(J) = \{(0,0,\lambda,\gamma)\}_{\lambda,\gamma\in\mathbb{R}}$. Bearing this in mind, let us write

$$J' = \begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding a general element of $N(J)$ in the non-zero columns of J', we obtain

$$J'' = \begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha'_{31} & 0 & \alpha'_{33} & 0 \\ \alpha'_{41} & 0 & \alpha'_{43} & 0 \end{pmatrix}.$$

Now, we shall complete the zero columns of J'' with arbitrary parameters, except for the i−rows of these colums with

$$i \in \{r + m_1, r + m_2\} = \{3, 4\}.$$

Notice that the zero columns of J'' are the j−columns with

$$j \in \{r+1\} = \{2\}.$$

Hence,

$$\tilde{J} = \begin{pmatrix} \frac{1}{5} & \alpha'_{12} & 0 & \alpha'_{14} \\ 0 & \alpha'_{22} & 1 & \alpha'_{24} \\ \alpha'_{31} & 0 & \alpha'_{33} & 0 \\ \alpha'_{41} & 0 & \alpha'_{43} & 0 \end{pmatrix}.$$

Let us denote $\hat{J} = (\alpha_{ij})$ and, maintaining the notation, $\tilde{J} = (\alpha'_{ij})$. The entries of $\hat{J}$ left to compute are

$$\alpha_{32} = \begin{pmatrix} \alpha'_{31} & 0 & \alpha'_{33} & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{12} \\ \alpha'_{22} \\ 0 \\ 0 \end{pmatrix} = 5\alpha'_{31} \cdot \alpha'_{12} + \alpha'_{33} \cdot \alpha'_{22}.$$

$$\alpha_{34} = \begin{pmatrix} \alpha'_{31} & 0 & \alpha'_{33} & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{14} \\ \alpha'_{24} \\ 0 \\ 0 \end{pmatrix} = 5\alpha'_{31} \cdot \alpha'_{14} + \alpha'_{33} \cdot \alpha'_{24},$$

$$\alpha_{42} = \begin{pmatrix} \alpha'_{41} & 0 & \alpha'_{43} & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{12} \\ \alpha'_{22} \\ 0 \\ 0 \end{pmatrix} = 5\alpha'_{41} \cdot \alpha'_{12} + \alpha'_{43} \cdot \alpha'_{22},$$

$$\alpha_{44} = \begin{pmatrix} \alpha'_{41} & 0 & \alpha'_{43} & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{14} \\ \alpha'_{24} \\ 0 \\ 0 \end{pmatrix} = 5\alpha'_{41} \cdot \alpha'_{14} + \alpha'_{43} \cdot \alpha'_{24}.$$

Therefore,

$$\hat{J} = \begin{pmatrix} \frac{1}{5} & \alpha'_{12} & 0 & \alpha'_{14} \\ 0 & \alpha'_{22} & 1 & \alpha'_{24} \\ \alpha'_{31} & (5\alpha'_{31} \cdot \alpha'_{12} + \alpha'_{33} \cdot \alpha'_{22}) & \alpha'_{33} & (5\alpha'_{31} \cdot \alpha'_{14} + \alpha'_{33} \cdot \alpha'_{24}) \\ \alpha'_{41} & (5\alpha'_{41} \cdot \alpha'_{12} + \alpha'_{43} \cdot \alpha'_{22}) & \alpha'_{43} & (5\alpha'_{41} \cdot \alpha'_{14} + \alpha'_{43} \cdot \alpha'_{24}) \end{pmatrix}.$$

To conclude, $\hat{A}$ is a reflexive generalized inverse of A if and only if:

$$\hat{A} = \begin{pmatrix} 3 & -2 & 1 & 0 \\ -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 11 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{5} & \alpha'_{12} & 0 & \alpha'_{14} \\ 0 & \alpha'_{22} & 1 & \alpha'_{24} \\ \alpha'_{31} & \alpha_{32} & \alpha'_{33} & \alpha_{34} \\ \alpha'_{41} & \alpha_{42} & \alpha'_{43} & \alpha_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 5 & -3 & 1 \end{pmatrix},$$

with $\alpha'_{12}, \alpha'_{14}, \alpha'_{22}, \alpha'_{24}, \alpha'_{31}, \alpha'_{33}, \alpha'_{42}, \alpha'_{43} \in \mathbb{R}$ and $\alpha_{32}, \alpha_{34}, \alpha_{42}, \alpha_{44}$ being the ones previously calculated. Notice that in this case we have that dim $N(A) = 2$ and $rk(A) = 2$ and from Theorem 4.3 we know that $A(1,2) \simeq \mathbb{R}^8$. Readers can easily check that this example is compatible with this result.

