Filomat 38:14 (2024), 5061–5082
https://doi.org/10.2298/FIL2414061N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On orthogonal interpolative iterative mappings with applications in multiplicative calculus

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Abstract. In this paper, some orthogonal interpolative versions of celebrated iterative mappings are reported. Various concrete conditions on the real valued functions $T, S : (0, \infty) \to (-\infty, \infty)$ for the existence of fixed-points of (T, S)-orthogonal interpolative iterative mappings are studied. We obtain fixed point theorems for Kannan type, Chatterjea type, Ciric-Reich-Rus type and Hardy-Rogers type (T, S)-contractions via interpolation in orthogonal multiplicative metric space. The fixed point theorem for Banach (T, S)-orthogonal contraction via interpolation is applied to show the existence of solutions to integral equation and fractional differential equation. The reported theory is supported by non-trivial examples.

1. Introduction

Metric fixed point theory deals with the study of fixed points of mappings in metric spaces. A fixed point of a function is a point that remains unchanged when the function is applied to it. In metric fixed point theory, the focus lies on proving the existence and uniqueness of these fixed points, along with establishing conditions under which such fixed points exist. Key concepts involve mappings satisfying certain contraction conditions, such as Banach's Contraction Mapping Principle, where a contraction mapping in a complete metric space possesses a unique fixed point. Various generalizations and extensions of this principle have been developed, including fixed point theorems for mappings with weaker contraction properties like Kannan and Nadler's fixed point theorems.

Applications span various disciplines like economics, computer science, and engineering, finding relevance in solving optimization problems, equilibrium computations, and modeling iterative processes. Metric fixed point theory continues to evolve through the exploration of diverse spaces, non-self mappings, and novel contraction conditions, contributing significantly to mathematical analysis and its applications.

The contraction principle appeared in [12] generalizes the Rakotch [43] contraction concept. Furthermore, Matkowski [34], Samet et al. [51], Karapinar et al. [22], and Pasicki [40] have all generalized the Boyd-Wong notion. The concept of F-contraction [57] is another notable generalization of the Banach contraction principle (BCP), and several research articles focusing on fixed points (common fixed points)

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 47H04
Keywords. Multiplicative Integral equation; fractional differential equation; (T, S)-orthogonal interpolative iterative mapping; complete orthogonal multiplicative metric space
Received: 25 May 2023; Accepted: 06 February 2024
Communicated by Erdal Karapınar
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of F-contractions have been published in the previous decade (see [9, 11, 52, 56], and references therein). Proinov [42](2020) offered various fixed-point theorems that built on previous work in [12, 22, 34, 40, 46–48, 51, 57].

The interpolative contraction principles consist of product of distances having exponents satisfying some conditions. The term "interpoltive contraction was introduced by the renowned mathematician Erdal Karapinar in his paper [30] published in 2018. The interpoltive contraction is defined as follows:

A self-mapping S, defined on a metric space $(\mathcal{A}, d)$, satisfying the following inequality

$$d(Sx, Sy) \leq K\left(d(x, y)\right)^{\nu}, \forall x, y \in \mathcal{A}, \tag{1}$$

is called an interpolative contraction, where $\nu \in (0, 1]$ and $K \in [0, 1)$. Note that for $\nu = 1$, S is a Banach contraction. If the mapping S defined on a metric space $(\mathcal{A}, d)$ satisfies the following inequalities:

$$\begin{aligned}
(i).\ d(Sx, Sy) &\leq K\left(d(x, Sx)\right)^{\nu}\left(d(y, Sy)\right)^{1-\nu},\\
(ii).\ d(Sx, Sy) &\leq K\left(d(x, Sy)\right)^{\nu}\left(d(y, Sx)\right)^{1-\nu},\\
(iii).\ d(Sx, Sy) &\leq K\left(d(x, y)\right)^{\eta}\left(d(x, Sx)\right)^{\nu}\left(d(y, Sy)\right)^{1-\nu-\eta},\\
(iv).\ d(Sx, Sy) &\leq K\left(d(x, y)\right)^{\nu}\left(d(x, Sx)\right)^{\eta}\left(d(y, Sy)\right)^{\gamma},\\
&\quad \left(\frac{1}{2}(d(x, Sy) \cdot d(y, Sx))\right)^{1-\eta-\nu-\gamma},
\end{aligned}$$

for all $x, y \in \mathcal{A}$, then S is called Kannan type interpolative contraction (i), Chatterjea type interpolative contraction (ii), Ćirić-Reich-Rus type interpolative contraction (iii) and Hardy Rogers type interpolative contraction (iv), respectively. Recently, many classical and advanced contractions have been revisited via interpolation (see [6, 24, 26, 27, 29, 31, 32, 37, 49, 50] and references therein).

To provide a new generalization of the Banach fixed point theorem, Gordji *et al.* [19] (2017) presented the notion of orthogonal-set (a non-empty set whose elements follow a particular relation termed it orthogonal relation). Gordji *et al.* [19] (2017) presented several examples to clarify the concept of orthogonal-set (see [19]). The *orthogonal metric space* is a metric defined on the orthogonal-set. Baghani *et al.*[9] extended the study done in [19] to F-contractions, while Nazam *et al.* [36] broadened the investigation done in [9] (2021).

Grossman and Katz [18] switched the roles of subtraction and addition to division and multiplication by establishing a new calculus known as multiplicative calculus. Compared to the calculus of Newton and Leibniz, multiplicative calculus has a more limited range of applications. In fact, it exclusively includes positive functions. Since a well-developed tool with a wider scope has already been made, the question of whether it is fair to design a new tool with a limited scope arises. The interpretation of the multiplicative derivative leads us to believe that multiplicative calculus is a useful mathematical tool for economics and finance [10]. The multiplicative space or multiplicative metric space is the by-product of multiplicative calculus.

Ozavsar et al. [39] and Yamaod et al. [58] have utilized the multiplicative metric space to produce some fixed point theorems. Recently, many authors have enriched metric fixed point theory with fixed point results in multiplicative metric spaces (see [2, 41, 45, 53, 54] and references therein).

In this paper, we extend the results that appeared in [39, 58] by introducing (T, S)-orthogonal interpolative contractions, which generalize and unify numerous interpolative contractions in the orthogonal multiplicative metric space. Motivated by the contraction principles described in [19, 42], we demonstrate that every interpolative contraction is orthogonal, but not conversely. We look for various conditions on the functions T, S to prove the presence of fixed-points of (T, S)-orthogonal Kannan type interpolative contraction, (T, S)-orthogonal Chatterjea type interpolative contraction, (T, S)-orthogonal Ćirić-Reich-Rus type interpolative contraction and (T, S)-orthogonal Hardy-Rogers type interpolative contraction. We also show how to resolve a fractional differential type equation as an application and some examples to back up our findings.

2. Preliminaries

Let $\mathcal{A}$ be a non-empty set and $\perp \subset \mathcal{A} \times \mathcal{A}$ be a binary relation satisfying the property (P),

$$(P) : \exists\, x_0 \in \mathcal{A} : \text{ either } (\forall u; x_0 \perp u) \text{ or } (\forall u; u \perp x_0).$$

We call the pair $(\mathcal{A}, \perp)$ an orthogonal set (abbreviated as, O-set). The concept of orthogonality in an inner product space is an example of orthogonal relation.

For the illustration of the orthogonal set and its examples [19, 38].

Example 2.1. *Let $\mathcal{A}$ be the set of all persons in the word. Define $x \perp e$ if x can give blood to e. According to the blood transfusion protocol, if x_0 is a person such that his (her) blood type is $O-$, then we have $x_0 \perp e$ for all $e \in \mathcal{A}$. This means that $(\mathcal{A}, \perp)$ is an O-set. In this O-set, x_0 is not unique. Note that, x_0 may be a person with blood type $AB+$. In this case, we have $e \perp x_0$ for all $e \in \mathcal{A}$.*

Definition 2.2. *Let $X \neq \emptyset$ and $b : X \times X \to [0, \infty)$ satisfies the following axioms:*

(i) $b(x, y) \geq 1$ *and* $b(x, y) = 1$ *if and only if* $x = y$;

(ii) $b(x, y) = b(y, x)$;

(iii) $b(x, z) \leq b(x, y) \cdot b(y, z)$, *for all* $x, y, z \in X$.

Then the functional b is known as multiplicative metric and the pair (X, b) is called multiplicative metric space.

For example, $\mathcal{R}^+$ is a multiplicative metric space. The multiplicative metric space is not a generalization or extension of a metric space. Both have different structures. A multiplicative metric can never be a metric and vice versa, however, if d is a metric then the expression a^d is a multiplicative metric. A sequence $\{x_n\}$ in $\mathcal{R}^+$ converges to some number x in $\mathcal{R}^+$, if for every $\epsilon > 1$ there exists a positive integer K such that

$$b(x_n, x) < \epsilon, \text{ for all } n > K.$$

The convergence in $\mathcal{R}^+$ is comparable in both multiplicative and conventional interpretations. But in more widespread situations, they might be different.

According to the investigations done in [10], the multiplicative absolute value function $|.|_m : (-\infty, \infty) \to (0, \infty)$ is defined as follows:

$$|x|_m = \begin{cases} x & \text{if } x \geq 1, \\ \frac{1}{x} & \text{if } x < 1. \end{cases}$$

We note that the expression $\left|\frac{x}{y}\right|_m$ defines a multiplicative metric for all $x, y \in (-\infty, \infty)$.

Example 2.3. *Let $\check{Z} = [0, \infty)$ and define the function $b : \check{Z}^2 \to [1, \infty)$ by*

$$b(x, y) = v^{|x-y|}; \ \ v > 1. \tag{2}$$

Then, $(\check{Z}, b)$ is a multiplicative metric space. A special case is $b(x, y) = e^{|x-y|}$.

