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The solution of the Yang-Baxter-like operator equation for rank-one operators

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Abstract. Let $A \in \mathcal{B}(\mathcal{H})$ be a rank-one operator, solutions of the Yang-Baxter-like operator equation $AXA = XAX$ on Hilbert spaces are investigated. We derive necessary and sufficient conditions for an operator $X \in \mathcal{B}(\mathcal{H})$ being a solution of the equation. Further, a necessary and sufficient condition that the equation has a rank-one solution is obtained for an arbitrary operator A.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ stand for the set of all bounded linear operators on Hilbert space $\mathcal{H}$. Let $A \in \mathcal{B}(\mathcal{H})$ be a given operator, the quadratic operator equation

$$AXA = XAX \tag{1}$$

is called the Yang-Baxter-like operator equation, where $X \in \mathcal{B}(\mathcal{H})$ is the unknown operator to be determined. In the finite space, the equation (1) is the Yang-Baxter-like matrix equation, which arises from the classical Yang-Baxter equation[1, 12].

For the Yang-Baxter-like matrix equation, its all solutions are not easy to be found for an arbitrary matrix A. Using eigenvalues and the corresponding generalized eigenspaces, authors in [4] and [5] obtain infinitely many commuting spectral solutions of the equation if eigenvalues of the matrix A are semi-simple and non-semisimple, respectively. Further, the results are extended to an arbitrary square matrix A[14], and explicit commuting spectral solutions are constructed. When A is a general singular matrix, infinitely many solutions of the equation are found in [6] by splitting the equation into the system of linear matrix equations. Moreover, by the generalized inverses, infinitely many new nontrivial non-commuting solutions are also derived in [2] for both regular and singular matrix A. When $A^2 = I$ and A is a square matrix with general Jordan structure forms, methods of solving all commuting solutions ($AX = XA$) to the equation are proposed in [8] and [10], respectively. When A is a rank-one matrix, all solutions of the equation are

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constructed by utilizing the special structures of the Jordan canonical forms of A in [11], and all solutions are also expressed concisely based on the sufficient and necessary conditions derived for a matrix to be a solution of the equation in [7]. Moreover, when A is an idempotent matrix and rank-two matrix, the equation is completely solved in [9, 13].

The main aim of the present article is to study nontrivial solutions ($X \neq 0, A$) of the operator equation (1) on Hilbert spaces, which is inspired by the work of [7]. It is easy to see that $X = 0, A$ are solutions of the equation, and we call them trivial solution. In this paper, we present some necessary and sufficient conditions for an operator $X \in \mathcal{B}(\mathcal{H})$ being a solution of the equation (1) when A is a rank-one operator. Further, the necessary and sufficient condition that the equation has nontrivial solutions is obtained for the rank-one operator A. Based on the characteristic of the equation, we also give a necessary and sufficient condition that the equation (1) has rank-one solutions for an arbitrary operator $A \in \mathcal{B}(\mathcal{H})$.

We first give the definition and properties of the rank-one operator.

Definition 1.1. Let $A \in \mathcal{B}(\mathcal{H})$. If $\dim \mathcal{R}(A) < \infty$, then we say that A is of finite rank. In particular, A is called a rank-one operator if $\dim \mathcal{R}(A) = 1$.

Note that a rank-one operator is always of the form $f \otimes g$ for nonzero vectors f, g in $\mathcal{H}$. Here, the operator $f \otimes g$ is defined by

$$(f \otimes g)h = \langle h, g\rangle f, \quad \text{for all } h \in \mathcal{H}.$$

Lemma 1.2. Let $A \in \mathcal{B}(\mathcal{H})$, $f, g, f_1, f_2 \in \mathcal{H}$, and a, b are complex numbers, then the following holds.

(1) $A(f \otimes g) = (Af) \otimes g,\ (f \otimes g)A = f \otimes (A^*g)$;

(2) $(af_1 + bf_2) \otimes g = a(f_1 \otimes g) + b(f_2 \otimes g),\ g \otimes (af_1 + bf_2) = \overline{a}(g \otimes f_1) + \overline{b}(g \otimes f_2)$;

(3) $f \otimes g = 0$ if and only if $f = 0$ or $g = 0$.

2. Solutions of the equation $AXA = XAX$ for the rank-one operator A

This section is devoted to investigate necessary and sufficient conditions for an operator $X \in \mathcal{B}(\mathcal{H})$ being a solution of the equation $AXA = XAX$ when A is a rank-one operator.

Theorem 2.1. *Let $A = f \otimes g$, where f, g are nonzero vectors in $\mathcal{H}$. If $\langle Xf, g\rangle = 0$ for an operator $X \in \mathcal{B}(\mathcal{H})$, then X is a solution of the equation $AXA = XAX$ if and only if $Xf = 0$ or $X^*g = 0$.*

Proof Sufficiency. If $Xf = 0$, then

$$AXA = (f \otimes g)X(f \otimes g) = (f \otimes g)((Xf) \otimes g) = 0,$$
$$XAX = X(f \otimes g)X = ((Xf) \otimes g)X = 0.$$

If $X^*g = 0$, then

$$AXA = (f \otimes (X^*g))(f \otimes g) = 0,$$
$$XAX = X(f \otimes (X^*g)) = 0.$$

It is obvious that we get $AXA = XAX$ if $Xf = 0$ or $X^*g = 0$.

