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L_p geominimal Gaussian surface area

Shuang Mou[a]

[a]*School of Mathematics and Statistics, Shaanxi Normal University Xi'an, 710119, China*

Abstract. In this paper, we study the L_p geominimal Gaussian surface area for $p \geq 1$. We prove some properties of the L_p geominimal Gaussian surface area, such as continuity and Santaló style inequality. Moreover, we obtain that the opposite question of continuity of the L_p geominimal Gaussian surface area is also continuous.

1. Introduction

In this paper, we will work in the n-dimensional Euclidean space, $\mathbb{R}^n$, and write $x = (x_1, \cdots, x_n)$ for $x \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we write $x \cdot y = x_1y_1 + \cdots + x_ny_n$ for the standard inner product of x and y, let $|x| = \sqrt{x \cdot x}$ for the Euclidean norm of x. A set $K \subset \mathbb{R}^n$ is convex if $\lambda x + (1 - \lambda)y \in K$ for all $x, y \in K$ and each $\lambda \in [0, 1]$. A convex subset $K \subset \mathbb{R}^n$ is a convex body if K is compact and has nonempty interior. Let $\mathcal{K}^n$, $\mathcal{K}_0^n$, $\mathcal{K}_c^n$ and $\mathcal{K}_s^n$ denote the class of all convex bodies in $\mathbb{R}^n$, the class of all convex bodies containing the origin o in their interiors, the class of all convex bodies with their centroid at origin o, and the set of all convex bodies with their Santaló point at origin o. Let $|K|$ denote the n-dimensional volume of a convex body K. For $K \in \mathcal{K}_0^n$, the geominimal surface area $G(K)$ of K was firstly introduced by Petty [18] more than five decades ago, i.e.,

$$G(K) = \inf\left\{\int_{S^{n-1}} h(Q, u)dS(K, u) : Q \in \mathcal{K}_s^n \text{ and } |Q^*| = \omega_n\right\}, \tag{1}$$

where Q^* is the polar body of Q, $S(K, \cdot)$ is the surface area of the convex body K and $h(Q, \cdot)$ is the support function (see Section 2). In [18], Petty proved the existence and uniqueness to the solution of the optimal problem (1). As Petty stated the geominimal surface area serves as a bridge connecting affine differential geometry, relative differential geometry and Minkowski geometry. This implies that the geominimal surface area is one of the basic concepts in Brunn-Minkowski theory.

With the development of L_p Brunn-Minkowski theory and motivated by the L_p mixed volume, the classical geominimal surface area has been extended to L_p case by Lutwak [13] (for $p \geq 1$). Using the L_p affine surface area integral formula, Ye [20] introduced the L_p geominimal surface area for all $-n \neq p < 1$. Moreover, Zhu, Zhou and Xu [25] extended the L_p geominimal surface area to the L_p mixed geominimal

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Email address: shuangmou92@163.com (Shuang Mou)

surface area and obtained Blaschke-Santaló inequality for the L_p mixed geominimal surface area. Ye, Zhu and Zhou [22] also obtained affine isoperimetric inequalities for the L_p mixed geominimal surface area with respect to multiple convex bodies. Recently, Li, Wang and Zhou [6] introduced (p, q) mixed geominimal surface area and (p, q) mixed affine surface area. For more results for the geominimal surface area, the reader can refer to [1, 5, 6, 8, 9, 11, 12, 15–17, 21, 23, 24, 26].

The Gaussian volume of Borel set K is denoted by γ_n as follows,

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n}\int_K e^{-\frac{|x|^2}{2}}dx.$$

Huang, Xi and Zhao [4] established the variational formula for the Gaussian volume γ_n. Recently, Liu [7] extended the variational formula for the Gaussian volume γ_n to the L_p $(p \geq 1)$ case:

$$\lim_{\epsilon\to 0}\frac{\gamma_n(K +_p \epsilon\cdot L) - \gamma_n(L)}{\epsilon} = \frac{1}{p}\int_{S^{n-1}} h(L, u)^p dS_{\gamma_n,p}(K, u), \tag{2}$$

where $S_{\gamma_n,p}(K,\cdot)$ is the L_p Gaussian surface area of K (see Section 2).

Utilizing the relationship between the geominimal surface area and the L_p variational formula for the Gaussian volume γ_n, we discuss the geominimal L_p Gaussian surface area, i.e., whether there exists $M \in \mathcal{K}_0^n$ with $|M^*| = \omega_n$ such that it is the unique solution for the following optimal problem

$$\inf\left\{\int_{S^{n-1}} h(Q, u)^p dS_{\gamma_n,p}(K, u) : Q \in \mathcal{K}_0^n \text{ and } |Q^*| = \omega_n\right\}? \tag{3}$$

The main aim of this paper is to solve the solution of the optimal problem (3).

Theorem 1.1. *Let $K \in \mathcal{K}_0^n$. Then there is a unique convex body $M \in \mathcal{K}_0^n$ with $|M^*| = \omega_n$ such that*

$$\int_{S^{n-1}} h(M, u)^p dS_{\gamma_n,p}(K, u) = \inf\left\{\int_{S^{n-1}} h(Q, u)^p dS_{\gamma_n,p}(K, u) : Q \in \mathcal{K}_0^n, |Q^*| = \omega_n\right\}.$$

The solution M of Theorem 1.1 is usually called the L_p Gaussian Petty body. Let $K \in \mathcal{K}_c^n$ be a simplex, then the L_p Gaussian Petty body M and K are homothetic. For $K \in \mathcal{K}_0^n$, let

$$\widetilde{G}_{\gamma_n,p}(K) = \inf\left\{\int_{S^{n-1}} h_L^p(u)dS_{\gamma_n,p,K}(u) : L \in \mathcal{K}_c^n,\ |L^*| = \omega_n\right\}.$$

Since the Gaussian volume γ_n has neither affine invariance nor homogeneity, we obtain that the L_p geominimal Gaussian surface area has neither affine invariance nor homogeneity. Fortunately, the L_p geominimal Gaussian surface area is continuous. Moreover, we obtain that the opposite question of continuity is also continuous.