Another example of multiplicative metric space is the space of positive definite $n \times n$ matrices W. In this space the multiplicative metric is defined as follow:

$$b(A, B) = \left\|AB^{-1}\right\|_m, \text{ for all } A, \ B \in W, \tag{3}$$

where $\|A\|_m = \prod_{i=1}^{n} \left|\eta_i\right|_m$, $\eta_1, \eta_2, \cdots, \eta_n$ are eigenvalues of matrix A. For more examples and details, we refer to [39, 58].

Definition 2.4. *The O-set* $(\mathcal{A}, \perp)$ *with a multiplicative metric b structure is called an O-multiplicative metric space (in short, OMMS). We denote it by* $(\mathcal{A}, \perp, b)$.

Definition 2.5. *[39] A mapping* $f : \mathcal{A} \to \mathcal{A}$ *satisfying the following inequality*

$$b(fx, fy) \leq b(x, y)^k \ \forall x, y \in \mathcal{A},$$

is called a multiplicative contraction, where $(\mathcal{A}, b)$ *is a multiplicative metric space and k is a some number in* $[0, 1)$.

There are many mappings that satisfy multiplicative contraction and admits a fixed point in multiplicative space. For example, defining the mapping $b : X^2 \to R$ by

$$b(u, v) = \sqrt{\left|\frac{x_1}{x_2}\right|_m \cdot \left|\frac{y_1}{y_2}\right|_m} \text{ for all } u = (x_1, y_1), v = (x_2, y_2) \in X,$$

where $X = \{(r, 1) \in R^2 : 1 \leq r \leq 2\} \cup \{(1, r) \in R^2 : 1 \leq r \leq 2\}$, we obtain a complete multiplicative metric space. The mapping $q : X \to X$ defined by $q(r, 1) = (1, \sqrt{r})$ and $q(1, r) = (\sqrt{r}, 1)$ is a multiplicative contraction with $k = 0.5$ and $(1, 1)$ is a fixed point of q. Similarly, for $X = \left[\frac{1}{10}, 1\right]$ endowed with $b(x, v) = \left|\frac{x}{v}\right|_m$, (X, b) is a complete multiplicative metric space. The mapping $q : X \to X$ defined by $q(r) = e^{r-1-\frac{r^3}{10}}$ is a multiplicative contraction for $k = 0.997$ and admits a fixed point $r = 0.7411317711$.

Definition 2.6. *A mapping* $f : \mathcal{A} \to \mathcal{A}$ *satisfying the following inequality,*

$$b(fx, fy) \leq b(x, y)^k \ \forall x, y \in \mathcal{A} \text{ with } x \perp y,$$

is called an orthogonal multiplicative contraction, where $(\mathcal{A}, \perp, b)$ *is an orthogonal multiplicative metric space and k is a some number in* $[0, 1)$.

Remark 2.7. *The multiplicative-contraction is orthogonal multiplicative-contraction but converse is not true.*

Let $\mathcal{A} = [0, 10)$ with multiplicative-metric b as defined in Example 2.3, then the pair $(\mathcal{A}, b)$ represents a multiplicative metric space. Define $\perp \subseteq \mathcal{A}^2$ by

$$x \perp y \text{ if } xy \leq x \vee y.$$

Then $(\mathcal{A}, \perp, b)$ is an O-multiplicative-metric space. Let the mapping $g : \mathcal{A} \to \mathcal{A}$ is given by $g(x) = \frac{x}{3}$ (for $x \leq 3$) and $g(x) = 0$ (for $x > 3$). We note that $b(g(4), g(3)) > b(4, 3)^k$, so, g is not a multiplicative-contraction. However, for all orthogonal elements of $\mathcal{A}$, g is an orthogonal multiplicative-contraction.

Now we prove the following Lemma that states the conditions on the sequence for not being Cauchy.

Lemma 2.8. *Let* (X, b) *be a multiplicative-metric space and* $\{x_n\} \subset X$ *be a sequence verifying* $\lim_{n\to\infty} b(x_n, x_{n+1}) = 1$. *If the sequence* $\{x_n\}$ *is not Cauchy, then there are* $\{x_{n_k}\}$, $\{x_{m_k}\}$ *and* $\xi \geq 0$ *such that*

$$\lim_{k\to\infty} b(x_{n_k+1}, x_{m_k+1}) = (1 + \xi). \tag{4}$$

$$\lim_{k\to\infty} b(x_{n_k}, x_{m_k}) = b(x_{n_k+1}, x_{m_k}) = b(x_{n_k}, x_{m_k+1}) = 1 + \xi. \tag{5}$$

Proof. Let (X, b) be a multiplicative metric space. Given $\{x_n\}$ is not Cauchy and $\lim_{n\to\infty} b(x_n, x_{n+1}) = 1$. Thus, for every $\xi > 0$, there exists a natural number K_0 such that for smallest $m \geq n$ we have

$$b(x_{n+1}, x_m) \leq 1 + \xi \text{ and } b(x_{n+1}, x_{m+1}) > (1 + \xi) \ \forall n, m > K_0.$$

As a result, we can construct two subsequences of $\{x_n\} : \{x_{n_k}\}$ and $\{x_{m_k}\}$ verifying the following inequalities:

$$b(x_{n_k+1}, x_{m_k}) \leq 1 + \xi \text{ and } b(x_{n_k+1}, x_{m_k+1}) > (1 + \xi) \ \forall n_k, m_k > K_0.$$

By axiom (iii) of the multiplicative metric, we have the following information:

$$\begin{aligned} 1+\xi &< b(x_{n_k+1}, x_{m_k+1}) \leq b(x_{n_k+1}, x_{m_k}) \cdot b(x_{m_k}, x_{m_k+1}) \\ &\leq (1+\xi) b(x_{m_k}, x_{m_k+1}). \end{aligned}$$

This implies that

$$\lim_{k\to\infty} b(x_{n_k+1}, x_{m_k+1}) = (1+\xi).$$

Again by axiom (iii) of the multiplicative metric, we have the following information:

$$\frac{b(x_{n_k+1}, x_{m_k+1})}{b(x_{m_k}, x_{m_k+1})} \leq b(x_{n_k+1}, x_{m_k}) \leq 1+\xi.$$

It leads to have

$$\lim_{k\to\infty} b(x_{n_k+1}, x_{m_k}) = (1+\xi).$$

Since, $b(x_{n_k+1}, x_{m_k}) \leq b(x_{n_k+1}, x_{n_k}) \cdot b(x_{n_k}, x_{m_k})$, we have the following inequality:

$$\frac{b(x_{n_k+1}, x_{m_k})}{b(x_{n_k+1}, x_{n_k})} \leq b(x_{n_k}, x_{m_k}) \leq b(x_{m_k}, x_{n_k+1}) \cdot b(x_{n_k+1}, x_{n_k}).$$

This suggests that

$$\lim_{k\to\infty} b(x_{n_k}, x_{m_k}) = (1+\xi).$$

Since,

$$\begin{aligned} 1+\xi &< b(x_{n_k+1}, x_{m_k+1}) \leq b(x_{n_k+1}, x_{n_k}) \cdot b(x_{n_k}, x_{m_k+1}) \\ \frac{1+\xi}{b(x_{n_k+1}, x_{n_k})} &\leq b(x_{n_k}, x_{m_k+1}) \leq b(x_{n_k}, x_{n_k+1}) \cdot b(x_{n_k+1}, x_{m_k+1}). \end{aligned}$$

This suggests that

$$\lim_{k\to\infty} b(x_{n_k}, x_{m_k+1}) = (1+\xi).$$

This completes the proof. □

Definition 2.9. *The OMMS $(\mathcal{A}, \perp, b)$ satisfying the property (R) is called a $\perp$-regular space.*
(R). For any O-sequence $\{x_n\} \subseteq \mathcal{A}$ converging to x, we have either $x_n \perp x$, or $x \perp x_n$ for all $n \in N$.

3. Main Results

In this section, we present the new results for orthogonal intplv. contrs. involving the functions $T, S : (0,\infty) \to (-\infty,\infty)$. The orthogonal interpolative contraction is a generalization of interpolative contraction [30]. The intplv. contrs. can be categorized as a multiplicative versions of classical contractions. The intplv. contrs. have applications in Economics and Finance. The example below demonstrates that orthogonal intplv. Kannan contr. (OIKC) is not equivalent to intplv. Kannan contr. (IKC).

Example 3.1. *Let $\mathcal{A} = \{1,2,3,4,5,6,7,8\}$ and define the relation $\perp \subseteq \mathcal{A}^2$ by*

$$x \perp w \text{ if } xw \leq x \vee w \text{ for all } x \neq w.$$

Consider the multiplicative-metric b as defined in Example 2.3, then $(\mathcal{A}, \perp, b)$ is an O-multiplicative metric space. Define $L : \mathcal{A} \to \mathcal{A}$ by $L(x) = 5$ (if $x = 1$) and $L(x) = x - 1$ (if $x \neq 1$). Since, for any $\nu \in (0,1)$ and $k \in [0,1)$ we have

$b(L(4), L(3)) > k[b(4, L(4))]^{\nu} \cdot [b(3, L(3)]^{1-\nu}$, *so, L is not an IKC, however, L is an OIKC.*
For if $x = 3$ *and* $w = 1$, *then* $x \perp w$ *and*

$$b(L(3), L(1)) \leq k[b(3, L(3))]^{\nu} \cdot [b(1, L(1)]^{1-\nu} \text{ for some } k \in [0, 1) \text{ and } \nu \in (0, 1).$$

For if $x = 4$ *and* $w = 1$, *then* $x \perp w$ *and*

$$b(L(4), L(1)) \leq k[b(4, L(4))]^{\nu} \cdot [b(1, L(1)]^{1-\nu} \text{ for some } k \in [0, 1) \text{ and } \nu \in (0, 1).$$

Thus, in view of [30, Example 2.3] and Example 3.1, we infer that:

$$\text{Kannan contraction } \rightarrow \text{ IKC } \rightarrow \text{ OIKC},$$

but not conversely.

