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Topological pseudo orbit tracing property, topological sensitivity and topological entropy

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Abstract. We introduce and study pseudo orbit tracing property on general topological spaces termed as topological pseudo orbit tracing property. We prove that on a compact Hausdorff space, a topologically sensitive dynamical system having topological pseudo orbit tracing property is cofinitely topologically sensitive and has positive topological entropy. Moreover, we also prove that such a dynamical system is locally uncountable.

1. Introduction

The notion of pseudo orbit tracing property(also known as shadowing property) was defined by R. Bowen in 1975 [5]. This concept can be used in computer simulations. For any map on a space to calculate the value of the map at a point a computer takes a value near to the actual image of the point. So, any orbit calculated by a computer is a pseudo orbit. Pseudo orbit tracing property(POTP) guarantees that any pseudo orbit can be approximated by a real orbit in the system. Hence, it becomes natural to study about the POTP. In this paper, we study relations among the topological version of POTP, sensitivity and entropy on a compact Hausdorff topological space.

Looking at the importance of POTP many authors have worked on it. In particular, for any map on a compact interval the fact that sensitivity implies cofinite sensitivity is proved in [19]. It is proved that if a continuous map on a compact, connected metric space has POTP and has dense set of periodic points then the map has positive entropy[4]. Relation between POTP and entropy has been studied by many authors[1, 9, 11, 15, 17, 21]. Moothathu proved that on a compact metric space if a continuous map has POTP then the restriction of that map to non-wandering set has POTP[20]. Moreover, he also obtained relations between sensitive points and entropy points on compact metric spaces. S.A. Ahmadi et al. generalized the result by Moothathu to uniform compact spaces[2]. More results on POTP can be found in [12, 16, 23]. Some results relating sensitivity and POTP on uniform spaces are proved in [1, 3].

As mentioned in above paragraph, for a continuous map on a compact interval sensitivity implies cofinite sensitivity, so there is a natural question that can we generalize this result to general topological spaces. We find out that POTP plays an important role in this. We prove that on a compact Hausdorff topological space a continuous map having topological sensitivity and topological POTP is cofinitely topologically

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sensitive, has positive topological entropy and the space is locally uncountable. In section 2, we provide necessary prerequisites required for the remaining sections of the paper. In section 3, we prove that the restriction of a map on a compact Hausdorff space having topological POTP to non wandering set also has topological POTP. We also prove that on a compact Hausdorff space if a continuous map has topological POTP then topological sensitivity of the map implies that the map is cofinitely topologically sensitive and finally we prove that the space with such a map is locally uncountable. In section 4 we prove results relating topological entropy and topological sensitivity in presence of topological POTP. We prove that on a compact Hausdorff topological space if a map has topological POTP and topological sensitivity then every point is an entropy point. We also obtain relations among various types of mixings in presence of topological POTP for compact Hausdorff spaces. We show that on a compact Hausdorff space if a map f has topological POTP and there is a sensitive point such that the same point belongs to the interior of the closure of the set of recurrent points of the map $f \times f$ then that point is an entropy point.

2. Preliminaries

Throughout the paper, we denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$. For any $r \in \mathbb{R}$, $\lceil r \rceil$ denotes the smallest natural number greater then or equal to r. A subset A of natural numbers is called *syndetic* if there exists a $k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $\{n, n+1, \dots, n+k\} \cap A \neq \emptyset$. A subset A of natural numbers is called *cofinite* if $\mathbb{N}/A$ is a finite set.

Let (X, f) be any dynamical system over a metric space X. A sequence $\overline{x} = (x_0, x_1, x_2, \dots)$ in X is called a $\delta-$ *pseudo orbit* for f if $d(f(x_i), x_{i+1}) < \delta$ for every $i \in \mathbb{N} \cup \{0\}$. An orbit $\overline{y} = (y_0, y_1, y_2, \dots)$ is said to be $\epsilon-$ *traced* if there exists a $z \in X$ such that $d(f^i(z), y_i) < \epsilon$ for every $i \in \mathbb{N} \cup \{0\}$. A dynamical system (X, f) is said to have *pseudo orbit tracing property(POTP)* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every $\delta-$ pseudo orbit is $\epsilon-$ traced.

Let X be any topological space and $f : X \to X$ be any map. A space X is called *locally uncountable* if every nonempty open set $G \subset X$ is uncountable. A point $z \in X$ is called a *periodic point* if there exists an $n \in \mathbb{N}$ such that $f^n(z) = z$. For any $x \in X$ and any open set G, $N(x, G) = \{n \in \mathbb{N} : f^n(x) \in G\}$.A point $z \in X$ is called *minimal point* if for any open set G containing z, $N(z, G)$ is syndetic. The set of minimal points of f is denoted by $M(f)$. A point $z \in X$ is called a *recurrent point* if for any open set G containing z there exists an $n \in \mathbb{N}$ such that $f^n(z) \in G$. The set of recurrent points of f is denoted by $R(f)$. A point $z \in X$ is called a *non wandering point* if for any open set G containing z there exists an $n \in \mathbb{N}$ such that $f^n(G) \cap G \neq \emptyset$. The set of non wandering points of a map f is denoted by $\Omega(f)$. A map f is called *non wondering* if for any open set $G \subset X$ there exists an $n \in \mathbb{N}$ such that $f^n(G) \cap G \neq \emptyset$. A map f is called *transitive* if for any pair of nonempty open sets U, V there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. A map f is called *totally transitive* if for any $k \in \mathbb{N}$, f^k is transitive. We say that f is *weakly mixing* if for any collection of nonempty open sets $U_1, U_2; V_1, V_2$ there exists an $n \in \mathbb{N}$ such that $f^n(U_1) \cap U_2 \neq \emptyset$ and $f^n(V_1) \cap V_2 \neq \emptyset$. We say that f is *topologically mixing* if for any pair of nonempty open sets U, V, $N(U, V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$ is a cofinite set.