Theorem 1.2. *For $p \geq 1$. Let $K, K_i \in \mathcal{K}_c^n$ be simplexes and let $\alpha > 0$ satisfying $|K_i^*| = \alpha$ and $|K^*| = \alpha$. If $\widetilde{G}_{\gamma_n,p}(K_i) \to \widetilde{G}_{\gamma_n,p}(K)$, then $K_i \to K$.*

The organization of this paper is as follows. Section 2 collects some basic concepts and various facts that will be used in the proofs of our results. Section 3 includes the basic properties of the L_p variation formula (2) and proves the main result Theorem 1.1. Section 4 proves the continuity of the L_p geominimal Gaussian surface area and the L_p Gaussian Petty body. Moreover, we will prove the main result Theorem 1.2.

2. Background and Notation

We now introduce the basic well-known facts and standard notations in this section. For more details and more concepts in convex geometry, please see [2, 3, 19].

The Minkowski sum of two Borel sets K, L is defined by $K + L := \{x + y : x \in K,\ y \in L\}$. The scalar product of $\lambda \in \mathbb{R}$ and Borel K is defined by $\lambda K := \{\lambda x : x \in K\}$. For $K \in \mathcal{K}^n$, the volume radius of K is defined by

$$\text{vrad}(K) = \left(\frac{|K|}{\omega_n}\right)^{1/n}.$$

The origin-centered unit ball in $\mathbb{R}^n$ is denoted by B_2^n, i.e., $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and let ω_n denote the volume of B_2^n. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Let $C(S^{n-1})$ be the set of continuous functions on S^{n-1}. Let ∂K, $convK$ denote the boundary and the convex hull of K, respectively.

For convex set K, it is uniquely determined by the support function h_K of K which is defined by $h_K(u) = \max\{x \cdot u : x \in K\}$, $\forall u \in S^{n-1}$. For $\lambda > 0$ and $K, L \in \mathcal{K}^n$, we obtain $h_{\lambda K} = \lambda h_K$. Moreover, $K \subset L$ if and only if

$$h_K(u) \leq h_L(u), \quad \forall u \in S^{n-1}. \tag{4}$$

Let $H(u,t) = \{x : x \cdot u = t\}$ and $H^-(u,t) = \{x : x \cdot u \leq t\}$ denote the hyperplane and the closed halfspace. Let $H(u, h(K,u))$ and $H^-(u, h(K,u))$ be respectively, the support plane and the support halfspace of K, with outer normal vector u. Obviously, for $K \in \mathcal{K}^n$, one has $K = \bigcap_{u \in S^{n-1}} H^-(u, h(K,u))$. If there are finite outer normal vectors such that $K = \bigcap_{i=1}^m H^-(u_i, h(K,u_i))$, then K is called polytope. Given a polytope P, a face of P is the intersection of P with a supporting hyperplane. A face of $(n-1)$-dimension is called a facet, and the outer normal vector of the facet is called the facet normal vector.

For the class $\mathcal{K}^n$, we consider the topology generated by the Hausdorff metric $d_H(\cdot,\cdot)$. Here $d_H(K,L)$ is defined by

$$d_H(K,L) = \|h_K - h_L\|_\infty = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|,$$

for $K, L \in \mathcal{K}^n$. Given $K \in \mathcal{K}^n$ and any Borel set $\omega \subset S^{n-1}$, the surface area measure $S(K,\cdot)$ is defined by

$$S(K,\omega) = \int_{\nu_K^{-1}(\omega)} d\mathcal{H}^{n-1},$$

where $\nu_K^{-1}(\omega)$ is the reverse Gauss image of ω and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Moreover, for any $K \in \mathcal{K}^n$, there is an integral formula for volume, that is,

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K(u)\,dS(K,u). \tag{5}$$

Let $l_u = \{tu : t \geq 0\}$ for $u \in S^{n-1}$. The set $L \subset \mathbb{R}^n$ is star-shaped with respect to the origin, if for each $u \in S^{n-1}$, the set $L \cap l_u$ is a closed line segment containing the origin. The radial function $\rho_L : S^{n-1} \to [0,\infty)$ of star-shaped set L, with respect to the origin, is defined by

$$\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\}, \quad \forall u \in S^{n-1}.$$

A compact star-shaped set with respect to the origin is uniquely determined by its radial function. If ρ_L is positive and continuous on S^{n-1}, then star-shaped set L is called star body with respect to the origin. Let $\mathcal{I}_0^n$ be the set of all star bodies with respect to the origin. Clearly, the radial function of $K \in \mathcal{K}_0^n$ is continuous and positive, i.e., $\mathcal{K}_0^n \subset \mathcal{I}_0^n$. If $K \in \mathcal{I}_0^n$, then

$$\partial K = \{\rho_K(u)u : u \in S^{n-1}\}.$$

It is well known that (e.g., see [19]) for any $K \in \mathcal{K}_0^n$ and for all $u \in S^{n-1}$,

$$h_{K^*}(u) = \frac{1}{\rho_K(u)} \quad \text{and} \quad \rho_{K^*}(u) = \frac{1}{h_K(u)}, \tag{6}$$

where K^* is the polar body of K, is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1,\ \text{for all } y \in K\}.$$

The radial metric is defined by

$$d_\rho(K,L) = \|\rho_K - \rho_L\|_\infty = \sup_{u\in S^{n-1}} |\rho_K(u) - \rho_L(u)|,$$

for $K, L \in \mathcal{I}_0^n$. We shall use the fact that the Hausdorff metric and the radial metric are topologically equivalent on $\mathcal{K}_0^n$. We say that the sequence $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ converges to a convex body $K_0 \in \mathcal{K}_0^n$ (write as $K_i \to K_0$), if $d_H(K_i, K_0) \to 0$ as $i \to \infty$. Obviously, according to the fact that $d_H(K_i, K_0) \to 0$ if and only if $d_\rho(K_i, K_0) \to 0$, and (6), we have

$$K_i \to K_0 \Leftrightarrow K_i^* \to K_0^*. \tag{7}$$

For each real $p \geq 1$ and $a, b > 0$, the compact set $a \cdot K +_p b \cdot L \in \mathcal{K}_0^n$ is called the Minkowski-Firey L_p combination of $K, L \in \mathcal{K}_0^n$, which is introduced by Firey (see, e.g., [19]) and is defined by

$$h^p_{a\cdot K+_p b\cdot L}(\cdot) = a h^p_K(\cdot) + b h^p_L(\cdot). \tag{8}$$

For each $p \in \mathbb{R}\backslash\{0\}$ and $a, b > 0$, the compact set $a \diamond K \tilde{+}_p b \diamond L \in \mathcal{I}_0^n$ is called the radial L_p combination of $K, L \in \mathcal{I}_0^n$, which is defined by

$$\rho^p_{a\diamond K\tilde{+}_p b\diamond L}(\cdot) = a\rho^p_K(\cdot) + b\rho^p_L(\cdot). \tag{9}$$

