Filomat 38:14 (2024), 5051–5060
https://doi.org/10.2298/FIL2414051A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Reconstruction of the Sturm-Liouville operator from nodal data

Rauf Amirov[a]

[a] *Sivas Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140 Turkey*

Abstract. In this study, the inverse nodal problem for second order differential operators on a finite interval with discontinuity conditions inside the interval is studied. For each discontinuity point $d \in \Re = \{r\pi, r \in (0,1) \cap \mathbb{Q}\}$, the existence of the solution of the inverse nodal problem has been proven and a constructive procedure for the solution is provided.

1. Introduction

One of the solution methods for the inverse problems of the Sturm-Liouville operators is to use the zeros of the eigenfunctions. These zeros are also called nodal points. Trying to reconstruct the coefficients of the operator from the asymptotic formula of the nodal points is known as inverse nodal problem. This problem for Sturm-Liouville operator was first investigated by McLaughlin in [1] .She accomplished to prove that this type of inverse problem has a unique solution. Same further numerical calculations for potential reconstruction are give in [2]. In 1997, Ch.-F.Yang [8] obtained a definite algorithm for the solution of inverse nodal problems with separated boundary conditions. Later, similar results for various boundary conditions were obtained in (see $[3-7, 9-16]$ and references there in). Next, inverse nodal problems for Sturm-Liouville operators with discontinuous conditions was first investigated by Chung-Tsun Shieh and V.A. Yurko in [11]. Later, the result obtained in the [11] generalized the by C.F. Yang in [10] using the same method. In particular, in the study [10], the uniqueness theorems and the stability feature of these problems were examined according to different characteristics regarding the solution of the inverse nodal problem for the Sturm-Liouville operator with discontinuity at $d = \dfrac{1}{2} \in [0,1]$ (midpoint of the part). In [21] studied a boundary value problem consisting of a Sturm-Liouville equation with conditions dependent of the spectral parameter and discontinuous conditions in point $x = \dfrac{\pi}{2}$ is investigated. Additionally, in studies $[22-25]$, inverse nodal problems and their important properties were examined for regular Sturm-Liouville and Dirac operators given with different properties. In this paper proved the uniquenessed theorem for solution of the inverse nodal problem, present constructive procedure for the potential function by using nodal lengths and studied Lipschitz stability for the inverse problem. In this study, the uniqueness theorem is proved for the solution of the inverse nodal problem to determine the potential function when the discontinuity point is the midpoint of the segment. However, the solution of the nodal inverse problem is not given when the point of discontinuity is any point in the interval.

2020 *Mathematics Subject Classification*. Primary 34A55; Secondary 34B24, 34L05
Keywords. Discontinuous conditions, nodal points, inverse nodal problem
Received: 23 May 2023; Revised: 15 November 2023; Accepted: 29 November 2023
Communicated by Ljubiša D. R. Kočinac
Email address: emirov@cumhuriyet.edu.tr (Rauf Amirov)

In this paper, unlike previous studies on this subject, when the discontinuity point $(0,\pi)$ is any of the countable number of irrational points in the form of $d_r = r\pi$, $(r \in (0,1)\cap\mathbb{Q})$, the proof of the uniqueness theorem is given for the solution of the inverse nodal problem and give an algorithm for the reconstruction of the coefficients of the problem using asymptotic of the nodal points.

Consider the following boundary value problem $L = L(q,h,H,a_1,a_2,d)$ with discontinuity conditions inside the interval:

$$-y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, V(y) := y'(\pi) + Hy(\pi) = 0, \tag{2}$$

$$y(d+0) = a_1 y(d-0), \;\; y'(d+0) = a_1^{-1} y'(d-0) + a_2\, y(d-0). \tag{3}$$

Here $d \in \mathfrak{R} := \{r\pi, r \in (0,1)\cap\mathbb{Q}\}$, λ is the spectral parameter, $q(x)$ is a real valued function, h, H, a_1, a_2 are real numbers, $q(x) \in L(0,\pi)$ and $a_1 > 0$.

Without loss of generality we assume that

$$\int_0^\pi q(x)dx = 0. \tag{4}$$

2. Main results

In the first part of this section, the asymptotics of the nodal points of the problem L are given. Then, we obtain uniqueness theorem and a procedure of recovering the potential $q(x)$ on the whole interval $(0,\pi)$ from a dense subset of nodal points.

Let $y(x)$ and $z(x)$ be continuously differentiable functions on $[0,d]$ and on $[d,\pi]$. Denote $\langle y,z\rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the matching conditions (3), then

$$\langle y,z\rangle |_{x=d-0} = \langle y,z\rangle |_{x=d+0}\,. \tag{5}$$

Let $\varphi(x,\lambda)$ be the solution of equation (1) satisfying the initial conditions $y(0,\lambda) = 1$, $y'(0,\lambda) = h$ and the discontinuity conditions (3). Then $U(\varphi) = 0$. Denote $\Delta(\lambda) := -V(\varphi)$. The function $\Delta(\lambda)$ is entire in λ of order $\frac{1}{2}$ and its zeros $\{\lambda_n\}_{n\geq 0}$ coincide with the eigenvalues of L. The function $\Delta(\lambda)$ is called the characteristic function for L. Since the boundary value problem L is self-adjoint, all zeros of $\Delta(\lambda)$ are real and simple (see [1])