Take a family $\mathcal{F}$ of continuos functions from X to Y and $x \in X$. Let $\mathcal{N}_x$ denote the collection of all open sets containing x. We say that $\mathcal{F}$ is *topologically equicontinuous at* (x, y) if for any $O \in \mathcal{N}_y$ there exist a $U \in \mathcal{N}_x$ and a $V \in \mathcal{N}_y$ such that $f(U) \cap V \neq \emptyset$ then $f(U) \subset O$. If $\mathcal{F}$ is topologically equicontinuous at (x, y) for every $y \in Y$ then we say $\mathcal{F}$ *is topologically equicontinuous at* x and if $\mathcal{F}$ is topologically equicontinuous at x for every $x \in X$ then we say that $\mathcal{F}$ is *topologically equicontinuous*[22].

Definition 2.1. ([8]) Let (X, f) be any dynamical system. An open cover $\mathscr{V}$ of X is called a *sensitivity cover* if for any open set G there exist an $n \in \mathbb{N}$ and $x, y \in G$ such that $(f^n(x), f^n(y)) \notin V \times V$ for any $V \in \mathscr{V}$. A dynamical system (X, f) is called *topologically sensitive* if there exists a sensitivity cover.

Let $D(\mathscr{V}, G) = \{n \in \mathbb{N} : f^n(G) \not\subset V$ for any $V \in \mathscr{V}\}$. If there exists an open cover $\mathscr{V}$ such that for any open set $G \subset X$, $D(\mathscr{V}, G)$ is cofinite then (X, f) is called *cofinitely topologically sensitive*[7].

Take any dynamical system (X, f) where X is a compact space and $f : X \to X$ is a continuous map. For any open covers $\mathscr{V}$ and $\mathscr{U}$, $\mathscr{V}$ is called a refinement of $\mathscr{U}$ if for any $V \in \mathscr{V}$ there exists a $U \in \mathscr{U}$ such that $V \subset U$. For open covers $\mathscr{V}, \mathscr{U}$ of X, $\mathscr{V} \vee \mathscr{U} = \{U \cap V : U \in \mathscr{U}, V \in \mathscr{V}\}$. For open cover $\mathscr{U}$ of X,

$f^{-1}(\mathscr{U}) = \{f^{-1}(U) : U \in \mathscr{U}\}$ and for any $n \in \mathbb{N}$ $f^{-(n+1)}(\mathscr{U}) = f^{-1}(f^{-n}(\mathscr{U}))$. For any open cover $\mathscr{V}$ of X and any $n \in \mathbb{N}$ take open cover $\mathscr{V} \vee f^{-1}(\mathscr{V}) \vee \cdots \vee f^{-(n)}(\mathscr{V})$ and for any nonempty open set $G \subset X$, define $N(\overline{G}, \mathscr{V})$ to be the minimum cardinality of any subcover of $\mathscr{V}$. Define $H(\overline{G}, \mathscr{V}) = \lim_{n\to\infty} \frac{\log N(\overline{G}, \mathscr{V} \vee f^{-1}(\mathscr{V}) \vee \cdots \vee f^{-(n)}(\mathscr{V}))}{n+1}$. We say that *entropy of f on $\overline{G}$* is $h(f, \overline{G}) = \sup\{H(\overline{G}, \mathscr{V}) : \mathscr{V}$ is any open cover of $\overline{G}\}$. We define *entropy of f* as $h(f) = \sup\{H(X, \mathscr{V}) : \mathscr{V}$ is an open cover of $X\}$. We say that $x \in X$ is an *entropy point* if for any open set G containing x, $h(f, \overline{G}) > 0$.

3. Topological sensitivity and cofinite topological sensitivity

Das et al. generalized the definitions of $\delta-$ pseudo orbit, $\epsilon-$ tracing and POTP for a dynamical system on a metric space to a uniform space[6]. Here we provide a topological version of all these definitions and then we will prove that on compact Hausdorff space, in presence of topological POTP, topological sensitivity implies cofinite topological sensitivity. Moreover, the space will be locally uncountable.

Definition 3.1. Consider the dynamical system (X, f), where X is any topological space and $f : X \to X$ is a continuous map.

1. Let $\overline{x} = (x_0, x_1, x_2, \dots)$ be a sequence in X and $\mathscr{U}$ be an open cover of X then we say that $\overline{x}$ is a $\mathscr{U}$*−pseudo orbit* if for any $n \in \mathbb{N} \cup \{0\}$, $(f(x_n), x_{n+1}) \in U \times U$ for some $U \in \mathscr{U}$.

2. Let $\mathscr{V}$ be any open cover of X and $\overline{y} = (y_0, y_1, y_2, \dots)$ be a sequence in X then we say that $\overline{y}$ is $\mathscr{V}-$ *traced* if there exists a $z \in X$ such that for any $n \in \mathbb{N} \cup \{0\}$, $(y_n, f^n(z)) \in V \times V$ for some $V \in \mathscr{V}$.

3. The dynamical system (X, f) is said to have *topological POTP*, if for any open cover $\mathscr{V}$, there exists an open cover $\mathscr{U}$ such that every $\mathscr{U}-$ pseudo orbit is $\mathscr{V}-$ traced.

First we see that the topological POTP is preserved under restriction of the map to the non-wandering set only. The result for compact metric spaces is proved in [20]. Here, we provide a simpler proof for generalization of this result to compact topological spaces. First we state a lemma to be used in the next proposition.

Lemma 3.1. ([13]) *Let X be any uniform space and $\mathscr{U}$ be an open cover of X, then there exists an open cover $s(\mathscr{U})$ of X such that for any $U_1, U_2 \in s(\mathscr{U})$, if $U_1 \cap U_2 \neq \emptyset$ then $U_1 \cup U_2 \subset U$ for some $U \in \mathscr{U}$.*

Note that the open cover $s(\mathscr{U})$ in lemma above need not be unique. We denote $s(s(\mathscr{U}))$ by $s^2(\mathscr{U})$ and $s(s^{n-1}(\mathscr{U}))$ by $s^n(\mathscr{U})$ for any $n \in \mathbb{N}, n \geq 2$.

It is worth mentioning that any space is a uniform space if and only if it is completely regular(See [10]). As any compact Hausdorff space is completely regular, so is a uniform space also.

Proposition 3.1. *Let (X, f) be a compact Hausdorff dynamical system having topological POTP. Then $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ also has topological POTP.*

Proof. Let $\mathscr{V}$ be any open cover of X and $\mathscr{U}$ be an open cover of X such that any $\mathscr{U}-$ pseudo orbit is $s(\mathscr{V})-$ traced.

