Filomat 38:14 (2024), 5083–5097
https://doi.org/10.2298/FIL2414083N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Partition−Nekrasov type matrices: A new subclass of nonsingular $H-$matrices and applications

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Abstract. Nonsingular $H-$matrices have proven to be a source of many interesting results in different research areas in numerical linear algebra and also in applications in economy, engineering, ecology. In this paper, a new criteria for identifying some special $H-$matrices is presented. It is based on attributing different partition of the index set to each row of the matrix in consideration and testing inequalities that are associated to these partitions and involve recursively defined Nekrasov row sums. The new subclass of $H-$matrices introduced in this way is then analyzed with respect to its relation to well-known matrix classes. We used our new condition to estimate norm of the inverse matrix and errors in linear complementarity problems that involve matrices of this type.

1. Preliminaries

The extensive study of $H-$matrix theory has been motivated by numerous possibilities for applications of the obtained results in different fields of research, see [2]. In literature, one can find many novel results, which stem from analysis of special subclasses of nonsingular $H-$matrices, on spectra and pseudo-spectra localizations, norm bounds for the inverse, (see [5, 8, 21, 24, 28]), error bounds for linear complementarity problems, (see [14, 25, 27]), Schur complement properties,(see [7, 10, 11, 30, 34]), etc.

There are several different approaches when it comes to defining subclasses of $H-$matrices through relaxing the strict diagonal dominance condition. One of these approaches would be to replace strict inequalities with nonstrict inequalities with additional demands such as irreducibility or chain property. Another approach is to consider classes that are not closed under similarity permutations and modify these conditions by introducing permutation matrices. One of the ideas that was widely applied is to fix a partition of the index set and then define conditions that involve parts of row sums corresponding to the chosen partition.

2020 *Mathematics Subject Classification*. 65F35, 15B99
Keywords. Partition−Nekrasov type condition, $H-$matrices, Norm estimates for the inverse, Error estimates in linear complementarity problems.
Received: 13 April 2023; Revised: 27 September 2023; Accepted: 19 December 2023
Communicated by Marko Petković
This work is partly supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grant No. 451-03-68/2022-14/200125 and Grant No. 451-03-68/2022-14/200156).
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If we observe more closely some of the recently defined classes, such as $\Pi-$Nekrasov, see [1, 8], or CKV$-$type matrices, see [5], the idea behind these conditions is quite similar. In the definition of $\Pi-$Nekrasov. matrices, we allow that for each row in the matrix there exists a permutation that will repair that row, i.e., transform that row to a Nekrasov dominant row. This is a relaxed condition compared to Gudkov condition, where we need one permutation that will repair all rows at once, transforming them all to Nekrasov dominant rows. In the case of CKV$-$type matrices, for each row of the matrix there should be a partition of the index set such that certain inequalities hold with respect to that partition. This is also a relaxed condition compared to CKV condition, see [6, 15], where we need to find one partition of the index set such that all the inequalities are satisfied with respect to that one fixed partition.

We start with the class of Partition$-$Nekrasov matrices (in literature also known as $\Sigma-$Nekrasov class, see [9, 34]). The definition of this class requires the existence of one fixed partition of the index set such that certain inequalities (involving parts of recursively defined Nekrasov row sums with respect to the chosen partition) hold. We relax this condition by allowing that each row has its own corresponding partition of the index set, that satisfies inequalities involving parts of recursive row sums. This relaxation can also be viewed as a change in the order of quantifiers. We define Partition$-$Nekrasov type matrices, that will be denoted shortly as PN$-$type matrices, through a generalization of Partition$-$Nekrasov condition and show that new condition is sufficient to guarantee nonsingularity and, moreover, the membership to the class of $H-$matrices. Then, we estimate the infinity norm of the inverse matrix for PN$-$type matrices and use that result to estimate errors in linear complementarity problems.

In the remainder of this preliminary section, we recall well known subclasses of $H-$matrices, in Section 2, we introduce PN$-$type matrices and prove that these matrices are nonsingular and, moreover, $H-$matrices. In Section 3, we give two estimates for the infinity norm of the inverse of PN$-$type matrices. In Section 4, we estimate errors in linear complementarity problems when a PN$-$type matrix is involved. In Section 5, we give conclusions and remarks.

We commence with the usual notations and well-known definitions.

By $N = \{1, 2, \ldots, n\}$ we denote the set of indices. Given a nonempty proper subset S of N, $\overline{S} = N \setminus S$ stands for the complement of S in N. The $i-$th deleted absolute row sum is given by

$$r_i(A) = \sum_{j\in N, j\neq i} |a_{ij}|,$$

and its part that corresponds to the nonempty proper subset S is

$$r_i^S(A) = \sum_{j\in S, j\neq i} |a_{ij}|$$

Provided that for each $i \in N$ it holds that

$$|a_{ii}| > r_i(A),$$

we say that $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a strictly diagonally dominant (SDD) matrix.

We will denote the set of indices of non$-$SDD rows by

$$N_A = \{i \in N | |a_{ii}| \leq r_i(A)\}.$$

It is known that SDD property implies nonsingularity see [36]).

Next, one may define the comparison matrix $\langle A\rangle = [m_{ij}]$ of $A \in \mathbb{C}^{n,n}$, by

$$m_{ii} = |a_{ii}| \;, \;\; m_{ij} = -|a_{ij}| \;, \;\; i, j \in N \;, \;\; i \neq j$$

and in case that $\langle A\rangle$ is invertible and inverse non-negative (that is, an $M-$matrix), we say that A belongs to $H-$matrices.

Considering the convergence of the Gauss$-$Seidel iteration, (see [19]), Gudkov defined the Nekrasov class in his paper [17]. Later on, many generalizations preserving nonsingularity property followed, together with new results on norms of the inverse, eigenvalues and Schur complements of matrices in new classes, see [33], [20], [26], [10], [9], [8].