Let $\lambda = \rho^2$, $\tau := \operatorname{Im}\rho$. For $|\lambda| \to \infty$ uniformly in x on has (see [1] and [17]);

$$\varphi(x,\lambda) = \cos\rho x + (h + Q(x))\frac{\sin\rho x}{\rho} + o\left(\frac{1}{\rho}\exp(|\tau| x)\right), x < d, \tag{6}$$

$$\varphi(x,\lambda) = \left(a_1^+\cos\rho x + a_1^-\cos\rho(2d-x)\right) + Q_1(x)\frac{\sin\rho x}{2\rho} + Q_2(x)\frac{\sin\rho(2d-x)}{2\rho} + o\left(\frac{1}{\rho}\exp(|\tau| x)\right), x > d, \tag{7}$$

$$\varphi'(x,\lambda) = -\rho\sin\rho x + (h + Q(x))\cos\rho x + o(\exp(|\tau| x)), x < d, \tag{8}$$

$$\varphi'(x,\lambda) = \rho(-a_1^+\sin\rho x + a_1^-\sin\rho(2d-x)) + \frac{1}{2}Q_1(x)\cos\rho x - \frac{1}{2}Q_2(x)\cos\rho(2d-x) + o(\exp(|\tau| x)), x > d, \tag{9}$$

where

$$Q(x)=\frac{1}{2}\int_0^x q(t)\,dt,\;\; Q_1(x)=a_1^+(2h+2Q(x))+a_2,\;\; Q_2(x)=a_1^-(2h-2Q(x)+4Q(d))-a_2,$$

$$a_1^+=\frac{1}{2}\left(a_1+a_1^{-1}\right),\;\; a_1^-=\frac{1}{2}\left(a_1-a_1^{-1}\right).$$

Theorem 2.1. *Let $\{\lambda_n\}_{n\geq 1}$ be the eigenvalues and $\varphi(x,\lambda_n)$ be the eigenfunctions of problem L. For sufficiently large n, the following asymptotic relations hold:*

$$\rho_n=\rho_n^0+\frac{d_n}{\rho_n^0}+o\left(\frac{1}{\rho_n^0}\right), \tag{10}$$

$$\varphi(x,\lambda_n)=\cos\rho_n^0x+(h+Q(x)-d_nx)\frac{\sin\rho_n^0x}{2\rho_n^0}+o\left(\frac{1}{\rho_n^0}\exp(|\tau|x)\right),x<d \tag{11}$$

$$\begin{aligned}\varphi(x,\lambda_n)=\;& a_1^+\cos\rho_n^0x+a_1^-\cos\rho_n^0(2d-x)+[Q_1(x)-d_nx]\frac{\sin\rho_n^0x}{2\rho_n^o}\\ &+[Q_2(x)-d_n(2d-x)]\frac{\sin\rho_n^0(2d-x)}{2\rho_n^0}+o\left(\frac{1}{\rho_n^0}\exp(|\tau|x)\right),x>d,\end{aligned} \tag{12}$$

where ρ_n^0's are the zeros of the $\Delta_0(\lambda)=\rho\left(-a_1^+\sin\rho\pi+a_1^-\sin\rho(2d-\pi)\right)$ function and

$$d_n=\frac{\left(Ha_1^++\frac{1}{2}Q_1(\pi)\right)\cos\rho_n^0\pi+\left(Ha_1^--\frac{1}{2}Q_2(\pi)\right)\cos\rho_n^0(2d-\pi)}{a_1^+\pi\cos\rho_n^0\pi-a_1^-(2d-\pi)\cos\rho_n^0(2d-\pi)}.$$

Proof. It follows from (7) and (9) that for $|\lambda|\to\infty$

$$\begin{aligned}\Delta(\lambda)=\;&\rho\left(-a_1^+\sin\rho\pi+a_1^-\sin\rho(2d-\pi)\right)+\left(Ha_1^++\frac{1}{2}Q_1(\pi)\right)\cos\rho\pi\\ &+\left(Ha_1^+-\frac{1}{2}Q_2(\pi)\right)\cos\rho(2d-\pi)+o(\exp(|\tau|\pi)).\end{aligned} \tag{13}$$

Let

$$\Delta_0(\lambda)=\rho\left(-a_1^+\sin\rho\pi+a_1^-\sin\rho(2d-\pi)\right).$$

Denote $G_\delta:=\left\{\rho:\left|\rho-\rho_n^0\right|\geq\delta>0,n=0,1,2,...\right\}$, where δ is sufficiently small number. As shown in [2] that for $\rho\in\overline{G_\delta}$, $\left|\Delta_0(\rho)\right|\geq C(\delta)\left|\rho\right|\exp(|\tau|\pi)$.

Since the function $\rho^{-1}\Delta_0(\lambda)$ is type of "sine" ([13] p.119), the number γ_δ exists such that for all $n\geq 1$, $\left|\dot{\Delta}_0\left(\rho_n^0\right)\right|\geq\gamma_\delta>0$. If the study [10] is used then we get that $\rho_n^0=n+h_n$, where $\sup_n|h_n|\leq M$. Also, $\lambda_0^0=0$.

