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g-Drazin inverse and group inverse for the anti-triangular block-operator matrices

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Abstract. We present the generalized Drazin inverse for certain anti-triangular operator matrices. Let $E, F, EF^{\pi} \in \mathcal{B}(X)^d$. If $EFEF^{\pi} = 0$ and $F^2EF^{\pi} = 0$, we prove that $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ has g-Drazin inverse and its explicit representation is established. Moreover, necessary and sufficient conditions are given for the existence of the group inverse of M under the condition $FEF^{\pi} = 0$. The group inverse for the anti-triangular block-operator matrices with two identical subblocks is thereby investigated. These extend the results of Zhang and Mosić (Filomat, 32(2018), 5907–5917) and Zou, Chen and Mosić (Studia Scient. Math. Hungar., 54(2017), 489–508).

1. Introduction

Let $\mathcal{B}(X)$ be a Banach algebra of all bounded linear operators over a Banach space X. An operator T in $\mathcal{B}(X)$ has generalized Drazin inverse (i.e., g-Drazin inverse) provided that there exists some $S \in \mathcal{B}(X)$ such that $ST = TS, S = STS, T - T^2S$ is quasinilpotent. Such S is unique, if it exists, and we denote it by T^d. If we replace the quasinilpotent by nilpotent in $\mathcal{B}(X)$, S is called the Drazin inverse of T, denoted by T^D. Equivalently, an operator T in $\mathcal{B}(X)$ has Drazin inverse T^D if and only if $TT^D = T^DT, T^D = T^DTT^D, T^n = T^{n+1}T^D$ for some $n \in \mathbb{N}$. Such smallest n is called the Drazin index of T and denote it by $ind(T)$.

Let $E, F \in \mathcal{B}(X)$ and I be the identity operator over the Banach space X. It is attractive to investigate the g-Drazin (Drazin) inverse of the block-operator matrix $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. It was firstly posed by Campbell that the solutions to singular systems of differential equations are determined by the Drazin inverse of the preceding operator matrix M (see [3]).

Since M is similar to the matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$, i.e., $M = \begin{pmatrix} 0 & I \\ I & -E \end{pmatrix}\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & -E \end{pmatrix}^{-1}$, it attracts many authors to investigate the g-Drazin (Drazin) inverse of M or $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$. In 2005, Castro-González and Dopazo

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gave the representations of the Drazin inverse for a class of operator matrix $\begin{pmatrix} I & I \\ F & 0 \end{pmatrix}$ (see [6, Theorem 3.3]). In 2011, Bu et al. investigate the Drazin inverse of the operator matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ under the condition $EF = FE$ (see [2, Theorem 3.2]). In 2016, Zhang investigated the g-Drazin inverse of M under the conditions $F^dEF^\pi = 0, F^\pi FE = 0$ and $F^\pi EF^d = 0, EFF^\pi = 0$ (see [17, Theorem 2.6, Theorem 2.8]). In [18, Theorem 2.3], Zhang and Mosić considered the g-Drazin inverse of M under the wider condition $FEF^\pi = 0$. We refer the reader to [19, 20, 22] for further recent progresses on the g-Drazin (Drazin) inverse of M or $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$.

The motivation of this paper is to further study the generalized inverse of the block-operator matrix M under the conditions that are wider than $FEF^\pi = 0$. In Section 2, we present the g-Drazin inverse for the operator matrix M under the condition $EFEF^\pi = 0$ and $F^2EF^\pi = 0$, which extend [18, Theorem 2.3] to a wider case. As the Drazin and g-Drazin inverses coincide with each other for a complex matrix, our results indeed provide algebraic method to find all function solutions of a new class of singular differential equations posed by Campbell (see [3]).

If T has Drazin index 1, T is said to have group inverse T^D, and denote its group inverse by $T^\#$. The group inverse of the block-operator matrices over a Banach space has interesting applications of resistance distances to the bipartiteness of graphs (see [16]). Many authors have studied such problems from many different views, e.g., [1, 4, 5, 7, 14, 15, 22].

In [23, Theorem 2.10], Zou et al. studied the group inverse for M under the condition $EF = 0$. In Section 3, we present necessary and sufficient condition on the existence of the group inverse and its representation for the block operator matrix M under the condition $FEF^\pi = 0$.

In [5, Theorem 5], Cao et al. considered the group inverse for a block matrix with identical subblocks over a right Ore domain. Finally, in the last section, we further investigate the necessary and sufficient condition for the existence of the group inverse of a 2×2 block-operator matrix $\begin{pmatrix} E & F \\ F & 0 \end{pmatrix}$ with identical subblocks. The explicit formula of its group inverse is thereby given under the same condition $FEF^\pi = 0$.

Throughout this paper, $\mathbb{C}^{n\times n}$ denotes the Banach algebra of all $n \times n$ complex matrices. Let $T \in \mathcal{B}(X)^d$. We use T^π to stand for the spectral idempotent operator $I - TT^d$. Let $p^2 = p, X \in \mathcal{B}(X)$. Then $X = pXp + pXp^\pi + p^\pi Xp + p^\pi Xp^\pi$, and we write X as a Pierce representation given by the matrix form $X = \begin{pmatrix} pXp & pXp^\pi \\ p^\pi Xp & p^\pi Xp^\pi \end{pmatrix}_p$.

2. g-Drazin inverse of anti-triangular block matrices

The aim of this section is to investigate the g-Drazin invertibility of the block-operator matrix $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. The representation of the g-Drazin inverse of M is given under some new kind of conditions. We begin with

Lemma 2.1. *([9, Theorem 5.1]) Let $p^2 = p \in \mathcal{B}(X)$ and $X = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}_p$, where $A, B \in \mathcal{B}(X)^d$ and $C \in \mathcal{B}(X)$. Then $X \in \mathcal{B}(X)^d$ and $X^d = \begin{pmatrix} A^d & 0 \\ Z & B^d \end{pmatrix}_p$, where $Z = \sum_{i=0}^{\infty}(B^d)^{i+2}CA^iA^\pi + \sum_{i=0}^{\infty}B^iB^\pi C(A^d)^{i+2} - B^dCA^d$.*

Lemma 2.2. *([11, Theorem 4.2.2]) Let $P, Q \in \mathcal{B}(X)^d$. If $PQP = 0$ and $Q^2P = 0$, then $P + Q \in \mathcal{B}(X)^d$. In this case,*

$$(P + Q)^d = \sum_{i=0}^{\infty}(P + Q)(P^d)^{i+2}Q^iQ^\pi + \sum_{i=0}^{\infty}P^{i+1}P^\pi(Q^d)^{i+2} + \sum_{i=0}^{\infty}QP^iP^\pi(Q^d)^{i+2} - (P + Q)P^dQ^d.$$

Using the previous lemmas, We now investigate the g-Drazin inverse of the block-operator matrix M.

Theorem 2.3. *Let $E,F\in\mathcal{B}(X)^d$. If $EFE=0$ and $F^2E=0$, then $M=\begin{pmatrix} E & I\\ F & 0\end{pmatrix}$ has g-Drazin inverse. In this case, $M^d=\begin{pmatrix}\Lambda & \Sigma\\ \Gamma & \Delta\end{pmatrix}$, where*

$$\begin{aligned}
\Lambda &= EE^{\pi}F^d-FE^dF^d+\sum_{i=0}^{\infty}[I+F(E^d)^2](E^d)^{2i+1}F^iF^{\pi}+\sum_{i=0}^{\infty}E^{2i+3}E^{\pi}(F^d)^{i+2}+\sum_{i=0}^{\infty}FE^{2i+1}E^{\pi}(F^d)^{i+2},\\
\Sigma &= -EE^dF^d-F(E^d)^2F^d+\sum_{i=0}^{\infty}[I+F(E^d)^2](E^d)^{2i+2}F^iF^{\pi}+\sum_{i=0}^{\infty}E^{2i+2}E^{\pi}(F^d)^{i+2}+\sum_{i=0}^{\infty}FE^{2i}E^{\pi}(F^d)^{i+2},\\
\Gamma &= FE^{\pi}F^d+\sum_{i=0}^{\infty}F(E^d)^{2i+2}F^iF^{\pi}+\sum_{i=0}^{\infty}FE^{2i+2}E^{\pi}(F^d)^{i+2},\\
\Delta &= -FE^dF^d+\sum_{i=0}^{\infty}F(E^d)^{2i+3}F^iF^{\pi}+\sum_{i=0}^{\infty}FE^{2i+1}E^{\pi}(F^d)^{i+2}.
\end{aligned}$$

Proof. Let

$$P=\begin{pmatrix} E^2 & E\\ 0 & 0\end{pmatrix}, Q=\begin{pmatrix} F & 0\\ FE & F\end{pmatrix}.$$

