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Skew-symmetric solutions of the classical Yang-Baxter equation and $\mathcal{O}$-operators of Malcev algebras

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Abstract. We study connections between skew-symmetric solutions of the classical Yang-Baxter equation (CYBE) and $\mathcal{O}$-operators of Malcev algebras. We prove that a skew-symmetric solution of the CYBE on a Malcev algebra can be interpreted as an $\mathcal{O}$-operator associated to the coadjoint representation. We show that this connection can be enhanced with symplectic forms when considering non-degenerate skew-symmetric solutions. We also show that $\mathcal{O}$-operators associated to a general representation could give skew-symmetric solutions of the CYBE on certain semi-direct product of Malcev algebras. We reveal the relationship between invertible $\mathcal{O}$-operators and compatible pre-Malcev algebra structures on a Malcev algebra. We finally obtain several analogous results on connections between the CYBE and $\mathcal{O}$-operators in the case of pre-Malcev algebras.

1. Introduction

The classical Yang-Baxter equation (CYBE) on a finite-dimensional nonassociative algebra of characteristic zero occupies a central place in connecting mathematics and mathematical physics. The study of the CYBE on a Lie algebra $\mathfrak{g}$ has substantial ramifications and applications in the areas of symplectic geometry, quantum groups, integrable systems, and quantum field theory, whereas characterizing specific solutions of the CYBE for a given $\mathfrak{g}$ is an indispensable and challenging task in terms of the viewpoint of pure mathematics; see for example [5, 21]. As a natural generalization of Lie algebras, Malcev algebras have been studied extensively since Malcev's work in the 1950s ([15]). Our primary objective is to give a systematic study on skew-symmetric solutions of the CYBE on Malcev algebras, stemming from the point of view of Kupershmidt in [11, Section 2] that regards solutions of the CYBE as $\mathcal{O}$-operators. Our approach exposes some interesting connections between the CYBE, $\mathcal{O}$-operators, and pre-Malcev algebras.

Let A be a Malcev algebra over a field $\mathbb{F}$ of characteristic zero and $r = \sum_i x_i \otimes y_i \in A \otimes A$. The equation

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0 \tag{1.1}$$

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is called the classical Yang-Baxter equation on A, where

$$r_{12}r_{13} = \sum_{i,j} x_i x_j \otimes y_i \otimes y_j, r_{13}r_{23} = \sum_{i,j} x_i \otimes x_j \otimes y_i y_j, r_{23}r_{12} = \sum_{i,j} x_j \otimes x_i y_j \otimes y_i.$$

Recall that for a vector space V, an element $r \in V \otimes V$ is called **skew-symmetric** if $\sigma(r) = -r$, where σ denotes the twist map on $V \otimes V$. Comparing with $\mathcal{O}$-operators of Lie algebras and introducing the notion of $\mathcal{O}$-operators of Malcev algebras, our first main theorem provides a sufficient and necessary condition for a skew-symmetric element $r \in A \otimes A$ being a solution of the CYBE on A. To articulate this result, we write $\mathcal{S}(A)$ for the set of all solutions of the CYBE on A and denote by $\mathcal{O}_A(V, \rho)$ the set of all $\mathcal{O}$-operators associated to the representation $\rho : A \longrightarrow \mathrm{End}(V)$. For a finite-dimensional vector space V over $\mathbb{F}$, V^* refers to the dual space of V and for $r \in V \otimes V$, we define T_r to be the linear map from V^* to V by

$$\langle \xi, T_r(\eta) \rangle = \langle \xi \otimes \eta, r \rangle \tag{1.2}$$

for all $\xi, \eta \in V^*$, where $\langle -, - \rangle : V^* \times V \longrightarrow \mathbb{F}$ denotes the natural pairing.

Theorem 1.1. *Let A be a finite-dimensional Malcev algebra over a field $\mathbb{F}$ of characteristic zero and r be a skew-symmetric element in $A \otimes A$. Then $r \in \mathcal{S}(A)$ if and only if $T_r \in \mathcal{O}_A(A^*, \mathrm{ad}^*)$, where (A^*, ad^*) denotes the coadjoint representation of A.*

More significantly, specializing in non-degenerate skew-symmetric element $r \in A \otimes A$, we could associate r with a bilinear form $\mathcal{B}_r$ defined by T_r^{-1} and the natural pairing. Our second major result demonstrates that such r is a solution of the CYBE on A if and only if $\mathcal{B}_r$ is a symplectic form on A. This result provides a possible way to explicitly describe all non-degenerate skew-symmetric solutions in $\mathcal{S}(A)$ for some specific Malcev algebras; see Example 3.5. To state this result, we recall that an element $r \in A \otimes A$ is **non-degenerate** if T_r defined by Eq. (1.2) is invertible; a bilinear form $\mathcal{B}$ on A is **symplectic** if $\mathcal{B}(xy, z) + \mathcal{B}(yz, x) + \mathcal{B}(zx, y) = 0$ for all $x, y, z \in A$. Given a non-degenerate element $r \in A \otimes A$, we define the bilinear form $\mathcal{B}_r$ on A by $\mathcal{B}_r(x, y) := \left\langle T_r^{-1}(x), y \right\rangle$ for all $x, y \in A$. Lemma 3.4 below shows that r is skew-symmetric if and only if $\mathcal{B}_r$ is skew-symmetric.

Theorem 1.2. *Let A be a finite-dimensional Malcev algebra over a field $\mathbb{F}$ of characteristic zero and $r \in A \otimes A$ be skew-symmetric and non-degenerate. Then $r \in \mathcal{S}(A)$ if and only if $\mathcal{B}_r$ is a symplectic form on A.*

Our third result provides a construction of a skew-symmetric solution of the CYBE on the semi-direct product Malcev algebra $A \ltimes_{\rho^*} V^*$ by an arbitrary $\mathcal{O}$-operator associated to a given representation (V, ρ) of a Malcev algebra A, which reveals an inverse procedure of Theorem 1.1 by loosing the restriction of coadjoint representations; compared with the case of Lie algebra ([2, Section 2]). Using the tensor-hom adjunction, we identify $\mathrm{Hom}(V, A)$ with $A \otimes V^*$, and we identify an arbitrary element of $A \otimes V^*$ with the image in $(A \oplus V^*) \otimes (A \oplus V^*)$ under the tensor product of the standard embeddings $A \longrightarrow A \oplus V^*$ and $V^* \longrightarrow A \oplus V^*$. Hence, given a linear map $T : V \longrightarrow A$, we define an element $\widetilde{T}$ (see Eq. (3.2) below) in $(A \oplus V^*) \otimes (A \oplus V^*)$ via this two identifications. Then we define a skew-symmetric element $r_T := \widetilde{T} - \sigma(\widetilde{T})$.

Theorem 1.3. *Let (V, ρ) be a representation of a finite-dimensional Malcev algebra A over a field $\mathbb{F}$ of characteristic zero and $T : V \longrightarrow A$ be a linear map. Then $r_T \in \mathcal{S}(A \ltimes_{\rho^*} V^*)$ if and only if $T \in \mathcal{O}_A(V, \rho)$.*

Moreover, invertible elements in $\mathcal{O}_A(V, \rho)$ have a close relationship with compatible pre-Malcev algebra structures on A; see [13, Section 2] for more details on pre-Malcev algebras. As pre-Lie algebras are Lie-admissible, pre-Malcev algebras are Malcev-admissible algebras in the sense of [17]. Let $\mathcal{A}$ be a pre-Malcev algebra. Then the commutator $xy = x \cdot y - y \cdot x$ for all $x, y \in \mathcal{A}$ defines a Malcev algebra $[\mathcal{A}]$, which is called the subadjacent Malcev algebra of $\mathcal{A}$, and we call $\mathcal{A}$ a compatible pre-Malcev algebra of $[\mathcal{A}]$. For an element $x \in \mathcal{A}$, the left multiplication operator $L_x : \mathcal{A} \longrightarrow \mathcal{A}$ sends $y \in \mathcal{A}$ to $x \cdot y$. Then the linear map $L : [\mathcal{A}] \longrightarrow \mathrm{End}(\mathcal{A})$ with $x \mapsto L_x$ gives a representation of the Malcev algebra $[\mathcal{A}]$. Now our fourth theorem can be summarized as follows.

Theorem 1.4. *Let (V, ρ) be a representation of a finite-dimensional Malcev algebra A over a field $\mathbb{F}$ of characteristic zero. For an invertible element $T \in \mathcal{O}_A(V, \rho)$, there exists a compatible pre-Malcev algebra structure $\mathcal{A}_T$ on A defined by $x \cdot y := T(\rho(x)T^{-1}(y))$ for all $x, y \in A$. Conversely, if there exists a compatible pre-Malcev algebra $\mathcal{A}$ on A, then the identity map id_A belongs to $\mathcal{O}_A(\mathcal{A}, L)$.*

We also provide two applications of the existence of compatible pre-Malcev algebra structures on Malcev algebras to construct skew-symmetric solutions of the CYBE; see Corollary 4.3. Our last several results concerned with the CYBE on pre-Malcev algebras and $\mathcal{O}$-operators can be regarded as an analogue of the theorems mentioned above. Compared with the case of pre-Lie algebras ([3, Section 2]), these results focus on revealing links between symmetric solutions of the CYBE, $\mathcal{O}$-operators and bilinear forms on pre-Malcev algebras; see Theorems 4.5, 4.6 and Proposition 4.8.

This paper is organized as follows. In Section 2, we present some fundamental results on representations of Malcev algebras and $\mathcal{O}$-operators, and then we develop two lemmas to prove Theorem 1.1. Section 3 contains the proofs of Theorems 1.2 and 1.3, which are both closely related to Theorem 1.1. Theorem 1.2 specializes in the case of Malcev algebras admitting a symplectic form and Theorem 1.3 extends $\mathcal{O}$-operators associated with the coadjoint representation (A^*, ad^*) to those associated with an arbitrary representation (V, ρ). In Section 4, we establish connections between invertible $\mathcal{O}$-operators and compatible pre-Malcev structures on a Malcev algebra. We prove Theorem 1.4 and produce several results about symmetric solutions of the CYBE on pre-Malcev algebras.

Throughout this article we assume that the ground field $\mathbb{F}$ is a field of characteristic zero and all algebras, vector spaces and representations are finite-dimensional over $\mathbb{F}$. The multiplication in a Malcev algebra A is denoted by xy for all $x, y \in A$, while the multiplication in a pre-Malcev algebra $\mathcal{A}$ is denoted by $x \cdot y$ for all $x, y \in \mathcal{A}$.

2. Malcev Algebras and $\mathcal{O}$-operators

We recall some fundamental concepts on representations of Malcev algebras. Comparing with $\mathcal{O}$-operators of Lie algebras, we introduce the notion of $\mathcal{O}$-operators of Malcev algebras and present concrete examples on some specific Malcev algebras. We close this section by giving a proof of Theorem 1.1.

2.1. Representations of Malcev algebras

Recall that a nonassociative anti-commutative algebra A over a field $\mathbb{F}$ is called a **Malcev algebra** provided that

$$(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y \qquad \text{(Malcev identity)}$$

for all $x, y, z \in A$. Compared with the relationship between Lie algebras and Lie groups, Malcev algebras appear as the tangent spaces of smooth Moufang loops at the identities; see for example [15] for more backgrounds. It was proved in [18, Proposition 2.21] that Malcev identity is also equivalent to

$$(xz)(yt) = ((xy)z)t + ((yz)t)x + ((zt)x)y + ((tx)y)z \qquad \text{(Sagle identity)}$$

for all $x, y, z, t \in A$. Note that each Lie algebra is a Malcev algebra, thus all Lie-admissible algebras are Malcev-admissible. Here we have an example of a 4-dimensional non-Lie Malcev algebra.