Let $\Lambda = \{(x, w) \in \mathcal{A}^2 : x \perp w\}$.

Definition 3.2. *The function* $L : \mathcal{A} \rightarrow \mathcal{A}$ *satisfying the condition* $L(x) \perp L(w)$ *for all* $x, w \in \mathcal{A}$ *with* $x \perp w$, *is called* $\perp$*-preserving, where* $\perp \subset \mathcal{A}^2$ *is an orthogonal-relation.*

Example 3.3. *Let* $\perp \subset [0, 1)^2$ *be defined by*

$$x \perp w \text{ if } xw \leq x \vee w.$$

Then $([0, 1), \perp)$ *is an O-set. Define* $L : [0, 1) \rightarrow [0, 1)$ *by*

$$L(x) = \begin{cases} \frac{x}{2} & \text{if } x \in Q \cap \mathcal{A}, \\ 0 & \text{if } x \in Q^c \cap \mathcal{A}. \end{cases}$$

Then L is a $\perp$*-preserving mapping. Indeed, for* $x = \frac{1}{5}, w = \frac{1}{3}$, *we have* $x \perp w$ *and since* $L\left(\frac{1}{5}\right) L\left(\frac{1}{3}\right) = \frac{1}{60} < L\left(\frac{1}{5}\right) \vee L\left(\frac{1}{3}\right)$, *so,* $L\left(\frac{1}{5}\right) \perp L\left(\frac{1}{3}\right)$. *Similarly for all the other cases, it is evident that L is a* $\perp$*-preserving mapping.*

3.1. Kannan type (T, S)- orthogonal interpolative contraction

Let $T, S : (0, \infty) \rightarrow (-\infty, \infty)$ be two functions. A mapping $L : \mathcal{A} \rightarrow \mathcal{A}$ defined on OMMS $(\mathcal{A}, \perp, b)$ will be called a Kannan type (T, S)- orthogonal interpolative contraction, if there exists $\nu \in (0, 1)$ verifying

$$T(b(Lx, Ly)) \leq S\left((b(x, L(x)))^{\nu} (b(y, Ly))^{1-\nu}\right), \tag{6}$$

for all $(x, y) \in \Lambda$, $b(Lx, Ly) > 0$.

Remark 3.4. *The following observations indicate the generality of Kannan type* (T, S) *orthogonal interpolative contraction for the specific definitions of the mappings* T, S.

i. If $T(x) = x$ *and* $S(x) = \lambda x$, *where* $0 \leq \lambda < 1$, *then L is an orthogonal interpolative Kannan type contraction [30].*

ii. If $T(x) = x$, *then L is an orthogonal interpolative Boyd-Wong-Kannan type contraction.*

iii. If T is lower semicontinuous and S is upper semicontinuous, then L is an orthogonal interpolative Kannan version of the contraction introduced in [4].

iv. If $S(x) = F(T(x))$, *then L is an orthogonal intplv. Kannan version of the contraction introduced in [35].*

v. If $S(x) = \alpha(x)T(x)$ *and* $T(x) = x$, *then L is an orthogonal interpolative Kannan version of the contraction introduced in [17].*

vi. *If $S(x) = \lambda T(x)$, then L is an orthogonal interpolative Kannan version of the contraction introduced in [55].*

vii. *If $S(x) = F(T(x))$ and $F(x) = x^{\alpha}$, then L is an orthogonal interpolative Kannan version of the contraction introduced in [21].*

viii. *If $S(x) = T(x) - \tau$, then L is an orthogonal interpolative Kannan version of the contraction introduced in [57].*

Example 3.5. *Let $\mathcal{A} = [1, 7)$ and the multiplicative-metric b on $\mathcal{A}$ be defined by $b(t, y) = e^{|t-y|}$. Let $\perp \subset [1, 7)^2$ be defined by*

$$x \perp w \text{ if } xw \in \{x, w\}.$$

Then $(\mathcal{A}, \perp, b)$ is an O-multiplicative metric space. Define $T, S : R^+ \to R$ by

$$T(x) = \begin{cases} 2x + 1 & \text{if } x \in \left\{e^{1.9}, e^{3.2}\right\}, \\ \frac{x+1}{3} & \text{if } x \in R^+ - \left\{e^{1.9}, e^{3.2}\right\}, \end{cases} \qquad S(x) = \begin{cases} 3x^2 + 1 & \text{if } x \in \left\{e^{1.9}, e^{3.2}\right\}, \\ 3x + 5 & \text{if } x \in R^+ - \left\{e^{1.9}, e^{3.2}\right\}. \end{cases}$$

Let $L : \mathcal{A} \to \mathcal{A}$ be defined by

$$L(x) = \begin{cases} 5 & \text{if } 1 \le x < 2, \\ 3.1 & \text{if } 2 \le x < 3, \\ 1.8 & \text{if } 3 \le x < 7. \end{cases}$$

Observe that $b(Lx, Lw) = e^{1.9}$, $b(x, Lx) = e^4$ and $b(w, Lw) = e^{1.1}$ if $x = 1, w = 2$ $(1 \perp 2)$. This information shows that

$$b(Lx, Lw) > \lambda[b(x, Lx)]^{\nu}[b(w, Lw)]^{1-\nu} \text{ for some } \lambda = \frac{1}{3}, \nu = 0.8.$$

Thus, L is not Kannan type orthogonal interpolative contraction. However, L is a (T, S)-Kannan type orthogonal interpolative contraction. Indeed,

$$T(b(Lx, Lw)) \le S\left([b(x, Lx)]^{\nu}[b(w, Lw)]^{1-\nu}\right).$$

We obtain the same conclusions for $x = 1, w = 3$ $(1 \perp 3)$; $x = 1, w = 4$ $(1 \perp 4)$; $x = 1, w = 5$ $(1 \perp 5)$ and $x = 1, w = 6$ $(1 \perp 6)$.

Remark 3.6. *Example 3.1 and Example 3.5 show that interpolative contraction is orthogonal interpolative contraction and then orthogonal interpolative contraction is (T, S)-orthogonal interpolative contraction but converse is not true.*

For the orthogonal relation $\perp$, self-mapping S and functions $T, S : (0, \infty) \to (-\infty, \infty)$, we state the following conditions:

(i) for each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_1 \perp \hbar_0$ or $\hbar_0 \perp \hbar_1$.

(ii) T is nondecreasing and $S(x) < T(x)$ $\forall\, x > 0$.

(iii) $\limsup_{x \to \delta+} S(x) < T(\delta+)$ $\forall\, \delta > 0$.

(iv) $\limsup_{a \to 0} S(a) \le \lim \inf_{a \to \xi+} T(a)$.

(v) $T(a^{\nu} b^{\eta}) \le T(a)$ and $S(x) < T(x)$ $\forall\, x > 0$.

(vi) $\inf_{a > \xi} T(a) > -\infty$.

(vii) if $\{T(\hbar_n)\}$ and $\{S(\hbar_n)\}$ are converging to same limit and $\{T(\hbar_n)\}$ is strictly decreasing, then $\lim_{n\to\infty} \hbar_n = 0$.

(viii) $\limsup_{a\to 0} S(a) < \liminf_{a\to\xi} T(a)$ for all $\xi > 0$.

The upcoming first theorem states the requirements for the presence of fixed points of a self-mapping L satisfying (6).

Theorem 3.7. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (6) *and* (i)-(iv), *admits a fixed-point in $\mathcal{A}$.*

Proof. Choose an initial guess $x_0 \in \mathcal{A}$ such that $x_0 \perp x_1$ or $x_1 \perp x_0$ for each $x_1 \in \mathcal{A}$, then by using the $\perp$-preserving nature of L, we construct an orthogonal sequence $\{x_n\}$ such that $x_n = L(x_{n-1}) = L^n(x_0)$ and $x_{n-1} \perp x_n$ for each $n \in N$. Note that, if $x_n = L(x_n)$, then x_n is a fixed point of L for all $n \geq 0$. We assume that $x_n \neq x_{n+1}$ for all $n \in N \cup \{0\}$. Let $t_n = b(x_n, x_{n+1})$ for all $n \geq 0$. By the first part of (ii) and (6), we have

$$\begin{aligned} T(t_n) &\leq T(b(L(x_{n-1}), L(x_n))) \leq S\left((b(x_{n-1}, Lx_{n-1}))^{\nu}(b(x_n, Lx_n))^{1-\nu}\right) \\ &\leq S\left((t_{n-1})^{\nu}(t_n)^{1-\nu}\right). \end{aligned}$$

In light of (ii), we have

$$T(t_n) \leq S\left((t_{n-1})^{\nu}(t_n)^{1-\nu}\right) < T\left((t_{n-1})^{\nu}(t_n)^{1-\nu}\right). \tag{7}$$

Since T is non-decreasing, one gets $t_n < t_{n-1}$ for each $n \geq 1$, so there is $L > 0$ so that $\lim_{n\to\infty} t_n = L+$. If $L > 0$, by (7), we obtain the following information:

$$T(L+) = \lim_{n\to\infty} T(t_n) \leq \lim_{n\to\infty} \sup S((t_{n-1})^{\nu}(t_n)^{1-\nu}) \leq \lim_{a\to L+} \sup S(a).$$

This contradicts (iii), so $L = 1$.