Then $M^2=\begin{pmatrix} E^2+F & E\\ FE & F\end{pmatrix}=P+Q$. Using Lemma 2.1, we have

$$Q^d=\begin{pmatrix} F^d & 0\\ FE(F^d)^2 & F^d\end{pmatrix}, Q^{\pi}=\begin{pmatrix} F^{\pi} & 0\\ -FEF^d & F^{\pi}\end{pmatrix}.$$

Likewise, we obtain $P^d=\begin{pmatrix} (E^d)^2 & (E^d)^3\\ 0 & 0\end{pmatrix}, P^{\pi}=\begin{pmatrix} E^{\pi} & -E^d\\ 0 & I\end{pmatrix}$. One easily checks that

$$\begin{aligned}
PQP &= \begin{pmatrix} E^2F & EF\\ 0 & 0\end{pmatrix}\begin{pmatrix} E^2 & E\\ 0 & 0\end{pmatrix}=\begin{pmatrix} E^2FE^2 & E^2FE\\ 0 & 0\end{pmatrix}=0;\\
Q^2P &= \begin{pmatrix} F & 0\\ FE & F\end{pmatrix}\begin{pmatrix} FE^2 & FE\\ 0 & 0\end{pmatrix}=\begin{pmatrix} F^2E^2 & F^2E\\ FEFE^2 & FEFE\end{pmatrix}=0.
\end{aligned}$$

In light of Lemma 2.2, M^2 has g-Drazin inverse, and so M has g-Drazin inverse. In this case,

$$M^d=M(M^2)^d=\sum_{i=0}^{\infty}M^3(P^d)^{i+2}Q^iQ^{\pi}+\sum_{i=0}^{\infty}MP^{i+1}P^{\pi}(Q^d)^{i+2}+\sum_{i=0}^{\infty}MQP^iP^{\pi}(Q^d)^{i+2}-M^3P^dQ^d.$$

We compute that

$$\begin{aligned}
& M^3(P^d)^{i+2}Q^iQ^{\pi}\\
=& \begin{pmatrix} E & I\\ F & 0\end{pmatrix}^3\begin{pmatrix} (E^d)^2 & (E^d)^3\\ 0 & 0\end{pmatrix}^{i+2}\begin{pmatrix} F & 0\\ FE & F\end{pmatrix}^i\begin{pmatrix} F^{\pi} & 0\\ -FEF^d & F^{\pi}\end{pmatrix}\\
=& \begin{pmatrix} E^3+EF+FE & E^2+F\\ FE^2+F^2 & FE\end{pmatrix}\begin{pmatrix} (E^d)^{2i+4} & (E^d)^{2i+5}\\ 0 & 0\end{pmatrix}\begin{pmatrix} F^i & 0\\ FEF^{i-1} & F^i\end{pmatrix}\begin{pmatrix} F^{\pi} & 0\\ -FEF^d & F^{\pi}\end{pmatrix}\\
=& \begin{pmatrix} (E^d)^{2i+1}+F(E^d)^{2i+3} & (E^d)^{2i+2}+F(E^d)^{2i+4}\\ F(E^d)^{2i+2} & F(E^d)^{2i+3}\end{pmatrix}\begin{pmatrix} F^iF^{\pi} & 0\\ FEF^{i-1}F^{\pi} & F^iF^{\pi}\end{pmatrix}\\
=& \begin{pmatrix} [I+F(E^d)^2](E^d)^{2i+1}F^iF^{\pi} & [I+F(E^d)^2](E^d)^{2i+2}F^iF^{\pi}\\ F(E^d)^{2i+2}F^iF^{\pi} & F(E^d)^{2i+3}F^iF^{\pi}\end{pmatrix}(i\geq 1),
\end{aligned}$$

$$\begin{aligned}
&M^3(P^d)^2Q^\pi\\
=&\begin{pmatrix} E & I\\ F & 0\end{pmatrix}^3\begin{pmatrix} (E^d)^2 & (E^d)^3\\ 0 & 0\end{pmatrix}^2\begin{pmatrix} F^\pi & 0\\ -FEF^d & F^\pi\end{pmatrix}\\
=&\begin{pmatrix} E^3+EF+FE & E^2+F\\ FE^2+F^2 & FE\end{pmatrix}\begin{pmatrix} (E^d)^4 & (E^d)^5\\ 0 & 0\end{pmatrix}\begin{pmatrix} F^\pi & 0\\ -FEF^d & F^\pi\end{pmatrix}\\
=&\begin{pmatrix} E^d+F(E^d)^3 & (E^d)^2+F(E^d)^4\\ F(E^d)^2 & F(E^d)^3\end{pmatrix}\begin{pmatrix} F^\pi & 0\\ -FEF^d & F^\pi\end{pmatrix}\\
=&\begin{pmatrix} [I+F(E^d)^2]E^dF^iF^\pi & [I+F(E^d)^2](E^d)^2F^\pi\\ F(E^d)^2F^\pi & F(E^d)^3F^\pi\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&MP^{i+1}P^\pi(Q^d)^{i+2}\\
=&\begin{pmatrix} E & I\\ F & 0\end{pmatrix}\begin{pmatrix} E^2 & E\\ 0 & 0\end{pmatrix}^{i+1}\begin{pmatrix} E^\pi & -E^d\\ 0 & I\end{pmatrix}\begin{pmatrix} F^d & 0\\ FE(F^d)^2 & F^d\end{pmatrix}^{i+2}\\
=&\begin{pmatrix} E^{2i+3}E^\pi & -E^{2i+3}E^d+E^{2i+2}\\ FE^{2i+2}E^\pi & -FE^{2i+2}E^d+FE^{2i+1}\end{pmatrix}\begin{pmatrix} (F^d)^{i+2} & 0\\ FE(F^d)^{i+3} & (F^d)^{i+2}\end{pmatrix}\\
=&\begin{pmatrix} E^{2i+3}E^\pi(F^d)^{i+2} & E^{2i+2}E^\pi(F^d)^{i+2}\\ FE^{2i+2}E^\pi(F^d)^{i+2} & FE^{2i+1}E^\pi(F^d)^{i+2}\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&MQP^\pi(Q^d)^2\\
=&\begin{pmatrix} E & I\\ F & 0\end{pmatrix}\begin{pmatrix} F & 0\\ FE & F\end{pmatrix}\begin{pmatrix} E^\pi & -E^d\\ 0 & I\end{pmatrix}\begin{pmatrix} (F^d)^2 & 0\\ FE(F^d)^3 & (F^d)^2\end{pmatrix}\\
=&\begin{pmatrix} EFE^\pi(F^d)^2+FEE^\pi(F^d)^2 & F^d-FEE^d(F^d)^2\\ FF^d & 0\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&MQP^iP^\pi(Q^d)^{i+2}\\
=&\begin{pmatrix} EF+FE & F\\ F^2 & 0\end{pmatrix}\begin{pmatrix} E^2 & E\\ 0 & 0\end{pmatrix}^i\begin{pmatrix} E^\pi & -E^d\\ 0 & I\end{pmatrix}\begin{pmatrix} F^d & 0\\ FE(F^d)^2 & F^d\end{pmatrix}^{i+2}\\
=&\begin{pmatrix} FE^{2i+1}E^\pi(F^d)^{i+2} & FE^{2i}E^\pi(F^d)^{i+2}\\ 0 & 0\end{pmatrix}(i\geq 1),
\end{aligned}$$

$$\begin{aligned}
&M^3P^dQ^d\\
=&\begin{pmatrix} E^3+EF+FE & E^2+F\\ FE^2+F^2 & FE\end{pmatrix}\begin{pmatrix} (E^d)^2F^d & (E^d)^3F^d\\ 0 & 0\end{pmatrix}\\
=&\begin{pmatrix} E^2E^dF^d+FE^dF^d & EE^dF^d+F(E^d)^2F^d\\ FEE^dF^d & FE^dF^d\end{pmatrix}.
\end{aligned}$$

Therefore $M^d=\begin{pmatrix}\Lambda & \Sigma\\ \Gamma & \Delta\end{pmatrix}$, where $\Lambda,\Sigma,\Gamma,\Delta$ are as in the preceding stating. This completes the proof. □

Lemma 2.4. *([10, Theorem 2.3]) Let $P,Q\in\mathcal{B}(X)^d$. If $PQ=0$, then $P+Q\in\mathcal{B}(X)^d$. In this case,*

$$(P+Q)^d=\sum_{i=0}^{\infty}Q^iQ^\pi(P^d)^{i+1}+\sum_{i=0}^{\infty}(Q^d)^{i+1}P^iP^\pi.$$

We obtain the main result in this section, which is an extension of [18, Theorem 2.3] for block-operator matrices.