Example 2.1. Let A be a vector space over $\mathbb{F}$ with a basis $\{e_1, e_2, e_3, e_4\}$. A direct calculation verifies that these non-zero products: $e_1e_2 = -e_2, e_1e_3 = -e_3, e_1e_4 = e_4, e_2e_3 = 2e_4$, give rise to a non-Lie Malcev algebra structure on A; see [18, Section 3]. ◇

Let A be a Malcev algebra over $\mathbb{F}$. A pair (V, ρ) of a vector space V over $\mathbb{F}$ and a linear map $\rho : A \longrightarrow \mathfrak{gl}(V)$ is called a **representation** of A if

$$\rho((xy)z) = \rho(x)\rho(y)\rho(z) - \rho(z)\rho(x)\rho(y) + \rho(y)\rho(zx) - \rho(yz)\rho(x) \tag{2.1}$$

for all $x, y, z \in A$. Note that when A is a Lie algebra, a Malcev representation of A is not necessarily a Lie representation; see for example [22, Section 3] and [10]. Two representations (V_1, ρ_1) and (V_2, ρ_2) are **isomorphic** if there exists a linear isomorphism $\varphi : V_2 \longrightarrow V_1$ such that $\rho_1(x) \circ \varphi = \varphi \circ \rho_2(x)$ for all $x \in A$.

Given a representation (V, ρ) of A, there exists a Malcev algebra structure on the direct sum $A \oplus V$ of vector spaces given by

$$(x, u)(y, v) = (xy, \rho(x)v - \rho(y)u) \tag{2.2}$$

for all $x, y \in A$ and $u, v \in V$. This Malcev algebra is called the **semi-direct product** of A and V and denoted by $A \ltimes_\rho V$. Moreover, consider the dual space V^* of V and a natural pairing $\langle -, - \rangle : V^* \times V \longrightarrow \mathbb{F}$. The **dual representation** (V^*, ρ^*) of (V, ρ) is defined by $\langle \rho^*(x)\xi, v \rangle = -\langle \xi, \rho(x)v \rangle$ for all $x \in A, \xi \in V^*$ and $v \in V$. See [12] for a survey on structures and representations of Malcev algebras. The following two examples of representations are necessary to us.

Example 2.2. Let A be a Malcev algebra over $\mathbb{F}$. As in the case of Lie algebras, the linear map $\mathrm{ad} : A \longrightarrow \mathrm{End}(A)$ sending x to ad_x, where $\mathrm{ad}_x(y) = xy$ for all $y \in A$, together with A, forms a representation (A, ad) of A, which is called the **adjoint representation** of A. The corresponding dual representation (A^*, ad^*) is called the **coadjoint representation** of A. ◇

2.2. $\mathcal{O}$-operators of Malcev algebras

Comparing with $\mathcal{O}$-operators of Lie algebras [11, Section 2], we introduce the notion of an $\mathcal{O}$-operator of a Malcev algebra that also generalizes the concept of a Rota-Baxter operator (of weight zero) on a Malcev algebra appeared in [13, Definition 8].

Definition 2.3. Let A be a Malcev algebra over $\mathbb{F}$ and (V, ρ) be a representation of A. A linear map $T : V \longrightarrow A$ is called an **$\mathcal{O}$-operator** of A associated to (V, ρ) if

$$T(v)T(w) = T(\rho(T(v))w - \rho(T(w))v) \tag{2.3}$$

for all $v, w \in V$. As stated previously, we write $\mathcal{O}_A(V, \rho)$ for the set of all $\mathcal{O}$-operators of A associated to (V, ρ). In particular, Rota-Baxter operators (of weight 0) of A are nothing but $\mathcal{O}$-operators associated to (A, ad). ◇

Example 2.4. Continued with Example 2.1, we consider the coadjoint representation (A^*, ad^*) of A. Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ be the basis of A^* dual to $\{e_1, e_2, e_3, e_4\}$. With respect to the two bases, a linear map $T : A^* \longrightarrow A$ corresponds to a 4×4-matrix. A direct verification shows that the following matrices

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ -a & -b & -c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ c & d & e & f \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 2a^2/k & a & b \\ 0 & 2a & k & c \\ -a & -b & -c & d \end{pmatrix}$$

are $\mathcal{O}$-operators of A associated to (A^*, ad^*), where $a, b, c, d, e, f \in \mathbb{F}$ and $k \in \mathbb{F} \setminus \{0\}$. ◇

Example 2.5. Consider the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ spanned by $\{x, y, z\}$ with nontrivial relations $[x, y] = 2y, [x, z] = -2z$ and $[y, z] = x$, which can also be viewed as a Malcev algebra. Suppose that V is a vector space spanned by $\{u, v\}$. It was proved in [6, Section 6] that the action of $\mathfrak{sl}_2(\mathbb{C})$ on V given by

$$xu = -2u, xv = 2v, yu = 0, yv = -2u, zu = -2v, zv = 0$$

makes V become an irreducible non-Lie Malcev representation of $\mathfrak{sl}_2(\mathbb{C})$. One can verify that

$$\begin{pmatrix} a & 2b & 0 \\ b & c & 0 \end{pmatrix} \text{ and } \begin{pmatrix} a & 0 & 0 \\ b & 0 & -2a \end{pmatrix}$$

are $\mathcal{O}$-operators of $\mathfrak{sl}_2(\mathbb{C})$ associated to this representation, where $a, b, c \in \mathbb{C}$. ◇

Lemma 2.6. *Suppose that V is a vector space over $\mathbb{F}$ and $r = \sum_i x_i \otimes y_i \in V \otimes V$, $\xi \in V^*$. Then $T_r(\xi) = \sum_i \langle \xi, y_i \rangle x_i$. In particular, if r is skew-symmetric, then $T_r(\xi) = -\sum_i \langle \xi, x_i \rangle y_i$.*

Proof. We have $\langle \eta, T_r(\xi) \rangle = \langle \eta \otimes \xi, r \rangle = \sum_i \langle \eta \otimes \xi, x_i \otimes y_i \rangle = \sum_i \langle \eta, x_i \rangle \langle \xi, y_i \rangle = \langle \eta, \sum_i \langle \xi, y_i \rangle x_i \rangle$ for all $\eta \in V^*$. Thus $\langle \eta, T_r(\xi) - \sum_i \langle \xi, y_i \rangle x_i \rangle = 0$. As the natural pairing is non-degenerate, it follows that $T_r(\xi) - \sum_i \langle \xi, y_i \rangle x_i = 0$. For the second statement, recall that $\sigma(r) = -r$ and we see that $\langle \eta, T_r(\xi) \rangle = \langle \eta \otimes \xi, r \rangle = -\langle \eta \otimes \xi, \sigma(r) \rangle = -\sum_i \langle \eta \otimes \xi, y_i \otimes x_i \rangle = -\langle \eta, \sum_i \langle \xi, x_i \rangle y_i \rangle$. The same reason as before implies that $T_r(\xi) = -\sum_i \langle \xi, x_i \rangle y_i$, as desired. □

Lemma 2.7. *Let V be a vector space over $\mathbb{F}$ and $r = \sum_i x_i \otimes y_i \in V \otimes V$. Then r is skew-symmetric if and only if $\langle \xi, T_r(\eta) \rangle = -\langle \eta, T_r(\xi) \rangle$ for all $\xi, \eta \in V^*$.*

Proof. ($\Longrightarrow$) Since r is skew-symmetric, we see that

$$-\langle \eta, T_r(\xi) \rangle = -\langle \eta \otimes \xi, r \rangle = \langle \eta \otimes \xi, \sigma(r) \rangle = \sum_i \langle \eta \otimes \xi, y_i \otimes x_i \rangle = \left\langle \xi, \sum_i \langle \eta, y_i \rangle x_i \right\rangle = \langle \xi, T_r(\eta) \rangle .$$

The last equation follows from Lemma 2.6.

($\Longleftarrow$) Note that

$$\begin{aligned}
\langle \eta \otimes \xi, \sigma(r) + r \rangle &= \langle \eta \otimes \xi, \sigma(r) \rangle + \langle \eta \otimes \xi, r \rangle \\
&= \langle \xi, T_r(\eta) \rangle + \sum_i \langle \eta, x_i \rangle \langle \xi, y_i \rangle \quad \text{(by Lemma 2.6)} \\
&= -\langle \eta, T_r(\xi) \rangle + \sum_i \langle \eta, x_i \rangle \langle \xi, y_i \rangle \quad \text{(by the assumption)} \\
&= -\sum_i \langle \xi, y_i \rangle \langle \eta, x_i \rangle + \sum_i \langle \eta, x_i \rangle \langle \xi, y_i \rangle \\
&= 0.
\end{aligned}$$

As the natural pairing is non-degenerate, $\sigma(r) + r = 0$, i.e., r is skew-symmetric. □

We are ready to prove Theorem 1.1.

Proof. We first assume that $r = \sum_i x_i \otimes y_i \in A \otimes A$ is skew-symmetric. Note that $(A^{\otimes 3})^* = (A^*)^{\otimes 3}$ when A is finite dimensional. For arbitrarily chosen $\xi, \eta, \zeta \in A^*$, we consider the natural pairing on $A^{\otimes 3}$ and see that

$$\begin{aligned}
\langle \xi \otimes \eta \otimes \zeta, r_{12} r_{13} \rangle &= \sum_{i,j} \left\langle \xi \otimes \eta \otimes \zeta, x_i x_j \otimes y_i \otimes y_j \right\rangle \\
&= \sum_{i,j} \left\langle \xi, x_i x_j \right\rangle \langle \eta, y_i \rangle \left\langle \zeta, y_j \right\rangle \\
&= \sum_{i,j} \left\langle \xi, (\langle \eta, y_i \rangle x_i) \left(\left\langle \zeta, y_j \right\rangle x_j \right) \right\rangle .
\end{aligned}$$

On the other hand, it follows from Lemma 2.6 that $\langle \xi, T_r(\eta) T_r(\zeta) \rangle = \sum_{i,j} \left\langle \xi, (\langle \eta, y_i \rangle x_i) \left(\left\langle \zeta, y_j \right\rangle x_j \right) \right\rangle$. Thus

$$\langle \xi \otimes \eta \otimes \zeta, r_{12} r_{13} \rangle = \langle \xi, T_r(\eta) T_r(\zeta) \rangle .$$

Similarly, we have $\langle \xi \otimes \eta \otimes \zeta, r_{13} r_{23} \rangle = \langle \zeta, T_r(\xi) T_r(\eta) \rangle$ and $\langle \xi \otimes \eta \otimes \zeta, r_{23} r_{12} \rangle = \langle \eta, T_r(\xi) T_r(\zeta) \rangle$. Hence,

$$\langle \xi \otimes \eta \otimes \zeta, r_{12} r_{13} + r_{13} r_{23} - r_{23} r_{12} \rangle = \langle \xi, T_r(\eta) T_r(\zeta) \rangle + \langle \zeta, T_r(\xi) T_r(\eta) \rangle - \langle \eta, T_r(\xi) T_r(\zeta) \rangle . \tag{2.4}$$