It can be shown using classical methods in [1] that the sequence $\{\rho_n\}_{n\geq 1}$ satisfies the asymptotic relation (10), i.e. first part of theorem is provided.

Let $\varphi(x,\lambda_n)$ be the eigenfunctions of the problem L. From (6), (7) and (10), we can see easily the asymptotic formulas (11) and (12) are valid for sufficiently large n. Hence, the second part of the theorem is provided.

For the boundary value problem L an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunction $\varphi(x,\lambda_n)$ has exactly n (simple) zeros inside the interval $(0,\pi)$, namely: $0<x_n^1<x_n^2<...<x_n^n<\pi$.

The set $X(L):=\left\{x_n^j:n=1,2,...,j=1,2,\ldots n\right\}$ is called the set of nodal points of the boundary value problem L. □

Theorem 2.2. *The following asymtotic expression is provided*

$$x_n^{j(n)} = \frac{\left(j(n)-\frac{1}{2}\right)\pi}{\rho_n^0} + \left[h + Q\left(x_n^{j(n)}\right) - d_n x_n^{j(n)}\right]\frac{1}{\left(\rho_n^0\right)^2} + o\left(\frac{1}{\left(\rho_n^0\right)^2}\right), x_n^{j(n)} \in (0,d), \tag{14}$$

$$x_n^{j(n)} = \frac{\left(j(n)-\frac{1}{2}\right)\pi}{\rho_n^0} + \frac{1}{\rho_n^0}\arctan\left(\frac{a_1^- \sin 2d\rho_n^0}{a_1^+ + a_1^-\cos 2d\rho_n^0}\right) + A_n\frac{B\left(x_n^{j(n)}\right)}{2\left(\rho_n^0\right)^2} + o\left(\frac{1}{\left(\rho_n^0\right)^2}\right), x_n^{j(n)} \in (d,\pi), \tag{15}$$

for sufficiently large n, uniformly with respect to j, where

$$A_n = \frac{\left(a_1^+ + a_1^-\cos 2d\rho_n^0\right)^2}{\left(a_1^+ + a_1^-\cos 2d\rho_n^0\right)^2 + \left(b_2\sin 2d\rho_n^0\right)^2},$$

$$B_n(x) = \frac{Q_1(x) - Q_2(x)\cos 2d\rho_n^0 - d_n x + d_n(2d-x)\cos 2d\rho_n^0}{a_1^+ + a_1^-\cos 2d\rho_n^0} - \frac{a_1^-\left[Q_2(x) - d_n(2d-x)\right]}{\left(a_1^+ + a_1^-\cos 2d\rho_n^0\right)^2}\sin 2d\rho_n^0.$$

Proof. Use the asymptotic formulas for the case $x<d$ and $x>d$ respectively (11) and (12) to get

$$0 = \varphi\left(x_n^{j(n)},\lambda_n\right) = \cos\rho_n^0 x_n^{j(n)} + \left[h + Q\left(x_n^{j(n)}\right) - d_n x_n^{j(n)}\right]\frac{\sin\rho_n^0 x_n^{j(n)}}{\rho_n^0} + o\left(\frac{1}{\rho_n^0}\exp\left(|\tau|\, x_n^{j(n)}\right)\right), x_n^{j(n)} \in (0,d),$$

$$\begin{aligned} 0 = \varphi\left(x_n^{j(n)},\lambda_n\right) = \;& a_1^+\cos\rho_n^0 x_n^{j(n)} + a_1^-\cos\rho_n^0\left(2d - x_n^{j(n)}\right) + \left[Q_1\left(x_n^{j(n)}\right) - d_n x_n^{j(n)}\right]\frac{\sin\rho_n^0 x_n^{j(n)}}{2\rho_n^0} \\ & + \left[Q_2\left(x_n^{j(n)}\right) - d_n(2d - x_n^{j(n)})\right]\frac{\sin\rho_n^0(2d - x_n^{j(n)})}{2\rho_n^0} \\ & + o\left(\frac{1}{\rho_n^0}\exp\left(|\tau|\, x_n^{j(n)}\right)\right), x_n^{j(n)} \in (d,\pi), \end{aligned}$$

and so

$$\tan\left(\rho_n x_n^{j(n)} + \frac{\pi}{2}\right) = \left[h + Q\left(x_n^{j(n)}\right) - d_n x_n^{j(n)}\right]\frac{1}{\rho_n^0} + o\left(\frac{1}{\rho_n^0}\right), x_n^{j(n)} \in (0,d), \tag{16}$$

$$\tan\left(\rho_n x_n^{j(n)} + \frac{\pi}{2}\right) = \frac{a_1^-\sin 2d\rho_n^0}{a_1^+ + a_1^-\cos 2d\rho_n^0} + B_n\left(x_n^{j(n)}\right)\frac{1}{2\rho_n^0} + o\left(\frac{1}{\rho_n^0}\right), x_n^{j(n)} \in (d,\pi). \tag{17}$$