Theorem 2.5. *Let $E,F,EF^\pi\in\mathcal{B}(X)^d$. If $EFEF^\pi=0$ and $F^2EF^\pi=0$, then $M=\begin{pmatrix} E & I\\ F & 0\end{pmatrix}$ has g-Drazin inverse. In this case,*

$$M^d=\begin{pmatrix}\varepsilon & [\zeta-(\alpha\zeta+\beta\theta)]\delta^d+\sum\limits_{i=1}^{\infty}[\zeta_i(1-\gamma\zeta)-\varepsilon_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1}\\ \eta & [\theta+(1-\gamma\zeta)]\delta^d+\sum\limits_{i=1}^{\infty}[\theta_i(1-\gamma\zeta)-\eta_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1}\end{pmatrix},$$

where

$$\begin{array}{rcl}\alpha &=& EF^{\pi}, \beta = F^{\pi}EFF^d + F^{\pi}, \gamma = FF^{\pi},\\ \delta^d &=& F^d + FF^d - FF^dEF^d;\end{array}$$

$$\begin{array}{rcl}\varepsilon &=& (\alpha\Lambda + \Gamma)\Lambda + (\alpha\Sigma + \Delta)\Gamma,\\ \zeta &=& (\alpha\Lambda + \Gamma)\Sigma\beta + (\alpha\Sigma + \Delta)\Delta\beta,\\ \eta &=& \gamma\Lambda^2 + \gamma\Sigma\Gamma,\\ \theta &=& \gamma\Lambda\Sigma\beta + \gamma\Sigma\Delta\beta;\end{array}$$

$$\begin{array}{rcl}\varepsilon_{n+1} &=& \alpha\varepsilon_n + \beta\eta_n, \varepsilon_1 = \varepsilon,\\ \zeta_{n+1} &=& \alpha\zeta_n + \beta\theta_n, \zeta_1 = \zeta,\\ \eta_{n+1} &=& \gamma\varepsilon_n, \eta_1 = \eta,\\ \theta_{n+1} &=& \gamma\theta_n, \theta_1 = \theta;\end{array}$$

$$\begin{array}{rcl}\Lambda &=& \sum\limits_{i=0}^{\infty}[I + F(F^{\pi}E^dF^{\pi})^2](F^{\pi}E^dF^{\pi})^{2i+1}F^i,\\ \Sigma &=& \sum\limits_{i=0}^{\infty}[I + F(F^{\pi}E^dF^{\pi})^2](F^{\pi}E^dF^{\pi})^{2i+2}F^i,\\ \Gamma &=& \sum\limits_{i=0}^{\infty} F(F^{\pi}E^dF^{\pi})^{2i+2}F^iF^{\pi},\\ \Delta &=& \sum\limits_{i=0}^{\infty} F(F^{\pi}E^dF^{\pi})^{2i+3}F^iF^{\pi}.\end{array}$$

Proof. Let $p = \begin{pmatrix} F^{\pi} & 0\\ 0 & 0\end{pmatrix}$. Then $M = \begin{pmatrix} \alpha & \beta\\ \gamma & \delta\end{pmatrix}_p$, where

$$\begin{array}{rcl}\alpha &=& \begin{pmatrix} EF^{\pi} & 0\\ 0 & 0\end{pmatrix}, \beta = \begin{pmatrix} F^{\pi}EFF^d & F^{\pi}\\ 0 & 0\end{pmatrix},\\ \gamma &=& \begin{pmatrix} 0 & 0\\ FF^{\pi} & 0\end{pmatrix}, \delta = \begin{pmatrix} FF^dE & FF^d\\ F^2F^d & 0\end{pmatrix}.\end{array}$$

Then $M = P + Q$, where $P = \begin{pmatrix} 0 & 0\\ 0 & \delta\end{pmatrix}, Q = \begin{pmatrix} \alpha & \beta\\ \gamma & 0\end{pmatrix}$. Since $F^2EF^{\pi} = 0$, we have $FF^dEF^{\pi} = (F^d)^2F^2EF^{\pi} = 0$. By hypothesis, E, EF^{π} have g-Drazin inverses. In view of [18, Lemma 2.2], FF^dE has g-Drazin inverse and $(FF^dE)^d = FF^dE^d$. By using [18, Lemma 2.2] again, EF^{π} has g-Drazin inverse and $(EF^{\pi})^d = E^dF^{\pi}$. Moreover, $FF^dE^dF^{\pi} = FF^d(EF^{\pi})^d = (F^d)^2F^2EF^{\pi}[(EF^{\pi})^d]^2 = 0$, and then $F^{\pi}E^dF^{\pi} = E^dF^{\pi}$. Hence, α has g-Drazin inverse and $\alpha^d = \begin{pmatrix} F^{\pi}E^dF^{\pi} & 0\\ 0 & 0\end{pmatrix}$, and then

$$\begin{array}{rcl}\alpha^{\pi} &=& p - \begin{pmatrix} EF^{\pi} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} F^{\pi}E^dF^{\pi} & 0\\ 0 & 0\end{pmatrix}\\ &=& \begin{pmatrix} F^{\pi} & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix} F^{\pi}EF^{\pi} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} F^{\pi}E^dF^{\pi} & 0\\ 0 & 0\end{pmatrix}\\ &=& \begin{pmatrix} F^{\pi} - F^{\pi}E(F^{\pi}E^dF^{\pi}) & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix} F^{\pi} - F^{\pi}EE^dF^{\pi} & 0\\ 0 & 0\end{pmatrix}\\ &=& \begin{pmatrix} F^{\pi}E^{\pi}F^{\pi} & 0\\ 0 & 0\end{pmatrix}.\end{array}$$

One easily checks that $\beta\gamma = \begin{pmatrix} FF^{\pi} & 0\\ 0 & 0\end{pmatrix}, (\beta\gamma)^d = 0$. Obviously, we have

$$\begin{array}{rcl}\delta^d &=& \begin{pmatrix} 0 & F^d\\ FF^d & -FF^dEF^d\end{pmatrix}, \delta^{\pi} = \begin{pmatrix} FF^d & 0\\ 0 & I\end{pmatrix} - \begin{pmatrix} FF^dE & FF^d\\ F^2F^d & 0\end{pmatrix}\begin{pmatrix} 0 & F^d\\ FF^d & -FF^dEF^d\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 0 & F^{\pi}\end{pmatrix};\\ P^d &=& \begin{pmatrix} 0 & 0\\ 0 & \delta^d\end{pmatrix}, P^{\pi} = \begin{pmatrix} p & 0\\ 0 & p^{\pi}\end{pmatrix} - \begin{pmatrix} 0 & 0\\ 0 & \delta\end{pmatrix}\begin{pmatrix} 0 & 0\\ 0 & \delta^d\end{pmatrix} = \begin{pmatrix} p & 0\\ 0 & \delta^{\pi}\end{pmatrix}, P^iP^{\pi} = 0\ (i \geq 1).\end{array}$$

We compute that $\alpha(\beta\gamma)\alpha = 0$ and $(\beta\gamma)^2\alpha = 0$. According to Theorem 2.3, $\begin{pmatrix} \alpha & 1 \\ \beta\gamma & 0 \end{pmatrix}$ has g-Drazin inverse and $\begin{pmatrix} \alpha & 1 \\ \beta\gamma & 0 \end{pmatrix}^d = \begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix}$, where

$$\begin{array}{rcl}
\Lambda & = & \alpha(\beta\gamma)^d + \beta\gamma\alpha\alpha^\pi[(\beta\gamma)^d]^2 - \alpha^2\alpha^d(\beta\gamma)^d - \beta\gamma\alpha^d(\beta\gamma)^d + \sum\limits_{i=0}^{\infty}[I + \beta\gamma(\alpha^d)^2](\alpha^d)^{2i+1}(\beta\gamma)^i(\beta\gamma)^\pi \\
& + & \sum\limits_{i=0}^{\infty}\alpha^{2i+3}\alpha^\pi[(\beta\gamma)^d]^{i+2} + \sum\limits_{i=1}^{\infty}\beta\gamma\alpha^{2i+1}\alpha^\pi[(\beta\gamma)^d]^{i+2}, \\
\Sigma & = & (\beta\gamma)^d - \beta\gamma\alpha\alpha^d[(\beta\gamma)^d]^2 - \alpha\alpha^d(\beta\gamma)^d - \beta\gamma(\alpha^d)^2(\beta\gamma)^d + \sum\limits_{i=0}^{\infty}[1 + \beta\gamma(\alpha^d)^2](\alpha^d)^{2i+2}(\beta\gamma)^i(\beta\gamma)^\pi \\
& + & \sum\limits_{i=0}^{\infty}\alpha^{2i+2}\alpha^\pi[(\beta\gamma)^d]^{i+2} + \sum\limits_{i=1}^{\infty}\beta\gamma\alpha^{2i}\alpha^\pi[(\beta\gamma)^d]^{i+2}, \\
\Gamma & = & \beta\gamma(\beta\gamma)^d - \beta\gamma\alpha\alpha^d(\beta\gamma)^d + \sum\limits_{i=0}^{\infty}\beta\gamma(\alpha^d)^{2i+2}(\beta\gamma)^i(\beta\gamma)^\pi + \sum\limits_{i=0}^{\infty}\beta\gamma\alpha^{2i+2}\alpha^\pi[(\beta\gamma)^d]^{i+2}, \\
\Delta & = & -\beta\gamma\alpha^d(\beta\gamma)^d + \sum\limits_{i=0}^{\infty}\beta\gamma(\alpha^d)^{2i+3}(\beta\gamma)^i(\beta\gamma)^\pi + \sum\limits_{i=0}^{\infty}\beta\gamma\alpha^{2i+1}\alpha^\pi[(\beta\gamma)^d]^{i+2}.
\end{array}$$