Note that for convex bodies $K, L \in \mathcal{K}_0^n$ it follows from (6), (8) and (9) that

$$K \tilde{+}_{-p} L = (K^* +_p L^*)^* \ \ for \ \ p \geq 1. \tag{10}$$

The Gaussian volume of Borel set K is denoted by γ_n,

$$\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n}\int_K e^{-\frac{|x|^2}{2}} dx.$$

Huang, Xi and Zhao [4] established the variational formula for the Gaussian volume γ_n. Recently, Liu [7] extended the variational formula for the Gaussian volume γ_n to the L_p $(p \geq 1)$ case:

$$\lim_{\epsilon\to 0} \frac{\gamma_n(K +_p \epsilon \cdot L) - \gamma_n(L)}{\epsilon} = \frac{1}{p}\int_{S^{n-1}} h(L,u)^p dS_{\gamma_n,p}(K,u), \tag{11}$$

where $S_{\gamma_n,p}(K,\cdot)$ is the L_p Gaussian surface area measure of K, i.e., which is defined by

$$S_{\gamma_n,p}(K,\omega) = \frac{1}{(\sqrt{2\pi})^n}\int_{\nu_K^{-1}(\omega)} e^{-\frac{|x|^2}{2}} (x\cdot\nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x). \tag{12}$$

For $p = 1$, the L_1 Gaussian surface area measure of the convex body K is $S_{\gamma_n}(K,\cdot)$. The L_p Gaussian surface area measure $S_{\gamma_n,p}(K,\cdot)$ of K has some basic properties.

(1). $dS_{\gamma_n,p}(K,\cdot) = h(K,\cdot)^{1-p} dS_{\gamma_n}(K,\cdot)$.

(2). It is absolutely continuous with respect to $(n-1)$-dimensional Hausdorff measure.

(3). It is not concentrated on any closed hemisphere, i.e., for all $v \in S^{n-1}$, there exists a constant $c > 0$ such that

$$\int_{S^{n-1}} (u,v)_+ dS_{\gamma_n,p}(K,u) \geq c, \tag{13}$$

where $(u\cdot v)_+ = \max\{u\cdot v, 0\}$.

(4). It is a weakly convergent measure, i.e., for $K_i, K_0 \in \mathcal{K}_0^n$, if $K_i \to K_0$, then

$$\lim_{i\to\infty}\int_{S^{n-1}} f(u) dS_{\gamma_n,p}(K_i,u) = \int_{S^{n-1}} f(u) dS_{\gamma_n,p}(K_0,u),$$

for all $f \in C(S^{n-1})$.

Moreover, if $\{f_i\}_{i\geq 1} \subset C(S^{n-1})$ converges uniformly to $f_0 \in C(S^{n-1})$ and $K_i \in \mathcal{K}_0^n$ converges to $K_0 \in \mathcal{K}_0^n$ in the Hausdorff distance, and the L_p Gaussian surface area measure $S_{\gamma_n,p}(K_i,\cdot)$ converges weakly to $S_{\gamma_n,p}(K_0,\cdot)$, we see that

$$\lim_{i\to\infty}\int_{S^{n-1}} f_i(u)\,dS_{\gamma_n,p}(K_i,u) = \int_{S^{n-1}} f_0(u)\,dS_{\gamma_n,p}(K_0,u). \tag{14}$$

We also use the following lemmas in the proofs of our main results.

Lemma 2.1. *(see [10]) If $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ is a bounded sequence such that $\{|K_i^*|\}_{i\geq 1}$ is also a bounded sequence, there exists a subsequence $\{K_{i_j}\}_{j\geq 1}$ of $\{K_i\}_{i\geq 1}$ and $K \in \mathcal{K}_0^n$ such that $K_{i_j} \to K$. In addition, if $|K_i^*| = \omega_n$, then $|K^*| = \omega_n$.*

Lemma 2.2. *(see[13]) Let $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ be a convergent sequence with the limit K_0. If the sequence $\{|K_i^*|\}_{i\geq 1}$ is bounded, then $K_0 \in \mathcal{K}_0^n$.*

3. The L_p geominimal Gaussian surface area

Firstly, we will prove the continuity of the L_p variational formula for the Gaussian volume γ_n.

Lemma 3.1. *For $p \geq 1$. Let $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ and $\{L_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ be two sequences of convex bodies with $K_i \to K_0 \in \mathcal{K}_0^n$ and $L_i \to L_0 \in \mathcal{K}_0^n$ as $i \to \infty$. Then*

$$\int_{S^{n-1}} h(L_i,u)^p dS_{\gamma_n,p}(K_i,u) \to \int_{S^{n-1}} h(L_0,u)^p dS_{\gamma_n,p}(K_0,u), \quad \text{as } i \to \infty.$$

Proof. Since $K_i \to K_0 \in \mathcal{K}_0^n$ and $L_i \to L_0 \in \mathcal{K}_0^n$ as $i \to \infty$, then $S_{\gamma_n,p}(K_i,\cdot)$ converges weakly to $S_{\gamma_n,p}(K_0,\cdot)$ and the support function $h_{L_i}(\cdot)$ converges uniformly to $h_{L_0}(\cdot)$, as $i \to \infty$. Thus, an application of (14) completes the proof. □

Next, we will discuss the boundedness of convex bodies.