Let $(z_0, z_1, \dots, z_n)$ be any finite $\mathscr{U}-$ pseudo orbit of elements of $\Omega(f)$. Then for any $i \in \{1, \dots, n\}$ there exists a $U_i \in \mathscr{U}$ such that $(f(z_{i-1}), z_i) \in U_i \times U_i$. As, for any $i \in \{0, 1, \dots, n\}$, $z_i \in \Omega(f)$ and $z_i \in U_i \cap f^{-1}(U_{i+1})$ (taking U_{n+1} to be an open set in $\mathscr{U}$ containing $f(z_n)$ and U_0 to be an open set in $\mathscr{U}$ containing z_0), so there exist a large enough $k_i \in \mathbb{N}$ and $z'_i \in U_i \cap f^{-1}(U_{i+1})$ such that $(f^{k_i}(z'_i), z'_i) \in (U_i \cap f^{-1}(U_{i+1})) \times (U_i \cap f^{-1}(U_{i+1}))$. So, for any $i \in \{1, 2, \dots, n\}$, $(f^{k_i}(z'_i), f(z'_{i-1})) \in U_i \times U_i$ and as $(f(z_n), f(z'_n)) \in U_{n+1} \times U_{n+1}$ and $(f^{k_0}(z'_0), z_0) \in U_0 \times U_0$ therefore, $(z_n, f(z'_n), \dots, f^{k_n-1}(z'_n), f(z'_{n-1}), \dots, f^{k_{n-1}-1}(z'_{n-1}), \dots, f(z'_1), \dots, f^{(k_1-1)}(z'_1), f(z'_0), \dots, f^{k_0-1}(z'_0), z_0)$ is a $\mathscr{U}-$ pseudo orbit from z_n to z_0. Rewrite this orbit as $z_n = y_0, y_1, \dots, y_k = z_0$. Let $D = [z_0, z_1, \dots, z_n, y_1, \dots, y_{k-1}]$, then $DDD\dots$ is a $\mathscr{U}-$ pseudo orbit. Let $DDD\dots$ be $s(\mathscr{V})-$ traced by $y \in X$. Then, for any $t \in \mathbb{N}$ $(f^{t(n+k)}(y), z_0) \in V' \times V'$ for some $V' \in s(\mathscr{V})$. Since X is compact, so we can take z to be a limit point of the sequence $(f^{t(n+k)}(y))_{t\in\mathbb{N}}$. Then $(f^i(z), z_i) \in V \times V$ for some $V \in \mathscr{V}$. As for any open set G containing z there exist infinitely many $t \in \mathbb{N}$ such that $f^{t(n+k)}(y) \in G$, so $f^{t'}(G) \cap G \neq \emptyset$ for some $t' \in \mathbb{N}$. Hence, $z \in \Omega(f)$.

Now, let $(z'_0, z'_1, z'_2, \dots)$ be an infinite $\mathscr{U}-$ pseudo orbit of elements of $\Omega(f)$. Then for any $n \in \mathbb{N}$, let $(z'_0, z'_1, \dots, z'_n)$ be $s(\mathscr{V})-$ traced by $y'_n \in \Omega(f)$ and assume that z' is a limit point of the sequence (y'_n). As proceeded in above paragraph, $z' \in \Omega(f)$ and the sequence $(z'_0, z'_1, z'_2, \dots)$ is $\mathscr{V}-$ traced by z'. Hence, $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ has topological POTP. □

Next, we prove that topological sensitivity of (X, f) implies cofinite topological sensitivity of (X, f) for a map f having topological POTP on a compact Hausdorff space X. First, we will prove that the map is syndetically topologically sensitive and then we will prove that it is cofinitely topologically sensitive.

Theorem 3.1. *Let X be any compact Hausdorff topological space and $f : X \to X$ be a continuous map having topological POTP then topological sensitivity of f implies that f is cofinitely topologically sensitive.*

Proof. Let $\mathscr{B}$ be a sensitivity cover for X and G be any open set. Then there exist $x_1^1, x_2^1 \in G$ and $n \in \mathbb{N}$ such that $(f^n(x_1^1), f^n(x_2^1)) \notin B \times B$ for any $B \in \mathscr{B}$. For any $i_1 \in \{1, 2\}$, define sequence $A_{i_1}^1 = (x_{i_1}^1, f(x_{i_1}^1), \dots, f^{n-1}(x_{i_1}^1))$.

Let $\mathscr{V} = s^3(\mathscr{B})$ be an open cover of X and $\mathscr{W}$, a refinement of $\mathscr{V}$ such that if for some $W \in \mathscr{W}$, $\{x_1^1, x_2^1\} \cap W \neq \emptyset$ then $W \subset G$. By definition of topological POTP, there exists an open cover $\mathscr{U}$, a refinement of $\mathscr{W}$, such that any $\mathscr{U}-$ pseudo orbit is $\mathscr{W}-$ traced. . Since X is a compact space, so without loss of generality, we can assume that $\mathscr{U}$ is finite and hence there exists an $m \in \mathbb{N}$ such that for any $U \in \mathscr{U}$ there exist a $t \leq m$ and $x, y \in U$ such that $(f^t(x), f^t(y)) \notin B \times B$ for any $B \in \mathscr{B}$. Note that by our construction $\mathscr{B} < s^3(\mathscr{B}) = \mathscr{V} < \mathscr{W} < \mathscr{U}$.