Thus, we derive

$$\begin{array}{rcl}
\Lambda & = & \sum\limits_{i=0}^{\infty}[1 + \beta\gamma(\alpha^d)^2](\alpha^d)^{2i+1}(\beta\gamma)^i, \\
\Sigma & = & \sum\limits_{i=0}^{\infty}[1 + \beta\gamma(\alpha^d)^2](\alpha^d)^{2i+2}(\beta\gamma)^i, \\
\Gamma & = & \sum\limits_{i=0}^{\infty}\beta\gamma(\alpha^d)^{2i+2}(\beta\gamma)^i, \\
\Delta & = & \sum\limits_{i=0}^{\infty}\beta\gamma(\alpha^d)^{2i+3}(\beta\gamma)^i.
\end{array}$$

We compute that

$$\begin{array}{rl}
& (1 + \beta\gamma(\alpha^d)^2)(\alpha^d)^{2i+1}(\beta\gamma)^i \\
= & \begin{pmatrix} I + F(F^\pi E^d F^\pi)^2 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} (F^\pi E^d F^\pi)^{2i+1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^iF^\pi & 0 \\ 0 & 0 \end{pmatrix} \\
= & \begin{pmatrix} [I + F(F^\pi E^d F^\pi)^2](F^\pi E^d F^\pi)^{2i+1}F^i & 0 \\ 0 & 0 \end{pmatrix},
\end{array}$$

$$\begin{array}{rl}
& (1 + \beta\gamma(\alpha^d)^2)(\alpha^d)^{2i+2}(\beta\gamma)^i \\
= & \begin{pmatrix} I + F(F^\pi E^d F^\pi)^2 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} (F^\pi E^d F^\pi)^{2i+2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^iF^\pi & 0 \\ 0 & 0 \end{pmatrix} \\
= & \begin{pmatrix} [I + F(F^\pi E^d F^\pi)^2](F^\pi E^d F^\pi)^{2i+2}F^i & 0 \\ 0 & 0 \end{pmatrix}, \\
& \beta\gamma(\alpha^d)^{2i+2}(\beta\gamma)^i \\
= & \begin{pmatrix} FF^\pi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (F^\pi E^d F^\pi)^{2i+2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^iF^\pi & 0 \\ 0 & 0 \end{pmatrix} \\
= & \begin{pmatrix} F(F^\pi E^d F^\pi)^{2i+2}F^iF^\pi & 0 \\ 0 & 0 \end{pmatrix}, \\
& \beta\gamma(\alpha^d)^{2i+3}(\beta\gamma)^i \\
= & \begin{pmatrix} FF^\pi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (F^\pi E^d F^\pi)^{2i+3} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^iF^\pi & 0 \\ 0 & 0 \end{pmatrix} \\
= & \begin{pmatrix} F(F^\pi E^d F^\pi)^{2i+3}F^iF^\pi & 0 \\ 0 & 0 \end{pmatrix}.
\end{array}$$

Then we have

$$\begin{array}{rcl} \Lambda &=& \sum\limits_{i=0}^{\infty}[I+F(F^{\pi}E^{d}F^{\pi})^{2}](F^{\pi}E^{d}F^{\pi})^{2i+1}F^{i},\\ \Sigma &=& \sum\limits_{i=0}^{\infty}[I+F(F^{\pi}E^{d}F^{\pi})^{2}](F^{\pi}E^{d}F^{\pi})^{2i+2}F^{i},\\ \Gamma &=& \sum\limits_{i=0}^{\infty}F(F^{\pi}E^{d}F^{\pi})^{2i+2}F^{i}F^{\pi},\\ \Delta &=& \sum\limits_{i=0}^{\infty}F(F^{\pi}E^{d}F^{\pi})^{2i+3}F^{i}F^{\pi}. \end{array}$$

We easily verify that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$
$$\begin{pmatrix} \alpha & 1 \\ \beta\gamma & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ \gamma & 0 \end{pmatrix}.$$

Therefore it follows by Cline's formula (see [12, Theorem 2.2]) that

$$\begin{array}{rcl} Q^{d} &=& \begin{pmatrix} \alpha & 1 \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} \Lambda & \Sigma \\ \Gamma & \Delta \end{pmatrix}^{2}\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}\\ &=& \begin{pmatrix} \alpha\Lambda+\Gamma & \alpha\Sigma+\Delta \\ \gamma\Lambda & \gamma\Sigma \end{pmatrix}\begin{pmatrix} \Lambda & \Sigma\beta \\ \Gamma & \Delta\beta \end{pmatrix}\\ &=& \begin{pmatrix} \varepsilon & \zeta \\ \eta & \theta \end{pmatrix}, \end{array}$$

where

$$\begin{array}{rcl} \varepsilon &=& (\alpha\Lambda+\Gamma)\Lambda+(\alpha\Sigma+\Delta)\Gamma,\\ \zeta &=& (\alpha\Lambda+\Gamma)\Sigma\beta+(\alpha\Sigma+\Delta)\Delta\beta,\\ \eta &=& \gamma\Lambda^{2}+\gamma\Sigma\Gamma,\\ \theta &=& \gamma\Lambda\Sigma\beta+\gamma\Sigma\Delta\beta. \end{array}$$

Moreover, we have

$$\begin{array}{rcl} Q^{\pi} &=& \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}-\begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}\begin{pmatrix} \varepsilon & \zeta \\ \eta & \theta \end{pmatrix}\\ &=& \begin{pmatrix} p-\alpha\varepsilon-\beta\eta & -\alpha\zeta-\beta\theta \\ -\gamma\varepsilon & 1-p-\gamma\zeta \end{pmatrix}. \end{array}$$

Write $Q^{n}=\begin{pmatrix} \varepsilon_{n} & \zeta_{n} \\ \eta_{n} & \theta_{n} \end{pmatrix}$. Then

$$\begin{array}{rcl} \varepsilon_{n+1} &=& \alpha\varepsilon_{n}+\beta\eta_{n},\\ \zeta_{n+1} &=& \alpha\zeta_{n}+\beta\theta_{n},\\ \eta_{n+1} &=& \gamma\varepsilon_{n},\\ \theta_{n+1} &=& \gamma\theta_{n}. \end{array}$$

$$\begin{array}{rl} & Q^{i}Q^{\pi}(P^{d})^{i+1}\\ =& \begin{pmatrix} \varepsilon_{i} & \zeta_{i} \\ \eta_{i} & \theta_{i} \end{pmatrix}\begin{pmatrix} p-\alpha\varepsilon-\beta\eta & -\alpha\zeta-\beta\theta \\ -\gamma\varepsilon & 1-p-\gamma\zeta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & (\delta^{d})^{i+1} \end{pmatrix}\\ =& \begin{pmatrix} \varepsilon_{i} & \zeta_{i} \\ \eta_{i} & \theta_{i} \end{pmatrix}\begin{pmatrix} 0 & -(\alpha\zeta+\beta\theta)(\delta^{d})^{i+1} \\ 0 & (1-p-\gamma\zeta)(\delta^{d})^{i+1} \end{pmatrix}\\ =& \begin{pmatrix} 0 & \zeta_{i}(1-p-\gamma\zeta)(\delta^{d})^{i+1}-\varepsilon_{i}(\alpha\zeta+\beta\theta)(\delta^{d})^{i+1} \\ 0 & \theta_{i}(1-p-\gamma\zeta)(\delta^{d})^{i+1}-\eta_{i}(\alpha\zeta+\beta\theta)(\delta^{d})^{i+1} \end{pmatrix}. \end{array}$$