This key equation involves the CYBE. To establish links between this equation and O-operators, we consider the coadjoint representation (A^*, ad^*) of A and note that

$$\begin{aligned}\left\langle \xi, T_r(\mathrm{ad}^*_{T_r(\eta)}(\zeta))\right\rangle &= -\left\langle \mathrm{ad}^*_{T_r(\eta)}(\zeta), T_r(\xi)\right\rangle \quad \text{(by Lemma 2.7)}\\ &= \left\langle \zeta, \mathrm{ad}_{T_r(\eta)}(T_r(\xi))\right\rangle\\ &= \langle \zeta, T_r(\eta)T_r(\xi)\rangle\\ &= -\langle \zeta, T_r(\xi)T_r(\eta)\rangle .\end{aligned}$$

Similarly, $\left\langle \xi, T_r(\mathrm{ad}^*_{T_r(\zeta)}(\eta))\right\rangle = -\langle \eta, T_r(\xi)T_r(\zeta)\rangle$. Hence,

$$\begin{aligned}&\left\langle \xi, T_r(\eta)T_r(\zeta) - T_r(\mathrm{ad}^*_{T_r(\eta)}(\zeta)) + T_r(\mathrm{ad}^*_{T_r(\zeta)}(\eta))\right\rangle\\ =\ &\langle \xi, T_r(\eta)T_r(\zeta)\rangle + \langle \zeta, T_r(\xi)T_r(\eta)\rangle - \langle \eta, T_r(\xi)T_r(\zeta)\rangle \\ =\ &\langle \xi\otimes\eta\otimes\zeta, r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}\rangle .\end{aligned} \tag{2.5}$$

Here the last equation follows from Eq. (2.4).

Now we are in a position to complete the proof. In fact, if $r \in \mathcal{S}(A)$, then $r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0$. Thus it follows from Eq. (2.5) that $\left\langle \xi, T_r(\eta)T_r(\zeta) - T_r(\mathrm{ad}^*_{T_r(\eta)}(\zeta)) + T_r(\mathrm{ad}^*_{T_r(\zeta)}(\eta))\right\rangle = 0$. Since ξ is arbitrary, we see that $T_r(\eta)T_r(\zeta) - T_r(\mathrm{ad}^*_{T_r(\eta)}(\zeta)) + T_r(\mathrm{ad}^*_{T_r(\zeta)}(\eta)) = 0$ for all $\eta, \zeta \in A^*$, i.e., $T_r \in O_A(A^*, \mathrm{ad}^*)$. Conversely, assume that $T_r \in O_A(A^*, \mathrm{ad}^*)$. By Eq. (2.5), we see that $\langle \xi\otimes\eta\otimes\zeta, r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}\rangle = 0$. Therefore, we have $r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0$, showing that r is a solution of the CYBE on A. □

3. Bilinear Forms, the CYBE and Semi-direct Products

Specializing in Malcev algebras admitting non-degenerate invariant bilinear forms, we establish an analogue of Theorem 1.1 in which O-operators could be replaced by Rota-Baxter operators of weight zero; see Corollary 3.3. We also give detailed proofs of Theorems 1.2 and 1.3. Throughout this section we let A be a Malcev algebra over $\mathbb{F}$.

3.1. *Invariant bilinear forms*

A bilinear form $\mathcal{B} : A \times A \longrightarrow \mathbb{F}$ is called **invariant** if $\mathcal{B}(xy, z) = \mathcal{B}(x, yz)$ for all $x, y, z \in A$.

Proposition 3.1. *Let A be a Malcev algebra over $\mathbb{F}$. Then the adjoint representation (A, ad) and the coadjoint representation (A^*, ad^*) of A are isomorphic if and only if A admits a non-degenerate invariant bilinear form.*

Proof. ($\Longrightarrow$) Suppose $\varphi : A \longrightarrow A^*$ is a linear isomorphism such that $\mathrm{ad}^*_x \circ\varphi = \varphi \circ \mathrm{ad}_x$ for arbitrary $x \in A$. Thus $\varphi(\mathrm{ad}_x(y)) = \mathrm{ad}^*_x(\varphi(y))$ for all $y \in A$. We define a bilinear form $\mathcal{B}_\varphi : A \times A \longrightarrow \mathbb{F}$ by $(x, y) \mapsto \langle\varphi(x), y\rangle$. To see that $\mathcal{B}_\varphi$ is invariant, we take $z \in A$, then $\mathcal{B}_\varphi(xy, z) = -\mathcal{B}_\varphi(yx, z) = -\mathcal{B}_\varphi(\mathrm{ad}_y(x), z) = -\left\langle\varphi(\mathrm{ad}_y(x)), z\right\rangle = -\left\langle\mathrm{ad}^*_y(\varphi(x)), z\right\rangle = \left\langle\varphi(x), \mathrm{ad}_y(z)\right\rangle = \mathcal{B}_\varphi(x, yz)$, which means $\mathcal{B}_\varphi$ is invariant. As φ is bijective and the natural pairing on A is non-degenerate, it follows that $\mathcal{B}_\varphi$ is also non-degenerate.

($\Longleftarrow$) Assume that there exists a non-degenerate invariant bilinear form $\mathcal{B}$ on A. We define

$$\varphi_{\mathcal{B}} : A \longrightarrow A^* \text{ by } x \mapsto \mathcal{B}_x, \tag{3.1}$$

where $\mathcal{B}_x(y) := \mathcal{B}(x, y)$ for all $y \in A$. We first note that $\varphi_{\mathcal{B}}$ is linear as $\mathcal{B}$ is bilinear. To see that $\varphi_{\mathcal{B}}$ is bijective, assume that $x_1, x_2 \in A$ are two elements such that $\mathcal{B}_{x_1} = \mathcal{B}_{x_2}$. Then $\mathcal{B}(x_1, y) = \mathcal{B}(x_2, y)$ for all $y \in A$, i.e., $\mathcal{B}(x_1 - x_2, y) = 0$. As $\mathcal{B}$ is non-degenerate, we have $x_1 = x_2$. Thus $\varphi_{\mathcal{B}}$ is injective. This fact, together with $\dim(A) = \dim(A^*)$, implies that $\varphi_{\mathcal{B}}$ is surjective. Hence, $\varphi_{\mathcal{B}}$ is a linear isomorphism. Moreover, we choose a natural pairing on A such that $\langle\mathcal{B}_x, y\rangle = \mathcal{B}(x, y)$ for all $x, y \in A$ as $\mathcal{B}$ is non-degenerate. Since $\mathcal{B}$ is invariant, for all $x, y, z \in A$, we see that $\langle\varphi_{\mathcal{B}}(\mathrm{ad}_x(y)) - \mathrm{ad}^*_x(\varphi_{\mathcal{B}}(y)), z\rangle = \left\langle\mathcal{B}_{xy}, z\right\rangle - \left\langle\mathrm{ad}^*_x(\mathcal{B}_y), z\right\rangle = \left\langle\mathcal{B}_{xy}, z\right\rangle + \left\langle\mathcal{B}_y, \mathrm{ad}_x(z)\right\rangle = \left\langle\mathcal{B}_{xy}, z\right\rangle + \left\langle\mathcal{B}_y, xz\right\rangle = \mathcal{B}(xy, z) + \mathcal{B}(y, xz) = -\mathcal{B}(yx, z) + \mathcal{B}(y, xz) = -\mathcal{B}(y, xz) + \mathcal{B}(y, xz) = 0.$

Hence, $\varphi_{\mathcal{B}}(\mathrm{ad}_x(y)) = \mathrm{ad}_x^*(\varphi_{\mathcal{B}}(y))$ for all $y \in A$, i.e., $\varphi_{\mathcal{B}} \circ \mathrm{ad}_x = \mathrm{ad}_x^* \circ \varphi_{\mathcal{B}}$. Therefore, $\varphi_{\mathcal{B}}$ is an isomorphism between (A, ad) and (A^*, ad^*). □

Proposition 3.2. *Let $\varphi : (V_1, \rho_1) \longrightarrow (V_2, \rho_2)$ be an isomorphism of two representations of a Malcev algebra A over $\mathbb{F}$. Then for each $T \in \mathcal{O}_A(V_2, \rho_2)$, the composition $T \circ \varphi \in \mathcal{O}_A(V_1, \rho_1)$. In particular, there is a one-to-one correspondence between $\mathcal{O}_A(V_1, \rho_1)$ and $\mathcal{O}_A(V_2, \rho_2)$ when the representations (V_1, ρ_1) and (V_2, ρ_2) are isomorphic.*

Proof. Suppose $T \in \mathcal{O}_A(V_2, \rho_2)$ and $v, w \in V_1$ are arbitrary elements. Since φ is an isomorphism of representations, we have

$$
\begin{aligned}
& (T \circ \varphi)(\rho_1((T \circ \varphi)(v))w - \rho_1((T \circ \varphi)(w))v) \\
= \;& T(\varphi(\rho_1(T(\varphi(v)))w) - \varphi(\rho_1(T(\varphi(w)))v)) \\
= \;& T(\rho_2(T(\varphi(v)))\varphi(w) - \rho_2(T(\varphi(w)))\varphi(v)) \\
= \;& T(\varphi(v))T(\varphi(w)) = (T \circ \varphi)(v)(T \circ \varphi)(w),
\end{aligned}
$$

which implies that $T \circ \varphi \in \mathcal{O}_A(V_1, \rho_1)$. Similarly, for each $S \in \mathcal{O}_A(V_1, \rho_1)$, one can show that $S \circ \varphi^{-1} \in \mathcal{O}_A(V_2, \rho_2)$. We define a map $\Phi : \mathcal{O}_A(V_1, \rho_1) \longrightarrow \mathcal{O}_A(V_2, \rho_2)$ by sending each S to $S \circ \varphi^{-1}$ and another map $\Psi : \mathcal{O}_A(V_2, \rho_2) \longrightarrow \mathcal{O}_A(V_1, \rho_1)$ by sending every T to $T \circ \varphi$. Clearly, $\Psi \circ \Phi = 1_{\mathcal{O}_A(V_1, \rho_1)}$ and $\Phi \circ \Psi = 1_{\mathcal{O}_A(V_2, \rho_2)}$. Thus Ψ is a bijection and the proof is completed. □

Together with Theorem 1.1, Propositions 3.1 and 3.2, imply the following result.

Corollary 3.3. *Let A be a Malcev algebra admitting a non-degenerate invariant bilinear form $\mathcal{B}$ and r be a skew-symmetric element in $A \otimes A$. Then $r \in \mathcal{S}(A)$ if and only if $T_r \circ \varphi_{\mathcal{B}}$ is a Rota-Baxter operator of weight zero on A, where T_r and $\varphi_{\mathcal{B}}$ are defined as in Eqs.* (1.2) *and* (3.1) *respectively.*

3.2. Symplectic forms

A non-degenerate skew-symmetric bilinear form $\mathcal{B} : A \times A \longrightarrow \mathbb{F}$ is said to be **symplectic** if $\mathcal{B}(xy, z) + \mathcal{B}(yz, x) + \mathcal{B}(zx, y) = 0$ for all $x, y, z \in A$.