The sequence $\{x_n\}$ is Cauchy: Assume that $\{x_n\}$ is not a Cauchy-sequence, so that by following Lemma 2.8, there exist two sub-sequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$ and $\xi > 0$ such that (4) and (5) hold. By (4), we infer that $b(x_{n_k+1}, x_{m_k+1}) > 1 + \xi$. Since $x_n \perp x_{n+1}$ for all $n \geq 0$, by transitivity of $\perp$, we have $x_{n_k} \perp x_{m_k}$ for all $k \geq 1$. Letting $x = x_{n_k}$ and $y = x_{m_k}$ in (6), we have for each $k \geq 1$,

$$\begin{aligned} T(b(x_{n_k+1}, x_{m_k+1})) &\leq T(b(Lx_{n_k}, Lx_{m_k})) \\ &\leq S\left((b(x_{n_k}, Lx_{n_k}))^{\nu}(b(x_{m_k}, Lx_{m_k}))^{1-\nu}\right) \\ &\leq S\left((b(x_{n_k}, x_{n_k+1}))^{\nu}(b(x_{m_k}, x_{m_k+1}))^{1-\nu}\right). \end{aligned}$$

If $a_k = b(x_{n_k+1}, x_{m_k+1})$, $b_k = b(x_{n_k}, x_{n_k+1})$ and $c_k = b(x_{m_k}, x_{m_k+1})$, we have

$$T(a_k) \leq S\left((b_k)^{\nu}(c_k)^{1-\nu}\right), \text{ for all } k \geq 1. \tag{8}$$

By (4), we have $\lim_{k\to\infty} a_k = (1+\xi)\times$ and (8) implies

$$\liminf_{a\to(1+\xi)\times} T(a) \leq \liminf_{k\to\infty} T(a_k) \leq \limsup_{k\to\infty} S\left((b_k)^{\nu}(c_k)^{1-\nu}\right) \leq \limsup_{a\to 0} S(a). \tag{9}$$

The information obtained in (9) contradicts the assumption (iv), and thus stamping the sequence $\{x_n\}$ as Cauchy in the OCMMS $(\mathcal{A}, \perp, b)$, hence there is $a^* \in \mathcal{A}$ so that $x_n \to a^*$ as $n \to \infty$. Snce, $(\mathcal{A}, \perp, b)$ is a $\perp$-regular space, so, we write $x_n \perp a^*$ or $a^* \perp x_n$. We claim that $b(a^*, L(a^*)) = 1$. If $b(x_{n+1}, L(a^*)) > 1$, then by (6),

$$\begin{aligned} T(b(x_{n+1}, L(a^*))) &\leq T(b(L(x_n), L(a^*))) \leq S\left((b(x_n, Lx_n))^{\nu}(b(a^*, La^*))^{1-\nu}\right) \\ &< T\left((b(x_n, Lx_n))^{\nu}(b(a^*, La^*))^{\nu}\right) \\ &\leq T\left((b(x_n, x_{n+1}))^{\nu}(b(a^*, La^*))^{1-\nu}\right). \end{aligned}$$

By the first part of (ii), we get $b(x_{n+1}, L(a^*)) < (b(x_n, x_{n+1}))^{\nu}(b(a^*, La^*))^{1-\nu}$. Applying limit $n \to \infty$, we obtain $b(a^*, L(a^*)) \leq 1$. This implies that $b(a^*, L(a^*)) = 1$. Hence, $a^* = L(a^*)$. □

The second main theorem (6) establishes various requirements for the presence of fixed-points of L.

Theorem 3.8. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (6) *and* (i),(iv)-(viii), *admits a fixed-point in $\mathcal{A}$.*

Proof. Let $x_0 \in \mathcal{A}$ be such that $x_0 \perp x_1$ or $x_1 \perp x_0$ for each $x_1 \in \mathcal{A}$, then by using the $\perp$-preserving nature of L, we construct an orthogonal sequence $\{x_n\}$ such that $x_n = L(x_{n-1}) = L^n(x_0)$ and $x_{n-1} \perp x_n$ for each $n \in N$. Note that, if $x_n = L(x_n)$ then x_n is a fixed point of L for all $n \geq 0$. We assume that $x_n \neq x_{n+1}$ for all $n \in N \cup \{0\}$. By (ii) and (6), one writes

$$\begin{aligned} T(b(x_n, x_{n+1})) &\leq T(b(L(x_{n-1}), L(x_n))) \leq S((b(x_{n-1}, Lx_{n-1}))^{\nu}(b(x_n, Lx_n))^{1-\nu}) \\ &\leq S((b(x_{n-1}, x_n))^{\nu}(b(x_n, x_{n+1}))^{1-\nu}) \\ &< T\left((b(x_{n-1}, x_n))^{\nu}(b(x_n, x_{n+1}))^{1-\nu}\right) \\ &\leq T(b(x_{n-1}, x_n)). \end{aligned} \tag{10}$$

The inequality (10) shows that $\{T(b(x_{n-1}, x_n))\}$ is strictly decreasing. If it is not bounded below, in view of (iii), we get $\inf_{b(x_{n-1},x_n)>\xi} T(b(x_{n-1}, x_n)) > -\infty$. This implies that

$$\lim \inf_{b(x_{n-1},x_n)\to\xi+} T(b(x_{n-1}, x_n)) > -\infty.$$

Thus, $\lim_{n\to\infty} b(x_{n-1}, x_n) = 1$, otherwise we have

$$\lim \inf_{b(x_{n-1},x_n)\to\xi+} T(b(x_{n-1}, x_n)) = -\infty$$

(i.e., a contradiction to (iii)). If it is bounded below, then $\{T(b(x_{n-1}, x_n))\}$ is a convergent sequence and by (10), $\{S(b(x_{n-1}, x_n))\}$ also converges and both have same limit. Thus, by (iv), one gets $\lim_{n\to\infty} b(x_{n-1}, x_n) = 1$. Hence, L is asymptotically regular.

Now, we claim that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy sequence, so by Lemma 2.8, there exist $\{x_{n_k}\}$, $\{x_{m_k}\}$ and $\xi > 0$ such that (4) and (5) hold. By (4), we infer that $b(x_{n_k+1}, x_{m_k+1}) > 1 + \xi$. Since $x_n \perp x_{n+1}$ for all $n \geq 0$ so by transitivity of $\perp$, we have $x_{n_k} \perp x_{m_k}$. Letting $g = x_{n_k}$ and $e = x_{m_k}$ in (6), one writes for all $k \geq 1$,

$$\begin{aligned} T(b(x_{n_k+1}, x_{m_k+1})) &\leq T(b(Lx_{n_k}, Lx_{m_k})) \\ &\leq S\left((b(x_{n_k}, Lx_{n_k}))^{\nu}(b(x_{m_k}, Lx_{m_k}))^{1-\nu}\right) \\ &\leq S\left((b(x_{n_k}, x_{n_k+1}))^{\nu}(b(x_{m_k}, x_{m_k+1}))^{1-\nu}\right). \end{aligned}$$

If $a_k = b(x_{n_k+1}, x_{m_k+1})$, $b_k = b(x_{n_k}, x_{n_k+1})$ and $c_k = b(x_{m_k}, x_{m_k+1})$, we have

$$T(a_k) \leq S\left((b_k)^{\nu}(c_k)^{1-\nu}\right), \text{ for all } k \geq 1. \tag{11}$$

By (4), we have $\lim_{k\to\infty} a_k = (1 + \xi)\times$ and (11) implies

$$\liminf_{a\to(1+\xi)\times} T(a) \leq \liminf_{k\to\infty} T(a_k) \leq \limsup_{k\to\infty} S\left((b_k)^{\nu}(c_k)^{1-\nu}\right) \leq \limsup_{a\to 0} S(a). \tag{12}$$

The information obtained in (12) contradicts the assumption (v), and thus stamping the sequence $\{x_n\}$ as Cauchy in the OCMMS $(\mathcal{A}, \perp, b)$. The completeness of the space ensures the convergence of $\{x_n\}$, let it converges to $x^* \in \mathcal{A}$. **Case 1.** If $b(x_{n+1}, Lx^*) = 1$ for some $n \geq 0$, then

$$b(x^*, Lx^*) \leq b(x^*, x_{n+1}).b(x_{n+1}, Lx^*)$$

taking limit $n \to \infty$ on both sides, we have $b(i^*, Li^*) \leq 1$. This implies $b(i^*, L(i^*)) = 1$. Hence, $x^* = Lx^*$.
Case 2. If for all $n \geq 0$, $b(x_{n+1}, Lx^*) > 0$, then by $\perp$-regularity of $\mathcal{A}$, we find $x_n \perp x^*$ or $x^* \perp x_n$. By (6), one writes

$$T(b(x_{n+1}, Lx^*)) \leq T(b(Lx_n, Lx^*)) \leq S\left((b(x_n, Lx_n))^{\nu}(b(x^*, Lx^*)^{1-\nu}\right) \text{ for all } n \geq 0.$$

By taking $x_n = b(x_{n+1}, Lx^*)$ and $b_n = b(x_n, x_{n+1})$, one writes

$$T(x_n) \le S\left((b_n)^\nu (b(x^*, Lx^*)^{1-\nu}\right) \text{ for all } n \ge 0. \tag{13}$$

Take $\delta = b(x^*, Lx^*)$. Note that $a_n \to \delta$ and $b_n \to 1$ as $n \to \infty$. Applying limits on (13), we have

$$\lim \inf_{x\to\delta} T(x) \le \lim \inf_{n\to\infty} T(x_n) \le \lim \sup_{n\to\infty} S((b_n)^\nu \delta^{1-\nu}) \le \lim \inf_{x\to 0} S(x).$$

This contradicts (vi) if $\delta > 1$. Thus, we have $b(x^*, Lx^*) = 1$, that is, x^* is a fixed point of L. □

3.2. Chatterjea type (T, S)-orthogonal interpolative contraction

Let $(\mathcal{A}, \perp, b)$ be an OMMS and $T, S : (0, \infty) \to (-\infty, \infty)$. The mapping $L : \mathcal{A} \to \mathcal{A}$ satisfying the following inequality,

$$T(b(Lx, Lw)) \le S\left(\sqrt{b(x, Lw) \cdot b(w, Lx)}\right), \tag{14}$$

for all $(x, w) \in \Lambda$, $b(Lx, Lw) > 0$, is called a Chatterjea type (T, S)-orthogonal interpolative contraction.