Nekrasov row sums, denoted $h_i(A)$, are defined recursively:

$$\begin{aligned} h_1(A) &= \sum_{j\neq 1} |a_{1j}| \\ h_i(A) &= \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 2, 3, \ldots, n \end{aligned}$$

and for $\emptyset \neq S \subsetneq N$, quantities $h_i^S(A)$ are calculated as

$$\begin{aligned} h_1^S(A) &= r_1^S(A) \\ h_i^S(A) &= \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j^S(A)}{|a_{jj}|} + \sum_{j=i+1, j\in S}^{n} |a_{ij}|, \quad i = 2, 3, \ldots, n. \end{aligned}$$

A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2$ is a Nekrasov matrix if, for each $i \in N$

$$|a_{ii}| > h_i(A).$$

For an arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2$, and given any nonempty proper subset S of N, the authors in [9] defined another class as follows: matrix A is an $S-$Nekrasov matrix if

$$|a_{ii}| > h_i^S(A) \text{ for all } i \in S \text{ and}$$

$$\left(|a_{ii}| - h_i^S(A)\right)\left(|a_{jj}| - h_j^{\overline{S}}(A)\right) > h_i^{\overline{S}}(A) h_j^S(A) \text{ for all } i \in S, \ j \in \overline{S}\,.$$

If there exists a nonempty proper subset S of N, such that $A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2$ is an $S-$Nekrasov matrix, then A is a Partition$-$Nekrasov matrix.

We also use the following vector notations:

$$h(A) = [h_1(A), ..., h_n(A)]^T,$$

$$h^S(A) = [h_1^S(A), ..., h_n^S(A)]^T.$$

In a recent paper [5] the authors introduced a new subclass of $H-$matrices, named CKV$-$type matrices, as follows. Let $A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2$ be an arbitrary matrix. If A is SDD or, for any $i \in N_A$ there exists a proper subset S containing i such that

$$|a_{ii}| > r_i^S(A) \text{ and}$$

$$\left(|a_{ii}| - r_i^S(A)\right)\left(|a_{jj}| - r_j^{\overline{S}}(A)\right) > r_i^{\overline{S}}(A) r_j^S(A) \text{ for all } \ j \in \overline{S},$$

then A is a CKV$-$type matrix.

They used the following notations:

$$\Sigma(i) = \{S \subsetneq N : i \in S\},$$

and

$$S_i^*(A) = \{S \in \Sigma(i) : |a_{ii}| > r_i^S(A) \text{ and for all } j \in \overline{S}$$
$$\left(|a_{ii}| - r_i^S(A)\right)\left(|a_{jj}| - r_j^{\overline{S}}(A)\right) > r_i^{\overline{S}}(A) r_j^S(A)\ \}.$$

In papers [5, 6, 12, 23, 24, 37], one can find various applications of classes of CKV and CKV$-$type matrices and related DZ and DZ$-$type matrices. In Section 2, we are going to apply a similar idea, considering recursively defined Nekrasov row sums instead of ordinary deleted row sums.

2. Partition−Nekrasov type (PN−type) condition

In this section, we introduce a new condition defining a subclass of nonsingular H−matrices. Then, we discuss relations of the new matrix class to well-known and related matrix classes. The new class is obtained through a relaxation of Partition−Nekrasov condition and examples show that our new class is wider than Partition−Nekrasov class. Also, examples show that our new class of PN−type matrices and the CKV−type class stand in a general relation.

2.1. Introducing PN−type matrices

Let us denote the set of indices corresponding to non−Nekrasov rows in the following way

$$N^-(A) = \{i \in N \mid |a_{ii}| \leq h_i(A)\}.$$

Definition 2.1. *Given any matrix $A = [a_{i,j}] \in \mathbb{C}^{n,n}, n \geq 2$, a matrix A is a Partition−Nekrasov type matrix (PN−type matrix) if*
1. A is a Nekrasov matrix or
2. For each index $i \in N^-(A)$ there exists a proper subset $S \subset N$ containing i, such that

$$\begin{aligned} &|a_{ii}| > h_i^S(A) \ \ \text{and} \\ &\Big(|a_{ii}| - h_i^S(A)\Big)\Big(|a_{jj}| - h_j^{\overline{S}}(A)\Big) > h_i^{\overline{S}}(A)h_j^S(A), \ \ \text{for all} \ \ j \in \overline{S}. \end{aligned} \tag{1}$$

For a given index $i \in N$ let $S_i(A)$ denote the set of corresponding subsets S satisfying the condition (1) from the previous definition.

Theorem 2.2. *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2$ is a PN−type matrix, then A is nonsingular.*

Proof:
If the given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}, n \geq 2$ belongs to Nekrasov matrix class, then A is clearly nonsingular. Having this in mind, in further considerations we assume that A is not Nekrasov, meaning that $N^-(A) \neq \emptyset$, i.e. the set of indices denoting non−Nekrasov rows is not empty. Let us suppose that A is a singular PN−type matrix. Then, there exists a nonzero vector $x = [x_1, ..., x_n]^T$ such that $Ax = 0$. Consider the index $k \in N$ such that $\|x\|_\infty = |x_k| > 0$.

As the first possible case, assume that index k denotes a non−Nekrasov row, i.e., $k \in N^-(A)$. Then, from the definition of PN−type matrices, $S_k(A) \neq \emptyset$, i.e., there exists a nonempty proper subset S of N such that

$$k \in S, \ \ |a_{kk}| > h_k^S(A)$$

and for all $j \in \overline{S}$

$$\Big(|a_{kk}| - h_k^S(A)\Big)\Big(|a_{jj}| - h_j^{\overline{S}}(A)\Big) > h_k^{\overline{S}}(A)h_j^S(A).$$

Now, we denote the maximum component of $|x|$ inside the set $\overline{S}$, $|x_l| = \max_{j \in \overline{S}} |x_j|$.