Since $\delta\gamma = 0$, we have $PQ = 0$. In light of Lemma 2.4,

$$\begin{array}{rcl} M^d &=& \sum\limits_{i=0}^{\infty} Q^iQ^{\pi}(P^d)^{i+1} + \sum\limits_{i=0}^{\infty}(Q^d)^{i+1}P^iP^{\pi} \\ &=& Q^dP^{\pi} + Q^{\pi}P^d + \sum\limits_{i=1}^{\infty} Q^iQ^{\pi}(P^d)^{i+1} \\ &=& \begin{pmatrix} \varepsilon & \zeta\delta^d \\ \eta & \theta\delta^d \end{pmatrix} + \begin{pmatrix} 0 & -(\alpha\zeta+\beta\theta)\delta^d \\ 0 & (1-p-\gamma\zeta)\delta^d \end{pmatrix} \\ &+& \begin{pmatrix} 0 & \sum\limits_{i=1}^{\infty}[\zeta_i(1-p-\gamma\zeta)-\varepsilon_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1} \\ 0 & \sum\limits_{i=1}^{\infty}[\theta_i(1-p-\gamma\zeta)-\eta_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1} \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} \varepsilon & [\zeta-(\alpha\zeta+\beta\theta)]\delta^d + \sum\limits_{i=1}^{\infty}[\zeta_i(1-\gamma\zeta)-\varepsilon_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1} \\ \eta & [\theta+(1-\gamma\zeta)]\delta^d + \sum\limits_{i=1}^{\infty}[\theta_i(1-\gamma\zeta)-\eta_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1} \end{pmatrix}.$$

as required. □

Corollary 2.6. *Let $E, F, EF^{\pi} \in \mathcal{B}(X)^d$. If $EFEF^{\pi} = 0$ and $F^2EF^{\pi} = 0$, then $M = \begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ has g-Drazin inverse. In this case,*

$$M^d = \begin{pmatrix} E & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \varepsilon & [\zeta-(\alpha\zeta+\beta\theta)]\delta^d + \sum\limits_{i=1}^{\infty}[\zeta_i(1-\gamma\zeta)-\varepsilon_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1} \\ \eta & [\theta+(1-\gamma\zeta)]\delta^d + \sum\limits_{i=1}^{\infty}[\theta_i(1-\gamma\zeta)-\eta_i(\alpha\zeta+\beta\theta)](\delta^d)^{i+1} \end{pmatrix}^2 \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \varepsilon_n, \zeta_n, \eta_n$ and θ_n are given as in Theorem 2.5.

Proof. In view of Theorem 2.5, the block operator matrix $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ has g-Drazin inverse. We easily see that

$$\begin{pmatrix} E & I \\ F & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} E & I \\ I & 0 \end{pmatrix},$$

it follows by Cline's formula that $\begin{pmatrix} E & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix}$ has g-Drazin inverse. That is, $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ has g-Drazin inverse. Moreover, we have

$$\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}^d = \begin{pmatrix} E & I \\ I & 0 \end{pmatrix} [\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}^d]^2 \begin{pmatrix} I & 0 \\ 0 & F \end{pmatrix},$$

and so the proof is completed by Theorem 2.5. □

We note that the corresponding facts of the preceding lemmas in this section are valid for Drazin inverse. Construct $P, Q, \Lambda, \Sigma, \Gamma, \Delta$ as in Theorem 2.5. Then $M^D = Q^DP^{\pi} + Q^{\pi}P^D + \sum\limits_{i=1}^{\infty} Q^iQ^{\pi}(P^D)^{i+1}$. Explicitly, $Q = \begin{pmatrix} F^{\pi}E & F^{\pi} \\ FF^{\pi} & 0 \end{pmatrix}$. Since $F^{m+1}F^{\pi} = 0$, we see that $\Lambda, \Sigma, \Gamma, \Delta$ can be represented as the finite sums. Then by using the most of the technicalities that occur in the proof of Theorem 2.5, we have

Theorem 2.7. *Let $E, F, EF^{\pi} \in \mathcal{B}(X)^D$. If $EFEF^{\pi} = 0$ and $F^2EF^{\pi} = 0$, then $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ has Drazin inverse. In this case,*

$$M^D = \begin{pmatrix} \varepsilon & [\zeta - (\alpha\zeta + \beta\theta)]\delta^D + \sum_{i=1}^{k}[\zeta_i(1-\gamma\zeta) - \varepsilon_i(\alpha\zeta + \beta\theta)](\delta^D)^{i+1} \\ \eta & [\theta + (1-\gamma\zeta)]\delta^D + \sum_{i=1}^{k}[\theta_i(1-\gamma\zeta) - \eta_i(\alpha\zeta + \beta\theta)](\delta^D)^{i+1} \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= EF^{\pi}, \beta = F^{\pi}EFF^D + F^{\pi}, \gamma = FF^{\pi}, \\ \delta^D &= F^D + FF^D - FF^DEF^D; \end{aligned}$$

$$\begin{aligned} \varepsilon &= (\alpha\Lambda + \Gamma)\Lambda + (\alpha\Sigma + \Delta)\Gamma, \\ \zeta &= (\alpha\Lambda + \Gamma)\Sigma\beta + (\alpha\Sigma + \Delta)\Delta\beta, \\ \eta &= \gamma\Lambda^2 + \gamma\Sigma\Gamma, \\ \theta &= \gamma\Lambda\Sigma\beta + \gamma\Sigma\Delta\beta; \end{aligned}$$

$$\begin{aligned} \varepsilon_{n+1} &= \alpha\varepsilon_n + \beta\eta_n, \varepsilon_1 = \varepsilon, \\ \zeta_{n+1} &= \alpha\zeta_n + \beta\theta_n, \zeta_1 = \zeta, \\ \eta_{n+1} &= \gamma\varepsilon_n, \eta_1 = \eta, \\ \theta_{n+1} &= \gamma\theta_n, \theta_1 = \theta; \end{aligned}$$

$$\begin{aligned} \Lambda &= \sum_{i=0}^{m}[I + F(F^{\pi}E^dF^{\pi})^2](F^{\pi}E^dF^{\pi})^{2i+1}F^i, \\ \Sigma &= \sum_{i=0}^{m}[I + F(F^{\pi}E^dF^{\pi})^2](F^{\pi}E^dF^{\pi})^{2i+2}F^i, \\ \Gamma &= \sum_{i=0}^{m}F(F^{\pi}E^dF^{\pi})^{2i+2}F^iF^{\pi}, \\ \Delta &= \sum_{i=0}^{m}F(F^{\pi}E^dF^{\pi})^{2i+3}F^iF^{\pi}. \end{aligned}$$

where $k = ind\begin{pmatrix} F^{\pi}E & F^{\pi} \\ FF^{\pi} & 0 \end{pmatrix}, m = ind(F)$.

3. Group inverse of anti-triangular block matrices

The aim of this section is to provide necessary and sufficient conditions on E and F so that the block operator matrix $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ has group inverse. We now derive

Theorem 3.1. *Let $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ and E, F, EF^{π} have Drazin inverses. If $FEF^{\pi} = 0$, then the following are equivalent:*

(1) *M has group inverse.*
(2) *F has group inverse and $E^{\pi}F^{\pi} = 0$.*

In this case,

$$M^{\#} = \begin{pmatrix} E^DF^{\pi} & F^{\#} + (E^DF^{\pi})^2 - E^DF^{\pi}EF^{\#} \\ FF^{\#} & -FF^{\#}EF^{\#} \end{pmatrix}.$$

Proof. (1) ⇒ (2) Write $M^{\#} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Then $MM^{\#} = M^{\#}M$, and so

$$\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}.$$

Then we have

$$\begin{array}{c} EX_{11} + X_{21} = X_{11}E + X_{12}F, \\ FX_{12} = X_{21}. \end{array}$$

Since $MM^{\#}M = M$, we have

$$\begin{array}{c} EX_{11} + X_{21} = I, \\ FX_{11} = 0. \end{array}$$

Therefore

$$\begin{array}{rcl} F & = & (FE)X_{11} + FX_{21} \\ & = & (FE)F^{d}FX_{11} + FX_{21} \\ & = & (FEF^{d})(FX_{11}) + FX_{21} \\ & = & F^{2}X_{12}. \end{array}$$

In view of [23, Lemma 1.2], F has group inverse.

Let $e = \begin{pmatrix} FF^{\#} & 0 \\ 0 & I \end{pmatrix}$. Then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{e}$, where

$$\begin{array}{rcl} a & = & \begin{pmatrix} FF^{\#}E & FF^{\#} \\ F^{2}F^{\#} & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 \\ FF^{\pi} & 0 \end{pmatrix} = 0, \\ c & = & \begin{pmatrix} F^{\pi}EFF^{\#} & F^{\pi} \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} EF^{\pi} & 0 \\ 0 & 0 \end{pmatrix}. \end{array}$$

As in the proof of Theorem 2.5, we prove that $(EF^{\pi})^{D} = E^{D}F^{\pi}$. Then we compute that

$$a^{\#} = \begin{pmatrix} 0 & F^{\#} \\ FF^{\#} & -FF^{\#}EF^{\#} \end{pmatrix}, d^{D} = \begin{pmatrix} E^{D}F^{\pi} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore we have

$$a^{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & F^{\pi} \end{pmatrix}, d^{\pi} = \begin{pmatrix} E^{\pi}F^{\pi} & 0 \\ 0 & 0 \end{pmatrix}.$$

In view of [13, Theorem 2.1], we have $d^{\pi}ca^{\pi} = 0$, and so

$$\begin{pmatrix} E^{\pi}F^{\pi} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} F^{\pi}EFF^{D} & F^{\pi} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & F^{\pi} \end{pmatrix} = \begin{pmatrix} 0 & E^{\pi}F^{\pi} \\ 0 & 0 \end{pmatrix} = 0.$$

Therefore $E^{\pi}F^{\pi} = 0$, as required.