Let $U_1^1, U_2^1 \in \mathscr{U}$ be two open sets such that $f^n(x_1^1) \in U_1^1$ and $f^n(x_2^1) \in U_2^1$. Hence, there exist $n_1^2, n_2^2 \in \mathbb{N}$, $n_1^2, n_2^2 \leq m$ and $x_1^2, x_2^2 \in U_1^1$, $x_3^2, x_4^2 \in U_2^1$ such that $(f^{n_1^2}(x_1^2), f^{n_1^2}(x_2^2)) \notin B \times B$ for any $B \in \mathscr{B}$ and $(f^{n_2^2}(x_3^2), f^{n_2^2}(x_4^2)) \notin B \times B$ for any $B \in \mathscr{B}$. So, for every $i_2 \in \{1, 2, 3, 4\}$, we get the sequence $A_{i_2}^2 = (x_{i_2}^2, f(x_{i_2}^2), \dots, f^{n^2_{\lceil i_2/2 \rceil}-1}(x_{i_2}^2))$ such that $(f^n(x_{\lceil i_2/2 \rceil}^1), x_{i_2}^2) \in U \times U$ for some $U \in \mathscr{U}$ and $(f^{n_{i_1}^2}(x_{2i_1-1}^2), f^{n_{i_1}^2}(x_{2i_1}^2)) \notin B \times B$ for any $B \in \mathscr{B}$ and $i_1 \in \{1, 2\}$.

Inductively, for any $k \in \mathbb{N}$, $k \geq 2$ and any $i_k \in \{1, 2, \dots, 2^k\}$ we can find finite sequences $A_{i_k}^k = (x_{i_k}^k, f(x_{i_k}^k), \dots, f^{n^k_{\lceil i_k/2 \rceil}-1}(x_{i_k}^k))$ such that for any $k \in \mathbb{N}$ and $i_k \in \{1, 2, \dots, 2^k\}$ there exists $n_{i_k}^k \leq m$ such that for any $i_k \in \{2i_{k-1} - 1, 2i_{k-1}\}$, $(f^{n^{k-1}_{\lceil i_{k-1}/2 \rceil}}(x_{i_{k-1}}^{k-1}), x_{i_k}^k) \in U \times U$ for some $U \in \mathscr{U}$ and for any $i_{k-1} \in \{1, 2, \dots, 2^{k-1}\}$, $(f^{n_{i_{k-1}}^k}(x_{2i_{k-1}-1}^k), f^{n_{i_{k-1}}^k}(x_{2i_{k-1}}^k)) \notin B \times B$ for any $B \in \mathscr{B}$.

Note that $(A_{i_k}^k)_{k\in\mathbb{N}}$, $i_k \in \{2i_{k-1} - 1, 2i_{k-1}\}$ for any $k > 1$ and $i_1 \in \{1, 2\}$ forms a $\mathscr{U}-$ pseudo orbit. Consider two different $\mathscr{U}-$ pseudo orbits, $(A_{i_k}^k)_{k\in\mathbb{N}}, (B_{i_k}^k)_{k\in\mathbb{N}}$ satisfying above conditions. We will show that these orbits are $\mathscr{W}-$ traced by two different points.

Let $(A_{i_k}^k)_{k\in\mathbb{N}}$ be $\mathscr{W}-$ traced by y and $(B_{i_k}^k)_{k\in\mathbb{N}}$ be $\mathscr{W}-$ traced by z. If $A_{i_1}^1 \neq B_{i_1}^1$ then without loss of generality, we can assume that $A_{i_1}^1 = (x_1^1, f(x_1^1), \dots, f^{n-1}(x_1^1))$ and $B_{i_1}^1 = (x_2^1, f(x_2^1), \dots, f^{n-1}(x_2^1))$. As $(f^n(x_1^1), f^n(x_2^1)) \notin B \times B$ for any $B \in \mathscr{B}$, $(f^n(x_1^1), x_2^2) \in U \times U$ for some $U \in \mathscr{U}$, $(x_2^2, f^n(y)) \in V \times V$ for some $V \in \mathscr{V}$(this is true because $\mathscr{W}$ is a refinement of $\mathscr{V}$), $(f^n(x_2^1), x_4^2) \in U' \times U'$ for some $U' \in \mathscr{U}$, $(x_4^2, f^n(z)) \in V' \times V'$ for some $V \in \mathscr{V}$ and $\mathscr{V} = s^3(\mathscr{B})$ (note that there is no loss of generality in taking x_2^2, x_4^2 here). So, $(f^n(y), f^n(z)) \notin V \times V$ for any $V \in \mathscr{V}$. Hence, $y \neq z$.

Now, assume that $A_{i_k}^k = B_{i_k}^k$ for $k \in \{1, 2, \dots, k'\}$ and $A_{i_{k'+1}}^{k'+1} \neq B_{i_{k'+1}}^{k'+1}$. Let $A_{i_{k'}}^{k'} = B_{i_{k'}}^{k'} = (x_{i_{k'}}^{k'}, f(x_{i'_k}^{k'}), \dots, f^{n^{k'}_{\lceil i_{k'}/2 \rceil}-1}(x_{i_{k'}}^{k'}))$. Again, we can assume that $A_{i_{k'+1}}^{k'+1} = (x_{2i_{k'}-1}^{k'+1}, f(x_{2i_{k'}-1}^{k'+1}), \dots, f^{n_{i_{k'}}^{k'+1}-1}(x_{2i_{k'}-1}^{k'+1}))$ and $B_{i_{k'+1}}^{k'+1} = (x_{2i_{k'}}^{k'+1}, f(x_{2i_{k'}}^{k'+1}), \dots, f^{n_{i_{k'}}^{k'+1}-1}(x_{2i_{k'}}^{k'+1}))$. As $(f^{n_{i_{k'}}^{k'+1}}(x_{2i_{k'}-1}^{k'+1}), f^{n_{i_{k'}}^{k'+1}}(x_{2i_{k'}}^{k'+1})) \notin B \times B$ for any $B \in \mathscr{B}$, for $t = n + \sum_{t'=2}^{k'+1} n^{t'}_{\lceil i_{t'}/2 \rceil}$ $(f^{n_{i_{k'}}^{k'+1}}(x_{2i_{k'}-1}^{k'+1}), x_{2(2i_{k'}-1)}^{k'+2}) \in U \times U$ for some $U \in \mathscr{U}$, $(x_{2(2i_{k'}-1)}^{k'+2}, f^t(y)) \in V \times V$ for some $V \in \mathscr{V}$, $(f^{n_{i_{k'}}^{k'+1}}(x_{2i_{k'}}^{k'+1}), x_{4i_{k'}}^{k'+2}) \in U' \times U'$ for some $U' \in \mathscr{U}$, $(x_{4i_{k'}}^{k'+2}, f^t(z)) \in V' \times V'$ for some $V' \in \mathscr{V}$ and $\mathscr{V} = s^3(\mathscr{B})$, so $(f^t(y), f^t(z)) \notin V \times V$ for any $V \in \mathscr{V}$. Hence, $y \neq z$. Therefore, for any two different $\mathscr{U}-$ pseudo orbits, there exist two different points $\mathscr{V}-$ tracing those orbits.