From $Ax = 0$, if $A = D - L - U$ is the standard splitting of the matrix A into diagonal, strictly lower and strictly upper triangular part, it follows

$$(D - L - U)x = 0$$

and by triangular inequality

$$(|D| - |L|)|x| \leq |U||x|.$$

As $|D| - |L|$ is a nonsingular M−matrix, (from the definition of PN−type matrices it follows that diagonal entries are positive), we obtain

$$|x| \leq (|D| - |L|)^{-1}|U||x|$$

and also

$$|D||x| \leq |D|(|D|-|L|)^{-1}|U||x| \leq |D|(|D|-|L|)^{-1}|U||x_k|e^S + |D|(|D|-|L|)^{-1}|U||x_l|e^{\overline{S}} \leq$$
$$\leq |x_k|h^S(A) + |x_l|h^{\overline{S}}(A),$$

as it is shown in [22] that

$$h^S(A) = |D|(|D|-|L|)^{-1}|U|e^S,$$

where

$$e_i^S = \begin{cases} 1, & i \in S, \\ 0, & i \in \overline{S} \end{cases}$$

Let us observe the $k-th$ row in the previous inequality and obtain

$$|a_{kk}||x_k| \leq |x_k|h_k^S(A) + |x_l|h_k^{\overline{S}}(A),$$

which implies

$$(|a_{kk}| - h_k^S(A))|x_k| \leq |x_l|h_k^{\overline{S}}(A). \tag{2}$$

Also, as $(|a_{kk}| - h_k^S(A))|x_k| > 0$ we have $|x_l| > 0$.

In a similar manner, considering the $l-th$ row, we have

$$|a_{ll}||x_l| \leq |x_k|h_l^S(A) + |x_l|h_l^{\overline{S}}(A),$$

which implies

$$(|a_{ll}| - h_l^{\overline{S}}(A))|x_l| \leq |x_k|h_l^S(A). \tag{3}$$

Multiplying inequalities (2) and (3) and dividing by $|x_k||x_l| > 0$, we obtain

$$(|a_{kk}| - h_k^S(A))(|a_{ll}| - h_l^{\overline{S}}(A)) \leq h_k^{\overline{S}}(A)h_l^S(A),$$

which is a contradiction with the assumption. This completes the proof for the first case.

As the second case, let us now assume that k corresponds to a Nekrasov row, i.e., $k \in N \backslash N^-(A)$. It is enough to show that $S_k(A) \neq \emptyset$ and then proceed as in the first case.

Suppose that A has only one Nekrasov row, i.e., $N \backslash N^-(A) = \{k\}$. Then for each $j \in N \backslash \{k\}$ it holds that $N \backslash \{k\} \in S_j(A)$ and it is easy to see that $S = \{k\} \in S_k(A)$.

On the other hand, if A has at least two Nekrasov rows, i.e., if $N \backslash N^-(A) \supseteq \{k, j\}$, then let us take $S = \{k\} \cup N^-(A)$. It is easy to see that $S \in S_k(A)$.

This completes the proof. □

Theorem 2.3. *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is a PN−type matrix, then A is an H−matrix.*

Proof: Let $A = D - L - U$ be the standard splitting of the matrix A. Then,

$$\langle A \rangle = |D| - |L| - |U| = (|D| - |L|)(I - (|D| - |L|)^{-1}|U|).$$

Suppose that

$$\rho((|D| - |L|)^{-1}|U|) \geq 1,$$

i.e., there exists an eigenvalue $\lambda \in \sigma((|D| - |L|)^{-1}|U|)$, such that $|\lambda| \geq 1$. Then the matrix $\lambda I - (|D| - |L|)^{-1}|U|$ is singular, and so is the matrix $B = \lambda(|D| - |L|) - |U|$. Let $B = D_B - L_B - U_B$ be the standard splitting of B and let e be the vector with all components equal to 1. If we consider now the diagonal entries and Nekrasov row sums of the matrix B, we obtain that the following relations hold for each index i in N.

$$|b_{ii}| = |\lambda||a_{ii}| \geq |a_{ii}|,$$

$$h_i(B) = |b_{ii}|((|D_B| - |L_B|)^{-1}|U_B|e)_i =$$
$$= |\lambda||a_{ii}|((|\lambda||D| - |\lambda||L|)^{-1}|U|e)_i =$$
$$= |\lambda||a_{ii}|\frac{1}{|\lambda|}((|D| - |L|)^{-1}|U|e)_i =$$
$$= |a_{ii}|((|D| - |L|)^{-1}|U|e)_i = h_i(A).$$

Also, for the given nonempty proper subset S of the index set N, it holds

$$h_i^S(B) = |b_{ii}|((|D_B| - |L_B|)^{-1}|U_B|e^S)_i = |\lambda||a_{ii}|((|\lambda||D| - |\lambda||L|)^{-1}|U|e^S)_i =$$
$$= |\lambda||a_{ii}|\frac{1}{|\lambda|}((|D| - |L|)^{-1}|U|e^S)_i = |a_{ii}|((|D| - |L|)^{-1}|U|e^S)_i = h_i^S(A).$$

Now, it is easy to see that if A is a PN−type matrix, so is B, therefore B is nonsingular, which is a contradiction.

Hence,

$$\rho((|D| - |L|)^{-1}|U|) < 1,$$

which implies

$$\langle A\rangle^{-1} = (I - (|D| - |L|)^{-1}|U|)^{-1}(|D| - |L|)^{-1} =$$
$$= \sum_{k\geq 0}((|D| - |L|)^{-1}|U|)^k(|D| - |L|)^{-1} \geq 0.$$

Therefore, A is an $H-$matrix. □

Let us now present some additional remarks on relations of this class to other subclasses of $H-$matrices, together with another definition of the new class.