(2) ⇒ (1) Since EF^{π} has Drazin inverse and $FEF^{\pi} = 0$, EF^{π} has g-Drazin inverse and $FF^{\#}EF^{\pi} = F^{\#}(FEF^{\pi}) = 0$. It follows by [18, Lemma 2.2] that $FF^{\#}E$ has g-Drazin inverse and $(FF^{\#}E)^{d} = FF^{\#}E^{D}$. Then $FE^{D}F^{\pi} = F(FF^{\#}E^{D})F^{\pi} = F(FF^{\#}E)^{d}F^{\pi} = F[(FF^{\#}E)^{d}]^{2}(FF^{\#}E)F^{\pi} = F[(FF^{\#}E)^{d}]^{2}F^{\#}(FEF^{\pi}) = 0$.

Set

$$N = \begin{pmatrix} E^{D}F^{\pi} & F^{\#} + (E^{D}F^{\pi})^{2} - E^{D}F^{\pi}EF^{\#} \\ FF^{\#} & -FF^{\#}EF^{\#} \end{pmatrix}.$$

Then we directly check that

$$\begin{array}{rcl} MN & = & \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}\begin{pmatrix} E^{D}F^{\pi} & F^{\#} + (E^{D}F^{\pi})^{2} - E^{D}F^{\pi}EF^{\#} \\ FF^{\#} & -FF^{\#}EF^{\#} \end{pmatrix} \\ & = & \begin{pmatrix} EE^{D}F^{\pi} + FF^{\#} & FF^{\#}EF^{\#} + EE^{D}F^{\pi}E^{D}F^{\pi} \\ 0 & FF^{\#} \end{pmatrix} \\ & = & \begin{pmatrix} I & E^{D}F^{\pi} \\ 0 & FF^{\#} \end{pmatrix} \\ & = & \begin{pmatrix} E^{D}F^{\pi}EF^{\pi} + FF^{\#} & E^{D}F^{\pi} \\ FF^{\#}EF^{\pi} & FF^{\#} \end{pmatrix} \\ & = & \begin{pmatrix} E^{D}F^{\pi} & F^{\#} + (E^{D}F^{\pi})^{2} - E^{D}F^{\pi}EF^{\#} \\ FF^{\#} & -FF^{\#}EF^{\#} \end{pmatrix}\begin{pmatrix} E & I \\ F & 0 \end{pmatrix} \\ & = & NM, \end{array}$$

$$M(1-MN) = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}\begin{pmatrix} 0 & -E^DF^\pi \\ 0 & F^\pi \end{pmatrix} = 0,$$

$$\begin{aligned} & (1-MN)N \\ = & \begin{pmatrix} 0 & -E^DF^\pi \\ 0 & F^\pi \end{pmatrix}\begin{pmatrix} E^DF^\pi & F^\# + (E^DF^\pi)^2 - E^DF^\pi EF^\# \\ FF^\# & -FF^\# EF^\# \end{pmatrix} \\ = & 0. \end{aligned}$$

Therefore $M^\# = N$, as asserted. □

Corollary 3.2. *Let* $M = \begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ *and* E, F, EF^π *have Drazin inverses. If* $FEF^\pi = 0$*, then the following are equivalent:*

(1) *M has group inverse.*
(2) *F has group inverse and* $E^\pi F^\pi = 0$.

In this case,

$$M^\# = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned} \Gamma &= F^\pi E^D F^\pi, \\ \Delta &= I - F^\pi E^D F^\pi E, \\ \Lambda &= F^\# + (E^DF^\pi)^2 - E^DF^\pi EF^\#, \\ \Xi &= E^DF^\pi - F^\# E - (E^DF^\pi)^2 E + E^DF^\pi EF^\# E, \end{aligned}$$

Proof. Let $N = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. Then

$$M = P^{-1}NP, P = \begin{pmatrix} 0 & I \\ I & -E \end{pmatrix}.$$

Therefore M has group inverse if and only if so does N, if and only if F has group inverse and $E^\pi F^\pi = 0$, by Theorem 3.1. In this case,

$$\begin{aligned} M^\# &= P^{-1}N^\# P = \begin{pmatrix} E & I \\ I & 0 \end{pmatrix} N^\# \begin{pmatrix} 0 & I \\ I & -E \end{pmatrix} \\ &= \begin{pmatrix} I & F^\pi E^D F^\pi \\ E^DF^\pi & F^\# + (E^DF^\pi)^2 - E^DF^\pi EF^\# \end{pmatrix}\begin{pmatrix} 0 & I \\ I & -E \end{pmatrix} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Gamma &= F^\pi E^D F^\pi, \\ \Delta &= I - F^\pi E^D F^\pi E, \\ \Lambda &= F^\# + (E^DF^\pi)^2 - E^DF^\pi EF^\#, \\ \Xi &= E^DF^\pi - F^\# E - (E^DF^\pi)^2 E + E^DF^\pi EF^\# E, \end{aligned}$$

as asserted. □

Theorem 3.3. *Let* $M = \begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ *and* E, F, EF^π *have Drazin inverse. If* $F^\pi EF = 0$*, then the following are equivalent:*

(1) *M has group inverse.*
(2) *F has group inverse and* $F^\pi E^\pi = 0$.

In this case,

$$M^{\#} = \begin{pmatrix} F^{\pi}E^{D} & FF^{\#} \\ F^{\#} + (F^{\pi}E^{D})^{2} - F^{\#}EF^{\pi}E^{D} & -F^{\#}EFF^{\#} \end{pmatrix}.$$

Proof. We consider the transpose $M^T = \begin{pmatrix} E^T & I \\ F^T & 0 \end{pmatrix}$ of M. Then M has group inverse if and only if so does M^T. Applying Theorem 3.1, M has group inverse if and only if F^T has group inverse and $(E^T)^{\pi}(F^T)^{\pi} = 0$, i.e., F has group inverse and $F^{\pi}E^{\pi} = 0$. In this case, we have

$$M^{\#} = [(M^T)^{\#}]^T = \begin{pmatrix} (E^T)^D(F^T)^{\pi} & (F^T)^{\#} + ((E^T)^D(F^T)^{\pi})^2 - (E^T)^D(F^T)^{\pi}E^T(F^T)^{\#} \\ F^T(F^T)^{\#} & -F^T(F^T)^{\#}E^T(F^T)^{\#} \end{pmatrix}^T,$$

as desired. □

Corollary 3.4. *Let $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ and E, F have Drazin inverses. If $F^{\pi}EF = 0$, then the following are equivalent:*

(1) *M has group inverse.*
(2) *F has group inverse and $F^{\pi}E^{\pi} = 0$.*

In this case,

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned} \Gamma &= F^{\pi}E^{D}F^{\pi}, \\ \Delta &= F^{\#} + (F^{\pi}E^{D})^{2} - F^{\#}EF^{\pi}E^{D}, \\ \Lambda &= I - EF^{\pi}E^{D}F^{\pi}, \\ \Xi &= F^{\pi}E^{D} - EF^{\#} - E(F^{\pi}E^{D})^{2} + EF^{\#}EF^{\pi}E^{D}, \end{aligned}$$

Proof. Let $N = \begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$. Then

$$M = P^{-1}NP, P = \begin{pmatrix} E & I \\ I & 0 \end{pmatrix}.$$

In view of Theorem 3.3,

$$N^{\#} = \begin{pmatrix} F^{\pi}E^{D} & FF^{\#} \\ F^{\#} + (F^{\pi}E^{D})^{2} - F^{\#}EF^{\pi}E^{D} & -F^{\#}EFF^{\#} \end{pmatrix}.$$

Hence, M has group inverse if and only if so does N, if and only if F has group inverse and $F^{\pi}E^{\pi} = 0$, by Theorem 3.3. Moreover, we have

$$M^{\#} = P^{-1}N^{\#}P = \begin{pmatrix} 0 & I \\ I & -E \end{pmatrix} N^{\#} \begin{pmatrix} E & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned} \Gamma &= F^{\pi}E^{D}F^{\pi}, \\ \Delta &= F^{\#} + (F^{\pi}E^{D})^{2} - F^{\#}EF^{\pi}E^{D}, \\ \Lambda &= I - EF^{\pi}E^{D}F^{\pi}, \\ \Xi &= F^{\pi}E^{D} - EF^{\#} - E(F^{\pi}E^{D})^{2} + EF^{\#}EF^{\pi}E^{D}, \end{aligned}$$

as asserted. □

We come now to extend [23, Theorem 2.10] to wider cases as follows.

Corollary 3.5. *Let* $M = \begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ *and* E, F, EF^{π} *have Drazin inverses. If* $EF = \lambda FE (\lambda \in \mathbb{C})$ *or* $EF^2 = FEF$*, then the following are equivalent:*

(1) *M has group inverse.*
(2) *F have group inverse and* $F^{\pi}E^{\pi} = 0$.