Lemma 3.2. *Let $\{K_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ and $K \in \mathcal{K}_0^n$ be such that $K_i \to K$ as $i \to \infty$. If $\{L_i\}_{i\geq 1} \subset \mathcal{K}_0^n$ is a sequence such that $\int_{S^{n-1}} h^p(L_i,u)dS_{\gamma_n,p}(K_i,u)$ is uniformly bounded , then the sequence $\{L_i\}_{i\geq 1}$ is uniformly bounded.*

Proof. Let $R_i(u_i) = \max\{\rho(L_i,u) : u \in S^{n-1}\}$. Obviously, $[o, R(u_i)u_i] \subset L_i$ and using (4) shows that $R_i(u_i)(u_i{\cdot}u)_+ \leq h(L_i,u)$. Suppose the sequence $\{L_i\}_{i\geq 1}$ is not uniformly bounded, then $R_i(u_i)$ converges to infinity. Thus, given a sufficiently large $M > 0$, there is an integer N such that $R_i \geq M$ for all $i > N$. For each sequence $\{u_i\}_{i\geq 1} \subset S^{n-1}$, since the sphere S^{n-1} is a compact set, there is a vector $u_0 \in S^{n-1}$ and the subsequence u_{i_j} of u_i such that $\lim_{j\to\infty} u_{i_j} = u_0$. Using Lemma 3.1, Jessen's inequality, (13) and the boundness of the sequence $\int_{S^{n-1}} h(L_i,u)^p dS_{\gamma_n,p}(K_i,u)$ we obtain that there is a constant $C > 0$ such that

$$\begin{aligned}
C &\geq \lim_{i\to\infty}\int_{S^{n-1}} h_{L_i}^p(u)dS_{\gamma_n,p}(K_i,u)\\
&\geq \lim_{i\to\infty}\int_{S^{n-1}} R_i(u_i)^p(u_i,u)_+^p dS_{\gamma_n,p}(K_i,u)\\
&\geq \lim_{j\to\infty} M^p\int_{S^{n-1}} (u,u_{i_j})_+^p dS_{\gamma_n,p}(K_{i_j},u)\\
&= M^p S_{\gamma_n,p}(K,S^{n-1})\left(\frac{1}{S_{\gamma_n,p}(K,S^{n-1})}\int_{S^{n-1}}(u,u_0)_+ dS_{\gamma_n,p}(K,u)\right)^p\\
&\geq M^p S_{\gamma_n,p}^{1-p}(K,S^{n-1})c^p.
\end{aligned}$$

By the arbitrariness of M, let $M \to \infty$, we obtain a contradiction $C \geq \infty$. The proof of lemma is complished quickly. □

Now, we define the L_p geominimal Gaussian surface area $G_{\gamma_n,p}(K)$ of the convex body K.

Definition 3.3. *Suppose $K \in \mathcal{K}_0^n$. The L_p geominimal Gaussian surface area $G_{\gamma_n,p}(K)$ of the convex body K, is defined by*

$$G_{\gamma_n,p}(K) = \inf\left\{\int_{S^{n-1}} h(Q,u)^p dS_{\gamma_n,p}(K,u) : Q \in \mathcal{K}_0^n \text{ and } |Q^*| = \omega_n\right\}. \tag{15}$$

Remark: For each convex body $L \in \mathcal{K}_0^n$, we have $|(vrad(L^*)L)^*| = \omega_n$. Thus, the foluma (15) is equivalent to

$$G_{\gamma_n,p}(K) = \inf_{L\in \mathcal{K}_0^n}\left\{\int_{S^{n-1}} h((vrad(L^*)L,u)^p dS_{\gamma_n,p}(K,u)\right\}. \tag{16}$$

Next, we will prove the main result Theorem 1.1.

Proof. Firstly, we prove the existence of Theorem 1.1. By Definition 3.3, there exists a sequence $\{M_i\}_{i=1}^\infty \subset \mathcal{K}_0^n$ with $|M_i^*| = \omega_n$ such that

$$S_{\gamma_n,p}(K,S^{n-1}) = \int_{S^{n-1}} h^p_{B_2^n}(u)dS_{\gamma_n,p}(K,u) \geq \int_{S^{n-1}} h^p_{M_i}(u)dS_{\gamma_n,p}(K,u) > 0, \text{ for all } i \geq 1.$$

Obviously, the sequence $\int_{S^{n-1}} h^p_{M_i}(u)dS_{\gamma_n,p}(K,u)$ is uniformly bounded. By Lemma 3.2, the sequence $\{M_i\}_{i=1}^\infty$ is uniformly bounded. Thus, using the Blaschke selection theorem, there is a subsequence of $\{M_i\}_{i=1}^\infty$, for convenience, still recorded as $\{M_i\}_{i=1}^\infty$ which converges to a compact convex set M. Using Lemma 2.2 and $M_i \in \mathcal{K}_0^n$ shows that $M \in \mathcal{K}_0^n$. And using (7) implies that M_i^* converges to M^* as $i \to \infty$. Thus, we obtain $|M^*| = \omega_n$. From Lemma 3.1, we show that

$$\lim_{i\to\infty}\int_{S^{n-1}} h^p_{M_i}(u)dS_{\gamma_n,p}(K,u) = \int_{S^{n-1}} h^p_M(u)dS_{\gamma_n,p}(K,u).$$

Now we prove the uniqueness of Theorem 1.1. Assume that there exist two convex bodies $M_1, M_2 \in \mathcal{K}_0^n$ with $|M_1^*| = |M_2^*| = \omega_n$ such that

$$G_{\gamma_n,p}(K) = \int_{S^{n-1}} h^p_{M_1}(u)dS_{\gamma_n,p}(K,u) = \int_{S^{n-1}} h^p_{M_2}(u)dS_{\gamma_n,p}(K,u).$$

For $p \geq 1$, the new compact set M with respect to $M_1, M_2 \in \mathcal{K}_0^n$ is defined by

$$M = \frac{1}{2}\cdot M_1 +_p \frac{1}{2}\cdot M_2.$$

Using (10) shows that

$$M^* = \frac{1}{2}\diamond M_1^* \tilde{+}_{-p} \frac{1}{2}\diamond M_2^*.$$

Using the L_p Brunn-Minkowski inequality (see, e.g., [19]) we obtain that $|M^*| \leq \omega_n$ and the equality holds if and only if M_1^* and M_2^* are dilates. If M_1^* and M_2^* are dilates, for convenience, we set $M_1^* = sM_2^*$ for real number $s > 0$. Since $|M_1^*| = s^n|M_2^*| = \omega_n$, we deduce that $s = 1$. Then $M_1 = M_2$ if and only if $\text{vrad}(M^*) = 1$. By (16), we have

$$\begin{aligned} G_{\gamma_n,p}(K) &\leq \int_{S^{n-1}} h^p_{(vrad(M^*))M}(u)dS_{\gamma_n,p}(K,u) \\ &\leq \int_{S^{n-1}} h^p(M,u)dS_{\gamma_n,p}(K,u) \\ &= \int_{S^{n-1}} \Big(\frac{1}{2}h^p(M_1,u) + \frac{1}{2}h^p(M_2,u)\Big)dS_{\gamma_n,p}(K,u) \\ &= G_{\gamma_n,p}(K). \end{aligned}$$