Remark 2.4. *From the proof of Theorem 2.2 we see that, for the given PN−type matrix A, it holds:*
1. If $j \in N^-(A)$, then for all $S \in S_j(A)$, $N^-(A) \subseteq S$.
2. If $k \in N\backslash N^-(A)$ and $card(N\backslash N^-(A)) \geq 2$, then $\{k\} \cup N^-(A) \in S_k(A)$.
3. If $\{k\} = N\backslash N^-(A)$, then $\{k\} \in S_k(A)$ and for each $j \in N^-(A)$, $N^-(A)$ is the only element in $S_j(A)$.

Remark 2.5. *From the proof of Theorem 2.2 we see that A is a PN−type matrix if and only if for all $i \in N$, $S_i(A) \neq \emptyset$. This is an equivalent definition for Definition 2.1.*

Remark 2.6. *Notice that the class of Nekrasov matrices is a subclass of the class of PN−type matrices. Also, if A is a Nekrasov matrix, then for all $i \in N$ all proper subsets S that contain index i belong to $S_i(A)$.*

Remark 2.7. *It is easy to show that PN−type matrix class contains the whole Partition−Nekrasov class of matrices, using the same arguments as done for CKV and CKV−type matrices in [5]. The matrix F in the following example is a PN−type matrix, but it is not in Partition−Nekrasov class. This shows that the class of PN−type matrices is wider than Partition−Nekrasov class of matrices.*

$$F = \begin{bmatrix} 0.5847 & 0.446 & 0.2388 & 0.17 \\ 0.0603 & 0.875 & 0.2045 & 0.0203 \\ 0.1315 & 0.1178 & 0.742 & 0.5853 \\ 0.1308 & 0.0704 & 0.0597 & 0.3143 \end{bmatrix}$$

Remark 2.8. *We could identify other well-known subclasses of $H-$matrices that are contained in the new $PN-$type matrix class. It is well-known that CKV class is contained in Partition$-$Nekrasov class, see [6, 9]. Therefore, CKV is a subclass in $PN-$type matrix class. The same conclusion holds for DZ class, see [12], as DZ is a subclass in CKV class.*

Remark 2.9. *Examples show that neither CKV$-$type class is a subclass of the class of PN$-$type matrices, nor the class of PN$-$type matrices is a subclass of CKV$-$type class. These two classes have nonempty intersection, as the whole SDD class is in both of them. In the examples that follow, matrix B is a PN$-$type matrix, but it is not a CKV$-$type matrix. On the other hand, matrix D is a CKV$-$type matrix, while it is not a PN$-$type matrix.*

$$B = \begin{bmatrix} 2 & -1 & -1 & 1 \\ 1 & 4 & 0 & -1 \\ 1 & 1 & 4 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1.2 & 0 \\ 0.5 & 1.1 & 0.5 \\ 0.5 & 0 & 1 \end{bmatrix}$$

3. Application to infinity norm estimation for the inverse matrix

Estimation of the norm of the inverse matrix could be very useful in numerical linear algebra, as this type of results can be applied to estimation of the condition number, pseudospectra problems and in analysis of errors of the solution and sensitivity to perturbations in linear complementarity problems. Starting with the well-known result of Varah for SDD matrices, see [35], different authors developed upper bounds for the norm of the inverse for many other special matrix classes. In this section, we give two different estimates for the infinity norm of the inverse for PN$-$type matrices. As the new class is wider than Partition$-$Nekrasov, our bounds can be applied in some cases where already known bounds for Partition$-$Nekrasov matrices cannot be applied. A similar observation stands for CKV$-$type matrices. Numerical examples show that if a matrix in consideration is both CKV$-$type and PN$-$type matrix, in some cases our new norm bounds give more precise estimate, closer to the exact value of the norm, than already known bounds formulated for CKV$-$type matrices.

Theorem 3.1. *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is a PN$-$type matrix, then*

$$\|A^{-1}\|_\infty \leq \max \left\{ \max_{i \in N \setminus N^-(A)} \frac{z_i(A)}{|a_{ii}| - h_i(A)}, \max_{i \in N^-(A)} \min_{S \in S_i} \max_{j \in \overline{S}} \gamma_{ij}^S(A) \right\},$$

where

$$\gamma_{ij}^S(A) = \frac{z_j(A) h_i^{\overline{S}}(A) + z_i(A)(|a_{jj}| - h_j^{\overline{S}}(A))}{(|a_{jj}| - h_j^{\overline{S}}(A))(|a_{ii}| - h_i^S(A)) - h_i^{\overline{S}}(A) h_j^S(A)}.$$

Proof: We know that

$$\|A^{-1}\|_\infty^{-1} = \min_{\|x\|_\infty = 1} \|Ax\|_\infty = \|Ax^*\|_\infty = \max_{i \in N} |(Ax^*)_i|$$

for some $\|x^*\|_\infty = 1$. Therefore, for all $i \in N$ it holds that

$$\|A^{-1}\|_\infty^{-1} \geq |(Ax^*)_i|.$$

Let k be the index such that $\|x^*\|_\infty = |x_k^*| = 1$.

First, suppose that k corresponds to a Nekrasov dominant row, i.e., $k \in N \setminus N^-(A)$, $|a_{kk}| > h_k(A)$.

Then, from

$$Ax^* = Dx^* - Lx^* - Ux^*$$

it follows that

$$(|D| - |L|)|x^*| - |U||x^*| \leq |Ax^*|.$$

If we multiply both sides of this inequality with $|D|(|D|-|L|)^{-1}$ from the left, this gives

$$|D||x^*| - |D|(|D|-|L|)^{-1}|U||x^*| \leq$$

$$\leq |D|(|D|-|L|)^{-1}|Ax^*| \leq |D|(|D|-|L|)^{-1}||A^{-1}||_\infty^{-1} e.$$

Now, as $|x^*| \leq e$ and $|x_k^*| = 1$, we observe closely the $k-th$ row of the inequality, and obtain:

$$||A^{-1}||_\infty^{-1}(|D|(|D|-|L|)^{-1}e)_k \geq |a_{kk}||x_k^*| - (|D|(|D|-|L|)^{-1}|U|e)_k = |a_{kk}| - h_k(A).$$

As $(|D|(|D|-|L|)^{-1}e)_k = z_k(A)$ and $|a_{kk}| - h_k(A) > 0$, clearly, we have

$$||A^{-1}||_\infty \leq \frac{z_k(A)}{|a_{kk}| - h_k(A)} \leq \max_{i \in N \setminus N^-(A)} \frac{z_i(A)}{|a_{ii}| - h_i(A)}.$$

This completes the first case, where k stands for a Nekrasov dominant row.