In this case,

$$M^{\#} = \begin{pmatrix} F^{\pi}E^{D} & FF^{\#} \\ F^{\#} + (F^{\pi}E^{D})^2 - F^{\#}EF^{\pi}E^{D} & -F^{\#}EFF^{\#} \end{pmatrix}.$$

Proof. If $EF = \lambda FE (\lambda \in \mathbb{C})$, then $F^{\pi}EF = \lambda F^{\pi}FE = 0$. If $EF^2 = FEF$, then $F^{\pi}EF = F^{\pi}EF^2F^{\#} = F^{\pi}FEFF^{\#} = 0$. This completes the proof by Theorem 3.3. □

4. Block-operator matrices with identical subblocks

In [4], Cao et al. considered the group inverse for block matrices with identical subblocks over a right Ore domain. In this section we are concerned with the group inverse for block-operator matrices with identical subblocks over a Banach space.

Theorem 4.1. *Let* $M = \begin{pmatrix} E & F \\ F & 0 \end{pmatrix}$ *and* E, EF^{π} *have Drazin inverse and F has group inverse. If* $FEF^{\pi} = 0$*, then the following are equivalent:*

(1) *M has group inverse.*
(2) $EE^{\pi}F^{\pi} = 0$.

In this case,

$$M^{\#} = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned} \Gamma &= [I - E^{\pi}F^{\pi}][E^{D}F^{\pi} + E^{\pi}F^{\pi}E(F^{\#})^2] + E^{\pi}F^{\pi}E(F^{\#})^2, \\ \Delta &= [I - E^{\pi}F^{\pi}][F^{\#} - E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#} - E^{D}F^{\pi}EF^{\#}] \\ &- E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#}, \\ \Lambda &= F[E^{D}F^{\pi} + E^{\pi}F^{\pi}E(F^{\#})^2]^2 + F^{\#} - FE^{\pi}F^{\pi}[E(F^{\#})^2]^2 \\ &- FE^{D}F^{\pi}E(F^{\#})^2, \\ \Xi &= [FE^{D}F^{\pi} + FE^{\pi}F^{\pi}E(F^{\#})^2][F^{\#} - E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#} \\ &- E^{D}F^{\pi}EF^{\#}] - [F^{\#} - FE^{\pi}F^{\pi}E(F^{\#})^2E(F^{\#})^2 \\ &- FE^{D}F^{\pi}E(F^{\#})^2]EF^{\#}. \end{aligned}$$

Proof. (1) ⇒ (2) Obviously, we have

$$M = \begin{pmatrix} F^{\pi}E & F \\ F & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ F^{\#}E & I \end{pmatrix},$$
$$M^2 = \begin{pmatrix} EF^{\pi}E + F^2 & EF \\ 0 & F^2 \end{pmatrix}\begin{pmatrix} I & 0 \\ F^{\#}E & I \end{pmatrix}.$$

Write $M^{\#} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$. Then $M^{\#}M^2 = M$, and so

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\begin{pmatrix} EF^{\pi}E + F^2 & EF \\ 0 & F^2 \end{pmatrix} = \begin{pmatrix} F^{\pi}E & F \\ F & 0 \end{pmatrix}.$$

Therefore

$$X_{11}EF^{\pi}E + X_{11}F^2 = F^{\pi}E,$$

hence,

$$X_{11}EF^{\pi}EF^{\pi} + X_{11}F^2F^{\pi} = F^{\pi}EF^{\pi}.$$

It follows that

$$X_{11}(EF^{\pi})^2 = F^{\pi}EF^{\pi} = EF^{\pi}.$$

In light of [23, Lemma 1.2], EF^{π} has group inverse. By virtue of [18, Lemma 2.2], $(EF^{\pi})^{\#} = E^DF^{\pi}$. Therefore

$$\begin{array}{rcl} EF^{\pi} & = & (EF^{\pi})^{\#}(EF^{\pi})^2 = E^DF^{\pi}EF^{\pi}EF^{\pi} = E^D(I-F^DF)EF^{\pi}EF^{\pi} \\ & = & E^DEF^{\pi}EF^{\pi} = E^DE(I-F^DF)EF^{\pi} = E^DE^2F^{\pi} \\ & = & E(EE^D)F^{\pi} = E(I-E^{\pi})F^{\pi}; \end{array}$$

hence, $EE^{\pi}F^{\pi} = 0$.

(2) ⇒ (1) Since $EE^{\pi}F^{\pi} = 0$, we have $E^DF^{\pi}(EF^{\pi})^2 = E^2E^DF^{\pi} = E(1-E^{\pi})F^{\pi} = EF^{\pi}$, and so EF^{π} has group inverse by [23, Lemma 1.2].

Let $N = \begin{pmatrix} E & I \\ F^2 & 0 \end{pmatrix}$. Choose $e = \begin{pmatrix} FF^{\#} & 0 \\ 0 & I \end{pmatrix}$. Then

$$a = \begin{pmatrix} FF^{\#}E & FF^{\#} \\ F^2 & 0 \end{pmatrix}, c = \begin{pmatrix} F^{\pi}EFF^{\#} & F^{\pi} \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} EF^{\pi} & 0 \\ 0 & 0 \end{pmatrix}$$

and $b = 0$. Then

$$N = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}_e.$$

Moreover, we have

$$a^{\#} = \begin{pmatrix} 0 & (F^{\#})^2 \\ FF^{\#} & -FF^{\#}E(F^{\#})^2 \end{pmatrix}, d^{\#} = \begin{pmatrix} E^DF^{\pi} & 0 \\ 0 & 0 \end{pmatrix}.$$

We compute that

$$a^{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & F^{\pi} \end{pmatrix}, d^{\pi} = \begin{pmatrix} E^{\pi}F^{\pi} & 0 \\ 0 & I \end{pmatrix}.$$

Obviously, $a^{\pi}cd^{\pi} = 0$. In light of [13, Theorem 2.1], N has group inverse. Moreover, we have

$$N^{\#} = \begin{pmatrix} a^{\#} & 0 \\ z & d^{\#} \end{pmatrix},$$

where

$$z = d^{\pi}c(a^{\#})^2 + (d^{\#})^2ca^{\pi} - d^{\#}ca^{\#}.$$

Clearly,

$$\begin{array}{rcl} d^{\pi}c(a^{\#})^2 & = & \begin{pmatrix} E^{\pi}F^{\pi}E(F^{\#})^2 & -E^{\pi}F^{\pi}E(F^{\#})^2E(F^{\#})^2 \\ 0 & 0 \end{pmatrix}, \\ (d^{\#})^2ca^{\pi} & = & \begin{pmatrix} 0 & E^DF^{\pi}E^DF^{\pi} \\ 0 & 0 \end{pmatrix}, d^{\#}ca^{\#} = \begin{pmatrix} 0 & E^DF^{\pi}E(F^{\#})^2 \\ 0 & 0 \end{pmatrix}. \end{array}$$

Hence we compute that $z = (z_{ij})$, where

$$\begin{array}{rcl} z_{11} & = & E^{\pi}F^{\pi}E(F^{\#})^2, \\ z_{12} & = & E^DF^{\pi}E^DF^{\pi} - E^{\pi}F^{\pi}E(F^{\#})^2E(F^{\#})^2 - E^DF^{\pi}E(F^{\#})^2, \\ z_{21} & = & 0, z_{22} = 0. \end{array}$$

Therefore

$$N^{\#}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

where

$$\begin{array}{lll}\alpha & = & E^DF^{\pi}+E^{\pi}F^{\pi}E(F^{\#})^2,\\ \beta & = & (F^{\#})^2+E^DF^{\pi}E^DF^{\pi}-E^{\pi}F^{\pi}E(F^{\#})^2E(F^{\#})^2-E^DF^{\pi}E(F^{\#})^2,\\ \gamma & = & FF^{\#},\\ \delta & = & -FF^{\#}E(F^{\#})^2.\end{array}$$

Hence, we have

$$\begin{array}{rl} & NN^{\#}=N^{\#}N\\ = & \begin{pmatrix}E & I\\ F^2 & 0\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\\ = & \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}E & I\\ F^2 & 0\end{pmatrix}\\ = & \begin{pmatrix}EE^DF^{\pi}+FF^{\#} & \alpha\\ F^2\alpha & FF^{\#}\end{pmatrix}.\end{array}$$

Thus we have

$$N^{\pi}=\begin{pmatrix}E^{\pi}F^{\pi} & -\alpha\\ -F^2\alpha & F^{\pi}\end{pmatrix}.$$

Obviously, one checks that

$$M=\begin{pmatrix}E & I\\ F & 0\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & F\end{pmatrix},$$
$$N=\begin{pmatrix}I & 0\\ 0 & F\end{pmatrix}\begin{pmatrix}E & I\\ F & 0\end{pmatrix}.$$