This implies that $\text{vrad}(M^*) = 1$. This completes the proof of the theorem. □

4. Continuity

The optimal solution M of Theorem 1.1 is called the L_p Gaussian Petty body. And the set of solutions is denoted by

$$M_{\gamma_n,p}(K) = \Big\{M \in \mathcal{K}_0^n : G_{\gamma_n,p}(K) = \int_{S^{n-1}} h_M^p(u)dS_{\gamma_n,p}(K,u) \text{ and } |M^*| = \omega_n\Big\}. \tag{17}$$

Obviously, the uniqueness of Theorem 1.1 implies that the set $M_{\gamma_n,p}(K)$ can define an operator on $\mathcal{K}_0^n$. We now prove the continuity of $G_{\gamma_n,p}(K)$ and the operator $M_{\gamma_n,p}(K)$.

Theorem 4.1. *Let $\{K_i\}_{i=1}^\infty \subset \mathcal{K}_0^n$ and $K \in \mathcal{K}_0^n$ with $K_i \to K$ as $i \to \infty$. Then*
(i) $\lim_{i\to\infty} G_{\gamma_n,p}(K_i) = G_{\gamma_n,p}(K)$; (ii) $\lim_{i\to\infty} M_{\gamma_n,p}(K_i) = M_{\gamma_n,p}(K)$.

Proof. Firstly, we give the proof of (i). Let $\{K_i\}_{i=1}^\infty \subset \mathcal{K}_0^n$ and $K \in \mathcal{K}_0^n$ such that $K_i \to K$ as $i \to \infty$. For sufficiently small $\varepsilon > 0$, using Definition 3.3 and Lemma 3.1, we obtain that there is a convex body $M_\varepsilon \in \mathcal{K}_0^n$ with $|M_\varepsilon^*| = \omega_n$ such that

$$G_{\gamma_n,p}(K) + \varepsilon \geq \int_{S^{n-1}} h_{M_\varepsilon}^p(u)dS_{\gamma_n,p}(K,u) = \lim_{i\to\infty}\int_{S^{n-1}} h_{M_\varepsilon}^p(u)dS_{\gamma_n,p}(K_i,u) \geq \limsup_{i\to\infty} G_{\gamma_n,p}(K_i).$$

Let $\varepsilon \to 0^+$, we have

$$G_{\gamma_n,p}(K) \geq \limsup_{i\to\infty} G_{\gamma_n,p}(K_i). \tag{18}$$

Next, we assume $M_i \in \mathcal{K}_0^n$ and $|M_i^*| = \omega_n$ such that $G_{\gamma_n,p}(K_i) = \int_{S^{n-1}} h_{M_i}^p(u)dS_{\gamma_n,p}(K_i,u)$. Thus

$$0 < \int_{S^{n-1}} h_{M_i}^p(u)dS_{\gamma_n,p}(K_i,u) \leq \int_{S^{n-1}} h_{B_2^n}^p(u)dS_{\gamma_n,p}(K_i,u) < \infty.$$

This implies that the sequence $\int_{S^{n-1}} h_{M_i}^p(u)dS_{\gamma_n,p}(K_i,u)$ is bounded. Thus, the sequence $\{M_i\}_{i=1}^\infty$ is uniformly bounded. Using the Blaschke selection theorem shows that there exists a convergent subsequence of $\{M_i\}_{i=1}^\infty$, which is also written as $\{M_i\}_{i=1}^\infty$, and a compact convex set M' such that $\lim_{i\to\infty} M_i = M'$. Combining $|M_i^*| = \omega_n$, Lemma 2.2 and Lemma 2.1, we obtain $M' \in \mathcal{K}_0^n$ with $|(M')^*| = \omega_n$. By Definition 3.3, Lemma 3.1 and Theorem 1.1, we have

$$G_{\gamma_n,p}(K) \leq \int_{S^{n-1}} h_{M'}^p(u)dS_{\gamma_n,p}(K_i,u) = \liminf_{i\to\infty}\int_{S^{n-1}} h_{M_i}^p(u)dS_{\gamma_n,p}(K_i,u) = \liminf_{i\to\infty} G_{\gamma_n,p}(K_i). \tag{19}$$

Combining (18) with (19), this completes the proof of (i).

Now, we give the proof of (ii). By (17) and Theorem 1.1, we know that the set $M_{\gamma_n,p}(K)$ has only one element. For simplicity, we set $M = M_{\gamma_n,p}(K)$ and $M_i = M_{\gamma_n,p}(K_i)$. For $K_i \to K$ as $i \to \infty$, using the continuity of $G_{\gamma_n,p}(\cdot)$ we obtain that

$$G_{\gamma_n,p}(K) = \lim_{i\to\infty} G_{\gamma_n,p}(K_i) = \lim_{i\to\infty}\int_{S^{n-1}} h_{M_i}^p(u)dS_{\gamma_n,p}(K_i,u).$$

This implies that $\int_{S^{n-1}} h_{M_i}^p(u)dS_{\gamma_n,p}(K_i,u)$ is uniformly bounded. Thus, the sequence $\{M_i\}_{i=1}^\infty$ is bounded. By Lemma 2.1, there exists a subsequence $\{M_{i_j}\}_{j=1}^\infty \subset \{M_i\}_{i=1}^\infty$ and a convex body $M_0 \in \mathcal{K}_0^n$ such that $\lim_{j\to\infty} M_{i_j} = M_0 \in \mathcal{K}_0^n$ and $|M_0^*| = \omega_n$. By (i) of Theorem 4.1 and Lemma 3.1, one has

$$G_{\gamma_n,p}(K) = \lim_{j\to\infty} G_{\gamma_n,p}(K_{i_j}) = \int_{S^{n-1}} h_{M_0}^p(u)dS_{\gamma_n,p}(K,u).$$

Using Theorem 1.1 shows that

$$G_{\gamma_n,p}(K) = \int_{S^{n-1}} h_M^p(u)dS_{\gamma_n,p}(K,u) \ \text{ with } \ |M^*| = \omega_n.$$

Since the operator $M_{\gamma_n,p}(K)$ has only one element, this implies that $\lim_{i\to\infty} M_i = M$. This completes the proof of the theorem. □

Next, we will discuss the continuity of the solution for the L_p geominimal Gaussian surface area.