Lemma 3.4. *Let V be a vector space over $\mathbb{F}$ and $r \in V \otimes V$ be non-degenerate. Then r is skew-symmetric if and only if the bilinear form $\mathcal{B}_r : V \times V \longrightarrow \mathbb{F}$ defined by $(x, y) \mapsto \left\langle T_r^{-1}(x), y \right\rangle$ is skew-symmetric, where T_r is defined as in Eq.* (1.2).

Proof. Note that r is non-degenerate, thus for any $x, y \in V$, there exist unique $\xi, \eta \in V^*$ such that $x = T_r(\xi)$ and $y = T_r(\eta)$. Now we assume that r is skew-symmetric. Then $\mathcal{B}_r(x, y) + \mathcal{B}_r(y, x) = \left\langle T_r^{-1}(x), y \right\rangle + \left\langle T_r^{-1}(y), x \right\rangle = \langle \xi, T_r(\eta) \rangle + \langle \eta, T_r(\xi) \rangle = 0$, where the last equality follows from Lemma 2.7. Hence, $\mathcal{B}_r$ is skew-symmetric. Conversely, by Lemma 2.7, it suffices to show that $\langle \xi, T_r(\eta) \rangle + \langle \eta, T_r(\xi) \rangle = 0$ for all $\xi, \eta \in V^*$. In fact, $\langle \xi, T_r(\eta) \rangle + \langle \eta, T_r(\xi) \rangle = \left\langle T_r^{-1}(T_r(\xi)), T_r(\eta) \right\rangle + \left\langle T_r^{-1}(T_r(\eta)), T_r(\xi) \right\rangle = \mathcal{B}_r(T_r(\xi), T_r(\eta)) + \mathcal{B}_r(T_r(\eta), T_r(\xi)) = \mathcal{B}_r(x, y) + \mathcal{B}_r(y, x) = 0$, since $\mathcal{B}_r$ is skew-symmetric. □

We can prove Theorem 1.2 as follows.

Proof. Suppose that $x, y, z \in A$ are arbitrary elements. Note that r is non-degenerate, thus there exist unique $\xi, \eta \in A^*$ such that $x = T_r(\xi)$ and $y = T_r(\eta)$.

($\Longrightarrow$) As $r \in \mathcal{S}(A)$ is skew-symmetric, it follows from Theorem 1.1 that

$$
\begin{aligned}
\mathcal{B}_r(xy, z) &= \left\langle T_r^{-1}(xy), z \right\rangle = \left\langle T_r^{-1}(T_r(\xi)T_r(\eta)), z \right\rangle \\
&= \left\langle T_r^{-1}(T_r(\mathrm{ad}^*_{T_r(\xi)}(\eta) - \mathrm{ad}^*_{T_r(\eta)}(\xi)), z \right\rangle \\
&= \left\langle \mathrm{ad}^*_{T_r(\xi)}(\eta) - \mathrm{ad}^*_{T_r(\eta)}(\xi), z \right\rangle \\
&= \left\langle \mathrm{ad}^*_{T_r(\xi)}(\eta), z \right\rangle - \left\langle \mathrm{ad}^*_{T_r(\eta)}(\xi), z \right\rangle.
\end{aligned}
$$

Note that

$$\left\langle \mathrm{ad}^*_{T_r(\xi)}(\eta), z\right\rangle = -\left\langle \eta, \mathrm{ad}_{T_r(\xi)}(z)\right\rangle = -\langle \eta, T_r(\xi)z\rangle = -\left\langle T_r^{-1}(y), xz\right\rangle = -\mathcal{B}_r(y, xz).$$

Similarly, $\left\langle \mathrm{ad}^*_{T_r(\eta)}(\xi), z\right\rangle = -\mathcal{B}_r(x, yz)$. Thus $\mathcal{B}_r(xy,z) = \mathcal{B}_r(x, yz) - \mathcal{B}_r(y, xz) = \mathcal{B}_r(x, yz) + \mathcal{B}_r(y, zx)$. Lemma 3.4 asserts that $\mathcal{B}_r$ is skew-symmetric. Hence $\mathcal{B}_r(xy,z)+\mathcal{B}_r(yz,x)+\mathcal{B}_r(zx,y) = 0$, that is, $\mathcal{B}_r$ is a symplectic form.

($\Longleftarrow$) Now we assume that $\mathcal{B}_r$ is a symplectic form. By Theorem 1.1, it suffices to show that $T_r \in \mathcal{O}_A(A^*, \mathrm{ad}^*)$. We have seen from the previous proof that $\mathcal{B}_r(xy,z) = \left\langle T_r^{-1}(xy), z\right\rangle = \left\langle T_r^{-1}(T_r(\xi)T_r(\eta)), z\right\rangle$, $\mathcal{B}_r(yz, x) = -\mathcal{B}_r(x, yz) = \left\langle \mathrm{ad}^*_{T_r(\eta)}(\xi), z\right\rangle$ and $\mathcal{B}_r(zx, y) = -\mathcal{B}_r(xz, y) = \mathcal{B}_r(y, xz) = -\left\langle \mathrm{ad}^*_{T_r(\xi)}(\eta), z\right\rangle$. Thus

$$\begin{aligned} 0 &= \mathcal{B}_r(xy,z)+\mathcal{B}_r(yz,x)+\mathcal{B}_r(zx,y) \\ &= \left\langle T_r^{-1}(T_r(\xi)T_r(\eta)), z\right\rangle + \left\langle \mathrm{ad}^*_{T_r(\eta)}(\xi), z\right\rangle - \left\langle \mathrm{ad}^*_{T_r(\xi)}(\eta), z\right\rangle \\ &= \left\langle T_r^{-1}(T_r(\xi)T_r(\eta)) + \mathrm{ad}^*_{T_r(\eta)}(\xi) - \mathrm{ad}^*_{T_r(\xi)}(\eta), z\right\rangle \end{aligned}$$

for all $z \in A$. As the natural pairing on A is non-degenerate, it follows that $T_r^{-1}(T_r(\xi)T_r(\eta)) + \mathrm{ad}^*_{T_r(\eta)}(\xi) - \mathrm{ad}^*_{T_r(\xi)}(\eta) = 0$, i.e., $T_r^{-1}(T_r(\xi)T_r(\eta)) = \mathrm{ad}^*_{T_r(\xi)}(\eta) - \mathrm{ad}^*_{T_r(\eta)}(\xi)$. Multiplying the two sides of this equation with T_r, we see that $T_r(\xi)T_r(\eta) = T_r(\mathrm{ad}^*_{T_r(\xi)}(\eta) - \mathrm{ad}^*_{T_r(\eta)}(\xi))$. This means that $T_r \in \mathcal{O}_A(A^*, \mathrm{ad}^*)$, and therefore, $r \in \mathcal{S}(A)$, as desired. □

Example 3.5. Let A be the 4-dimensional Malcev algebra with the basis $\{e_1, \ldots, e_4\}$ defined in Example 2.1. With respect to this basis, a direct calculation shows that any symplectic form $\mathcal{B}$ on A has the following form

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & d \\ -b & c & 0 & e \\ -c & -d & -e & 0 \end{pmatrix},$$

where $a, b, c, d, e \in \mathbb{F}$ and $c^2 - ae \pm bd \neq 0$. By Theorem 1.2, one can construct all non-degenerate skew-symmetric solutions of the CYBE on A. For instance, if we set $c = 1$ and $a = b = d = e = 0$, then $r = e_1 \otimes e_4 - e_4 \otimes e_1 - e_2 \otimes e_3 + e_3 \otimes e_2$ is a non-degenerate skew-symmetric solution. ◇

3.3. The CYBE and semi-direct products

To give a proof of Theorem 1.3, we use notations appeared in Introduction. The identification of $A \otimes V^*$ with $\mathrm{Hom}(V, A)$ can be realized via the linear isomorphism τ defined by sending $x \otimes \xi$ to $\tau_{x\otimes\xi}$, where $x \in A, \xi \in V^*$ and $\tau_{x\otimes\xi}(v) := \xi(v)x$ for all $v \in V$. Let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{\xi_1, \ldots, \xi_n\}$ be the dual basis of V^*. Then we can identify an element $T \in \mathrm{Hom}(V, A)$ with the element

$$\widetilde{T} := \sum_{i=1}^{n} T(v_i) \otimes \xi_i \in A \otimes V^* \subseteq (A \oplus V^*) \otimes (A \oplus V^*). \tag{3.2}$$

We define

$$r_T := \widetilde{T} - \sigma(\widetilde{T}) = \sum_{i=1}^{n} (T(v_i) \otimes \xi_i - \xi_i \otimes T(v_i)) \in (A \oplus V^*) \otimes (A \oplus V^*). \tag{3.3}$$

To analyze equivalent conditions of r_T being a solution of the CYBE on $A \ltimes_{\rho^*} V^*$, we first note that

$$\begin{aligned} (r_T)_{12}(r_T)_{13} = \sum_{i,j=1}^{n} (&T(v_i)T(v_j) \otimes \xi_i \otimes \xi_j - \rho^*(T(v_i))\xi_j \otimes \xi_i \otimes T(v_j) \\ &+\rho^*(T(v_j))\xi_i \otimes T(v_i) \otimes \xi_j). \end{aligned} \tag{3.4}$$

As $\rho^*(T(v_i))\xi_j \in V^*$ for all $i, j \in \{1, \ldots, n\}$, we assume that $\rho^*(T(v_i))\xi_j = a_1(ij)\xi_1 + \cdots + a_n(ij)\xi_n$, where $a_s(ij) \in \mathbb{F}$. For any $k \in \{1, \ldots, n\}$, we have $a_k(ij) = \langle \sum_{s=1}^n a_s(ij)\xi_s, v_k \rangle = \left\langle \rho^*(T(v_i))\xi_j, v_k \right\rangle = -\left\langle \xi_j, \rho(T(v_i))v_k \right\rangle$. Thus

$$\rho^*(T(v_i))\xi_j = -\sum_{k=1}^n \left\langle \xi_j, \rho(T(v_i))v_k \right\rangle \xi_k. \tag{3.5}$$

Similarly, we observe that

$$\rho(T(v_i))v_j = \sum_{k=1}^n \left\langle \xi_k, \rho(T(v_i))v_j \right\rangle v_k. \tag{3.6}$$

Hence, it follows from Eqs. (3.5) and (3.6) that

$$\begin{aligned}
\sum_{i,j=1}^n \rho^*(T(v_i))\xi_j \otimes \xi_i \otimes T(v_j) &= \sum_{i,j=1}^n (-\sum_{k=1}^n \left\langle \xi_j, \rho(T(v_i))v_k \right\rangle \xi_k) \otimes \xi_i \otimes T(v_j) \\
&= -\sum_{i,k=1}^n \xi_k \otimes \xi_i \otimes T(\sum_{j=1}^n \left\langle \xi_j, \rho(T(v_i))v_k \right\rangle v_j) \\
&= -\sum_{i,j=1}^n \xi_j \otimes \xi_i \otimes T(\sum_{k=1}^n \left\langle \xi_k, \rho(T(v_i))v_j \right\rangle v_k) \\
&= -\sum_{i,j=1}^n \xi_j \otimes \xi_i \otimes T(\rho(T(v_i))v_j).
\end{aligned}$$

Further, a similar calculation shows that

$$\sum_{i,j=1}^n \rho^*(T(v_j))\xi_i \otimes T(v_i) \otimes \xi_j = -\sum_{i,j=1}^n \xi_i \otimes T(\rho(T(v_j))v_i) \otimes \xi_j.$$

Taking the previous two equations back to Eq. (3.4), we see that

$$(r_T)_{12}(r_T)_{13} = \sum_{i,j=1}^n ((T(v_i)T(v_j) \otimes \xi_i \otimes \xi_j + \xi_j \otimes \xi_i \otimes T(\rho(T(v_i))v_j) - \xi_i \otimes T(\rho(T(v_j))v_i) \otimes \xi_j).$$

We proceed in this way on $(r_T)_{13}(r_T)_{23}$ and $(r_T)_{23}(r_T)_{12}$ and eventually derive

$$\begin{aligned}
& (r_T)_{12}(r_T)_{13} + (r_T)_{13}(r_T)_{23} - (r_T)_{23}(r_T)_{12} \\
= & \sum_{i,j=1}^n (T(v_i)T(v_j) - T(\rho(T(v_i))v_j) + T(\rho(T(v_j))v_i)) \otimes \xi_i \otimes \xi_j \\
& + \sum_{i,j=1}^n \xi_i \otimes (T(v_j)T(v_i) - T(\rho(T(v_j))v_i) + T(\rho(T(v_i))v_j)) \otimes \xi_j \\
& + \sum_{i,j=1}^n \xi_i \otimes \xi_j \otimes (T(v_i)T(v_j) - T(\rho(T(v_i))v_j) + T(\rho(T(v_j))v_i)).
\end{aligned} \tag{3.7}$$

Now we are ready to give a proof to Theorem 1.3.