By virtue of Cline's formula, M has Drazin inverse. We see that

$$\begin{array}{rl} & \begin{pmatrix}E & I\\ F & 0\end{pmatrix}N^{\pi}\begin{pmatrix}I & 0\\ 0 & F\end{pmatrix}\\ = & \begin{pmatrix}E & I\\ F & 0\end{pmatrix}\begin{pmatrix}E^{\pi}F^{\pi} & -\alpha\\ -F^2\alpha & F^{\pi}\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & F\end{pmatrix}\\ = & \begin{pmatrix}-F^2\alpha & -E\alpha F\\ FE^{\pi}F^{\pi} & -F\alpha F\end{pmatrix}.\end{array}$$

We directly compute that

$$\begin{array}{rcl}F^2\alpha & = & F^2E^DF^{\pi}+F^2E^{\pi}F^{\pi}E(F^{\#})^2=F^2EF^{\pi}(E^DF^{\pi})^2+F^2(EF^{\pi})(E^DF^{\pi})E(F^{\#})^2=0,\\ E\alpha F & = & EE^{\pi}F^{\pi}EF^{\#}=0,\\ FE^{\pi}F^{\pi} & = & FE^DEF^{\pi}=F(E^DF^{\pi})(EF^{\pi})=F(EF^{\pi})(E^DF^{\pi})=0,\\ F\alpha F & = & FE^{\pi}F^{\pi}EF^{\#}=F(E^DF^{\pi})(EF^{\pi})EF^{\#}=F(EF^{\pi})(E^DF^{\pi})EF^{\#}=0.\end{array}$$

Hence $M=MM^DM$, i.e., M has group inverse. Thus we have $M^{\#}=M^D$.

Moreover, we have

$$\begin{array}{rcl}M^D & = & \begin{pmatrix}E & I\\ F & 0\end{pmatrix}(N^{\#})^2\begin{pmatrix}I & 0\\ 0 & F\end{pmatrix}\\ & = & \begin{pmatrix}E & I\\ F & 0\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^2\begin{pmatrix}I & 0\\ 0 & F\end{pmatrix}\\ & = & \begin{pmatrix}\Gamma & \Delta\\ \Lambda & \Xi\end{pmatrix},\end{array}$$

where

$$\begin{array}{rcl} \Gamma &=& (E\alpha+\gamma)\alpha+(E\beta+\delta)\gamma, \\ \Delta &=& (E\alpha+\gamma)\beta F+(E\beta+\delta)\delta F, \\ \Lambda &=& F(\alpha^2+\beta\gamma), \\ \Xi &=& F(\alpha\beta+\beta\delta)F. \end{array}$$

Therefore we complete the proof by the direct computation. □

Corollary 4.2. *Let* $M=\begin{pmatrix} E & F \\ F & 0 \end{pmatrix}$ *and* E, EF^{π} *have Drazin inverse and* F *has group inverse. If* $F^{\pi}EF=0$*, then the following are equivalent:*

(1) *M has group inverse.*
(2) $F^{\pi}E^{\pi}E=0$.

In this case,

$$M^{\#}=\begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{array}{rcl} \Gamma &=& [F^{\pi}E^D+(F^{\#})^2EF^{\pi}E^{\pi}][I-F^{\pi}E^{\pi}]+(F^{\#})^2EF^{\pi}E^{\pi}, \\ \Delta &=& [F^{\#}-F^{\#}E(F^{\#})^2EF^{\pi}E^{\pi}-F^{\#}EF^{\pi}E^D][I-F^{\pi}E^{\pi}]-F^{\#}E(F^{\#})^2EF^{\pi}E^{\pi}, \\ \Lambda &=& [F^{\pi}E^D+(F^{\#})^2EF^{\pi}E^{\pi}]^2F+F^{\#}-[(F^{\#})^2E]^2F^{\pi}E^{\pi}F-(F^{\#})^2EF^{\pi}E^DF, \\ \Xi &=& [F^{\#}-F^{\#}E(F^{\#})^2EF^{\pi}E^{\pi}-F^{\#}EF^{\pi}E^D][F^{\pi}E^DF+(F^{\#})^2EF^{\pi}E^{\pi}F]-F^{\#}E[F^{\#} \\ &-& (F^{\#})^2E(F^{\#})^2EF^{\pi}E^{\pi}F-(F^{\#})^2EF^{\pi}E^DF]. \end{array}$$

Proof. By virtue of Cline's formula, $F^{\pi}E$ has Drazin inverse. Then the proof is complete by applying Theorem 4.1 to the transpose $M^T=\begin{pmatrix} E^T & F^T \\ F^T & 0 \end{pmatrix}$. □

Corollary 4.3. *Let* $M=\begin{pmatrix} E & F \\ F & 0 \end{pmatrix}$ *and* E, F *have group inverse,* EF^{π} *has Drazin inverse. If* $F^{\pi}EF=0$*, then* M *has group inverse. In this case,*

$$M^{\#}=\begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{array}{rcl} \Gamma &=& [I-E^{\pi}F^{\pi}][E^{\#}F^{\pi}+E^{\pi}F^{\pi}E(F^{\#})^2]+E^{\pi}F^{\pi}E(F^{\#})^2, \\ \Delta &=& [I-E^{\pi}F^{\pi}][F^{\#}-E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#}-E^{\#}F^{\pi}EF^{\#}]-E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#}, \\ \Lambda &=& F[E^{\#}F^{\pi}+E^{\pi}F^{\pi}E(F^{\#})^2]^2+F^{\#}-FE^{\pi}F^{\pi}[E(F^{\#})^2]^2-FE^{\#}F^{\pi}E(F^{\#})^2, \\ \Xi &=& [FE^{\#}F^{\pi}+FE^{\pi}F^{\pi}E(F^{\#})^2][F^{\#}-E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#}-E^{\#}F^{\pi}EF^{\#}]-[F^{\#}-FE^{\pi}F^{\pi}E(F^{\#})^2E(F^{\#})^2 \\ &-& FE^{\#}F^{\pi}E(F^{\#})^2]EF^{\#}. \end{array}$$

Proof. Since E has group inverse, we see that $EE^{\pi}=0$, and so $EE^{\pi}F^{\pi}=0$. In light of Theorem 4.1, M has group inverse. Therefore we obtain the representation of $M^{\#}$ by the formula in Theorem 4.1. □

As an immediate consequence of Corollary 4.3, we have

Corollary 4.4. *Let* $M=\begin{pmatrix} E & F \\ F & 0 \end{pmatrix}$ *and* E, F *have group inverse,* EF^{π} *has Drazin inverse. If* $EF=\lambda FE$ $(\lambda\in\mathbb{C})$ *or* $EF^2=FEF$*, then* M *has group inverse. In this case,*

$$M^{\#}=\begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{array}{rcl}\Gamma & = & [I - E^{\pi}F^{\pi}][E^{\#}F^{\pi} + E^{\pi}F^{\pi}E(F^{\#})^2] + E^{\pi}F^{\pi}E(F^{\#})^2,\\ \Delta & = & [I - E^{\pi}F^{\pi}][F^{\#} - E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#} - E^{\#}F^{\pi}EF^{\#}] - E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#},\\ \Lambda & = & F[E^{\#}F^{\pi} + E^{\pi}F^{\pi}E(F^{\#})^2]^2 + F^{\#} - FE^{\pi}F^{\pi}[E(F^{\#})^2]^2 - FE^{\#}F^{\pi}E(F^{\#})^2,\\ \Xi & = & [FE^{\#}F^{\pi} + FE^{\pi}F^{\pi}E(F^{\#})^2][F^{\#} - E^{\pi}F^{\pi}E(F^{\#})^2EF^{\#} - E^{\#}F^{\pi}EF^{\#}] - [F^{\#} - FE^{\pi}F^{\pi}E(F^{\#})^2E(F^{\#})^2\\ & - & FE^{\#}F^{\pi}E(F^{\#})^2]EF^{\#}.\end{array}$$

Proof. As in proof of Corollary 3.5, we obtain the result by Corollary 4.3. □

We illustrate Theorem 4.1 by a numerical example.

Example 4.5. *Let* $M = \begin{pmatrix} E & F \\ F & 0 \end{pmatrix} \in \mathbb{C}^{6\times 6}$, *where* $E = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} i & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^{3\times 3}$, $i^2 = -1$. *Then*

$$M^{\#} = \begin{pmatrix} 0 & 1 & 0 & -i & -i & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -i & -i & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Obviously, we have

$$E^{\#} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E^{\pi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; F^{\#} = \begin{pmatrix} -i & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F^{\pi} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we check that $FEF^{\pi} = 0, EE^{\pi}F^{\pi} = 0$. Construct Γ, Δ, Λ and Ξ as in Theorem 4.1. Then we compute that

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Delta = \begin{pmatrix} -i & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} -i & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Xi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This completes the proof by Theorem 4.1. □

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