If we apply the identity

$$\arctan\alpha - \arctan\beta = \arcsin\left(\frac{\left|\alpha-\beta\right|}{\sqrt{(1+\alpha^2)(1+\beta^2)}}\right),$$

we get the asymptotic formulas (14) and (15) for the nodal points from the equations (16) and (17). Theorem 2.2 is provided. □

It is clear from the expression of $\left\{\rho_n^0\right\}_{n\geq 1}$ that $\{h_n\}_{n\geq 1}$ is a real sequence. Since $\sup_n |h_n| \leq M < +\infty$, let's choose subsequence $\{n_k\}_{k\geq 0} \subset \mathbb{N}$ as $\lim_{k\to\infty} h_{n_k} = h_o < +\infty$. Let's define the set $\mathfrak{R} = \left\{r\pi : r = \frac{p}{q},\ p<q,\ p,\ q \in \mathbb{N}\right\}$.

It is clear that the set $\Re$ is dense in the range $(0,\pi)$ and consists of irrational numbers in the form $r\pi, r \in (0,1)\cap\mathbb{Q}$, in this range.

Let us take any point $d \in \Re \subset (0,\pi)$ and choose the sequence $\{n_k\}_{k\geq 0}$ with $n_k = qm_k, \left(m_k \in \mathbb{N}, \lim\limits_{k\to\infty} m_k = +\infty\right)$. In this case, since $\sin 2d\rho^0_{n_k} = \sin 2dh_{n_k}$, $\cos 2d\rho^0_{n_k} = \cos 2dh_{n_k}$ and $\frac{1}{\rho^0_{n_k}} = \frac{1}{n_k} - \frac{h_{n_k}}{(n_k)^2} + o\left(\frac{1}{(n_k)^2}\right)$, we get following asymptotic formulas for the nodal points of the problem L, for $k \to \infty$ uniformly in j:

$$x^{j(n_k)}_{n_k} = \frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k} - \frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{(n_k)^2}h_{n_k} + \left[h + Q\left(x^{j(n_k)}_{n_k}\right) - d_{n_k}x^{j(n_k)}_{n_k}\right]\frac{1}{(n_k)^2} + o\left(\frac{1}{(n_k)^2}\right), x^{j(n_k)}_{n_k} \in (0,d), \quad (18)$$

$$\begin{aligned} x^{j(n_k)}_{n_k} = & \frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k} - \frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{(n_k)^2}h_{n_k} + \frac{1}{n_k}\arctan\left(\frac{a_1^-\sin 2dh_{n_k}}{a_1^+ + a_1^-\cos 2dh_{n_k}}\right) \\ & -\frac{h_{n_k}}{(n_k)^2}\arctan\left(\frac{a_1^-\sin 2dh_{n_k}}{a_1^+ + a_1^-\cos 2dh_{n_k}}\right) + A_{n_k}\frac{B_{n_k}\left(x^{j(n_k)}_{n_k}\right)}{2(n_k)^2} + o\left(\frac{1}{(n_k)^2}\right), x^{j(n_k)}_{n_k} \in (d,\pi), \end{aligned} \quad (19)$$

where

$$A_n = \frac{\left(a_1^+ + a_1^-\cos 2dh_{n_k}\right)^2}{\left(a_1^+ + a_1^-\cos 2dh_{n_k}\right)^2 + \left(a_1^-\sin 2dh_{n_k}\right)^2},$$

$$\begin{aligned} B_n\left(x^{j(n_k)}_{n_k}\right) = & \frac{Q_1\left(x^{j(n_k)}_{n_k}\right) - Q_2\left(x^{j(n_k)}_{n_k}\right)\cos 2dh_{n_k} - d_{n_k}x^{j(n_k)}_{n_k} + d_{nk}\left(2d - x^{j(n_k)}_{n_k}\right)\cos 2dh_{n_k}}{a_1^+ + a_1^-\cos 2dh_{n_k}} \\ & - \frac{a_1^-\left[Q_2\left(x^{j(n_k)}_{n_k}\right) - d_{n_k}\left(2d - x^{j(n_k)}_{n_k}\right)\right]}{\left(a_1^+ + a_1^-\cos 2dh_{n_k}\right)^2}\left(\sin 2dh_{n_k}\right)^2. \end{aligned}$$

Let $X_0(L) = \left\{x^{j(n_k)}_{n_k} : n_k = 1,2,...,j(n_k) = 1,2,\dots,n_k\right\}$ be a subsequence of the numbers $x^{j(n)}_n$ that is dense on $(0,\pi)$. According to above result, the existence of such set is obvious.