In the second case, we assume the oposite, that k denotes a non-Nekrasov row, i.e., that $k \in N^-(A)$, $|a_{kk}| \leq h_k(A)$.

As A is a PN−type matrix, for $k \in N^-(A)$ there exists a nonempty proper subset S of N, such that $S \in S_k(A)$, i.e., it holds

$$k \in S, \ |a_{kk}| > h_k^S(A)$$

and for all $j \in \overline{S}$

$$(|a_{kk}| - h_k^S(A))(|a_{jj}| - h_j^{\overline{S}}(A)) > h_k^{\overline{S}}(A) h_j^S(A).$$

Let $l \in \overline{S}$ be the index such that $|x_l^*| = \max_{j \in \overline{S}} |x_j^*|$.

In a similar manner as done when discussing the first case, we obtain

$$|D||x^*| \leq |D|(|D|-|L|)^{-1}|Ax^*| + |D|(|D|-|L|)^{-1}|U||x^*| \leq$$

$$\leq |D|(|D|-|L|)^{-1}||A^{-1}||_\infty^{-1} e + |D|(|D|-|L|)^{-1}|U|e^S|x_k^*| + |D|(|D|-|L|)^{-1}|U|e^{\overline{S}}|x_l^*|.$$

In the $k-th$ row, we have

$$|a_{kk}| \leq ||A^{-1}||_\infty^{-1} z_k(A) + h_k^S(A) + h_k^{\overline{S}}(A)|x_l^*|,$$

which implies

$$|a_{kk}| - h_k^S(A) - ||A^{-1}||_\infty^{-1} z_k(A) \leq h_k^{\overline{S}}(A)|x_l^*|.$$

In the $l-th$ row, we have

$$|a_{ll}||x_l^*| \leq ||A^{-1}||_\infty^{-1} z_l(A) + h_l^S(A)|x_k^*| + h_l^{\overline{S}}(A)|x_l^*|,$$

which implies

$$(|a_{ll}| - h_l^{\overline{S}}(A))|x_l^*| \leq ||A^{-1}||_\infty^{-1} z_l(A) + h_l^S(A)$$

and also

$$|x_l^*| \leq \frac{||A^{-1}||_\infty^{-1} z_l(A) + h_l^S(A)}{|a_{ll}| - h_l^{\overline{S}}(A)}.$$

Therefore

$$|a_{kk}| - h_k^S(A) - ||A^{-1}||_\infty^{-1} z_k(A) \leq h_k^{\overline{S}}(A)|x_l^*| \leq h_k^{\overline{S}}(A) \frac{||A^{-1}||_\infty^{-1} z_l(A) + h_l^S(A)}{|a_{ll}| - h_l^{\overline{S}}(A)}.$$

Multiplying both sides with $|a_{ll}| - h_l^{\overline{S}}(A) > 0$ gives

$$(|a_{kk}| - h_k^S(A))(|a_{ll}| - h_l^{\overline{S}}(A)) - ||A^{-1}||_\infty^{-1} z_k(A)(|a_{ll}| - h_l^{\overline{S}}(A)) \leq$$

$$\leq \|A^{-1}\|_\infty^{-1} z_l(A) h_k^{\overline{S}}(A) + h_l^S(A) h_k^{\overline{S}}(A),$$

which implies

$$\|A^{-1}\|_\infty^{-1} \geq \frac{(|a_{kk}| - h_k^S(A))(|a_{ll}| - h_l^{\overline{S}}(A)) - h_l^S(A) h_k^{\overline{S}}(A)}{z_l(A) h_k^{\overline{S}}(A) + z_k(A)(|a_{ll}| - h_l^{\overline{S}}(A))}$$

and

$$\|A^{-1}\|_\infty \leq \frac{z_l(A) h_k^{\overline{S}}(A) + z_k(A)(|a_{ll}| - h_l^{\overline{S}}(A))}{(|a_{kk}| - h_k^S(A))(|a_{ll}| - h_l^{\overline{S}}(A)) - h_l^S(A) h_k^{\overline{S}}(A)} = \gamma_{kl}^S(A).$$

Now, we obtain

$$\|A^{-1}\|_\infty \leq \gamma_{kl}^S(A) \leq \max_{j \in \overline{S}} \gamma_{kj}^S(A) \leq \min_{S \in S_k(A)} \max_{j \in \overline{S}} \gamma_{kj}^S(A) \leq \max_{i \in N^-(A)} \min_{S \in S_i(A)} \max_{j \in \overline{S}} \gamma_{ij}^S(A).$$

This completes the second case.

Taking both cases into consideration, we now obtain the bound

$$\|A^{-1}\|_\infty \leq \max \left\{ \max_{i \in N \setminus N^-(A)} \frac{z_i(A)}{|a_{ii}| - h_i(A)}, \max_{i \in N^-(A)} \min_{S \in S_i(A)} \max_{j \in \overline{S}} \gamma_{ij}^S(A) \right\}.$$

□

According to Remark 2.5, one can easily obtain another estimate for the norm of the inverse matrix.

Theorem 3.2. *If a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is a PN−type matrix, then*

$$\|A^{-1}\|_\infty \leq \max_{i \in N} \min_{S \in S_i(A)} \max_{j \in \overline{S}} \gamma_{ij}^S(A),$$

where $\gamma_{ij}^S(A)$ is defined as in Theorem 3.1.