For any $k \in \mathbb{N}$, take finite orbits $(A^j_{i_j})_{j\in\{1,2,...,k\}}$ and $(B^j_{i_j})_{j\in\{1,2,...,k\}}$ such that $A^j_{i_j} = B^j_{i_j}$ for $j \in \{1, 2, \ldots, k-1\}$ and $A^k_{i_k} \neq B^k_{i_k}$. Then there exist $y, z \in G$ such that $(A^j_{i_j})_{j\in\{1,2,...,k\}}$ is $\mathscr{W}-$ traced by y and $(B^j_{i_j})_{j\in\{1,2,...,k\}}$ is $\mathscr{W}-$ traced by z and for $t = n + \sum_{t'=2}^{k} n^{t'}_{\lceil i_{t'}/2\rceil}$, $(f^t(y), f^t(z)) \notin V \times V$ for any $V \in \mathscr{V}$. Hence, for any $t \in \{n + \sum_{t'=2}^{k} n^{t'}_{\lceil i_{t'}/2\rceil} : i_{t'+1} \in \{2i_{t'}, 2i_{t'-1}\}, k \in \mathbb{N}\}$, there exist $y, z \in G, y \neq z$ such that $(f^t(y), f^t(z)) \notin V \times V$ for any $V \in \mathscr{V}$. Since, $\{n + \sum_{t'=2}^{k} n^{t'}_{\lceil i_{t'}/2\rceil} : i_{t'+1} \in \{2i_{t'}, 2i_{t'-1}\}, k \in \mathbb{N}\}$ is a syndetic set. So, (X, f) is syndetically topologically sensitive.

Take open cover $\mathscr{B}'$ such that for any $B \in \mathscr{B}'$ and any $i \in \{1, 2, \ldots, m\}$, $f^i(B) \subset V$ for some $V \in \mathscr{V}$. Then as $\{n + \sum_{t'=2}^{k} n^{t'}_{\lceil i_{t'}/2\rceil} : i_{t'+1} \in \{2i_{t'}, 2i_{t'-1}\}\}$ is syndetic with $n^{t'}_{\lceil i_{t'}/2\rceil} \leq m$, so for every $t > n$ there exist $x, y \in G$ such that $(f^t(x), f^t(y)) \notin B \times B$ for any $B \in \mathscr{B}'$. Hence, (X, f) is cofinitely topologically sensitive. □

From above theorem, we can deduce that if a dynamical system on a compact Hausdorff space is topologically sensitive but not cofinitely topologically sensitive then the dynamical system cannot have topological POTP (See [18, 19] for examples).

Theorem 3.2. *Let (X, f) be a topologically sensitive dynamical system having topological POTP with X being a completely regular space. Then X is locally uncountable.*

Proof. Using notations same as used in proof of Theorem 3.1, let $\mathscr{B}$ be a sensitivity cover and take any open set G. Let $x^1_1, x^1_2 \in G, n \in \mathbb{N}$ be such that $(f^n(x^1_1), f^n(x^1_2)) \notin B \times B$ for any $B \in \mathscr{B}$. Take open covers $\mathscr{W}, \mathscr{U}$ as in Theorem 3.1 such that every $\mathscr{U}-$ pseudo orbit is $\mathscr{W}-$ traced(Any open cover need not be finite here). Then there exists sequences $(A^k_{i_k})_{k\in\mathbb{N}}$ such that $\{(A^k_{i_k})_k \in \mathbb{N} : i_k \in \{2i_{k-1}, 2i_{k-1}\}$ for $k > 1\}$ forms a $\mathscr{U}-$ pseudo orbit and hence, as done in Theorem 3.1 for different such $\mathscr{U}-$ pseudo orbit there exist different $z_1, z_2 \in G$ such that the orbits are $\mathscr{W}-$ traced by z_1, z_2. Note that the set $\{(A^k_{i_k})_{k\in\mathbb{N}} : i_k \in \{2i_{k-1} - 1, 2i_{k-1}\}$ for $k > 1\}$ is an uncountable set. Hence, G is uncountable. □

Example 3.1. Let $X = Q \cap [0, 1]$ with usual topology and define $f : X \to X$ by $f(x) = \{2x : 0 \leq x < 1/2, 2-2x : 1/2 \leq x \leq 1\}$. It is well known that f is sensitive. Since our space is countable so, from above theorem, (X, f) cannot have topological POTP.

4. Topological entropy

We start with defining a localized version of topological sensitivity and topological entropy.

Definition 4.1. Let (X, f) be any dynamical system.

1. A point $z \in X$ is called a *topologically sensitive point* if there exists an open cover $\mathscr{U}$ such that for any open set G containing z there exist an $n \in \mathbb{N}$ and $x, y \in G$ such that $(f^n(x), f^n(y)) \notin U \times U$ for any $U \in \mathscr{U}$. The set of topologically sensitive points of f is denoted by $Sen(g)$.

2. A point $z \in X$ is called a *topological entropy point* if for any open set G containing z, $h(f, \overline{G}) > 0$. The set of topological entropy points of f is denoted by $Ent(f)$.

First, we will show that any topologically sensitive map on a compact Hausdorff space having topological POTP has positive topological entropy.

T. Arai and N. Chinen showed that a dynamical system on a compact, connected metric space having dense set of periodic points and POTP has positive entropy[4]. Moothathu showed that on an infinite compact, connected metric space if a non-wandering map having POTP has sensitivity as well then every point is an entropy point[20]. Here, we will show that for the above result the condition and property of the non-wandering map and connectedness of the space or denseness of periodic points is not required for a dynamical system on compact Hausdorff topological spaces.