Theorem 2.3. *For $x \in (0,\pi)$, let $X_0(L) \subset X(L)$ and $\lim\limits_{k\to\infty} x^{j(n_k)}_{n_k} = x$. Then, for any point $d \in \Re$ the following limits exist and are finite:*

$$f_1(x) := \lim_{k\to\infty}\left[n_k x^{j(n_k)}_{n_k} - \left(j(n_k) - \frac{1}{2}\right)\pi\right] = xh_0, x \in [0,d), \quad (20)$$

$$g_1(x) := \lim_{k\to\infty} n_k\left[n_k x^{j(n_k)}_{n_k} - \left(j(n_k) - \frac{1}{2}\right)\pi + \frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k}h_{n_k}\right] = h + Q(x) - xd_0, \ x \in [0,d) \quad (21)$$

$$f_2(x) := \lim_{k\to\infty}\left[n_k x^{j(n_k)}_{n_k} - \left(j(n_k) - \frac{1}{2}\right)\pi\right] = xh_0 + \arctan\left(\frac{a_1^-\sin 2dh_0}{a_1^+ + a_1^-\cos 2dh_0}\right), \ x \in (d,\pi], \quad (22)$$

$$\begin{aligned} g_2(x) := & \lim_{k\to\infty} n_k\left[n_k x^{j(n_k)}_{n_k} - \left(j(n_k) - \frac{1}{2}\right)\pi + \frac{\left(j(n_k)-\frac{1}{2}\right)\pi}{n_k}h_{n_k} - \arctan\left(\frac{a_1^-\sin 2dh_{n_k}}{a_1^+ + a_1^-\cos 2dh_{n_k}}\right)\right] \\ & = h_0\arctan\left(\frac{a_1^-\sin 2dh_0}{a_1^+ + a_1^-\cos 2dh_0}\right) + \frac{1}{2}A_0B_0(x), x \in (d,\pi], \end{aligned} \quad (23)$$

where

$$A_0 = \lim_{k\to\infty} A_{n_k} = \frac{\left(a_1^+ + a_1^- \cos 2dh_0\right)^2}{\left(a_1^+ + a_1^- \cos 2dh_o\right)^2 + \left(a_1^- \sin 2dh_0\right)^2},$$

$$B_0(x) = \lim_{k\to\infty} B_{n_k}\left(x_{n_k}^{j(n_k)}\right) = \frac{Q_1(x) - Q_2(x)\cos 2dh_0 - d_0 x + d_0(2d-x)\cos 2dh_0}{a_1^+ + a_1^- \cos 2dh_0}$$
$$- \frac{a_1^-\left[Q_2(x) - d_0(2d-x)\right]}{\left(a_1^+ + a_1^- \cos 2dh_0\right)^2}\left(\sin 2dh_0\right)^2,$$

$$d_0 = \lim_{k\to\infty} d_{n_k} = \frac{\left(Ha_1^+ + \frac{1}{2}Q_1(\pi)\right)\cos \pi h_0 + \left(Ha_1^- - \frac{1}{2}Q_2(\pi)\right)\cos(2d-\pi)h_0}{a_1^+\pi\cos\pi h_0 - a_1^-(2d-\pi)\cos(2d-\pi)h_0}.$$

Proof. Let $d \in \mathfrak{R} \subset (0,\pi)$ be any point. For each fixed $x \in [0,\pi]\setminus\{d\}$, there exists a sequence $\left(x_n^{j(n)}\right)$ converges to x. For $n_k = qm_k, m_k \in \mathbb{N}$, the subsequence $\left(x_{n_k}^{j(n_k)}\right)$ converges also to x. Since we get from the asymptotics from (18) and (19), the limits (20)-(23) exist and they are finite. Therefore the theorem is provided. □

Let us now state a uniqueness theorem and present a constructive procedure for solving inverse nodal problem.

Theorem 2.4. *Let $X_0(L) \subset X(L)$ be a subset of nodal points which is dense in $(0,\pi)$. Then, for any $d \in \mathfrak{R}$ the specification of $X_0(L)$ uniquely determines the potential $q(x)$ a.e. on $(0,\pi)$ and the coefficients h and H. The potential $q(x)$ and the number h can be constructed via the following algorithm:*

1. For each $x \in [0,\pi]$, we choose a sequence $\left\{x_n^{j(n)}\right\} \subset X_0(L)$ such that

$$\lim_{n\to\infty} x_n^{j(n)} = x.$$

2. From (21), we find the function $g_1(x)$ and from $g_1(0)$ we calculate

$$h = g_1(0). \tag{24}$$

3. From (23), we find the function $g_2(x)$ and from $g_2(\pi)$ we calculate

$$\int_0^d q(t)dt = 2G\left(H, g_1(0), h_0, a_1, a_2\right). \tag{25}$$

4. The function q(x) is determined from the equalities (21) and (23) as follows:

$$\begin{aligned} q(x) = \quad & 2g_1'(x) - \frac{\left[a_1^+(H+h) + \frac{a_2}{2}\right]\cos\pi h_0 + \left[a_1^-(H-h) + \frac{a_2}{2}\right]\cos(2d-\pi)h_0}{a_1^+\pi\cos\pi h_0 - a_1^-(2d-\pi)\cos(2d-\pi)h_0} \\ & + \frac{2a_1^-\cos(2d-\pi)h_0 G\left(H, g_1(0), h_0, a_1, a_2\right)}{a_1^+\pi\cos\pi h_0 - a_1^-(2d-\pi)\cos(2d-\pi)h_0}, x \in [0,d), \end{aligned} \tag{26}$$