Let us consider the following numerical example. The matrix

$$A = \begin{bmatrix} 5 & 2 & 0 & -1 \\ -2 & 4 & 0 & 1 \\ -1.5 & 2 & 5 & -1 \\ 0.5 & 0 & -2 & 5 \end{bmatrix}$$

is an SDD matrix. It is the same matrix considered in [5]. There, it was shown that of many well−known norm bounds for the inverse, the one developed for CKV−type matrices gives the sharpest result (0.9286), while the exact value of the norm of the inverse is $\|A^{-1}\|_\infty = 0.4358$. The bound obtained from Theorem 3.1 is 1, which is not better than CKV−type result, but if we apply Theorem 3.2, we obtain a sharper bound, 0.7895.

Remark 3.3. *One can consider block generalizations of H−matrices and subclasses and define infinity norm estimations for the inverse matrix in the block−case. Block H−matrices were researched in [31].*

4. Linear complementarity problem

The linear complementarity problem (LCP) is one of the most well-known problems in numerical mathematics. For formulation, applications and in-depth analysis, the eager reader may refer to [4]. Due to its iterative nature, a key to its successful analysis is the proposal of numerically acceptable and computationally cheap estimations for the error, which was introduced and studied in [3], and further polished for matrices having some special properties, (see [1, 16, 25, 27, 29, 32]).

For a given matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and a given vector $q \in \mathbb{R}^n$, to solve LCP(A, q) is to find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, Ax + q \geq 0, (Ax + q)^T x = 0,$$

or to show that such vector does not exist.

The unique solution of LCP(A, q) for any vector $q \in \mathbb{R}^n$ is guaranteed if and only if the matrix A is a $P-$matrix, see [4], the condition being satisfied for any $H-$matrix with positive diagonal entries.

Let A be a $P-$matrix, let x^* denote the solution of the LCP(A, q) and denote $r(x) = min(x, Ax + q)$, $D = diag(d_i), 0 \leq d_i \leq 1$. In [3] the estimate for the error of the solution was presented in the following form

$$\|x - x^*\|_\infty \leq \max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \|r(x)\|_\infty,$$

meaning that we could use new estimates for the norm of the inverse developed in this paper for PN$-$type matrices to bound errors in LCP.

In order to do that, we also need the following lemmas.

Lemma 4.1 ([25]). *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a matrix with $a_{ii} > 0$ for all $i \in N$ and let $B = I - D + DA = [b_{ij}]$, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$. Then*

$$z_i(B) \leq \eta_i(A),$$

and

$$\frac{z_i(B)}{b_{ii}} \leq \frac{\eta_i(A)}{\min(a_{ii}, 1)},$$

where

$$z_1(B) = \eta_1(A) = 1,$$

$$z_i(B) = \sum_{j=1}^{i-1} \frac{|b_{ij}|}{|b_{jj}|} z_j(B) + 1, \ i = 2, 3, ..., n,$$

and

$$\eta_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{\min(a_{jj}, 1)} \eta_j(A) + 1, \ i = 2, 3, ..., n.$$

Lemma 4.2 ([27]). *Let $\gamma > 0$, $\eta \geq 0$. Then, for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}}$$

and

$$\frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}.$$

Using these Lemmas and the arguments from [14], we will estimate errors for LCP with PN$-$type matrices.

First, let us consider the shifted matrix $B = I - D + DA$, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$, and prove that the matrix B inherits PN$-$type property from the matrix A.

Lemma 4.3. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \geq 2$, with $a_{ii} > 0$, for all $i \in N$ and let $B = I - D + DA$, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$. Then, for any nonempty proper subset S of the index set N, it holds*

$$h_i^S(B) \leq d_i h_i^S(A), \ i \in N,$$

and

$$\frac{h_i^S(B)}{b_{ii}} \leq \frac{h_i^S(A)}{a_{ii}}, \ i \in N.$$

Proof: Consider the matrix $B = I - D + DA$, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$. Then, $b_{ii} = 1 - d_i + d_i a_{ii} \geq d_i a_{ii}$. Also, $b_{ij} = d_i a_{ij}$, $i \neq j$.

For an arbitrary nonempty proper subset S of the index set, it holds that

$$h_i^S(B) \leq d_i h_i^S(A),\ i \in N.$$

We show this by mathematical induction as follows.

$$h_1^S(B) = r_1^S(B) = \sum_{j=2, j \in S}^{n} |b_{1j}| = \sum_{j=2, j \in S}^{n} d_1 |a_{1j}| = d_1 h_1^S(A).$$

Assume that inequality $h_j^S(B) \leq d_j h_j^S(A)$ holds for $j = 1, ..., i-1$.

Now, if all $d_j > 0$, $j = 1, ..., i-1$,

$$h_i^S(B) = \sum_{j=1}^{i-1} |b_{ij}| \frac{h_j^S(B)}{b_{jj}} + \sum_{j=i+1, j \in S}^{n} |b_{ij}| \leq$$

$$\leq \sum_{j=1}^{i-1} d_i |a_{ij}| \frac{d_j h_j^S(A)}{d_j a_{jj}} + \sum_{j=i+1, j \in S}^{n} d_i |a_{ij}| = d_i h_i^S(A).$$

Notice that inequality holds also when for some $j \leq i-1$, $d_j = 0$. Namely, if $d_j = 0$, then $h_j^S(B) = 0$ and $b_{jj} = 1$, therefore

$$\frac{h_j^S(B)}{b_{jj}} = \frac{0}{1} \leq \frac{h_j^S(A)}{a_{jj}},$$

and

$$h_i^S(B) = \sum_{j=1}^{i-1} |b_{ij}| \frac{h_j^S(B)}{b_{jj}} + \sum_{j=i+1, j \in S}^{n} |b_{ij}| \leq$$

$$\leq \sum_{j=1}^{i-1} d_i |a_{ij}| \frac{h_j^S(A)}{a_{jj}} + \sum_{j=i+1, j \in S}^{n} d_i |a_{ij}| = d_i h_i^S(A).$$