Proof. ($\Longrightarrow$) Assume that $T \in \mathcal{O}_A(V, \rho)$ is an $\mathcal{O}$-operator. Then $T(v_i)T(v_j) - T(\rho(T(v_i))v_j) + T(\rho(T(v_j))v_i) = 0$ for all $i, j \in \{1, \ldots, n\}$. Thus the right-hand side of Eq. (3.7) is zero. This implies that $(r_T)_{12}(r_T)_{13} + (r_T)_{13}(r_T)_{23} - (r_T)_{23}(r_T)_{12} = 0$, i.e., r_T is a solution of the CYBE on $A \ltimes_{\rho^*} V^*$.

($\Longleftarrow$) Suppose that $r_T \in \mathcal{S}(A \ltimes_{\rho^*} V^*)$ is a solution, that is $(r_T)_{12}(r_T)_{13} + (r_T)_{13}(r_T)_{23} - (r_T)_{23}(r_T)_{12} = 0$. Thus the right-hand side of Eq. (3.7) is equal to zero. Let $\{x_1, \ldots, x_m\}$ be a basis of A. Assume that $T(v_i)T(v_j) - T((\rho(T(v_i))v_j - \rho(T(v_j))v_i) = c_1(ij)x_1 + \cdots + c_m(ij)x_m$ for some $c_1(ij), \ldots, c_m(ij) \in \mathbb{F}$. Hence,

$$0 = \sum_{i,j=1}^{n} \sum_{k=1}^{m} c_k(ij)(x_k \otimes \xi_i \otimes \xi_j + \xi_j \otimes x_k \otimes \xi_i + \xi_i \otimes \xi_j \otimes x_k).$$

Since $\{x_k \otimes \xi_i \otimes \xi_j, \xi_j \otimes x_k \otimes \xi_i, \xi_i \otimes \xi_j \otimes x_k \mid 1 \leqslant i, j \leqslant n, 1 \leqslant k \leqslant m\}$ is a subset of a basis of $(A \ltimes_{\rho^*} V^*)^{\otimes 3}$, its elements are linearly independent over $\mathbb{F}$. Hence, $c_k(ij) = 0$ for all k, i and j, which means that

$$T(v_i)T(v_j) = T(\rho(T(v_i))v_j - \rho(T(v_j))v_i),$$

for all $i, j \in \{1, \ldots, n\}$. Therefore, $T \in \mathcal{O}_A(V, \rho)$ and the proof is completed. □

Example 3.6. We consider the 4-dimensional Malcev algebra A with the basis $\{e_1, \ldots, e_4\}$ and the skew-symmetric solution $r = e_1 \otimes e_4 - e_4 \otimes e_1 - e_2 \otimes e_3 + e_3 \otimes e_2$ of the CYBE on A described in Example 3.5. By Theorem 1.1, we see that the linear map $T : A^* \longrightarrow A$ defined by

$$T(\varepsilon_1) = -e_4, T(\varepsilon_2) = e_3, T(\varepsilon_3) = -e_2, T(\varepsilon_4) = e_1 \tag{3.8}$$

is an $\mathcal{O}$-operator of A associated to the coadjoint representation (A^*, ad^*), where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ is the dual basis of A^*.

Let $(A^*)^*$ be the dual space of A^* with a basis $\{x_1, \ldots, x_4\}$. We identify $(A^*)^*$ with A and thus $\{x_1, \ldots, x_4\}$ could be viewed as another basis of A. Take the $\mathcal{O}$-operator T in Theorem 1.3 as in Eq. (3.8). We note that all non-zero products in Malcev algebra $A \ltimes_{(\mathrm{ad}^*)^*} (A^*)^* = A \ltimes_{\mathrm{ad}} A$ are given by

$$\begin{aligned}
&e_1e_2 = -e_2, e_1e_3 = -e_3, e_1e_4 = e_4, e_2e_3 = 2e_4, e_1x_2 = -x_2, e_1x_3 = -x_3,\\
&e_1x_4 = x_4, e_2x_3 = 2x_4, e_2x_1 = x_2, e_3x_1 = x_3, e_4x_1 = -x_4, e_3x_2 = -2x_4.
\end{aligned}$$

It follows from Theorem 1.3 that $r_T = -e_4 \otimes x_1 + x_1 \otimes e_4 + e_3 \otimes x_2 - x_2 \otimes e_3 - e_2 \otimes x_3 + x_3 \otimes e_2 + e_1 \otimes x_4 - x_4 \otimes e_1$ is a skew-symmetric solution of CYBE on $A \ltimes_{\mathrm{ad}} A$. ◇

4. $\mathcal{O}$-operators and the CYBE on Pre-Malcev Algebras

After recalling basic facts on bimodules of pre-Malcev algebras, we study connections between $\mathcal{O}$-operators and compatible pre-Malcev structures on a Malcev algebra, giving a proof of Theorem 1.4 with two applications. Comparing with Theorems 1.1 and 1.3, we also derive several analogous results on symmetric solutions of the CYBE on pre-Malcev algebras.

4.1. $\mathcal{O}$-operators of Malcev algebras and Pre-Malcev algebras

Recall in [13, Definition 4] that a pre-Malcev algebra $\mathcal{A}$ is a vector space over $\mathbb{F}$ endowed with a binary product $\cdot$ satisfying an identity $P_M(x, y, z, t) = 0$, where

$$\begin{aligned}
&P_M(x, y, z, t)\\
= \;& (y \cdot z) \cdot (x \cdot t) - (z \cdot y) \cdot (x \cdot t) + ((x \cdot y) \cdot z) \cdot t - ((y \cdot x) \cdot z) \cdot t + (z \cdot (y \cdot x)) \cdot t\\
&-(z \cdot (x \cdot y)) \cdot t + y \cdot ((x \cdot z) \cdot t) - y \cdot ((z \cdot x) \cdot t) + z \cdot (x \cdot (y \cdot t)) - x \cdot (y \cdot (z \cdot t))
\end{aligned} \tag{4.1}$$

for all $x, y, z, t \in \mathcal{A}$. As Malcev-admissible algebras, pre-Malcev algebras extend the notion of pre-Lie algebras (or left-symmetric algebras) which have been studied extensively; see for example [1, 16, 19].

Example 4.1. Let A be the 4-dimensional Malcev algebra appeared in Example 2.1 with the basis $\{e_1, e_2, e_3, e_4\}$. A direct calculation verifies that the following non-zero noncommutative products:

$$e_1 \cdot e_2 = -e_2, e_1 \cdot e_3 = -e_3, e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = 2e_4,$$

give rise to a compatible pre-Malcev algebra structure $\mathcal{A}$ on A. In other words, $[\mathcal{A}] = A$. ◇

Let $(\mathcal{A}, \cdot)$ be a pre-Malcev algebra over $\mathbb{F}$. A triple $(V, \ell, \mathfrak{r})$ of a vector space V over $\mathbb{F}$ and two linear maps $\ell, \mathfrak{r} : \mathcal{A} \longrightarrow \mathrm{End}(V)(x \mapsto \ell_x, x \mapsto \mathfrak{r}_x)$ is called a **bimodule** of $\mathcal{A}$ if the following four equations hold:

$$\mathfrak{r}_x\mathfrak{r}_y\mathfrak{r}_z - \mathfrak{r}_x\mathfrak{r}_y\ell_z - \mathfrak{r}_x\ell_y\mathfrak{r}_z + \mathfrak{r}_x\ell_y\ell_z - \mathfrak{r}_{z\cdot(y\cdot x)} + \ell_y\mathfrak{r}_{z\cdot x} + \ell_{z\cdot y}\mathfrak{r}_x - \ell_{y\cdot z}\mathfrak{r}_x - \ell_z\mathfrak{r}_x\ell_y + \ell_z\mathfrak{r}_x\mathfrak{r}_y = 0, \tag{4.2}$$

$$\mathfrak{r}_x\mathfrak{r}_y\ell_z - \mathfrak{r}_x\mathfrak{r}_y\mathfrak{r}_z - \mathfrak{r}_x\ell_y\ell_z + \mathfrak{r}_x\ell_y\mathfrak{r}_z - \ell_z\mathfrak{r}_{y\cdot x} + \ell_y\ell_z\mathfrak{r}_x + \mathfrak{r}_{z\cdot x}\mathfrak{r}_y - \mathfrak{r}_{z\cdot x}\ell_y - \mathfrak{r}_{(y\cdot z)\cdot x} + \mathfrak{r}_{(z\cdot y)\cdot x} = 0, \tag{4.3}$$

$$\mathfrak{r}_x\ell_{y\cdot z} - \mathfrak{r}_x\ell_{z\cdot y} - \mathfrak{r}_x\mathfrak{r}_{y\cdot z} + \mathfrak{r}_x\mathfrak{r}_{z\cdot y} - \ell_y\ell_z\mathfrak{r}_x + \mathfrak{r}_{y\cdot(z\cdot x)} + \mathfrak{r}_{y\cdot x}\ell_z - \mathfrak{r}_{y\cdot x}\mathfrak{r}_z - \ell_z\mathfrak{r}_x\mathfrak{r}_y + \ell_z\mathfrak{r}_x\ell_y = 0, \tag{4.4}$$

$$\ell_{(x\cdot y)\cdot z} - \ell_{(y\cdot x)\cdot z} - \ell_{z\cdot(x\cdot y)} + \ell_{z\cdot(y\cdot x)} - \ell_x\ell_y\ell_z + \ell_z\ell_x\ell_y + \ell_{y\cdot z}\ell_x - \ell_{z\cdot y}\ell_x - \ell_y\ell_{z\cdot x} + \ell_y\ell_{x\cdot z} = 0, \tag{4.5}$$

where $x, y, z \in \mathcal{A}$. Equivalently, a triple $(V, \ell, \mathfrak{r})$ is an $\mathcal{A}$-bimodule if and only if the direct sum $\mathcal{A} \oplus V$ of vector spaces is turned into a pre-Malcev algebra, called the **semi-direct product** of $\mathcal{A}$ and V via $(\ell, \mathfrak{r})$, by defining the binary product on $\mathcal{A} \oplus V$ as

$$(x, u) \cdot (y, v) := (x \cdot y, \ell_x(v) + \mathfrak{r}_y(u)) \tag{4.6}$$

for all $x, y \in \mathcal{A}$ and $u, v \in V$. We denote this pre-Malcev algebra by $\mathcal{A} \ltimes_{\ell,\mathfrak{r}} V$.