Remark 3.9. *The following observations indicate the generality of Chatterjea type (T, S) orthogonal interpolative contraction for the specific definitions of the mappings T, S.*

- *i. If $T(x) = x$ and $S(x) = \lambda x$, where $0 \le \lambda < 1$, then L is an orthogonal interpolative Chatterjea type contraction.*
- *ii. If $T(x) = x$, then L is an orthogonal interpolative Boyd-Wong-Chatterjea type contraction.*
- *iii. If T is lower semicontinuous and S is upper semicontinuous, then L is an orthogonal interpolative Chatterjea version of the contraction introduced in [4].*
- *iv. If $S(x) = F(T(x))$, then L is an orthogonal interpolative Chatterjea version of the contraction introduced in [35].*
- *v. If $S(x) = \alpha(x)T(x)$ and $T(x) = x$, then L is an orthogonal interpolative Chatterjea version of the contraction introduced in [17].*
- *vi. If $S(x) = \lambda T(x)$, then L is an orthogonal interpolative Chatterjea version of the contraction introduced in [55].*
- *vii. If $S(x) = F(T(x))$ and $F(x) = x^\alpha$, then L is an orthogonal interpolative Chatterjea version of the contraction introduced in [21].*
- *viii. If $S(x) = T(x) - \tau$, then L is an orthogonal interpolative Chatterjea version of the contraction introduced in [57].*

Example 3.10. *Let $\mathcal{A} = [1, 7)$ and the multiplicative-metric b on $\mathcal{A}$ be defined by $b(t, y) = e^{|t-y|}$, then $(\mathcal{A}, b)$ is an incomplete multiplicative metric space. Define the relation $\perp$ on $\mathcal{A}$ by*

$$x \perp w \text{ if } xw \in \{x, w\}.$$

Define $T, S : R^+ \to R$ by

$$T(x) = \begin{cases} 2x + 1 & \text{if } x \in \left\{e^{3.5}, e^{4.5}\right\} \\ \frac{x+1}{2} & \text{if } x \in R^+ - \left\{e^{3.5}, e^{4.5}\right\} \end{cases} \qquad S(x) = \begin{cases} 2x^2 + 1 & \text{if } x \in \left\{e^{3.5}, e^{4.5}\right\} \\ 3x + 10 & \text{if } x \in R^+ - \left\{e^{3.5}, e^{4.5}\right\} \end{cases}$$

Let $L : \mathcal{A} \to \mathcal{A}$ be defined by

$$L(x) = \begin{cases} 5 & \text{if } 1 \le x < 2 \\ 1.5 & \text{if } 2 \le x < 3 \\ 0.5 & \text{if } 3 \le x < 7 \end{cases}$$

Observe that $b(Lx, Lw) = e^{3.5}$, $b(x, Lw) = \sqrt{e}$ *and* $b(w, Lx) = e^3$ *if* $x = 1, w = 2$ $(1 \perp 2)$. *This information shows that*

$$b(Lx, Lw) > \lambda \sqrt{b(x, Lw) \cdot b(w, Lx)}, \text{ for all } \lambda \in (0, 1).$$

Thus, L is not a Chatterjea type orthogonal interpolative contraction. However, L is a Chatterjea type (T, S)*-orthogonal interpolative contraction. Indeed,*

$$T\,(b(Lx, Lw)) \leq S\left(\sqrt{b(x, Lw) \cdot b(w, Lx)}\right)$$

We obtain the same conclusion for $x = 1, w = 3$ $(1 \perp 3)$*;* $x = 1, w = 4$ $(1 \perp 4)$*;* $x = 1, w = 5$ $(1 \perp 5)$*;* $x = 1, w = 6$ $(1 \perp 6)$.

The next two theorems deals with Chatterjea type (T, S)-orthogonal-interpolative contractions.

Theorem 3.11. *Let* $\perp$ *be a transitive orthogonal relation, then, every* $\perp$*-preserving self-mapping defined on a* $\perp$*-regular OCMMS* $(\mathcal{A}, \perp, b)$ *verifying* (14) *and* (i)-(iv)*, admits a fixed point in* $\mathcal{A}$.

Proof. Chasing the starting steps taken in proof of Theorem 3.7, we have

$$\begin{aligned} T(h_n) &\leq T(b(L(x_{n-1}), L(x_n))) \\ &\leq S\left(\sqrt{b(x_{n-1}, Lx_n) \cdot b(x_n, Lx_{n-1})}\right) \\ &\leq S\left(\sqrt{b(x_{n-1}, x_{n+1})}\right) \qquad (15) \\ &\leq S\left(\sqrt{b(x_{n-1}, x_n) \cdot b(x_n, x_{n+1})}\right). \qquad (16) \end{aligned}$$

Suppose that $b(x_{n-1}, x_n) < b(x_n, x_{n+1})$ for some $n \geq 1$, then by (16) and (ii), we have

$$T(h_n) \leq S(h_n) < T(h_n). \qquad (17)$$

The information obtained in (17) contradicts the definition of T, therefore we go with

$$T(h_n) \leq S\,(h_{n-1}) < T\,(h_{n-1})\ \forall\, n \geq 1.$$

Now crawling through the proof of Theorem 3.7, we reach to the statement $x_n \to o^*$ as $n \to \infty$, and then taking the support of $\perp$-regularity of the space $(\mathcal{A}, \perp, b)$, we achieve that $x_n \perp t^*$ or $t^* \perp x_n$. We need to have $b(t^*, L(t^*)) = 1$. Letting $b(x_{n+1}, L(t^*)) > 1$ and using (14),

$$\begin{aligned} T(b(x_{n+1}, L(t^*))) &\leq T(b(L(x_n), L(t^*))) \\ &\leq S\left(\sqrt{b(x_n, Lt^*) \cdot b(t^*, Lx_n)}\right) \\ &\leq S\left(\sqrt{b(x_n, Lt^*) \cdot b(t^*, x_{n+1})}\right) \\ &< T\left(\sqrt{b(x_n, Lt^*) \cdot b(t^*, x_{n+1})}\right). \end{aligned}$$

Given that the function T satisfies assumption (ii), thus

$$b(x_{n+1}, L(t^*)) < \sqrt{b(x_n, Lt^*) \cdot b(t^*, x_{n+1})}.$$

The last inequality implies that $b(t^*, L(t^*)) \leq \sqrt{b(t^*, Lt^*)}$ (for large n). Hence, $b(t^*, L(t^*)) = 1$, or $t^* = L(t^*)$. □

Theorem 3.12. *Let* $\perp$ *be a transitive orthogonal relation, then, every* $\perp$*-preserving self-mapping defined on a* $\perp$*-regular OCMMS* $(\mathcal{A}, \perp, b)$ *verifying* (14) *and* (i),(iv)-(viii)*, admits a fixed point in* $\mathcal{A}$.

Proof. Chasing the steps taken in the proof of Theorem 3.8 and Theorem 3.11, we achieve the objective. □

3.3. Ćirić-Reich-Rus type (T, S)-orthogonal interpolative contraction

The self-mapping L defined on OMMS $(\mathcal{A}, \perp, b)$ and satisfying the inequality (18) is called Ćirić-Reich-Rus type (T, S)-orthogonal interpolative contraction.

$$T(b(Lx, Lw)) \leq S\left(b(x, w)^{\nu} b(x, Lx)^{T} b(w, Lw)^{1-\eta-\nu}\right), \tag{18}$$

for all $(x, w) \in \Lambda$, $b(Lx, Lw) > 0$, where $\nu, \eta \in [0, 1)$ with $\nu + \eta < 1$.

Remark 3.13. *The following observations indicate the generality of Ćirić-Reich-Rus type (T, S) orthogonal interpolative contraction for the specific definitions of the mappings T, S .*

i. *If $T(x) = x$ and $S(x) = \lambda x$, where $0 \leq \lambda < 1$, then L is an orthogonal interpolative Ćirić-Reich-Rus type contraction [1].*

ii. *If $T(x) = x$, then L is an orthogonal interpolative Boyd-Wong-Ćirić-Reich-Rus type contraction.*

iii. *If T is lower semicontinuous and S is upper semicontinuous, then L is an orthogonal interpolative Ćirić-Reich-Rus version of the contraction introduced in [4].*

iv. *If $S(x) = F(T(x))$, then L is an orthogonal interpolative Ćirić-Reich-Rus version of the contraction introduced in [35].*

v. *If $S(x) = \alpha(x)T(x)$ and $T(x) = x$, then L is an orthogonal interpolative Ćirić-Reich-Rus version of the contraction introduced in [17].*

vi. *If $S(x) = \lambda T(x)$, then L is an orthogonal interpolative Ćirić-Reich-Rus version of the contraction introduced in [55].*

vii. *If $S(x) = F(T(x))$ and $F(x) = x^{\alpha}$, then L is an orthogonal interpolative Ćirić-Reich-Rus version of the contraction introduced in [21].*

viii. *If $S(x) = T(x) - \tau$, then L is an orthogonal interpolative Ćirić-Reich-Rus version of the contraction introduced in [57].*

ix. *For $\nu = 0$, we obtain Kannan type (T, S)-orthogonal interpolative contraction from (18) .*

The requirements for the presence of a fixed-point of Ćirić-Reich-Rus type (T, S)-orthogonal interpolative contraction are stated in the following two theorems.