$$q(x)=\frac{2\left[\left(a_1^+ + a_1^- \cos 2dh_0\right)^2+\left(a_1^- \sin 2dh_0\right)^2\right]}{\left(a_1^+ + a_1^- \cos 2dh_0\right)^2+a_1^- \sin^2 2dh_0}\left\{g_2'(x)\right.$$
$$+\,2A_0\frac{\left[a_1^+\left(H+g_1(0)\right)+\frac{a_2}{2}\right]\cos\pi h_0+\left[a_1^-\left(H-g_1(0)\right)+\frac{a_2}{2}\right]\cos(2d-\pi)h_0}{\left[a_1^+\pi\cos\pi h_0-a_1^-(2d-\pi)\cos(2d-\pi)h_0\right]\left(a_1^+ + a_1^-\cos 2dh_0\right)}\left.\right\} \tag{27}$$
$$+\frac{4\left[\left(a_1^+ + a_1^- \cos 2dh_0\right)^2+\left(a_1^- \sin 2dh_0\right)^2\right]a_1^-\cos(2d-\pi)h_0\sin^2 dh_0 G\left(H,g_1(0),h_0,a_1,a_2\right)}{\left[a_1^+\pi\cos\pi h_0-a_1^-(2d-\pi)\cos(2d-\pi)h_0\right]\left[\left(a_1^+ + a_1^-\cos 2dh_0\right)^2+a_1^-\sin^2 2dh_0\right]\left(a_1^+ + a_1^-\cos 2dh_0\right)},$$
$$x\in(d,\pi],$$

Proof. Formulas (24), (25), (26) and (27) can be derived from (21) and (23) step by step. We obtain the following reconstruction procedure:

i) By taking value of $g_1(x)$ at $x=0$, then we obtain $h=g_1(0)$.

ii) By taking value of $g_2(x)$ at $x=\pi$, then we obtain $\int_0^d q(t)dt = 2G\left(H,g_1(0),h_0,a_1,a_2\right)$.

iii) By taking derivatives of the functions $g_i(x),(i=1,2)$, we obtain (26) and (27).

Let the function $\psi(x,\lambda)$ be the solution of (1) under the initial conditions $\psi(\pi,\lambda)=1,\psi'(\pi,\lambda)=-H$, and discontinuity conditions (3). It is clear that $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$, where $\beta_n=\psi'(0,\lambda_n)$.

To complete the proof, consider a sequence $\left\{x_n^{j(n)}\right\}\subset X_0(L)$ that converges to π and write equation (1) for $\psi(x,\lambda_n)$ and $\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)$ as follows

$$-\widetilde{\psi}''\left(x,\widetilde{\lambda}_n\right)+q(x)\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)=\widetilde{\lambda}_n\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right),$$
$$-\psi''(x,\lambda_n)+q(x)\psi(x,\lambda_n)=\lambda_n\psi(x,\lambda_n).$$

If these equations are multiplied by $\psi(x,\lambda_n)$ and $\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)$, respectively, subtracted from each other and integrated over the interval $\left(x_n^{j(n)},\pi\right)$, the equality

$$\psi'(\pi,\lambda_n)\widetilde{\psi}\left(\pi,\widetilde{\lambda}_n\right)-\widetilde{\psi}'\left(x,\widetilde{\lambda}_n\right)\psi(\pi,\lambda_n)=\left(\lambda_n-\widetilde{\lambda}_n\right)\int_{x_n^{j(n)}}^{\pi}\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)\psi(x,\lambda_n)\,dx$$

is obtained. Using (10), we get the following estimate for sufficiently large n

$$H-\widetilde{H}=\left[2\left(d_n-\widetilde{d}_n\right)+o(1)\right]\int_{x_n^{j(n)}}^{\pi}\widetilde{\psi}\left(x,\widetilde{\lambda}_n\right)\psi(x,\lambda_n)\,dx.$$

Since the sequences (d_n) and $\left(\widetilde{d}_n\right)$ are bounded, then $H=\widetilde{H}$. This completes the proof. □

Corollary 2.5. *If $h_0=0$, then $f_1(x)=0$ and $f_2(x)=0$. In this case $h=g_1(0)$,*

$$a_1^-Q(d)=\left\{-\left(a_1^+ + a_1^-\right)g_2(\pi)+\left(a_1^+ - a_1^-\right)g_1(0)+a_2+(\pi-d)\frac{a_1^+(H+h)+a_1^-(H-h)+a_2}{\pi\left(a_1^+ + a_1^-\right)-2a_1^-d}\right\}\times$$
$$\times\frac{\pi\left(a_1^+ + a_1^-\right)-2a_1^-d}{(\pi-d)-2\left[\pi\left(\left(a_1^+ + a_1^-\right)-2a_1^-d\right)\right]},$$

$$
\begin{aligned}
q(x) = \; & 2\left[g_1^{'}(x) + \frac{a_1^+(H+h) + a_1^-(H-h) + a_2}{a_1^+\pi - a_1^-(2d-\pi)}\right] + \frac{1}{a_1^+\pi - a_1^-(2d-\pi)} \times \\
& \times \left\{ -\left(a_1^+ + a_1^-\right) g_2(\pi) + \left(a_1^+ - a_1^-\right) g_1(0) + a_2 + (\pi - d)\frac{a_1^+(H+h) + a_1^-(H-h) + a_2}{\pi\left(a_1^+ + a_1^-\right) - 2a_1^- d} \right\} \times \\
& \times \frac{\pi\left(a_1^+ + a_1^-\right) - 2a_1^- d}{(\pi - d) - 2\left[\pi\left(\left(a_1^+ + a_1^-\right) - 2a_1^- d\right)\right]}, x \in [0, d),
\end{aligned}
$$