Suppose that ℓ and $\mathfrak{r}$ are two linear maps from $\mathcal{A}$ to $\mathrm{End}(V)$. Consider the dual space V^* of V and $\mathrm{End}(V^*)$. We define two linear maps $\ell^*, \mathfrak{r}^* : \mathcal{A} \longrightarrow \mathrm{End}(V^*)$ by

$$\langle \ell^*_x(\xi), v\rangle := -\langle \xi, \ell_x(v)\rangle, \langle \mathfrak{r}^*_x(\xi), v\rangle := -\langle \xi, \mathfrak{r}_x(v)\rangle, \tag{4.7}$$

respectively, where $x \in \mathcal{A}, \xi \in V^*$ and $v \in V$. Moreover, if $(V, \ell, \mathfrak{r})$ is an $\mathcal{A}$-bimodule, then one can show via a direct check that $(V^*, \ell^* - \mathfrak{r}^*, -\mathfrak{r}^*)$ is also an $\mathcal{A}$-bimodule.

Let $(\mathcal{A}, \cdot)$ be a pre-Malcev algebra. For elements $x, y \in \mathcal{A}$, the left multiplication operator L_x is defined in the Introduction and we also define the right multiplication operator $R_x(y) := y \cdot x$. Let $L : \mathcal{A} \longrightarrow \mathrm{End}(\mathcal{A})$ with $x \mapsto L_x$ and $R : \mathcal{A} \longrightarrow \mathrm{End}(\mathcal{A})$ with $x \mapsto R_x$ for all $x \in \mathcal{A}$ be two linear maps. Then $(\mathcal{A}, L, R)$ is an $\mathcal{A}$-bimodule and hence $(\mathcal{A}^*, L^* - R^*, -R^*)$ is also an $\mathcal{A}$-bimodule.

A bimodule of a pre-Malcev algebra can be used to construct representations of the subadjacent Malcev algebra. In fact, if $(V, \ell, \mathfrak{r})$ is a bimodule of a pre-Malcev algebra $\mathcal{A}$, then it can be checked directly that (V, ℓ) and $(V, \ell - \mathfrak{r})$ are both representations of the Malcev algebra $[\mathcal{A}]$.

Before giving a proof of Theorem 1.4, we first reveal a general connection between $\mathcal{O}$-operators of Malcev algebras and pre-Malcev algebras, generalizing a link between Rota-Baxter operators and pre-Malcev algebras ([13, Proposition 9]); also see [2, Section 3] for the case of Lie algebras.

Proposition 4.2. *Let (V, ρ) be a representation of a Malcev algebra A. Given a $T \in \mathcal{O}_A(V, \rho)$, we define a binary product on V by $v * w := \rho(T(v))w$ for all $v, w \in V$. Then the following results hold.*

(1) $(V, *)$ is a pre-Malcev algebra.

(2) The binary product

$$T(v) \cdot T(w) := T(v * w) \tag{4.8}$$

gives rise to a pre-Malcev algebra structure on $T(V) := \{T(v) \mid v \in V\} \subseteq A$.

(3) In particular, if T is surjective, then there exists a compatible pre-Malcev algebra structure $\mathcal{A}_T$ on A.

Proof. (1) A direct verification on Eq. (4.1) applies to a proof of the first statement.

(2) We first show that Eq. (4.8) is well-defined. We only need to verify that, if $T(v) = 0$, then $T(v * w) = T(w * v) = 0$ for all $w \in V$. In fact, if $T(v)$=0, then for all $w \in W$, we have

$$\begin{aligned} T(v * w) &= T(\rho(T(v))w) = 0, \\ T(w * v) &= T(\rho(T(w))v) = T(\rho(T(w))v - \rho(T(v))w) = T(w)T(v) = 0. \end{aligned}$$

Hence Eq. (4.8) is well-defined. Then it is direct to verify Eq. (4.1) on $T(V)$.

(3) By the second statement, we obtain that $A = T(V)$ has a pre-Malcev algebra structure $\mathcal{A}_T$. It is sufficient to show that the pre-Malcev algebra $\mathcal{A}_T$ is compatible with A. In fact, $T(v) \cdot T(w) - T(w) \cdot T(v) = T(v * w) - T(w * v) = T(\rho(T(v))w) - T(\rho(T(w))v) = T(\rho(T(v))w - \rho(T(w))v) = T(v)T(w)$, where the last equation follows from the assumption that $T \in \mathcal{O}_A(V, \rho)$. Hence, $[\mathcal{A}_T] = A$, i.e., $\mathcal{A}_T$ is a compatible pre-Malcev algebra structure on A. □

Now we come to the proof of Theorem 1.4.

Proof. Since $T \in O_A(V,\rho)$ is invertible, for $x, y \in A$, there exist unique $v, w \in V$ such that $x = T(v)$ and $y = T(w)$. By the third statement of Proposition 4.2, there exists a compatible pre-Malcev algebra on A defined by

$$x \cdot y = T(v) \cdot T(w) = T(v * w) = T(\rho(T(v))w) = T(\rho(x)T^{-1}(y)).$$

Conversely, we have $\mathrm{id}_A(L_{\mathrm{id}_A(x)}y - L_{\mathrm{id}_A(y)}x) = \mathrm{id}_A(L_x(y) - L_y(x)) = \mathrm{id}_A(x \cdot y - y \cdot x) = \mathrm{id}_A(xy) = xy = \mathrm{id}_A(x)\,\mathrm{id}_A(y)$ for all $x, y \in A$, which means $\mathrm{id}_A \in O_A(\mathcal{A}, L)$. □

Theorem 1.4 has the following two direct applications for which the first one gives a way to construct a skew-symmetric solution of the CYBE on the semi-direct product of a Malcev algebra A and its representation $(\mathcal{A}^*, L^*)$; and the second one shows that a Malcev algebra admitting a non-degenerate symplectic form $\mathcal{B}$ must have a compatible pre-Malcev algebra structure; compared with [4] for the case of Lie algebras.

Corollary 4.3. *Let $\mathcal{A}$ be a pre-Malcev algebra with a basis $\{e_1, \ldots, e_n\}$ and $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the basis of A^* dual to $\{e_1, \ldots, e_n\}$. Then the element*

$$r := \sum_{i=1}^{n} (e_i \otimes \varepsilon_i - \varepsilon_i \otimes e_i)$$

is a skew-symmetric solution of the CYBE on the Malcev algebra $[\mathcal{A}] \ltimes_{L^*} \mathcal{A}^*$.

Proof. Consider the identity map id_A. By Theorem 1.4, we see that id_A is an O-operator of $[\mathcal{A}]$ associated to $(\mathcal{A}, L)$. It follows from Eq. (3.2) that $\widetilde{\mathrm{id}_A} = \sum_{i=1}^{n} \mathrm{id}_A(e_i) \otimes \varepsilon_i = \sum_{i=1}^{n} e_i \otimes \varepsilon_i$. Thus it follows from Theorem 1.3 that $r = \sum_{i=1}^{n}(e_i \otimes \varepsilon_i - \varepsilon_i \otimes e_i)$ is a skew-symmetric solution of the CYBE on the Malcev algebra $[\mathcal{A}] \ltimes_{L^*} \mathcal{A}^*$, as desired. □

Proposition 4.4. *Let A be a Malcev algebra admitting a non-degenerate symplectic form $\mathcal{B}$. Then there exists a compatible pre-Malcev algebra $(\mathcal{A}, \cdot)$ on A such that $\mathcal{B}(x \cdot y, z) = -\mathcal{B}(y, xz)$ for all $x, y, z \in A$.*

Proof. Since $\mathcal{B}$ is a non-degenerate symplectic form, we define an invertible linear map $T : A^* \longrightarrow A$ by $\left\langle T^{-1}(x), y \right\rangle = \mathcal{B}(x, y)$ for $x, y \in A$. A similar argument as in the proof of Theorem 1.2 shows that $T \in O_A(A^*, \mathrm{ad}^*)$. By Theorem 1.4, there exists a compatible pre-Malcev algebra $\mathcal{A}$ given by $x \cdot y = T(\mathrm{ad}_x^*(T^{-1}(y)))$ for all $x, y \in A$. Hence we derive

$$\mathcal{B}(x \cdot y, z) = \mathcal{B}(T(\mathrm{ad}_x^*(T^{-1}(y))), z) = \left\langle \mathrm{ad}_x^*(T^{-1}(y)), z \right\rangle = -\left\langle T^{-1}(y), xz \right\rangle = -\mathcal{B}(y, xz)$$

for all $x, y, z \in A$. This completes the proof. □

4.2. The CYBE on pre-Malcev algebras

Let $(\mathcal{A}, \cdot)$ be a pre-Malcev algebra over $\mathbb{F}$ and (V, ℓ, r) be an $\mathcal{A}$-bimodule. A linear map $T : V \longrightarrow \mathcal{A}$ is called an ***O*-operator** of $\mathcal{A}$ associated to (V, ℓ, r) if

$$T(v) \cdot T(w) = T(\ell_{T(v)}(w) + \mathrm{r}_{T(w)}(v)) \tag{4.9}$$

for all $v, w \in V$. We write $O_{\mathcal{A}}(V, \ell, \mathrm{r})$ for the set of all O-operators of $\mathcal{A}$ associated to (V, ℓ, r). We say that an element $r = \sum_i x_i \otimes y_i \in \mathcal{A} \otimes \mathcal{A}$ is a solution of the CYBE on $\mathcal{A}$ if $-r_{12} \cdot r_{13} + r_{12} \cdot r_{23} + r_{13} r_{23} = 0$, where

$$r_{12} \cdot r_{13} = \sum_{i,j} x_i \cdot x_j \otimes y_i \otimes y_j, r_{12} \cdot r_{23} = \sum_{i,j} x_i \otimes y_i \cdot x_j \otimes y_j,$$

$$r_{13} r_{23} = \sum_{i,j} x_i \otimes x_j \otimes y_i y_j = \sum_{i,j} x_i \otimes x_j \otimes (y_i \cdot y_j - y_j \cdot y_i),$$

denote the images of r under the three standard embeddings from $\mathcal{A} \otimes \mathcal{A}$ to $\mathcal{A}^{\otimes 3}$ respectively. We use $\mathcal{S}(\mathcal{A})$ to denote the set of all solutions of the CYBE on $\mathcal{A}$. We first obtain an analogue of Theorem 1.1 in the case of pre-Malcev algebras and symmetric solutions of the CYBE.