Theorem 3.14. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (18) *and* (i)-(iv)*, admits a fixed point in $\mathcal{A}$.*

Proof. Chasing the starting steps taken in the proof of Theorem 3.7, we have

$$\begin{aligned} T(h_n) &\leq T(b(L(x_{n-1}), L(x_n))) \\ &\leq S\left((b(x_{n-1}, x_n))^{\nu} (b(x_{n-1}, Lx_{n-1}))^{\eta} (b(x_n, Lx_n))^{1-\eta-\nu}\right) \\ &\leq S\left((b(x_{n-1}, x_n))^{\nu} (b(x_{n-1}, x_n))^{\eta} (b(x_n, x_{n+1}))^{1-\eta-\nu}\right) \\ &\leq S\left((b(x_{n-1}, x_n))^{\nu+\eta} (b(x_n, x_{n+1}))^{1-\eta-\nu}\right) \\ &< T\left((b(x_{n-1}, x_n))^{\nu+\eta} (b(x_n, x_{n+1}))^{1-\eta-\nu}\right). \end{aligned} \tag{19}$$

By (19) and monotonicity of T implies

$$(h_n)^{\nu+\eta} < (h_{n-1})^{\nu+\eta}, \forall n \geq 1.$$

Now taking steps as taken in the proof of Theorem 3.7, we get $x_n \to p^*$ as $n \to \infty$, and with the support of $\perp$-regularity of $(\mathcal{A}, \perp, b)$, we have $x_n \perp p^*$ or $p^* \perp x_n$. We need to prove that $b(p^*, L(p^*)) = 1$. Letting $b(x_{n+1}, L(p^*)) > 1$ and using (18), we have

$$\begin{aligned} T(b(x_{n+1}, L(p^*))) &\leq T(b(L(x_n), L(p^*))) \\ &\leq S\left((b(x_n, p^*))^{\nu} (b(x_n, Lx_n))^{\eta} (b(p^*, Lp^*))^{1-\eta-\nu}\right) \\ &\leq S\left((b(x_n, p^*))^{\nu} (b(x_n, x_{n+1}))^{\eta} (b(p^*, Lp^*))^{1-\eta-\nu}\right) \\ &< T\left((b(x_n, p^*))^{\nu} (b(x_n, x_{n+1}))^{\eta} (b(p^*, Lp^*))^{1-\eta-\nu}\right). \end{aligned}$$

Using (ii), we get

$$b(x_{n+1}, L(p^*)) < (b(x_n, p^*))^{\nu} (b(x_n, x_{n+1}))^{\eta} (b(p^*, Lp^*))^{1-\eta-\nu}.$$

Now for large n, the last inequality implies that $b(p^*, L(p^*)) \leq 1$. Hence, $b(p^*, L(p^*)) = 1$, or $p^* = L(p^*)$. □

Example 3.15. *Let $\perp \subset \mathcal{A}^2 = [1, 7)^2$ be defined by*

$$x \perp w \text{ if } xw \in \{x, w\},$$

so that $(\mathcal{A}, \perp)$ is an O-set. Consider the multiplicative metric defined in Example 2.3 and define $T, S : R^+ \to R$ by

$$T(x) = \begin{cases} 2x & \text{if } x = e^{2.5} \\ 3x + 7 & \text{if } x \in R^+ - \left\{e^{2.5}\right\}, \end{cases} \quad S(x) = \begin{cases} \frac{x+2}{3} & \text{if } x = e^{2.5} \\ 4x + 5 & \text{if } x \in R^+ - \left\{e^{2.5}\right\}. \end{cases}$$

Define the self-mapping L on $\mathcal{A}$ by

$$L(x) = \begin{cases} 4 & \text{if } 1 \leq x < 1.5 \\ 1.5 & \text{if } 1.5 \leq x < 7. \end{cases}$$

Observe that $b(Lx, Lw) = e^{2.5}$, $b(x, w) = e$, $b(x, Lx) = e^3$ and $b(w, Lw) = \sqrt{e}$ if $x = 1, w = 2$ $(1 \perp 2)$. This information shows that

$$b(Lx, Lw) > \lambda b(x, w)^{\nu} b(x, Lx)^{\eta} b(w, Lw)^{1-\eta-\nu} \text{ for some } \lambda = \frac{1}{3}, \nu = 0.45, \eta = 0.3.$$

Thus, L is not a Ćirić-Reich-Rus type orthogonal interpolative contraction. However, L is a Ćirić-Reich-Rus type (T, S)-orthogonal interpolative contraction. Indeed,

$$T(b(Lx, Lw)) \leq S\left(b(x, w)^{\nu} b(x, Lx)^{\eta} b(w, Lw)^{1-\eta-\nu}\right).$$

We obtain the same conclusions for $x = 1, w = 3$ $(1 \perp 3)$; $x = 1, w = 4$ $(1 \perp 4)$; $x = 1, w = 5$ $(1 \perp 5)$; $x = 1, w = 6$ $(1 \perp 6)$. The point $x = 1.5$ is a fixed point of the mapping L.

Theorem 3.16. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (18) *and* (i),(iv)-(viii), *admits a fixed point in $\mathcal{A}$.*

Proof. Chasing the steps taken in the proof of Theorem 3.8 and Theorem 3.14, we complete the proof of Theorem 3.16. □

3.4. Hardy-Rogers type (T,S)-orthogonal interpolative contraction

The self-mapping L defined on OMMS $(\mathcal{A}, \perp, b)$ and satisfying the inequality (20) is called an Hardy-Rogers type(T,S)-orthogonal interpolative contraction.

$$\begin{aligned} & T(b(Lx, Lw)) \\ \leq \quad & S\left(b(x,w)^{\nu} b(x,Lx)^{\eta} b(w,Lw)^{\gamma}\left(\frac{1}{2}(b(x,Lw)+b(w,Lx))\right)^{1-\eta-\nu-\gamma}\right), \end{aligned} \tag{20}$$

for all $(x,w) \in \Lambda$, $b(Lx, Lw) > 0$, where $\nu, \eta, \gamma \in [0,1)$ with $\nu + \eta + \gamma < 1$.

Remark 3.17. *The following observations indicate the generality of Hardy-Rogers type (T,S) orthogonal interpolative contraction for the specific definitions of the mappings T, S.*

i. *If $T(x) = x$ and $S(x) = \lambda x$, where $0 \leq \lambda < 1$, then L is an orthogonal interpolative Hardy-Rogers type contraction [28].*

ii. *If $T(x) = x$, then L is an orthogonal interpolative Boyd-Wong-Hardy-Rogers type contraction.*

iii. *If T is lower semicontinuous and S is upper semicontinuous, then L is an orthogonal interpolative Hardy-Rogers version of the contraction introduced in [4].*

iv. *If $S(x) = F(T(x))$, then L is an orthogonal interpolative Hardy-Rogers version of the contraction introduced in [35].*

v. *If $S(x) = \alpha(x)T(x)$ and $T(x) = x$, then L is an orthogonal interpolative Hardy-Rogers version of the contraction introduced in [17].*

vi. *If $S(x) = \lambda T(x)$, then L is an orthogonal interpolative Hardy-Rogers version of the contraction introduced in [55].*

vii. *If $S(x) = F(T(x))$ and $F(x) = x^{\alpha}$, then L is an orthogonal interpolative Hardy-Rogers version of the contraction introduced in [21].*

viii. *If $S(x) = T(x) - \tau$, then L is an orthogonal interpolative Hardy-Rogers version of the contraction introduced in [57].*

The requirements for the presence of a fixed-point of Hardy-Rogers type (T,S)-orthogonal interpolative contraction are stated in the following two theorems.

Theorem 3.18. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (20) *and* (i)-(iv), *admits a fixed point in $\mathcal{A}$.*

Proof. Let $x_0 \in \mathcal{A}$ be such that $x_0 \perp x_1$ or $x_1 \perp x_0$ for each $x_1 \in \mathcal{A}$, then by using the $\perp$-preserving nature of L, we construct an orthogonal sequence $\{x_n\}$ such that $x_n = L(x_{n-1}) = L^n(x_0)$ and $x_{n-1} \perp x_n$ for each $n \in N$. Note that, if $x_n = L(x_n)$ then x_n is a fixed point of L for all $n \geq 0$. We assume that $x_n \neq x_{n+1}$ for all $n \in N \cup \{0\}$. By (ii) and (6), we obtain

$$\begin{aligned} T(h_n) &\leq T(b(L(x_{n-1}), L(x_n))) \\ &\leq S\begin{pmatrix} b(x_{n-1}, x_n)^{\nu} b(x_{n-1}, L(x_{n-1}))^{\eta} b(x_n, L(x_n))^{\gamma} \\ \left(\frac{1}{2}(b(x_{n-1}, L(x_n)) + b(x_n, L(x_{n-1})))\right)^{1-\eta-\nu-\gamma} \end{pmatrix} \\ &\leq S\left((h_{n-1})^{\nu+\eta}(h_n)^{\gamma}\left(\frac{1}{2}b(x_{n-1}, x_{n+1})\right)^{1-\eta-\nu-\gamma}\right) \\ &< T\left((h_{n-1})^{\nu+\eta}(h_n)^{\gamma}\left(\frac{1}{2}(h_{n-1}+h_n)\right)^{1-\eta-\nu-\gamma}\right). \end{aligned} \tag{21}$$

Suppose that $h_{n-1} < h_n$ for some $n \geq 1$. By monotonicity of T and (21), we have $(h_n)^{\nu+\eta} < (h_n)^{\nu+\eta}$. This is not possible. Consequently, we have $h_n < h_{n-1}$ for all $n \geq 1$. Now taking steps as taken in the proof of Theorem 3.7, we get $x_n \to u^*$ as $n \to \infty$, and with the support of $\perp$-regularity of $(\mathcal{A}, \perp, b)$, we have $x_n \perp u^*$ or $u^* \perp x_n$. We need to prove that $b(u^*, L(u^*)) = 1$. Letting $b(x_{n+1}, L(u^*)) > 1$ and using (20), we have

$$\begin{aligned} & T(b(x_{n+1}, L(u^*))) \leq T(b(L(x_n), L(u^*))) \\ \leq \quad & S\left(b(x_n, u^*)^\nu b(x_n, L(x_n))^\eta b(u^*, L(u^*))^\gamma \left(\frac{1}{2}(b(x_n, L(u^*)) + b(u^*, L(x_n)))\right)^{1-\eta-\nu-\gamma}\right) \\ \leq \quad & S\left(b(x_n, u^*)^\nu b(x_n, x_{n+1})^\eta b(u^*, L(u^*))^\gamma \left(\frac{1}{2}(b(x_n, L(u^*)) + b(u^*, x_{n+1}))\right)^{1-\eta-\nu-\gamma}\right) \\ < \quad & T\left(b(x_n, u^*)^\nu b(x_n, x_{n+1})^\eta b(u^*, L(u^*))^\gamma \left(\frac{1}{2}(b(x_n, L(u^*)) + b(u^*, x_{n+1}))\right)^{1-\eta-\nu-\gamma}\right). \end{aligned}$$