Corollary 4.2. *Let $K_i, K \in \mathcal{K}_0^n$. For $p \geq 1$, let M_i be the solution of $G_{\gamma_n,p}(K_i)$ and M be the solution of $G_{\gamma_n,p}(K)$. If $G_{\gamma_n,p}(K_i)$ converges to $G_{\gamma_n,p}(K)$, then M_i converges to M.*

Proof. Using Theorem 1.1 we obtain that the solution for the L_p geominimal Gaussian surface area is unique. Hence, let M_i, M be such that $\int_{S^{n-1}} h^p_{M_i}(u)dS_{\gamma_n,p}(K_i,u) = G_{\gamma_n,p}(K_i)$, and $\int_{S^{n-1}} h^p_M(u)dS_{\gamma_n,p}(K,u) = G_{\gamma_n,p}(K)$. Since $G_{\gamma_n,p}(K_i)$ converges to $G_{\gamma_n,p}(K)$, this shows that there is an integer N such that

$$G_{\gamma_n,p}(K_i) \leq G_{\gamma_n,p}(K) + \varepsilon,\ i \geq N,$$

for all $\varepsilon > 0$. This implies that the sequence $\int_{S^{n-1}} h^p_{M_i}(u)dS_{\gamma_n,p}(K_i,u)$ is bounded. Thus, using Lemma 3.2 shows that M_i is uniformly bounded. Using the Blaschke selection theorem, we obtain that there is a subsequence M_{i_j} of M_i such that it converges to a compact convex set M'. Combining Lemma 2.1 with Lemma 2.2, we obtain that $M' \in \mathcal{K}_0^n$ and $|(M')^*| = \omega_n$. Since M_{i_j} is the solution of $G_{\gamma_n,p}(K_{i_j})$, taking the limit as $j \to \infty$, we obtain that M' is the solution of $G_{\gamma_n,p}(K)$. The uniqueness of the solution immediately get that $\lim_{j\to\infty} M_{i_j} = M$. This completes the proof of the corollary. □

Corollary 4.3. *Let $K \in \mathcal{K}_0^n$. For $p, p_i \geq 1$, let M_i be the solution of $G_{\gamma_n,p_i}(K)$ and M be the solution of $G_{\gamma_n,p}(K)$. If $G_{\gamma_n,p_i}(K)$ converges to $G_{\gamma_n,p}(K)$, then M_i converges to M.*

Proof. The proof is similar to Corollary 4.2. □

Lemma 4.4. *Let $K \in \mathcal{K}_0^n$ be a polytope. For $p_1, p_2 \geq 1$ and $p_1 \neq p_2$. If the convex body M_j ($j = 1,2$) is the solution to the L_{p_j} geominimal Gaussian surface area, then M_1 and M_2 are polytopes with the same facet normal vector.*

Proof. Let $K \in \mathcal{K}_0^n$ be a polytope with the facet normal vectors $u_1, \cdots, u_m$. Obviously, $\{u_1, u_2, \cdots, u_m\}$ is not concentrated on any closed hemisphere of S^{n-1} and $K = \bigcap_{i=1}^m H^-(u_i, h_K(u_i))$. Since the L_p Gaussian surface measure is absolutely continuous with respect to $(n-1)$-dimensional Hausdorff measure, this implies that the L_p Gaussian surface measure $S_{\gamma_n,p_1}(K,\cdot)$ is a discrete measure concentrated on $\{u_1, u_2, \cdots, u_m\} \subset S^{n-1}$. Let P be a polytope with the facet normal vectors $u_1, u_2, \cdots, u_m$, such that

$$P_j = \bigcap_{i=1}^m H^-(u_i, h_{M_j}(u_i)).$$

Therefore, one has $M_j \subset P_j$. Since $M_j \in M_{\gamma_n,p_j}(K)$ ($j = 1,2$), this implies that $vrad(P^*) \leq 1$. Combined Definition 3.3 with $M_j \in M_{\gamma_n,p_j}(K)$ ($j = 1,2$), we obtain

$$\begin{aligned}
G_{\gamma_n,p_j}(K) &\leq \int_{S^{n-1}} h^{p_j}(vard(P_j^*)P_j,u)dS_{\gamma_n,p_j}(K,u) \\
&\leq \int_{S^{n-1}} h^{p_j}(P_j,u)dS_{\gamma_n,p_j}(K,u) \\
&= \sum_{i=1}^m h^{p_j}(P_j,u_i)S_{\gamma_n,p_j}(K,\{u_i\}) \\
&= \sum_{i=1}^m h^{p_j}(M_j,u_i)S_{\gamma_n,p_j}(K,\{u_i\}) \\
&\leq \int_{S^{n-1}} h^{p_j}(M_j,u)dS_{\gamma_n,p_j}(K,u) \\
&= G_{\gamma_n,p_j}(K).
\end{aligned}$$

This shows that $vrad(P_j^*) = vrad(M_j^*) = 1$. We know that $|M_j| = |P_j|$ and $M_j \subset P_j$. Thus, $M_j = P_j$, which means that the optimal solution M_j is a polytope with the facet normal vectors $u_1, u_2, \cdots, u_m$. This completes the proof of the lemma. □

Let $conv\{x_1,\cdots,x_m\}$ be the convex hull of $x_1,\cdots,x_m \in \mathbb{R}^n$. If $x_1,\cdots,x_{n+1}$ are affine independent, then the convex hull $conv\{x_1,\cdots,x_{n+1}\}$ denotes the simplex. Moreover, the convex hull of n points $\{x_{i_1},\cdots,x_{i_n}\} \subset \{x_1,\cdots,x_{n+1}\}$ denote the facets of the simplex $conv\{x_1,\cdots,x_{n+1}\}$. Let $T = conv\{o,e_1,\cdots,e_n\}$ denotes standard simplex, where $e_1,\cdots,e_n$ denote the vectors of the standard bases of $\mathbb{R}^n$. Obviously, for each simplex K, there is a transformation $\phi \in GL(n)$ and $x \in \mathbb{R}^n$ such that $K = \phi T + x$. Let $K \in \mathcal{K}_0^n$, then the volume product $|K||K^*|$ is $GL(n)$ invariant. Moreover, if $K \in \mathcal{K}_c^n$, then

$$|K||K^*| \le \omega_n^2,$$

with equality if and only if K is an ellipsoid. The lower bound of the volume product is the Mahler conjecture, i.e., if $K \in \mathcal{K}_c^n$, then

$$\frac{(n+1)^{n+1}}{(n!)^2} \le |K||K^*|, \tag{20}$$

with equality if and only if K is a simplex. Recently, Meyer and Reisner [14] used the shadow system to prove an exact reverse Blaschke-Santaló inequality (20) for polytopes in $\mathbb{R}^n$ that have at most $n+3$ vertices.