Theorem 4.1. *Let (X, f) be a dynamical system over compact Hausdorff topological space having topological POTP then topological sensitivity of (X, f) implies that every point is a topological entropy point and hence (X, f) has nonzero topological entropy.*

Proof. Using notations used in proof of the Theorem 3.1, take collection of finite orbits $\{(A^k_{i_k})^j_{k=1} : i_1 \in \{1, 2\}, i_k \in \{2i_{k-1}, 2i_{k-1} - 1\}$ for every $k \in \{2, 3, \dots, j\}\}$. Then these orbits are $\mathscr{W}-$ traced by different points, say $\{z_1, z_2, \dots, z_{2^j}\}$.

Take two different points $z', z'' \in \{z_1, z_2, \dots, z_{2^j}\}$. We will show that there exists a $t \in \{n\} \cup \{n + \sum_{t'=2}^{j'} n^{t'}_{\lceil i_{t'}/2\rceil} : i_{t'+1} \in \{2i_{t'}, 2i_{t'} - 1\}, j' \leq j\}$ such that $(f^t(z'), f^t(z'')) \notin V \times V$ for any $V \in \mathscr{V}$. Let $((A^k_{i_k})^j_{k=1}), (B^k_{i_k})^j_{k=1}$ be $\mathscr{V}-$ traced by z', z'' respectively. If $A^1_{i_1} \neq B^1_{i_1}$ then for $t = n$, we have $(f^t(z'), f^t(z'')) \notin V \times V$ for any $V \in \mathscr{V}$. Now, suppose that $A^{k'}_{i_{k'}} = B^{k'}_{i_{k'}}$ for every $k' < k''$ and $B^{k''}_{i_{k''}} \neq A^{k''}_{i_{k''}}$. Then for $t = n + \sum_{t'=2}^{k''} n^{t'}_{\lceil i_{t'}/2\rceil}$, we have $(f^t(z'), f^t(z'')) \notin V \times V$ for any $V \in \mathscr{V}$. Hence, we have that for any two different points $z', z'' \in \{z_1, z_2, \dots, z_{2^j}\}$ there exists a $t \in \{n + \sum_{t'=2}^{j'} n^{t'}_{\lceil i_{t'}/2\rceil} : i_{t'+1} \in \{2i_{t'}, 2i_{t'} - 1\}, j' \leq j\}$ such that $(f^t(z'), f^t(z'')) \notin V \times V$ for any $V \in \mathscr{V}$.

Now, let $t_j = \max\{n + \sum_{t'=2}^{j} n^{t'}_{i_{t'}} : i_{t'+1} \in \{2i_{t'}, 2i_{t'} - 1\}, j' \leq j\}$ then as $n^{t'}_{i_{t'}} \leq m$ for every $t' \in \mathbb{N}$ so $t_j \leq n + (j-1)m$. Now, for open cover $\mathscr{V}$, cardinality of a subcover of $\mathscr{V} \vee f^{-1}(\mathscr{V}) \vee \cdots \vee f^{-(t_j)}(\mathscr{V})$ covering G is at least 2^j, so $H(\overline{G}, \mathscr{V}) \geq \lim_{j\to\infty} \frac{\log 2^j}{t_j+1} \geq \lim_{j\to\infty} \frac{\log 2^j}{n+(j-1)m+1} = \frac{\log 2}{m}$. Hence, x is a topological entropy point implying that $\mathrm{Ent}(f) > 0$. □

Theorem 4.2. *Let X be a compact Hausdorff topological space with topological POTP then we have*

1. *f is weakly mixing implies f is topologically mixing.*
2. *f is totally transitive implies f is weakly mixing.*
3. *X is connected and f is non wandering implies f is totally transitive.*

Proof. If X contains only one element then each part is obvious. Next, we prove the lemma assuming X contains more than one element.

1. Let $a, b \in X$ and $a \neq b$. Now, let U, V be any pair of nonempty disjoint open sets in X containing a, b respectively, then there exist open sets U_1, V_1 such that $a \in U_1 \subset \overline{U}_1 \subset U$ and $b \in V_1 \subset \overline{V}_1 \subset V$. Define open cover $\mathscr{V} = \{U, V, X/(\overline{U}_1 \cup \overline{V}_1)\}$. Note that for $\mathscr{V}$, there exists an open cover $\mathscr{U}$ such that any $\mathscr{U}-$ pseudo orbit is $\mathscr{V}-$ traced. Since X is compact, we can assume that $\mathscr{U}$ is finite, say $\mathscr{U} = \{W_1, W_2, \dots, W_k\}$. Without loss of generality, we can assume that $b \in W_k$. As (X, f) is weakly mixing, so there exists a $p \in \mathbb{N}$ such that $f^p(W_i) \cap W_k \neq \emptyset$ for all $i \in \{1, 2, \dots, k\}$. Now, for any $j \in \mathbb{N}$ take $i \in \{1, 2, \dots, k\}$ such that $f^j(a) \in W_i$ and consider $\mathscr{U}-$ pseudo orbit $\overline{z} = (a, f(a), \dots, f^{j-1}(a), z_i, f(z_i), \dots, f^{p-1}(z_i), b)$, where $z_i \in W_i \cap f^{-p}(W_k)$. Assume that $\overline{z}$ is $\mathscr{V}-$ traced by a'. Since $a \notin V, X/(\overline{U}_1 \cup \overline{V}_1)$. So $(a, a') \in U \times U$ and similarly $(f^{p+j}(a'), b) \in V \times V$. Therefore, $f^{j+p}(U) \cap V \neq \emptyset$ for any $j \in \mathbb{N}$. Hence, (X, f) is topologically mixing.

2. Suppose f is totally transitive and let U', V' be nonempty open sets and U_1, U_2 be nonempty open sets such that $U_2 \subset \overline{U}_2 \subset U_1 \subset \overline{U}_1 \subset U'$. Take open cover $\mathscr{V} = \{U', X/\overline{U}_1\}$ and let $\mathscr{U}$, a refinement of $\{U_1, X/\overline{U}_2\}$, be an open cover such that any $\mathscr{U}-$ pseudo orbit is $\mathscr{V}-$ traced. Then there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$ where $U \in \mathscr{U}$ is an open set contained in U_1. Let $t \in \mathbb{N}$ be such that $f^{nt}(U) \cap V' \neq \emptyset$. Take $a \in U \cap f^{-n}(U)$ and say, $\overline{a} = (a, f(a), f^2(a), \dots, f^{n-1}(a))$ then $\overline{a}\,\overline{a} \dots \overline{a}a$ ($\overline{a}$ is repeated t times) is a $\mathscr{U}-$ pseudo orbit and hence is $\mathscr{V}-$ traced by some $y \in X$. As $a \in U_1$ that is $a \notin X/\overline{U}_1$ so, $(a, y) \in U' \times U'$ and $(a, f^{nt}(y)) \in U' \times U'$ therefore $f^{nt}(U') \cap U' \neq \emptyset$ and as $U \subset U_1 \subset U'$ and $f^{nt}(U) \cap V' \neq \emptyset$ so, $f^{nt}(U') \cap V' \neq \emptyset$. Therefore, (X, f) is weakly mixing.