Theorem 4.5. *Let $\mathcal{A}$ be a finite-dimensional pre-Malcev algebra over a field $\mathbb{F}$ of characteristic zero and r be a symmetric element in $\mathcal{A}\otimes\mathcal{A}$. Then $r\in\mathcal{S}(\mathcal{A})$ if and only if $T_r\in\mathcal{O}_{\mathcal{A}}(\mathcal{A}^*, L^*-R^*, -R^*)$.*

Proof. Suppose that $r=\sum_i x_i\otimes y_i\in\mathcal{A}\otimes\mathcal{A}$ is symmetric. For all $\xi,\eta,\zeta\in\mathcal{A}^*$, we consider the natural pairing on $\mathcal{A}^{\otimes 3}$, and obtain

$$\begin{aligned}\langle\xi\otimes\eta\otimes\zeta, r_{12}\cdot r_{13}\rangle &= \sum_{i,j}\left\langle\xi\otimes\eta\otimes\zeta, x_i\cdot x_j\otimes y_i\otimes y_j\right\rangle\\ &= \sum_{i,j}\left\langle\xi, x_i\cdot x_j\right\rangle\langle\eta,y_i\rangle\left\langle\zeta,y_j\right\rangle\\ &= \sum_{i,j}\left\langle\xi,\langle\eta,y_i\rangle x_i\cdot\left\langle\zeta,y_j\right\rangle x_j\right\rangle\\ &= \langle\xi, T_r(\eta)\cdot T_r(\zeta)\rangle.\end{aligned}$$

Similarly, we have $\langle\xi\otimes\eta\otimes\zeta, r_{12}\cdot r_{23}\rangle=\langle\eta, T_r(\xi)\cdot T_r(\zeta)\rangle$ and $\langle\xi\otimes\eta\otimes\zeta, r_{13}r_{23}\rangle=\langle\zeta, T_r(\xi)T_r(\eta)\rangle$. We observe that

$$\begin{aligned}\left\langle\xi, T_r((L^*_{T_r(\eta)}-R^*_{T_r(\eta)})(\zeta))\right\rangle &= \left\langle(L^*_{T_r(\eta)}-R^*_{T_r(\eta)})(\zeta), T_r(\xi)\right\rangle\\ &= -\left\langle\zeta, L_{T_r(\eta)}(T_r(\xi))\right\rangle+\left\langle\zeta, R_{T_r(\eta)}(T_r(\xi))\right\rangle\\ &= \langle\zeta, T_r(\xi)\cdot T_r(\eta)-T_r(\eta)\cdot T_r(\xi)\rangle\\ &= \langle\zeta, T_r(\xi)T_r(\eta)\rangle,\end{aligned}$$

and an analogous argument shows that $\left\langle\xi, T_r(-R^*_{T_r(\zeta)}(\eta))\right\rangle=\langle\eta, T_r(\xi)\cdot T_r(\zeta)\rangle$. Hence

$$\begin{aligned}&\left\langle\xi, T_r(\eta)\cdot T_r(\zeta)-T_r((L^*_{T_r(\eta)}-R^*_{T_r(\eta)})(\zeta))-T_r(-R^*_{T_r(\zeta)}(\eta))\right\rangle\\ =\ &\langle\xi, T_r(\eta)\cdot T_r(\zeta)\rangle-\langle\zeta, T_r(\xi)T_r(\eta)\rangle-\langle\eta, T_r(\xi)\cdot T_r(\zeta)\rangle \qquad (4.10)\\ =\ &-\langle\xi\otimes\eta\otimes\zeta, -r_{12}\cdot r_{13}+r_{12}\cdot r_{23}+r_{13}r_{23}\rangle.\end{aligned}$$

($\Longrightarrow$) Now we suppose $r\in\mathcal{S}(\mathcal{A})$, that is, $-r_{12}\cdot r_{13}+r_{12}\cdot r_{23}+r_{13}r_{23}=0$. Thus it follows from Eq. (4.10) that $\left\langle\xi, T_r(\eta)\cdot T_r(\zeta)-T_r((L^*_{T_r(\eta)}-R^*_{T_r(\eta)})(\zeta))-T_r(-R^*_{T_r(\zeta)}(\eta))\right\rangle=0$. Since ξ is arbitrary, we see that $T_r(\eta)\cdot T_r(\zeta)-T_r((L^*_{T_r(\eta)}-R^*_{T_r(\eta)})(\zeta))-T_r(-R^*_{T_r(\zeta)}(\eta))=0$, i.e. $T_r\in\mathcal{O}_{\mathcal{A}}(\mathcal{A}^*, L^*-R^*, -R^*)$. ($\Longleftarrow$) Conversely, the assumption that $T_r\in\mathcal{O}_{\mathcal{A}}(\mathcal{A}^*, L^*-R^*, -R^*)$, together with Eq. (4.10), implies that $\langle\xi\otimes\eta\otimes\zeta, -r_{12}\cdot r_{13}+r_{12}\cdot r_{23}+r_{13}r_{23}\rangle=0$. Thus $-r_{12}\cdot r_{13}+r_{12}\cdot r_{23}+r_{13}r_{23}=0$, i.e., $r\in\mathcal{S}(\mathcal{A})$. The proof is completed. □

The following result is a symmetric element version of Theorem 1.3 for pre-Malcev algebras.

Theorem 4.6. *Let $(V,\ell,\mathfrak{r})$ be a representation of a finite-dimensional pre-Malcev algebra $\mathcal{A}$ over a field $\mathbb{F}$ of characteristic zero and $T: V\longrightarrow\mathcal{A}$ be a linear map. Suppose that $\{v_1,\dots,v_n\}$ is a basis of V and $\{\xi_1,\dots,\xi_n\}$ is the dual basis of V^*. Define*

$$\widetilde{T}:=\sum_{i=1}^{n}T(v_i)\otimes\xi_i\in\mathcal{A}\otimes V^*\subseteq(\mathcal{A}\oplus V^*)\otimes(\mathcal{A}\oplus V^*). \qquad (4.11)$$

Then $T\in\mathcal{O}_{\mathcal{A}}(V,\ell,\mathfrak{r})$ if and only if $s_T=\widetilde{T}+\sigma(\widetilde{T})\in\mathcal{S}(\mathcal{A}\ltimes_{\ell^*-\mathfrak{r}^*,-\mathfrak{r}^*}V^*)$.

Proof. Note that

$$\begin{aligned}-(s_T)_{12}\cdot(s_T)_{13} &= \sum_{i,j=1}^{n}(-T(v_i)\cdot T(v_j)\otimes\xi_i\otimes\xi_j-(\ell^*_{T(v_i)}-\mathfrak{r}^*_{T(v_i)})(\xi_j)\otimes\xi_i\otimes T(v_j) \qquad (4.12)\\ &\quad -(-\mathfrak{r}^*_{T(v_j)})(\xi_i)\otimes T(v_i)\otimes\xi_j).\end{aligned}$$

As $(\ell^*_{T(v_i)} - \mathfrak{r}^*_{T(v_i)})(\xi_j) \in V^*$ for all $i, j \in \{1, \dots, n\}$, we assume that $(\ell^*_{T(v_i)} - \mathfrak{r}^*_{T(v_i)})(\xi_j) = a_1(ij)\xi_1 + \cdots + a_n(ij)\xi_n$, where $a_s(ij) \in \mathbb{F}$. For any $k \in \{1, \dots, n\}$, we have

$$a_k(ij) = \left\langle \sum_{s=1}^n a_s(ij)\xi_s, v_k \right\rangle = \left\langle (\ell^*_{T(v_i)} - \mathfrak{r}^*_{T(v_i)})(\xi_j), v_k \right\rangle = -\left\langle \xi_j, (\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_k) \right\rangle.$$

Thus

$$(\ell^*_{T(v_i)} - \mathfrak{r}^*_{T(v_i)})(\xi_j) = -\sum_{k=1}^n \left\langle \xi_j, (\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_k) \right\rangle \xi_k. \tag{4.13}$$

Similarly,

$$\mathfrak{r}^*_{T(v_i)}(\xi_j) = -\sum_{k=1}^n \left\langle \xi_j, \mathfrak{r}_{T(v_i)}(v_k) \right\rangle \xi_k. \tag{4.14}$$

Hence, it follows from Eqs. (4.13) and (4.14) that

$$\begin{aligned}
\sum_{i,j=1}^n (\ell^*_{T(v_i)} - \mathfrak{r}^*_{T(v_i)})(\xi_j) \otimes \xi_i \otimes T(v_j) &= \sum_{i,j=1}^n \left(-\sum_{k=1}^n \left\langle \xi_j, (\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_k) \right\rangle \xi_k \right) \otimes \xi_i \otimes T(v_j) \\
&= -\sum_{i,k=1}^n \xi_k \otimes \xi_i \otimes T\left(\sum_{j=1}^n \left\langle \xi_j, (\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_k) \right\rangle v_j \right) \\
&= -\sum_{i,j=1}^n \xi_j \otimes \xi_i \otimes T\left(\sum_{k=1}^n \left\langle \xi_k, (\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_j) \right\rangle v_k \right) \\
&= -\sum_{i,j=1}^n \xi_j \otimes \xi_i \otimes T((\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_j)),
\end{aligned}$$

and

$$\sum_{i,j=1}^n \mathfrak{r}^*_{T(v_j)}(\xi_i) \otimes T(v_i) \otimes \xi_j = -\sum_{i,j=1}^n \xi_i \otimes T(\mathfrak{r}_{T(v_j)}(v_i)) \otimes \xi_j.$$

Now Eq. (4.12) reads

$$\begin{aligned}
& -(s_T)_{12} \cdot (s_T)_{13} \\
= & \sum_{i,j=1}^n (-T(v_i) \cdot T(v_j) \otimes \xi_i \otimes \xi_j + \xi_j \otimes \xi_i \otimes T((\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_j)) - \xi_i \otimes T(\mathfrak{r}_{T(v_j)}(v_i)) \otimes \xi_j).
\end{aligned}$$

We do similar calculations on $(s_T)_{12} \cdot (s_T)_{23}$ and $(s_T)_{13}(s_T)_{23}$ and conclude

$$\begin{aligned}
& -(s_T)_{12} \cdot (s_T)_{13} + (s_T)_{12} \cdot (s_T)_{23} + (s_T)_{13}(s_T)_{23} \\
= & \sum_{i,j=1}^n (-T(v_i) \cdot T(v_j) + T(\ell_{T(v_i)}(v_j)) + T(\mathfrak{r}_{T(v_j)}(v_i))) \otimes \xi_i \otimes \xi_j \\
& + \sum_{i,j=1}^n \xi_i \otimes (T(v_i) \cdot T(v_j) - T(\ell_{T(v_i)}(v_j)) - T(\mathfrak{r}_{T(v_j)}(v_i))) \otimes \xi_j \\
& + \sum_{i,j=1}^n \xi_i \otimes \xi_j \otimes (T(v_i)T(v_j) - T((\ell_{T(v_i)} - \mathfrak{r}_{T(v_i)})(v_j)) + T((\ell_{T(v_j)} - \mathfrak{r}_{T(v_j)})(v_i))).
\end{aligned} \tag{4.15}$$