Corollary 4.5. *Let $p \ge 1$. If $K \in \mathcal{K}_0^n$ is the simplex and M is the solution of $G_{\gamma_n,p}(K)$, then M and K are homothetic.*

Proof. For each simplex K, there is a transformation $\phi \in GL(n)$ and $x \in \mathbb{R}^n$ such that $K = \phi T + x$. From Lemma 4.1, we obtain that M and K are simplexes with the same facet normal vector. Thus, there is $\lambda > 0$ and $x_2 \in \mathbb{R}^n$ such that $M = \lambda\phi T + x_2$. This implies that $M = \lambda K + x$, where $\lambda > 0$ and $x \in \mathbb{R}^n$ with $|(\lambda K + x)^*| = \omega_n$. This completes the proof of the corollary. □

Next, we will establish the main result Theorem 1.2.

Proof. Let the convex body M_i be the solution to the geominimal L_p style Gaussian surface area $\widetilde{G}_{\gamma_n,p}(K)$. Using Corollary 4.5 we obtain that there are $\lambda_i > 0$ and $x_i \in \mathbb{R}^n$ such that $M_i = \lambda_i K + x_i$. Since $M_i, K \in \mathcal{K}_c^n$, one has $x_i = o$ for all i. Combining $|K_i^*| = \alpha$ with $|M_i^*| = \omega_n$, this implies that $M_i = \left(\frac{\alpha}{\omega_n}\right)^{\frac{1}{n}} K_i$. Similarly, we have $M = \left(\frac{\alpha}{\omega_n}\right)^{\frac{1}{n}} K$. Thus, an application of Corollary 4.2 completes the proof. □

Corollary 4.6. *For $p \ge 1$. Let $K, K_i \in \mathcal{K}_c^n$ be simplexes. Assume $|K_i^*| = |K^*| = \alpha$ for $\alpha > 0$, and if $\int_{S^{n-1}} h^p(K_i,u)dS_{\gamma_n,p,K_i}(u)$ converges to $\int_{S^{n-1}} h^p(K,u)dS_{\gamma_n,p,K}(u)$, then K_i converges to K.*

Proof. Let the convex body M_i be the solution to the geominimal L_p style Gaussian surface area $\widetilde{G}_{\gamma_n,p}(K)$. Using Corollary 4.5 implies that there are $\lambda_i > 0$ and $x_i \in \mathbb{R}^n$ such that $M_i = \lambda_i K_i + x_i$. Since $M_i, K_i \in \mathcal{K}_c^n$, one has $x_i = o$ for all i. Combining $|K_i^*| = \alpha$ with $|M_i^*| = \omega_n$, this implies that $M_i = \left(\frac{\alpha}{\omega_n}\right)^{\frac{1}{n}} K_i$. Similarly, we have $M = \left(\frac{\alpha}{\omega_n}\right)^{\frac{1}{n}} K$. Thus, $\widetilde{G}_{\gamma_n,p}(K_i) = \left(\frac{\alpha}{\omega_n}\right)^{\frac{p}{n}} \int_{S^{n-1}} h^p(K_i,u)dS_{\gamma_n,p,K_i}(u)$ and $\widetilde{G}_{\gamma_n,p}(K) = \left(\frac{\alpha}{\omega_n}\right)^{\frac{p}{n}} \int_{S^{n-1}} h^p(K,u)dS_{\gamma_n,p,K}(u)$. Since $\int_{S^{n-1}} h^p(K_i,u)dS_{\gamma_n,p,K_i}(u)$ converges to $\int_{S^{n-1}} h^p(K,u)dS_{\gamma_n,p,K}(u)$, we now obtain $\widetilde{G}_{\gamma_n,p}(K_i) \to \widetilde{G}_{\gamma_n,p}(K)$. Thus, an application of Theorem 1.2 completes the proof of the corollary. □

Next, we will establish the Blaschke-Santaló style inequality for the L_p geominimal Gaussian surface area.

Corollary 4.7. *For $p \ge 1$. If $K \in \mathcal{K}_c^n$ is a simplex, then*

$$G_{\gamma_n,p}(K)G_{\gamma_n,p}(K^*) \le \frac{n^2}{(2\pi)^n \omega_n^{2p/n}} \left(\frac{(n+1)^{n+1}}{(n!)^2}\right)^{\frac{p+n}{n}}.$$

Proof. Let M be the solution to the L_p geominimal Gaussian surface area $\tilde{G}_{\gamma_n,p}(K)$. Using Corollary 4.6 shows that $M = (\frac{|K^*|}{\omega_n})^{1/n}K$. Thus $\tilde{G}_{\gamma_n,p}(K) = \left(\frac{|K^*|}{\omega_n}\right)^{p/n} \int_{S^{n-1}} h(K,u)dS_{\gamma_n,K}(u)$. Similarly, we have $\tilde{G}_{\gamma_n,p}(K^*) = \left(\frac{|K|}{\omega_n}\right)^{p/n} \int_{S^{n-1}} h(K^*,u)dS_{\gamma_n,K^*}(u)$. By using (20), one has

$$\frac{(n+1)^{n+1}}{(n!)^2} \leq |K||K^*|,$$

with equality if and only if $K \in \mathcal{K}_c^n$ is a simplex. Thus, we have $\frac{(n+1)^{n+1}}{(n!)^2} = |K||K^*|$. Combining (11) with (12), this implies that $\int_{S^{n-1}} h(K,u)dS_{\gamma_n,K}(u) \leq \frac{1}{(\sqrt{2\pi})^n}n|K|$. Similarly, $\int_{S^{n-1}} h(K^*,u)dS_{\gamma_n,K^*}(u) \leq \frac{1}{(\sqrt{2\pi})^n}n|K^*|$. Thus,

$$\tilde{G}_{\gamma_n,p}(K)\tilde{G}_{\gamma_n,p}(K^*) \leq \frac{n^2}{(2\pi)^n\omega_n^{2p/n}}\left(\frac{(n+1)^{n+1}}{(n!)^2}\right)^{\frac{p+n}{n}}.$$

Together with $\tilde{G}_{\gamma_n,p}(K) \geq G_{\gamma_n,p}(K)$, this completes the proof of the corollary. □

We now discuss the monotonicity of the L_p geominimal Gaussian surface area with respect to p.