3. Let $a, b \in X$, U and V be nonempty disjoint open sets in X such that $a \in U, b \in V$. Let U_1, V_1 be nonempty open sets such that $a \in U_1 \subset \overline{U}_1 \subset U$ and $b \in V_1 \subset \overline{V}_1 \subset V$ and take open cover $\mathscr{V} = \{U, V, X/(\overline{U}_1 \cup \overline{V}_1)\}$. Let $\mathscr{U}$

be an open cover such that any $\mathscr{U}-$ pseudo orbit is $\mathscr{V}-$ traced. Since X is connected, therefore we can find a sequence $a = a_0, a_1, \ldots, a_{n-1}, a_n = b$ such that for every $i \in \{0, 1, \ldots, n-1\}$, $(a_i, a_{i+1}) \in U'_i \times U'_i$ for some $U'_i \in s(\mathscr{U})$. Now, for every $i \in \{0, 1, \ldots, n-1\}$ we can find out a $k_i \in \mathbb{N}$ such that $(a_i, f^{k_i}(a_i)) \in U' \times U'$ for some $U' \in s(\mathscr{U})$. Consider $A_i = (a_i, f(a_i), f^2(a_i), \ldots, f^{k_i-1}(a_i))$ for every $i \in \{0, 1, \ldots, n-1\}$. Note that $(a_i, f^{k_i}(a_i)) \in U' \times U'$ for some $U' \in s(\mathscr{U})$ and $(a_i, a_{i+1}) \in U'_i \times U'_i$ for some $U'_i \in s(\mathscr{U})$. Hence, $(f^{k_i}(a_i), a_{i+1}) \in U \times U$ for some $U \in \mathscr{U}$. So, for any $p \in \mathbb{N}$, $(A_0A_0 \ldots A_0A_1 \ldots A_1 \ldots A_{n-1} \ldots A_{n-1}b)$ (where each A_i is repeated p times) is a $\mathscr{U}-$ pseudo orbit. So, there exists a $y \in U$ such that $f^{kp}(y) \in V$, where $k = k_0 + k_1 + \cdots + k_{n-1}$, implying that f^p is a transitive map and hence f is totally transitive. □

From above Theorem, Theorem 3.1 and Theorem 4.1, we can deduce that on a compact Hausdorff space any totally transitive map having topological POTP or when the space is connected, a non wandering map having topological POTP is cofinitely topologically sensitive and has positive topological entropy.

Proposition 4.1. *Let (X, f) be a compact, Hausdorff dynamical system having topological POTP. Then we have*

1. *If $z \in R(f)$ then $(z, z) \in R(f \times f)$.*
2. *If $z \in R(f)/M(f)$ then $z \in Sen(f)$.*

Proof. 1. Let U and V be any open sets containing z, then $z \in U \cap V$ and hence there exists an $n \in \mathbb{N}$ such that $f^n(z) \in U \cap V$ implying that $(f \times f)^n(z, z) \in (U \times V)$. Hence, $(z, z) \in R(f \times f)$.

2. Since $z \notin M(f)$ so there exists an open set G containing z such that for any $n \in \mathbb{N}$ there exists some $k' \in \mathbb{N}$ such that $\{k', k'+1, \ldots, k'+n\} \cap N(z, G) = \emptyset$.

Let G_1, G_2 be nonempty open sets such that $z \in G_1 \subset \overline{G}_1 \subset G_2 \subset \overline{G}_2 \subset G$. We show that open cover $\mathscr{V} = \{G, X/\overline{G}_2\}$ is a sensitivity cover for z. Let $\mathscr{U}$, a refinement of $\mathscr{V}' = \{G_2, X/\overline{G}_1\}$, be an open cover such that any $\mathscr{U}-$ pseudo orbit is $\mathscr{V}'-$ traced. Let $U \in \mathscr{U}$ be an open set such that $z \in U$. As $z \in R(f)$ so there exists an $n \in \mathbb{N}$ such that $f^n(z) \in U$. Now consider $\mathscr{U}-$ pseudo orbit, $\overline{z} = (z, f(z), f^2(z), \ldots, f^{n-1}(z), z, f(z), f^2(z), \ldots, f^{n-1}(z), \ldots)$ and let $y \in G_2$ be an element such that $\overline{z}$ is $\mathscr{V}'-$ traced by y. So for any $k \in \mathbb{N}$ $f^{nk}(y) \in G_2$. Since for every $n \in \mathbb{N}$ there exists a k' such that $\{k', k', +1, \ldots, k'+n\} \cap N(z, G) = \emptyset$, therefore we can find a $k \in \mathbb{N}$ such that $f^{kn}(z) \notin G$. Hence, there exist $y, z \in G_1$ such that $(f^{nk}(y), f^{nk}(z)) \notin V \times V$ for any $V \in \mathscr{V}$. Thus, $z \in Sen(f)$. □

In Theorem 4.1 we proved that if a dynamical system having topological POTP is topologically sensitive then every point is an entropy point. So, the question is whether every sensitive point is an entropy point. In the next theorem, we find required condition for a sensitive point to be an entropy point in a dynamical system having topological POTP. The metric version of this theorem for compact spaces is proved in [20].