Example 5.5. *Let $\mathbb{F}_7$ be the field with seven elements. We shall now characterize the set of reflexive generalized inverses of the matrix*

$$A = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 6 & 4 & 1 & 3 \\ 1 & 6 & 2 & 6 \\ 3 & 0 & 3 & 2 \end{pmatrix} \in Mat_{4\times 4}(\mathbb{F}_7).$$

Since the Jordan canonical form of A is

$$A = \begin{pmatrix} 1 & 1 & 6 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 6 & 2 & 4 \\ 0 & 0 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 & 6 & 4 \\ 3 & 1 & 4 & 5 \\ 3 & 0 & 3 & 2 \\ 3 & 0 & 3 & 3 \end{pmatrix},$$

similarly to that above, let us make clear the notation in every step of the algorithm. Hence,

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case,one has that $C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in Mat_{2\times 2}(\mathbb{F}_7)$ *and* $\{m_1\}$ *with* $m_1 = 2$*, such that the* $(2+2) = (4)-$*column of* J *is zero. Bearing in mind that* $\frac{1}{2} = 4$ *and* $\frac{1}{3} = 5$ *in* $\mathbb{F}_7$*, one has that* $C^{-1} = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$ *and that* $N(J) = \{(0,0,0,\lambda)\}_{\lambda\in\mathbb{F}_7}$*. Bearing this in mind, let us set*

$$J' = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding a general element of $N(J)$ *in the non-zero columns of* J'*, we obtain*

$$J'' = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha'_{41} & \alpha'_{42} & 0 & \alpha'_{44} \end{pmatrix}.$$

Now, we shall complete the zero columns of J'' *with arbitrary parameters, except for the* $i-$*rows of these colums with*

$$i \in \{r + m_1\} = \{4\}.$$

Notice that the zero columns of J'' *are the* $j-$*columns with*

$$j \in \{r+1\} = \{3\}.$$

Accordingly,

$$\tilde{J} = \begin{pmatrix} 4 & 0 & \alpha'_{13} & 0 \\ 0 & 5 & \alpha'_{23} & 0 \\ 0 & 0 & \alpha'_{33} & 1 \\ \alpha'_{41} & \alpha'_{42} & 0 & \alpha'_{44} \end{pmatrix}.$$

Let us denote $\hat{J} = (\alpha_{ij})$ *and* $\tilde{J} = (\alpha'_{ij})$*. The entry of* $\hat{J}$ *left to calculate is*

$$\alpha_{43} = \begin{pmatrix} \alpha'_{41} & \alpha'_{42} & 0 & \alpha'_{44} \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha'_{13} \\ \alpha'_{23} \\ \alpha'_{33} \\ 0 \end{pmatrix} = 2\alpha'_{41} \cdot \alpha'_{13} + 3\alpha'_{42} \cdot \alpha'_{23} + \alpha'_{44} \cdot \alpha'_{33}.$$

Therefore,

$$\hat{J} = \begin{pmatrix} 4 & 0 & \alpha'_{13} & 0 \\ 0 & 5 & \alpha'_{23} & 0 \\ 0 & 0 & \alpha'_{33} & 1 \\ \alpha'_{41} & \alpha'_{42} & (2\alpha'_{41} \cdot \alpha'_{13} + 3\alpha'_{42} \cdot \alpha'_{23} + \alpha'_{44} \cdot \alpha'_{33}) & \alpha'_{44} \end{pmatrix}.$$

So we conclude that $\hat{A}$ is a generalized inverse of A if and only if

$$\hat{A} = \begin{pmatrix} 1 & 1 & 6 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 6 & 2 & 4 \\ 0 & 0 & 6 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 & \alpha'_{13} & 0 \\ 0 & 5 & \alpha'_{23} & 0 \\ 0 & 0 & \alpha'_{33} & 1 \\ \alpha'_{41} & \alpha'_{42} & \alpha_{43} & \alpha'_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 & 6 & 4 \\ 3 & 1 & 4 & 5 \\ 3 & 0 & 3 & 2 \\ 3 & 0 & 3 & 3 \end{pmatrix},$$

with $\alpha'_{13}, \alpha'_{23}, \alpha'_{33}, \alpha'_{41}, \alpha'_{42}, \alpha'_{44} \in \mathbb{F}_7$ *and* α_{43} *being the one previously calculated. Notice that* $\dim N(A) = 1$ *and* $rk(A) = 3$ *so we have that* $A(1,2) \simeq (\mathbb{F}_7)^6$*, which is coherent with the computations made in this example.*

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