Corollary 4.8. *Let $K \in \mathcal{K}_0^n$. If $p_1, p_2 \geq 1$ and $p_1 \leq p_2$, then*

$$G_{\gamma_n,p_1}^{p_2}(K) \leq G_{\gamma_n,p_2}^{p_1}(K)\left(\int_{S^{n-1}} h_K(u)dS_{\gamma_n}(K,u)\right)^{p_2-p_1}.$$

Proof. Let the convex body M_i be the solution of $G_{\gamma_n,p_i}(K)$ for $i = 1, 2$. Combining Definition 3.3 with Hölder inequality, one has

$$\begin{aligned} G_{\gamma_n,p_1}(K) &= \int_{S^{n-1}} h_{M_1}^{p_1}(u)dS_{\gamma_n,p_1}(K,u) \\ &\leq \int_{S^{n-1}} \left(\frac{h_{M_2}(u)}{h_K(u)}\right)^{p_1} h_K(u)dS_{\gamma_n}(K,u) \\ &\leq \left(\int_{S^{n-1}} \left(\frac{h_{M_2}(u)}{h(K,u)}\right)^{p_2} h_K(u)dS_{\gamma_n}(K,u)\right)^{\frac{p_1}{p_2}} \left(\int_{S^{n-1}} h_K(u)dS_{\gamma_n}(K,u)\right)^{\frac{p_2-p_1}{p_2}} \\ &= G_{\gamma_n,p_2}^{\frac{p_1}{p_2}}(K)\left(\int_{S^{n-1}} h_K(u)dS_{\gamma_n}(K,u)\right)^{\frac{p_2-p_1}{p_2}}. \end{aligned}$$

This completes the proof. □

References

[1] Y. Feng, W. Wang, L_p-dual mixed geominimal surface area, Glasg. Math. J. **56** (2014), 229-239.
[2] R.J. Gardner, Geometric tomography, Cambridge Univ. Press, Cambridge, (1995).
[3] P.M. Gruber, Convex and discrete geometry, Springer-Verlag, Berlin Heidelberg, (2007).
[4] Y. Huang, D. Xi, Y. Zhao, The Minkowski problem in Gaussian probability space, Adv. Math., **385** (2021), Paper No. 107769, 36pp.
[5] N. Li, S. Mou, The General Dual Orlicz Geominimal Surface Area, J. Function Spaces, 2020(1):1-6.
[6] X. Li, H. Wang, J. Zhou, (p,q)-mixed geominimal surface area and (p,q)-mixed affine surface area, J. Math. Anal. Appl., **475** (2019), 1472-1492.
[7] J. Liu, The L_p-Gaussian Minkowski problem, Calc. Var. Partial Differential Equations, **66** (2022), Paper No. 28, 23 pp.
[8] M. Ludwig, General affine surface areas, Adv. Math., **224** (2010), 2346-2360.

[9] M. Ludwig, M. Reitzner, A characterization of affine surface area, Adv. Math., **147** (1999), 138-172.
[10] X. Luo, D. Ye, B. Zhu, On the polar Orlicz-Minkowski problems and the p-capacitary Orlicz-Petty bodies, Indiana Univ. Math. J., **69** (2020), 385-420.
[11] E. Lutwak, Dual mixed volume, Pacific Journal of Mathematics, **58** (1975), 531-538.
[12] E. Lutwak, Centroid bodies and dual mixed volumes, Pro. London Math. Soc., **2** (1990), 365-391.
[13] E. Lutwak, The Brunn-Minkowski-Firey theory II. Affine and geominimal surface areas, Adv. Math., **118** (1996), 244-294.
[14] M. Meyer, S. Reisner, Shadow systems and volumes of polar convex bodies, Mathematika, **53** (2006), 129-148.
[15] S. Mou, The geominimal integral curvature, AIMS Math., **7** (2022), 14338-14353.
[16] S. Mou, J. Dai, Geominimal Lp integral curvature, Acta Math. Sinica (Chinese Ser.) 66 (2023), 617-628.
[17] S. Mou, B. Zhu, The orlicz-minkowski problem for measure in R^n and Orlicz geominimal measures , Int. J. Math., **30** (2019), 26pp.
[18] C.M. Petty, Geominimal surface area , Geom. Dedicata, **3** (1974), 77-97.
[19] R. Schneider Convex Bodies: The Brunn-Minkowski theory , Second edition, Cambridge Univ., (2014).
[20] D. Ye, L_p geominimal surface areas and their inequalities , Int. Math. Res. Not., **2015** (2015), 2465-2498.
[21] D. Ye, Dual Orlicz-Brunn-Minkowski theory: dual Orlicz L_ϕ affine and geominimal surface areas , J. Math. Anal. Appl., **443** (2016), 352-371.
[22] D. Ye, B. Zhu, J. Zhou, The mixed L_p geominimal surface area for multiple convex bodies, Indiana Univ. Math. J., **64** (2015), 1513-1552.
[23] S. Yuan, H. Jin, G. Leng, Orlicz geominimal surface areas, Math. Ineq. Appl., **18** (2015), 353-362.
[24] B. Zhu, N. Li, J. Zhou, Isoperimetric inequalities for L_p geominimal surface area, Glasg. Math. J., **53** (2011), 717-726.
[25] B. Zhu, J. Zhou, W. Xu, L_p mixed geominimal surface area, J. Math. Anal. Appl., **422** (2015), 1247-1263.
[26] B. Zhu, H. Hong, D. Ye, The Orlicz Petty bodies, Int. Math. Res. Not., **14** (2018), 4356-4403.