Using (ii), we get

$$\begin{aligned} & b(x_{n+1}, L(u^*)) \\ < \quad & b(x_n, u^*)^\nu b(x_n, x_{n+1})^\eta b(u^*, L(u^*))^\gamma \left(\frac{1}{2}(b(x_n, L(u^*)) + b(u^*, x_{n+1}))\right)^{1-\eta-\nu-\gamma}. \end{aligned}$$

Now for large n, the last inequality implies that $b(u^*, L(u^*)) \leq 1$. Hence, $b(u^*, L(u^*)) = 1$, or $u^* = L(u^*)$. □

Theorem 3.19. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (20) *and* (i),(iv)-(viii), *admits a fixed point in $\mathcal{A}$.*

Proof. Following the steps as taken in the proof of Theorem 3.8 and Theorem 3.18, we complete this proof. □

Definition 3.20. *A self-mapping L on OMMS $(\mathcal{A}, \perp, b)$ and satisfying the inequality (22) is called Banach type (T, S)-orthogonal interpolative contraction.*

$$T(b(Lx, Lw)) \leq S\left(b(x, w)^\nu\right), \tag{22}$$

for all $(x, w) \in \Lambda$ with $b(Lx, Lw) > 0$, where $\nu \in (0, 1]$.

Remark 3.21. *The following observations indicate the generality of (T, S) orthogonal interpolative contraction for the specific definitions of the mappings T, S.*

i. If $T(x) = x$ and $S(x) = \lambda x$, where $0 \leq \lambda < 1$, then L is an orthogonal interpolative contraction.

ii. If $T(x) = x$, then L is an orthogonal interpolative Boyd-Wong type contraction.

iii. If T is lower semicontinuous and S is upper semicontinuous, then L is an orthogonal interpolative version of the contraction introduced in [4].

iv. If $S(x) = F(T(x))$, then L is an orthogonal interpolative version of the contraction introduced in [35].

v. If $S(x) = \alpha(x)T(x)$ and $T(x) = x$, then L is an orthogonal interpolative version of the contraction introduced in [17].

vi. If $S(x) = \lambda T(x)$, then L is an orthogonal interpolative version of the contraction introduced in [55].

vii. If $S(x) = F(T(x))$ and $F(x) = x^\alpha$, then L is an orthogonal interpolative version of the contraction introduced in [21].

viii. If $S(x) = T(x) - \tau$, then L is an orthogonal interpolative version of the contraction introduced in [57].

The next two theorems deals with Banach type (T, S)-orthogonal interpolative contraction.

Theorem 3.22. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (22) *and* (i)-(iv)*, admits a fixed point in $\mathcal{A}$.*

Theorem 3.23. *Let $\perp$ be a transitive orthogonal relation, then, every $\perp$-preserving self-mapping defined on a $\perp$-regular OCMMS $(\mathcal{A}, \perp, b)$ verifying* (22) *and* (i),(iv)-(viii)*, admits a fixed point in $\mathcal{A}$.*

Remark 3.24. *If $\nu = 1$, the proofs of Theorem 3.22 and Theorem 3.23 are directly followed by [36]. If $0 < \nu < 1$, the proofs of Theorem 3.22 and Theorem 3.23 are similar to precedent ones.*

Remark 3.25. *For $S(w) = T(w) - \tau$ and orthogonal relation $\perp$ as follows:*

$$x \perp w \text{ifandonlyif} \alpha(x, w) \geq 1.$$

The Theorem 3.22 with $\nu = 1$ is the main result presented in [15].
For $S(w) = T(w) = w$, $L : \mathcal{A} \to \mathcal{A}$ and orthogonal relation $\perp$ as follows:

$$x \perp w \text{ifandonlyif} x \leq w.$$

The Theorem 3.22 with $\nu = 1$ is the main result presented in [44].

4. Applications

4.1. An application to resolve a fractional differential equation

A variety of useful fractional differential features were postulated and researched by Lacroix (1819). Caputo and Fabrizio announced [13], a new fractional technique, in 2015. The need to characterise a class of non-local systems that cannot be properly represented by traditional local theories or fractional models with singular kernel [13] sparked interest in this description. The different kernels that can be selected to satisfy the requirements of different applications are the fundamental differences among fractional derivatives. The Caputo fractional derivative [14], the Caputo-Fabrizio derivative [13], and the Atangana-Baleanu fractional derivative [5], for example, are determined by power laws, the Caputo-Fabrizio derivative by an exponential decay law, and the Atangana-Baleanu derivative by a Mittag-Leffler law. Various new Caputo-Fabrizio derivative (CFD) models were recently investigated in [3, 7, 8]. In multplicative-metric spaces, we will look at one of these models.

Let $b : C^2_{(I,\ R)} \to [1, \infty)$ be defined by

$$b(u, v) = e^{\|u-v\|_\infty} = e^{\sup_{l \in I} |u(l)-v(l)|}, \text{ for all } u, v \in C_{(I,\ R)}.$$

Then $(C_{(I,\ R)}, b)$ is a complete multiplicative metric space, where $I = [0, 1]$ and

$$C_{(I,\ R)} = \{u | u : I \to \ R \text{ and u is continuous}\}.$$

The relation $\perp$ on $C_{(I,\ R)}$ given as follows:

$$u \perp v \text{ if and only if } u(l)v(l) \geq u(l) \vee v(l), \text{ for all } u, v \in C_{(I,\ R)},$$

is an orthogonal relation and $(C_{(I,\ R)}, \perp, b)$ is an OCMMS. Let the function $K_1 : I \times \ R \to \ R$ be taken as $K_1(s, r) \geq 0$ for all $s \in I$ and $r \geq 0$. We shall apply Theorem 3.22 to resolve the following CFDE:

$$^{C}D^{\upsilon} w(s) = K_1(s, w(s)); \ w \in C_{(I,\ R)}; \tag{23}$$

$$w(0) = 0, Iw(1) = w'(0).$$

We denote CFD of order υ by $^{C}D^{\upsilon}$ and for $\upsilon \in (m-1, m); m = [\upsilon] + 1$, we have

$$^{C}D^{\upsilon} w(s) = \frac{1}{\Gamma(m-\upsilon)} \int_0^s (s-z)^{m-\upsilon-1} w(z))\, dz.$$

The notation $I^{v}w$ is interpreted as follows:

$$I^{v}w(s)=\frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1}w(z)\,dz \ : v>0.$$

We can represent (23) as follows:

$$\begin{aligned} w(s) &= \frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1}K_1(z,w(z))\,dz \\ &+ \frac{s}{\Gamma(v)}\int_0^1\int_0^z (z-p)^{v-1}K_1(p,w(p))\,dpdz. \end{aligned}$$

For the mappings $K_1 : I\times R\to R$ and $u_0\in C_{(I,\,R)}$, we state the following conditions:

(A) For $\tau>0$, let

$$|K_1(s,w(s))-K_1(s,u(s))|\le \frac{\Gamma(v+1)}{\Gamma(v)}|w(s)-u(s)|,$$

for all $w,u\in C_{(I,\,R)}$ following the order $w\perp u$.

(B) there exists $u_0\in C_{(I,\,R)}$ such that

$$\begin{aligned} u_0(s) &\le \frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1}K_1(z,u_0(z))\,dz \\ &+ \frac{l}{\Gamma(v)}\int_0^1\int_0^z (z-p)^{v-1}K_1(p,u_0(p))\,dpdz. \end{aligned}$$

We noticed that $K_1 : I\times R\to R$ is not necessarily Lipschitz continuous. For instant, K_1 given by

$$K_1(s,w(s))=\begin{cases} sw(s) & \text{if } w(s)\le \frac{1}{2}, \\ 0 & \text{if } w(s)>\frac{1}{2}. \end{cases}$$

follows (A) however, K_1 is not continuous and monotone. Moreover, for $s=e^{-\tau}\Gamma(v+1)$,

$$b(K_1(s,w(s)),K_1(t,u(t)))=e^{\left|K_1\left(l,\frac{1}{2}\right)-K_1\left(l,\frac{2}{3}\right)\right|}=e^{\frac{s}{2}}>e^{\frac{s}{6}}=e^{s\left|\frac{1}{2}-\frac{2}{3}\right|}=b(w,u).$$

Theorem 4.1. *Let the mappings $K_1 : I\times R\to R$ and $u_0\in C_{(I,\,R)}$ satisfies conditions (A)-(B), then the equation (23) admits a solution in $C_{(I,\,R)}$.*

Proof. Let $X=\left\{J\in C_{(I,\,R)} : J(s)\ge 0 \text{ for all } s\in I\right\}$ and define $\psi : X\to X$ by

$$(\psi J)(s)=\frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1}K_1(z,J(z))\,dz+\frac{s}{\Gamma(v)}\int_0^1\int_0^p (p-z)^{v-1}K_1(z,J(z))\,dzdp.$$

We define an orthogonal relation $\perp$ in X by

$u\perp v$ if and only if $u(s)v(s)\ge u(s)\vee v(s),\ \forall\ u,v\in X.$

According to above definitions, ψ is $\perp$-preserving and there is $u_0\in C_{(I,\,R)}$ verifying (B) such that $u_n=R^n(u_0))$ with $u_n\perp u_{n+1}$ or $u_{n+1}\perp u_n$ for all $n\ge 0$. We work on the validation of (22) in the next lines.