In order to prove the second part of this lemma, we notice that in the case $d_i > 0$,

$$\frac{h_i^S(B)}{b_{ii}} \leq \frac{d_i h_i^S(A)}{d_i a_{ii}} = \frac{h_i^S(A)}{a_{ii}}.$$

Also, if $d_i = 0$, then $\frac{h_i^S(B)}{b_{ii}} = 0 \leq \frac{h_i^S(A)}{a_{ii}}$. This completes the proof. □

Theorem 4.4. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \geq 2$, with $a_{ii} > 0$, for all $i \in N$. Let $B = I - D + DA$, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$. If A is a PN−type matrix, then B is also a PN−type matrix. Furthermore, for each $i \in N$, $S_i(A) \subseteq S_i(B)$.*

Proof: Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \geq 2$, with $a_{ii} > 0$, for all $i \in N$ be a PN−type matrix. Then, for each $i \in N$, $S_i(A) \neq \emptyset$, i.e., there is a proper subset S containing i such that

$$|a_{ii}| > h_i^S(A)$$

and for all $j \in \overline{S}$,

$$\Big(|a_{ii}| - h_i^S(A)\Big)\Big(|a_{jj}| - h_j^{\overline{S}}(A)\Big) > h_i^{\overline{S}}(A) h_j^S(A).$$

Now, we prove that $S \in S_i(B)$. If $d_i \neq 0$, then,

$$b_{ii} \geq d_i a_{ii} > d_i h_i^S(A) \geq h_i^S(B),$$

and if $d_i = 0$, then $b_{ii} = 1 > 0 = h_i^S(B)$.

As $S \in S_i(A)$, for all $j \in \overline{S}$

$$\left(a_{ii} - h_i^S(A)\right)\left(a_{jj} - h_j^{\overline{S}}(A)\right) > h_i^{\overline{S}}(A) h_j^S(A).$$

If $d_i > 0$ and $d_j > 0$ for all $j \in \overline{S}$, by multiplying previous inequality with $d_i d_j$, we obtain that

$$\left(b_{ii} - h_i^S(B)\right)\left(b_{jj} - h_j^{\overline{S}}(B)\right) \geq d_i\left(a_{ii} - h_i^S(A)\right) d_j\left(a_{jj} - h_j^{\overline{S}}(A)\right) >$$

$$> d_i h_i^{\overline{S}}(A) d_j h_j^S(A) \geq h_i^{\overline{S}}(B) h_j^S(B).$$

It is easy to see that

$$\left(b_{ii} - h_i^S(B)\right)\left(b_{jj} - h_j^{\overline{S}}(B)\right) > h_i^{\overline{S}}(B) h_j^S(B)$$

holds also when $d_i = 0$ or, for some $j \in \overline{S}$, $d_j = 0$. Therefore, $S \in S_i(B)$.

This proves that if A is a PN−type matrix with positive diagonal entries, then $B = I - D + DA$ where $D = diag(d_i)$ with $0 \leq d_i \leq 1$ is also a PN−type matrix and, moreover, for each index i it holds that $S_i(A) \subseteq S_i(B)$.

□

Theorem 4.5. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \geq 2$, $a_{ii} > 0$, $i = 1, ..., n$, be a PN−type matrix. Then,*

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq \max_{i \in N} \min_{S \in S_i(A)} \max_{j \in \overline{S}} \delta_{ij}^S(A),$$

where

$$\delta_{ij}^S(A) = \frac{\frac{\eta_j(A) h_i^{\overline{S}}(A)(a_{jj} - h_j^{\overline{S}}(A))}{\min\{a_{jj} - h_j^{\overline{S}}(A), 1\}} + \frac{\eta_i(A)(a_{ii} - h_i^S(A))(a_{jj} - h_j^{\overline{S}}(A))}{\min\{a_{ii} - h_i^S(A), 1\}}}{(a_{ii} - h_i^S(A))(a_{jj} - h_j^{\overline{S}}(A)) - h_i^{\overline{S}}(A) h_j^S(A)}.$$

Proof:

Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \geq 2$, $a_{ii} > 0$, $i = 1, ..., n$, be a PN−type matrix. Then, for any diagonal matrix D, $D = diag(d_i)$ with $0 \leq d_i \leq 1$, the matrix $B = I - D + DA$ is also a PN−type matrix (as shown in Theorem 4.4).

According to previous Lemmas, we obtain that the following inequality holds for each $i \in N, S \in S_i(A)$.

$$\frac{1}{b_{ii} - h_i^S(B)} \leq \frac{1}{1 - d_i + d_i a_{ii} - d_i h_i^S(A)} =$$

$$= \frac{1}{1 - d_i + d_i(a_{ii} - h_i^S(A))} \leq \frac{1}{\min\{a_{ii} - h_i^S(A), 1\}}.$$

Then, we can apply the norm bound for the inverse matrix from Theorem 3.2 to the matrix B, as follows:

$$\|B^{-1}\|_\infty \leq \max_{i \in N} \min_{S \in S_i(B)} \max_{j \in \overline{S}} \gamma_{ij}^S(B).$$