$$
\begin{aligned}
q(x) = \; & 2\left[g_2^{'}(x) + \frac{a_1^+(H+h) + a_1^-(H-h) + a_2}{(a_1^+ + a_1^-)\left[\pi\left(a_1^+ + a_1^-\right) - 2a_1^- d\right]}\right] + \frac{2}{(a_1^+ + a_1^-)\left[\pi\left(a_1^+ + a_1^-\right) - 2a_1^- d\right]} \times \\
& \times \left\{ -\left(a_1^+ + a_1^-\right) g_2(\pi) + \left(a_1^+ - a_1^-\right) g_1(0) + a_2 + (\pi - d)\frac{a_1^+(H+h) + a_1^-(H-h) + a_2}{\pi\left(a_1^+ + a_1^-\right) - 2a_1^- d} \right\} \times \\
& \times \frac{\pi\left(a_1^+ + a_1^-\right) - 2a_1^- d}{(\pi - d) - 2\left[\pi\left(\left(a_1^+ + a_1^-\right) - 2a_1^- d\right)\right]}, x \in (d, \pi].
\end{aligned}
$$

Corollary 2.6. *Let $d = \frac{\pi}{2}$. Then $h_n = 0$ for all n, consequently $h_0 = 0$. In this case, we get the following equalities:*

$$g_1(0) = h,$$

$$a_1^- Q(\tfrac{\pi}{2}) = \frac{2a_1^+}{\left(1 - 4a_1^+\right)} \left\{ -\left(a_1^+ + a_1^-\right) g_2(\pi) + \left(a_1^+ - a_1^-\right) g_1(0) + a_2 + \frac{1}{2a_1^+}\left[a_1^+(H + g_1(0)) + a_1^-(H - g_1(0)) + a_2\right] \right\},$$

$$
\begin{aligned}
q(x) = \; & \left\{ 2g_1^{'}(x) + \frac{2}{\pi\left(4a_1^+ - 1\right)}\left[-\left(a_1^+ + a_1^-\right) g_2(\pi) + \left(a_1^+ - a_1^-\right) g_1(0) + a_2\right] \right. \\
& \left. + \frac{4}{\pi\left(4a_1^+ - 1\right)}\left[a_1^+(H + g_1(0)) + a_1^-(H - g_1(0)) + a_2\right] \right\}, x \in \left[0, \frac{\pi}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
q(x) = \; & 2\left\{ g_2^{'}(x) + \frac{4}{\pi\left(1 - 4a_1^+\right)\left(a_1^+ + a_1^-\right)}\left[a_1^+(H + g_1(0)) + a_1^-(H - g_1(0)) + a_2\right] \right. \\
& \left. + \frac{2\left[-\left(a_1^+ + a_1^-\right) g_2(\pi) + \left(a_1^+ - a_1^-\right) g_1(0) + a_2\right]}{\pi\left(4a_1^+ - 1\right)\left(a_1^+ + a_1^-\right)} \right\}, x \in \left(\frac{\pi}{2}, \pi\right].
\end{aligned}
$$

Corollary 2.7. *In the (1)-(3) problem, if the interval $[0, 1]$ is taken instead of the interval $[0, \pi]$, it must be $d \in (0, 1) \cap \mathbb{Q}$ for the inverse nodal problem to be solvable.*

Corollary 2.8. *In the (1)-(3) problem, if the interval $[0, T]$ is taken instead of the interval $[0, \pi]$, it must be $\frac{d}{T} \in (0, 1) \cap \mathbb{Q}$ for the inverse nodal problem to be solvable.*

Example 2.9. We consider the following discontinuity boundary value problem:

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in \Omega = [0, \pi] \setminus \{d\},$$

$$y'(0) - hy(0) = 0, y'(\pi) + Hy(\pi) = 0,$$

$$y(d + 0) = a_1 y(d - 0), \;\; y'(d + 0) = a_1^{-1} y'(d - 0) + a_2\, y(d - 0),$$

where $q(x) \in C^1[0, \pi]$, h and H are unknown coefficients of the problem. If we take $d = \frac{\pi}{2}$, then it is clear that $h_0 = 0$.