$$b((\psi J)(s),(\psi U)(s))=\exp\left(\sup_{s\in I}\left|\begin{array}{l} \frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1}K_1(z,J(z))\,dz \\ -\frac{1}{\Gamma(v)}\int_0^s (s-z)^{v-1}K_1(z,U(z))\,dz \\ +\frac{s}{\Gamma(v)}\int_0^1\int_0^p (p-z)^{v-1}K_1(z,J(z))\,dzdp \\ -\frac{s}{\Gamma(v)}\int_0^1\int_0^p (p-z)^{v-1}K_1(z,U(z))\,dzdp \end{array}\right|\right)$$

$$\le \exp\left(\sup_{s,z\in I}\left\{\begin{array}{l} \frac{1}{\Gamma(v)}\Gamma(v+1)\cdot\int_0^s (s-z)^{v-1}|J(z)-U(z)|\,dz \\ -\frac{s}{\Gamma(v)}\Gamma(v+1)\cdot\int_0^1\int_0^p (p-z)^{v-1}|J(z)-U(z)|\,dzdp \end{array}\right\}\right)$$

$$
\begin{aligned}
& b((\psi J)(s), (\psi U)(s)) \\
\leq\ & \exp\left(\sup_{s,z\in I}\left\{\begin{array}{l} \frac{1}{\Gamma(v)}\Gamma(v+1)\,|J(z)-U(z)|\cdot\int_0^s (s-z)^{v-1}\,dz \\ -\frac{s}{\Gamma(v)}\Gamma(v+1)\,|J(z)-U(z)|\cdot\int_0^1\int_0^p (p-z)^{v-1}\,dzdp \end{array}\right\}\right) \\
\leq\ & \exp\left(\begin{array}{l} \frac{\Gamma(v+1)}{\Gamma(v)}\sup_{z\in I}|J(z)-U(z)| \\ \sup_{s\in I}\left\{\int_0^s (s-z)^{v-1}\,dz - s\int_0^1\int_0^z (p-z)^{v-1}\,dzdp\right\} \end{array}\right) \\
\leq\ & \exp\left(\begin{array}{l} \frac{\Gamma(v+1)}{\Gamma(v)}\sup_{z\in I}|J(z)-U(z)| \\ \sup_{s\in I}\left\{\int_0^s (s-z)^{v-1}\,dz - s\int_0^1\int_0^z (p-z)^{v-1}\,dzdp\right\} \end{array}\right) \\
\leq\ & \exp\left(\begin{array}{l} \frac{\Gamma(v)\cdot\Gamma(v+1)}{\Gamma(v)\cdot\Gamma(v+1)}\sup_{z\in I}|J(z)-U(z)| \\ -sB(v+1,1)\,\frac{\Gamma(v)\cdot\Gamma(v+1)}{\Gamma(v)\cdot\Gamma(v+1)}\sup_{s,z\in I}|J(z)-U(z)| \end{array}\right) \\
\leq\ & \exp\left((1-sB(v+1,1))\sup_{s,z\in I}|J(z)-U(z)|\right) \\
\leq\ & \exp\left((1-sB(v+1,1))\sup_{s,z\in I}|J(z)-U(z)|\right) \\
=\ & \left(\exp\left(\sup_{s,z\in I}|J(z)-U(z)|\right)\right)^{1-sB(v+1,1)} \\
=\ & (b(J(z),U(z)))^{1-sB(v+1,1)};\ \ \text{where B is a beta function.}
\end{aligned}
$$

By defining $T(w)=\ln(w)$ and $S(w)=DT(w)$; $w>0$, $\tau>0$, and putting $1-sB(v+1,1)=D<1$, the last inequality gets the form:

$$T(b(\psi(J)(s), \psi(U)(s))) \leq S(b(J,U)).$$

□

4.2. Application to Volterra type integral equation

There are several types of integral equations but generally they are used to model scientific phenomena in which the value, or the rate of change of the value, of some quantity (or quantities) depends on past history. This is in contrast to differential equations, in which only the current value determines the rate at which a quantity is evolving. Just as for differential equations, integral equations need to be "solved" to describe and predict how a physical quantity is going to behave as time passes. For solving integral equations, there are things like Fredholm theorems, fixed point methods, boundary element methods, and Nystrom methods. In this paper we apply Theorem 3.22 to show the existence of solution of multiplicative Volterra type integral equation given below:

$$f(k)=\int_0^k L(k,h,f)^{dh}, \tag{24}$$

for all $k\in[0,1]$ and L is a mapping from $[0,1]\times[0,1]\times C_{(I,\ R)}$ to R. We show the existence of the solution to (24).

Let $b: C^2_{(I,\ R)} \to [1,\infty)$ be defined by

$$b(u,v)=\left|\frac{u(l)}{v(l)}\right|_m,\ \text{for all } u,v\in C_{(I,\ R)}.$$

Then $(C_{(I,\ R)}, b)$ is a complete multiplicative metric space, where $I=[0,1]$ and

$$C_{(I,\ R)}=\{u|u: I\to R \text{ and u is continuous}\}.$$

The relation $\perp$ on $C_{(I,\ R)}$ given as follows

$$u \perp v \text{ if and only if } u(l)v(l) \geq u(l) \vee v(l), \text{ for all } u, v \in C_{(I,\ R)},$$

is an orthogonal relation and $(C_{(I,\ R)}, \perp, b)$ is an OCMMS.
The following is the existence theorem for integral equation (24).

Theorem 4.2. *Assume that the following conditions are satisfied.*

(a) Assume that $L : [0,1] \times [0,1] \times C_{(I,\ R)} \rightarrow\ R$ *is continuous.*

(b) Suppose there exists $\tau > 0$, *such that*

$$\left|\frac{L(k,h,f)}{L(k,h,q)}\right|_m \leq \frac{b(f,q)}{(\tau\sqrt{b(f,q)}+1)^2}, \tag{25}$$

for all $k, h \in [0,1]$ *and* $f, q \in C_{(I,\ R^+)}$. *Then, integral equation (24) admits a solution in* $C_{(I,\ R^+)}$.

Proof. Let $Q = C_{(I,\ R)}$ and endow it with the relation $\perp$ and multiplicative metric b. Define the mapping $\Psi : Q \rightarrow Q$ by

$$(\Psi f)(k) = \int_0^k (L(k,h,f))^{dh}, \tag{26}$$

so that the fixed point of Ψ is a solution to integral equation (24). According to above definitions, ψ is $\perp$-preserving and there is $u_0 \in C_{(I,\ R)}$ verifying $u_n = R^n(u_0))$ with $u_n \perp u_{n+1}$ or $u_{n+1} \perp u_n$ for all $n \geq 0$. We work on the validation of (22) in the next lines. By assumption (b), we have

$$\begin{aligned}
b(\Psi(f), \Psi(q)) &= \left|\frac{(\Psi f)(k)}{(\Psi q)(k)}\right|_m = \left|\frac{\int_0^k (L(k,h,f))^{dh}}{\int_0^k (L(k,h,q))^{dh}}\right|_m \\
&\leq \int_0^k \left|\frac{L(k,h,f)}{L(k,h,q)}\right|_m^{dh} \\
&\leq \int_0^k \left(\frac{b(f,q)}{(\tau\sqrt{b(f,q)}+1)^2}\right)^{dh} \\
&= \exp\left(\int_0^k \ln\left(\frac{b(f,q)}{(\tau\sqrt{b(f,q)}+1)^2}\right) dh\right) \\
&= \exp\left(\ln\left(\frac{b(f,q)}{(\tau\sqrt{b(f,q)}+1)^2}\right)\int_0^k dh\right) \\
&\leq \exp\left(\ln\left(\frac{b(f,q)}{(\tau\sqrt{b(f,q)}+1)^2}\right)\right) \\
&= \frac{b(f,q)}{(\tau\sqrt{b(f,q)}+1)^2}
\end{aligned}$$

This implies

$$\sqrt{b(\Psi(f), \Psi(q))} \leq \frac{\sqrt{b(f,q)}}{\sqrt{b(f,q)}+1}. \tag{27}$$

$$\frac{\tau\sqrt{b(f,q)}+1}{\sqrt{b(f,q)}} \leq \frac{1}{\sqrt{b(\Psi(f),\Psi(q))}}. \tag{28}$$

$$\tau + \frac{1}{\sqrt{b(f,q)}} \leq \frac{1}{\sqrt{b(\Psi(f),\Psi(q))}}, \tag{29}$$

which further implies

$$\tau - \frac{1}{\sqrt{b(\Psi(f),\Psi(q))}} \leq \frac{-1}{\sqrt{b(f,q)}}. \tag{30}$$

So all the conditions of Theorem 3.22 are satisfied for $T(q) = \frac{-1}{\sqrt{q}}$; $q > 0$, $S(q) = T(q) - \tau$ and $\nu = 1$. Hence, the integral equation (24) admits at most one solution. □

5. Conclusion and future work

This work presents explicit criteria for the existence of fixed-points in (T,S)-orthogonal interpolative iterative mappings. It also introduces and investigates orthogonal interpolative versions of well-known iterative mappings. The proven fixed-point theorems for several forms of contractions (Kannan, Chatterjea, Ciric-Reich-Rus, Hardy-Rogers) in the context of orthogonal multiplicative metric spaces show how flexible and useful the suggested method is. The fixed point theorem's usefulness in solving integral equations and fractional differential equations is further expanded by its application to Banach (T,S)-orthogonal contractions via interpolation. Non-trivial examples are used to support theoretical conclusions, highlighting the published results practical value within the larger field of mathematical analysis.

Declarations

Ethical Approval
Not Applicable

Competing interest
The authors declare that they have no competing interests.

Author's contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Funding
This paper does not receive any external funding.

Availability of data and materials
Not applicable.

Acknowledgements
We would like to express our sincere gratitude to the anonymous referee for his/her helpful comments that will help to improve the quality of the manuscript.

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