Here, if $d_i > 0$ and $d_j > 0$,

$$\gamma_{ij}^S(B) = \frac{z_j(B) h_i^{\overline{S}}(B) + z_i(B)(|b_{jj}| - h_j^{\overline{S}}(B))}{(|b_{jj}| - h_j^{\overline{S}}(B))(|b_{ii}| - h_i^S(B)) - h_i^{\overline{S}}(B) h_j^S(B)} =$$

$$= \frac{\frac{z_j(B)h_i^{\overline{S}}(B)}{(b_{jj}-h_j^{\overline{S}}(B))(b_{ii}-h_i^S(B))} + \frac{z_i(B)}{b_{ii}-h_i^S(B)}}{1 - \frac{h_i^{\overline{S}}(B)h_j^S(B)}{(b_{jj}-h_j^{\overline{S}}(B))(b_{ii}-h_i^S(B))}} \leq$$

$$\leq \frac{\frac{\eta_j(A)d_j h_i^{\overline{S}}(A)}{\min\{a_{jj}-h_j^{\overline{S}}(A),1\}d_i(a_{ii}-h_i^S(A))} + \frac{\eta_i(A)}{\min\{a_{ii}-h_i^S(A),1\}}}{1 - \frac{d_i h_i^{\overline{S}}(A)d_j h_j^S(A)}{d_j(a_{jj}-h_j^{\overline{S}}(A))d_i(a_{ii}-h_i^S(A))}} =$$

$$= \frac{\frac{\eta_j(A)h_i^{\overline{S}}(A)(a_{jj}-h_j^{\overline{S}}(A))}{\min\{a_{jj}-h_j^{\overline{S}}(A),1\}} + \frac{\eta_i(A)(a_{ii}-h_i^S(A))(a_{jj}-h_j^{\overline{S}}(A))}{\min\{a_{ii}-h_i^S(A),1\}}}{(a_{ii}-h_i^S(A))(a_{jj}-h_j^{\overline{S}}(A)) - h_i^{\overline{S}}(A)h_j^S(A)} = \delta_{ij}^S(A).$$

The conclusion also holds when $d_i = 0$ or $d_j = 0$.
Therefore,

$$\|B^{-1}\|_\infty \leq \max_{i\in N} \min_{S\in S_i(B)} \max_{j\in\overline{S}} \gamma_{ij}^S(B) \leq \max_{i\in N} \min_{S\in S_i(A)} \max_{j\in\overline{S}} \delta_{ij}^S(A).$$

□

The special case for matrices with all diagonal entries equal to 1 is given in the following Corollary.

Corollary 4.6. *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$, $n \geq 2$, $a_{ii} = 1$, $i = 1, ..., n$, be a PN−type matrix. Then,*

$$\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq \max_{i\in N} \min_{S\in S_i(A)} \max_{j\in\overline{S}} \delta_{ij}^S(A),$$

where

$$\delta_{ij}^S(A) = \frac{\eta_j(A)h_i^{\overline{S}}(A) + \eta_i(A)(1 - h_j^{\overline{S}}(A))}{(1 - h_i^S(A))(1 - h_j^{\overline{S}}(A)) - h_i^{\overline{S}}(A)h_j^S(A)}.$$

Benefits of these new error estimates are twofold.
Firstly, there are matrices that belong to PN−type class, but do not belong to Partition−Nekrasov class (see matrix F from Remark 2.7). For such matrices neither of the well-known bounds defined for SDD, DZ, CKV, Nekrasov nor Partition−Nekrasov matrices can be applied, while our new estimate given for PN−type matrices can be applied.
Secondly, even if the given matrix belongs both to PN−type class and some other well-known class, in some cases our new bounds give tighter estimates, as the following example shows.

Let us consider the matrix C,

$$C = \begin{bmatrix} 0.5 & 1 & -1 \\ 0 & 2 & 1 \\ 0.1 & 1 & 2 \end{bmatrix}.$$

The matrix C belongs to CKV class and it does not belong to Nekrasov matrix class. It is also a PN−type matrix, as CKV is a subclass of PN−type class. If we apply error bounds for linear complementarity problems for CKV matrices given in [18] we obtain 16.6667. If we apply our new bounds for PN−type matrices from Theorem 4.5 we obtain tighter estimate 10.8.

5. Conclusions and remarks

We conclude our considerations with the following remarks and observations. Note that there are several benefits of the new PN−type condition, introduced in this paper. First, this new condition guarantees nonsingularity of the given matrix. Also, we proved that matrices satisfying this condition belong to H−matrices. When analyzing the relation of the new class to other subclasses of H−matrices, one can see that the new class of PN−type matrices contains the class of Partition−Nekrasov matrices, as the new condition is more relaxed compared to Partition−Nekrasov condition. This is due to allowing for each index to have its own corresponding partition of the index set, instead of demanding that there exists one fixed partition of the index set such that all the inequalities hold with respect to that same partition. Numerical examples given in Section 2 show that the new class and CKV−type class stand in a general relation, meaning that neither is a subclass of the other and they have a nonempty intersection, as they both contain SDD matrices. We used the new condition to estimate the infinity norm of the inverse for PN−type matrices. Numerical examples illustrate that we sometimes obtain better estimates using new results instead of using the bounds developed for some other matrix classes, such as CKV−type matrices, when the matrix in consideration belongs both to CKV−type and the new class. Also, there are matrices that do not belong to Partition−Nekrasov or CKV−type matrices, but do belong to PN−type matrices. For these matrices previously known bounds cannot be applied, while our new estimates for PN−type matrices can be applied. One more benefit is briefly presented as a remark on the possibility for application of these results to block H−matrices. There are matrices that do not belong to H−matrices in the classical sense (for instance, if they have one or more zero diagonal entries) but they belong to block generalizations of different subclasses of H−matrices. For these matrices, classical estimates for the norm of the inverse cannot be applied, but block results can be used instead. The same consideration is valid for our new condition, as it can be used to deal with another class of block H−matrices. As the relation between subclasses of block H−matrices is analogous to the relation between corresponding classes in the point−wise case, norm estimates for block PN−type matrices could be applicable in some cases where already known bounds (point−wise or block) cannot be applied. One more possible application that could be easily derived lies in defining PN−type B-matrices via a special rank one perturbation, as done for many other matrix classes. In this way, one could obtain a new subclass of P−matrices and estimate corresponding error bounds for LCP. Also, for further research it might be interesting to consider different sparsity patterns and their effect on proposed bounds and computational costs, as done for DZ−type and CKV−type matrices (see [13, 23]).

Acknowledgments

This work is partly supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grant No. 451-03-68/2022-14/200125 and Grant No. 451-03-68/2022-14/200156).

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