For sufficiently large n, the nodal points provide the following asymptotic:

$$x_n^{j(n)} = \frac{\left(j(n)-\frac{1}{2}\right)\pi}{n} + \left(h - d_n x_n^{j(n)}\right)\frac{1}{n^2} - \frac{1}{6}\frac{\left(j(n)-\frac{1}{2}\right)\pi}{n^3} + \frac{1}{6}\frac{\left(j(n)-\frac{1}{2}\right)^3}{n^5} + o\left(\frac{1}{n^5}\right), x_n^{j(n)} \in \left(0, \frac{\pi}{2}\right),$$

$$x_n^{j(n)} = \frac{\left(j(n)-\frac{1}{2}\right)\pi}{n} - \left\{\frac{d_n\left(2x_n^{j(n)}-\pi\right)}{a_1^+ + a_1^-} + \frac{2h\left(a_1^+ - a_1^-\right)+2a_2}{a_1^+ + a_1^-} - \frac{2a_1^-}{a_1^+ + a_1^-}\int_0^{\frac{\pi}{2}} q(t)dt\right\}\frac{1}{n^2}$$
$$+\frac{a_1^+ - a_1^-}{a_1^+ + a_1^-}\left\{-\frac{1}{6}\frac{\left(j(n)-\frac{1}{2}\right)\pi}{n^3} + \frac{1}{6}\frac{\left[\left(j(n)-\frac{1}{2}\right)\pi\right]^3}{n^5}\right\} + o\left(\frac{1}{n^5}\right), x_n^{j(n)} \in \left(\frac{\pi}{2}, \pi\right).$$

According to these data, we can calculate $q(x)$ and h. Let $n_k = 2k, k \in \mathbb{Z}$, one can calculate that

$$\lim_{k\to\infty} 4k^2\left(x_{2k}^{j(2k)} - \frac{\left(j(2k)-\frac{1}{2}\right)\pi}{2k}\right) = g_1(x) = h - d_0 x - \frac{1}{6}x + \frac{1}{6}x^3, x \in \left(0, \frac{\pi}{2}\right),$$

$$\lim_{k\to\infty} 4k^2\left(x_{2k}^{j(2k)} - \frac{\left(j(2k)-\frac{1}{2}\right)\pi}{2k}\right) = g_2(x) = \frac{h\left(a_1^+ - a_1^-\right)}{a_1^+ + a_1^-} - \frac{a_1^-}{a_1^+ + a_1^-}\int_0^{\frac{\pi}{2}} q(t)dt - \frac{d_0(x-\pi)}{a_1^+ + a_1^-}$$
$$+\frac{a_1^+ - a_1^-}{a_1^+ + a_1^-}\left(-\frac{1}{6}x + \frac{1}{6}x^3\right).$$

By the formulas (24), (25) and (26);

$$h = g_1(0),$$

$$\begin{aligned} q(x) &= 2g_1'(x) + \frac{4}{\pi\left(4a_1^+ - 1\right)}\left[-\left(a_1^+ - a_1^-\right)(2H + g_2(\pi)) - \left(a_1^+ - a_1^-\right)g_1(0) - a_2\right] \\ &= x^2 - \frac{1}{3} - 2d_0 + \frac{4}{\pi\left(4a_1^+ - 1\right)}\left[-\left(a_1^+ - a_1^-\right)(2H + g_2(\pi)) - \left(a_1^+ - a_1^-\right)g_1(0) - a_2\right], x \in \left[0, \frac{\pi}{2}\right], \end{aligned}$$

$$\begin{aligned} q(x) = &\ 2g_2'(x) + \frac{4}{\pi\left(4a_1^+ - 1\right)\left(a_1^+ + a_1^-\right)}\left[-\left(a_1^+ + a_1^-\right)g_2(\pi) + \left(a_1^+ - a_1^-\right)g_1(0) + a_2\right. \\ &\left. -2a_1^+(H+h) - 2a_1^-(H-h) - 2a_2\right] \\ = &\ \frac{a_1^+ - a_1^-}{a_1^+ + a_1^-}\left(x^2 - \frac{1}{3}\right) - \frac{2d_0}{a_1^+ + a_1^-} \\ &+ \frac{4}{\pi\left(4a_1^+ - 1\right)\left(a_1^+ + a_1^-\right)}\left[-\left(a_1^+ - a_1^-\right)(g_2(\pi) + 2H) - \left(a_1^+ - a_1^-\right)g_1(0) - a_2\right], x \in \left[\frac{\pi}{2}, \pi\right], \end{aligned}$$

where

$$d_0 = \frac{1}{\pi a_1^+}\left\{H\left(a_1^+ + a_1^-\right) + \frac{1}{2}\left(1 + \frac{a_1^-}{a_1^+}\right)H + \frac{1}{2}\left(1 - \frac{a_1^-}{a_1^+}\right)g_1(0) - \frac{a_2}{2a_1^+} + \frac{1}{2}g_2(\pi)\right\},$$

$$\frac{a_1^-}{a_1^+ + a_1^-}\int_0^{\frac{\pi}{2}} q(t)dt = -g_2(\pi) + \frac{h(a_1^+ - a_1^-)}{a_1^+ + a_1^-} + \frac{\pi\left(a_1^+ - a_1^-\right)\left(\pi^2 - 1\right)}{6\left(a_1^+ + a_1^-\right)} - \frac{\pi d_0}{a_1^+ + a_1^-}.$$

Acknowledgements

The paper was presented at the International Conference "Modern Problems of Mathematics and Mechanics" dedicated to the 100-th anniversary of the National Leader Heydar Aliyev, which was held under the organization of the Institute of Mathematics and Mechanics Ministry of Science and Education of the Azerbaijan Republic, April 26-28, 2023, Baku.

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