Theorem 4.3. *Let X be a compact Hausdorff space and $f : X \to X$ be a continuous map having topological POTP. Let $Y \subset X$ be a closed and invariant subset of X and $g = f|_Y$. If there exists a point $z \in Y$ such that $z \in Sen(g)$ and $(z, z) \in int(\overline{R(g \times g)})$ then $z \in Ent(f)$ and hence $h(f) > 0$.*

Proof. Let $G_1 \subset X$ be any open set containing z and G_2 be an open set such that $z \in G_2 \subset \overline{G}_2 \subset G_1$. Then $\mathscr{W} = \{G_1, X/\overline{G}_2\}$ is an open cover of X. Let $\mathscr{V}$, a refinement of $\mathscr{W}$, be an open cover of X such that $\mathscr{V}' = \{V \cap Y | V \in \mathscr{V}\}$ is a sensitivity cover for z.

As (X, f) has topological POTP so let $\mathscr{U}$, a refinement of $s(s(\mathscr{V}))$, be an open cover such that any $\mathscr{U}-$ pseudo orbit can be $s(s(\mathscr{V}))-$ traced. Let $G \subset U \cap G_1$, for some $U \in \mathscr{U}$, be an open set containing z such that $(G \cap Y) \times (G \cap Y) \subset int(\overline{R(g \times g)})$. Take $n \in \mathbb{N}$ and $(a, b) \in R(g \times g) \cap (G \cap Y \times G \cap Y)$ such that $(f^n(a), f^n(b)) \notin V \times V$ for any $V \in \mathscr{V}$ and since $(a, b) \in R(g \times g)$, so there exists a $k > n$ such that $(f^k(a), f^k(b)) \in G \times G$.

Let $A = (a, f(a), f^2(a), \ldots, f^{k-1}(a))$ and $B = (b, f(b), f^2(b), \ldots, f^{k-1}(b))$. Now, for any $n \in \mathbb{N}$, let S be the set of finite sequences constructed by n permutations of A, B that is $S = \{A, B\}^n$. Then S contains 2^n elements and every sequence in S is a $\mathscr{U}-$ pseudo orbit. Let C and D be two different sequences in S. Assume C and D are $s(s(\mathscr{V}))$-traced by x_C and x_D, then for every $t \in \{0, 1, \ldots, k-1\}$ there exists $p_t \leq nk - 1$ such that $(f^{p_t}(x_C), f^t(a)) \in V''_1 \times V''_1$ for some $V''_1 \in s(s(\mathscr{V}))$ and $(f^{p_t}(x_D), f^t(b)) \in V''_2 \times V''_2$ for some $V''_2 \in s(s(\mathscr{V}))$

(If $(f^{p_t}(x_D), f^t(a)) \in V_1'' \times V_1''$ and $(f^{p_t}(x_C), f^t(b)) \in V_2'' \times V_2''$ then we will have a similar case). As for $t' = n \in \{0, 1, \dots, k-1\}$ we have $(f^{t'}(a), f^{t'}(b)) \notin V \times V$ for any $V \in \mathscr{V}$ and $(f^{p'_t}(x_D), f^{t'}(b)) \in V_2'' \times V_2''$, $(f^{p_{t'}}(x_D), f^{t'}(b)) \in V_2'' \times V_2''$ for some $V_1'', V_2'' \in s(\mathscr{V})$ so, $(f^{p_{t'}}(x_C), f^{p_{t'}}(x_D)) \notin V'' \times V''$ for any $V'' \in s(s(\mathscr{V}))$. Hence, $x_C \neq x_D$.

Now, we will show that $x_C, x_D \in G_1$. Without loss of generality, we can assume that the $\mathscr{U}-$ pseudo orbit $s(s(\mathscr{V}))$ traced by x_C starts with a. Then $a, x_C \in V_1''$ for some $V_1'' \in s(s(\mathscr{V}))$. Note that $a, z \in G \subset U$ for some $U \in \mathscr{U}$ and $\mathscr{U}$ is a refinement of $s(s(\mathscr{V}))$. So, $a, z \in V''$ for some $V'' \in s(s(\mathscr{V}))$. Therefore, $(z, x_C) \in V'$ for some $V' \in s(\mathscr{V})$. Since $\mathscr{V}$ is a refinement of $\mathscr{W} = \{G_1, X/\overline{G}_2\}$ and $z \notin X/\overline{G}_2$, so $x_C \in G_1$ and similarly $x_D \in G_1$ for any $C, D \in S$.

As for every sequence in S, there exists a different tracing point and there are 2^n sequences in S, so minimum cardinality of $s(s(\mathscr{V})) \vee f^{-1}(s(s(\mathscr{V}))) \vee \cdots \vee f^{-nk-1}(s(s(\mathscr{V})))$ that covers $\overline{G}_1$ is at least 2^n. Therefore, $N(\overline{G}_1, s(s(\mathscr{V})) \vee f^{-1}(s(s(\mathscr{V}))) \vee \cdots \vee f^{-nk-1}(s(s(\mathscr{V})))) \geq 2^n$ and so $h(f, \overline{G}_1) \geq \lim_{n\to\infty} \frac{\log 2^n}{nk} = \frac{\log 2}{k}$. Hence, $z \in Ent(f)$ and $h(f) > 0$. □

Kumar & Das proved that any minimal non topologically equicontinuous dynamical system is topologically sensitive[14]. We use this fact in next corollary.

Corollary 4.1. *Let (X, f) be a dynamical system having topological POTP with X being a compact Hausdorff space. If there exists a $z \in R(f)/M(f)$ or z belonging to a minimal non-topologically equicontinuous subsystem of (X, f) then $h(f) > 0$.*

Proof. For the first part, since $z \in R(f)/M(f)$, so by Proposition 4.1, $z \in Sen(f)$. Take $Y = \overline{O_f(z)}$ and $g = f|_{\overline{O_f(z)}}$. Then (Y, g) is transitive and hence $g \times g$ is non wandering. So, $(z, z) \in int(\overline{R(g \times g)})$. Hence, by Theorem 4.3, $h(f) > 0$.

For the second part, let $z \in (Y, g)$ where (Y, g) is a minimal non-topologically equicontinuous subsystem of (X, f). Since (Y, g) is minimal and non-topologically equicontinuous so (Y, g) is sensitive and as g is transitive so $g \times g$ is non wandering and hence, $z \in Sen(g)$ and $(z, z) \in int(\overline{R(g \times g)})$. Therefore, again by Theorem 4.3 $h(f) > 0$. □

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