We are ready to complete the proof. ($\Longrightarrow$) We assume that $T \in O_{\mathcal{A}}(V, \ell, \mathfrak{r})$, that is, $T(v_i) \cdot T(v_j) - T(\ell_{T(v_i)}(v_j)) - T(\mathfrak{r}_{T(v_j)}(v_i)) = 0$ for all $i, j \in \{1, \ldots, n\}$. Thus the right-hand side of Eq. (4.15) must be zero. This implies that $-(s_T)_{12} \cdot (s_T)_{13} + (s_T)_{12} \cdot (s_T)_{23} + (s_T)_{13}(s_T)_{23} = 0$, i.e., s_T is a solution of the CYBE on the pre-Malcev algebra $\mathcal{A} \ltimes_{\ell^*-\mathfrak{r}^*,-\mathfrak{r}^*} V^*$. ($\Longleftarrow$) Suppose that $s_T \in \mathcal{S}(\mathcal{A} \ltimes_{\ell^*-\mathfrak{r}^*,-\mathfrak{r}^*} V^*)$ is a solution of the CYBE. Thus the left-hand side of Eq. (4.15) is equal to zero. We write $\{x_1, \ldots, x_m\}$ for a basis of $\mathcal{A}$ and assume that $T(v_i) \cdot T(v_j) - T(\ell_{T(v_i)}(v_j) + \mathfrak{r}_{T(v_j)}(v_i)) = a_1(ij)x_1 + \cdots + a_m(ij)x_m$ for some $a_1(ij), \ldots, a_m(ij) \in \mathbb{F}$. Hence,

$$0 = \sum_{i,j=1}^{n} \sum_{k=1}^{m} (-a_k(ij)x_k \otimes \xi_i \otimes \xi_j + a_k(ij)\xi_j \otimes x_k \otimes \xi_i + (a_k(ij) - a_k(ji))\xi_i \otimes \xi_j \otimes x_k).$$

The fact that $\{x_k \otimes \xi_i \otimes \xi_j, \xi_j \otimes x_k \otimes \xi_i, \xi_i \otimes \xi_j \otimes x_k \mid 1 \leqslant i, j \leqslant n, 1 \leqslant k \leqslant m\}$ is a subset of a basis of $(\mathcal{A} \ltimes_{\ell^*-\mathfrak{r}^*,-\mathfrak{r}^*} V^*)^{\otimes 3}$ implies that these elements are linearly independent over $\mathbb{F}$. Hence, $a_k(ij) = 0$ for all k, i and j. Hence, $T(v_i) \cdot T(v_j) = T((\ell_{T(v_i)}(v_j) - \mathfrak{r}_{T(v_j)}(v_i))$ for all $i, j \in \{1, \ldots, n\}$. This shows that $T \in O_{\mathcal{A}}(V, \ell, \mathfrak{r})$ and we are done. □

Corollary 4.7. *Let $\mathcal{A}$ be a pre-Malcev algebra with a basis $\{e_1, \ldots, e_n\}$ and $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the basis of $\mathcal{A}^*$ dual to $\{e_1, \ldots, e_n\}$. Then the element*

$$s := \sum_{i=1}^{n} (e_i \otimes \varepsilon_i + \varepsilon_i \otimes e_i)$$

is a symmetric solution of the CYBE on the pre-Malcev algebra $\mathcal{A} \ltimes_{L^*,0} \mathcal{A}^*$.

Proof. Since $\mathrm{id}_{\mathcal{A}}$ is an O-operator of $\mathcal{A}$ associated to the bimodule $(\mathcal{A}, L, 0)$, we have $\widetilde{\mathrm{id}_{\mathcal{A}}} = \sum_{i=1}^{n} \mathrm{id}_{\mathcal{A}}(e_i) \otimes \varepsilon_i = \sum_{i=1}^{n} e_i \otimes \varepsilon_i$. Thus it follows from Theorem 4.6 that $s = \sum_{i=1}^{n} (e_i \otimes \varepsilon_i + \varepsilon_i \otimes e_i)$ is a symmetric solution of the CYBE on the pre-Malcev algebra $\mathcal{A} \ltimes_{L^*,0} \mathcal{A}^*$. □

We close this subsection by establishing connections between invertible O-operators and bilinear forms on a given pre-Malcev algebra $\mathcal{A}$.

Proposition 4.8. *Let $\mathcal{A}$ be a pre-Malcev algebra and $T : \mathcal{A}^* \longrightarrow \mathcal{A}$ be an invertible linear map. Suppose $\mathcal{B} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{F}$ is a bilinear form defined by $\mathcal{B}(x, y) = \left\langle T^{-1}(x), y \right\rangle$. For all $x, y, z \in \mathcal{A}$, we have the following results:*

(1) $T \in O_{\mathcal{A}}(\mathcal{A}^*, L^* - R^*, 0)$ if and only if $\mathcal{B}(x \cdot y, z) = -\mathcal{B}(y, x \cdot z - z \cdot x)$.

(2) $T \in O_{\mathcal{A}}(\mathcal{A}^*, L^* - R^*, -R^*)$ if and only if $\mathcal{B}(x \cdot y, z) = -\mathcal{B}(y, x \cdot z) + \mathcal{B}(y, z \cdot x) + \mathcal{B}(x, z \cdot y)$.

(3) $T \in O_{[\mathcal{A}]}(\mathcal{A}^*, L^*)$ if and only if $\mathcal{B}(xy, z) = \mathcal{B}(x, y \cdot z) - \mathcal{B}(y, x \cdot z)$.

Proof. We first note that T is invertible, thus for $x, y \in \mathcal{A}$, there exist unique $\xi, \eta \in \mathcal{A}^*$ such that $x = T(\xi)$ and $y = T(\eta)$. (1) Assume that T is an O-operator of $\mathcal{A}$ associated to the bimodule $(\mathcal{A}^*, L^* - R^*, 0)$. We note that

$$\begin{aligned}
\mathcal{B}(x \cdot y, z) + \mathcal{B}(y, x \cdot z - z \cdot x) &= \left\langle T^{-1}(x \cdot y), z \right\rangle + \left\langle T^{-1}(y), x \cdot z - z \cdot x \right\rangle \\
&= \left\langle T^{-1}(T(\xi) \cdot T(\eta)), z \right\rangle + \langle \eta, L_x(z) - R_x(z) \rangle \\
&= \left\langle T^{-1}(T(\xi) \cdot T(\eta)) - (L_x^* - R_x^*)(\eta), z \right\rangle \\
&= \left\langle T^{-1}(T(\xi) \cdot T(\eta) - T((L_x^* - R_x^*)(\eta))), z \right\rangle \\
&= \left\langle T^{-1}(T(\xi) \cdot T(\eta) - T((L_{T(\xi)}^* - R_{T(\xi)}^*)(\eta))), z \right\rangle \\
&= 0
\end{aligned}$$

Hence, $\mathcal{B}(x \cdot y, z) = -\mathcal{B}(y, x \cdot z - z \cdot x)$. Conversely, suppose that $\mathcal{B}(x \cdot y, z) = -\mathcal{B}(y, x \cdot z - z \cdot x)$. Then $0 = \mathcal{B}(x \cdot y, z) + \mathcal{B}(y, x \cdot z - z \cdot x) = \left\langle T^{-1}(T(\xi) \cdot T(\eta) - T((L_{T(\xi)}^* - R_{T(\xi)}^*)(\eta))), z \right\rangle$. Since z is arbitrary, we see that $T^{-1}(T(\xi) \cdot T(\eta) - T((L_{T(\xi)}^* - R_{T(\xi)}^*)(\eta))) = 0$. Therefore, $T_r \in O_{\mathcal{A}}(\mathcal{A}^*, L^* - R^*, 0)$.

(2) Suppose that $T \in \mathcal{O}_{\mathcal{A}}(\mathcal{A}^*, L^* - R^*, -R^*)$, then $x \cdot y = T(\xi) \cdot T(\eta) = T((L^*_{T(\xi)} - R^*_{T(\xi)})(\eta) - R^*_{T(\eta)}(\xi))$. We obtain that

$$\begin{aligned} \mathcal{B}(x \cdot y, z) &= \left\langle T^{-1}(x \cdot y), z \right\rangle = \left\langle (L^*_{T(\xi)} - R^*_{T(\xi)})(\eta) - R^*_{T(\eta)}(\xi), z \right\rangle \\ &= \left\langle (L^*_x - R^*_x)(T^{-1}(y)) - R^*_y(T^{-1}(x)), z \right\rangle \\ &= -\left\langle T^{-1}(y), (L_x - R_x)(z) \right\rangle + \left\langle T^{-1}(x), R_y(z) \right\rangle \\ &= -\mathcal{B}(y, x \cdot z) + \mathcal{B}(y, z \cdot x) + \mathcal{B}(x, z \cdot y). \end{aligned} \tag{4.16}$$

Conversely, assume that $\mathcal{B}(x \cdot y, z) = -\mathcal{B}(y, x \cdot z) + \mathcal{B}(y, z \cdot x) + \mathcal{B}(x, z \cdot y)$. By a discussion similar to Eq. (4.16), we see that $T(\xi) \cdot T(\eta) = T((L^*_{T(\xi)} - R^*_{T(\xi)})(\eta) - R^*_{T(\eta)}(\xi))$, that is, $T \in \mathcal{O}_{\mathcal{A}}(\mathcal{A}^*, L^* - R^*, -R^*)$, as desired.

(3) Assume that $T \in \mathcal{O}_{[\mathcal{A}]}(\mathcal{A}^*, L^*)$, then $T(\xi)T(\eta) = T(L^*_{T(\xi)}\eta - L^*_{T(\eta)}\xi)$. Hence we have

$$\begin{aligned} & \mathcal{B}(xy, z) - \mathcal{B}(x, y \cdot z) + \mathcal{B}(y, x \cdot z) \\ = & \left\langle T^{-1}(xy), z \right\rangle - \left\langle T^{-1}(x), y \cdot z \right\rangle + \left\langle T^{-1}(y), x \cdot z \right\rangle \\ = & \left\langle T^{-1}(xy), z \right\rangle - \left\langle T^{-1}(x), L_y(z) \right\rangle + \left\langle T^{-1}(y), L_x(z) \right\rangle \\ = & \left\langle T^{-1}(T(\xi)T(\eta)), z \right\rangle + \left\langle L^*_{T(\eta)}(\xi), z \right\rangle - \left\langle L^*_{T(\xi)}(\eta), z \right\rangle \\ = & \left\langle T^{-1}(T(\xi)T(\eta) - T(L^*_{T(\xi)}(\eta) - L^*_{T(\eta)}(\xi))), z \right\rangle = 0. \end{aligned}$$

For the converse statement, we suppose that $\mathcal{B}(xy, z) = \mathcal{B}(x, y \cdot z) - \mathcal{B}(y, x \cdot z)$. Thus $0 = \mathcal{B}(xy, z) - \mathcal{B}(x, y \cdot z) + \mathcal{B}(y, x \cdot z) = \left\langle T^{-1}(T(\xi)T(\eta) - T(L^*_{T(\xi)}(\eta) - L^*_{T(\eta)}(\xi))), z \right\rangle$. For the non-degeneration of the natural pairing, we see that $T(\xi)T(\eta) = T(L^*_{T(\xi)}(\eta) - L^*_{T(\eta)}(\xi))$, i.e., $T \in \mathcal{O}_{[\mathcal{A}]}(\mathcal{A}^*, L^*)$. □

Remark 4.9. Note that besides Malcev algebras, there are some other nonassociative algebras that contain Lie algebras as a subclass and have attracted many researchers' attention; such as Hom-Lie algebras [8, 14] and ω-Lie algebras [7, 9, 23]. The method of our article might be applied to a study of the CYBE on these nonassociative algebras; see [20] for the study on $\mathcal{O}$-operators of Hom-Lie algebras and the classical Hom-Yang-Baxter equation. ◇

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