Necessity. If X is a solution of the equation $AXA = XAX$, then, for any $h \in \mathcal{H}$, we have $AXAh = XAXh$, i.e., $(f \otimes g)X(f \otimes g)h = X(f \otimes g)Xh$. According to Lemma 1.2, it follows that

$$\begin{aligned}(f \otimes g)X(f \otimes g)h &= (f \otimes g)((Xf) \otimes g)h \\ &= \langle h, g\rangle (f \otimes g)Xf \\ &= \langle h, g\rangle \langle Xf, g\rangle f,\end{aligned}$$

and

$$X(f \otimes g)Xh = \langle Xh, g\rangle Xf.$$

Thus,

$$\langle Xh, g\rangle Xf = \langle h, g\rangle\langle Xf, g\rangle f. \tag{2}$$

From the assumption $\langle Xf, g\rangle = 0$, we obtain that $\langle Xh, g\rangle Xf = 0$, which shows $Xf = 0$ or $\langle Xh, g\rangle = 0$. Since h is arbitrary, it follows that $X^*g = 0$. We complete the proof. □

Corollary 2.2. Let $A = f \otimes g$, where f, g are nonzero vectors in $\mathcal{H}$. If $\langle Xf, g\rangle = 0$ for an operator $X \in \mathcal{B}(\mathcal{H})$, then X is a solution of the equation $AXA = XAX$ if and only if X has the eigenpair $(0, f)$ or X^* has the eigenpair $(0, g)$.

Theorem 2.3. Let $A = f \otimes g$, where f, g are nonzero vectors in $\mathcal{H}$. If $\langle Xf, g\rangle = 0$ for an operator $X \in \mathcal{B}(\mathcal{H})$, then

(1) X is a commuting solution of the equation $AXA = XAX$ if and only if $Xf = 0$ and $X^*g = 0$.

(2) X is a non-commuting solution of the equation $AXA = XAX$ if and only if one and only one of the equalities $Xf = 0$ and $X^*g = 0$ is satisfied.

Proof We only prove (1), and (2) is similar.

Sufficiency. If $Xf = 0$ and $X^*g = 0$, then X is a solution of the equation $AXA = XAX$ from Theorem 2.1. Note that

$$AX = (f \otimes g)X = f \otimes (X^*g) = 0,$$
$$XA = X(f \otimes g) = (Xf) \otimes g = 0,$$

which show that $AX = XA$, and hence, X is a commuting solution of the equation $AXA = XAX$.

Necessity. If X is a commuting solution of the equation $AXA = XAX$, then, by Theorem 2.1, we know $Xf = 0$ or $X^*g = 0$. Now, assume that $Xf = 0$ and $X^*g \neq 0$, and then

$$AX = (f \otimes g)X = f \otimes (X^*g) \neq 0$$

since $f \neq 0$, and

$$XA = X(f \otimes g) = (Xf) \otimes g = 0.$$

It is clear that $AX \neq XA$, a contradiction. Similarly, assume that $X^*g = 0$ and $Xf \neq 0$, we also obtain a contradiction. We complete the proof. □

Theorem 2.4. Let $A = f \otimes g$, where f, g are nonzero vectors in $\mathcal{H}$. If the solution of the equation $AXA = XAX$ satisfies $\langle Xf, g\rangle \neq 0$, then $\langle f, g\rangle \neq 0$.

Proof If $X \in \mathcal{B}(\mathcal{H})$ is a solution of $AXA = XAX$, then, by (2), it follows that

$$\langle Xh, g\rangle\langle Xf, g\rangle = \langle h, g\rangle\langle Xf, g\rangle\langle f, g\rangle$$

for any $h \in \mathcal{H}$. We take $h = f$, then $\langle f, g\rangle \neq 0$ is obtained since $\langle Xf, g\rangle \neq 0$. □

Theorem 2.5. Let $A = f \otimes g$, where f, g are nonzero vectors in $\mathcal{H}$. If $\langle Xf, g\rangle \neq 0$ for an operator $X \in \mathcal{B}(\mathcal{H})$, then X is a solution of the equation $AXA = XAX$ if and only if $Xf = \langle f, g\rangle f$ and $X^*g = \langle g, f\rangle g$.

Proof Sufficiency. Assume $Xf = \langle f, g\rangle f$ and $X^*g = \langle g, f\rangle g$. Then, by $\langle f, g\rangle f = (f \otimes g)f$, $\langle g, f\rangle g = (g \otimes f)g$, and Lemma 1.2,

$$\begin{aligned}
AXA &= (f \otimes g)((Xf) \otimes g) \\
&= \langle f, g\rangle(f \otimes g)(f \otimes g) \\
&= (\langle f, g\rangle)^2(f \otimes g) \\
&= (\langle f, g\rangle f) \otimes (\langle g, f\rangle g) \\
&= (Xf) \otimes (X^*g) \\
&= X(f \otimes g)X \\
&= XAX.
\end{aligned}$$

So, X is a solution of the equation $AXA = XAX$.

Necessity. Assume that X is a solution of the equation $AXA = XAX$. We take $h = f$ in (2), then we have

$$\langle Xf, g\rangle Xf = \langle f, g\rangle\langle Xf, g\rangle f.$$

This yields $Xf = \langle f, g\rangle f$ since $\langle Xf, g\rangle \neq 0$. Now, substituting $Xf = \langle f, g\rangle f$ into (2), it deduces

$$\langle Xh, g\rangle\langle f, g\rangle f = \langle h, g\rangle\langle f, g\rangle\langle f, g\rangle f.$$

According to Theorem 2.4, $\langle f, g\rangle \neq 0$. Thus,

$$\langle Xh, g\rangle f = \langle h, g\rangle\langle f, g\rangle f,$$

This gives

$$\langle h, X^* g - \langle g, f\rangle g\rangle f = 0,$$

Since h is arbitrary and $f \neq 0$, $X^* g = \langle g, f\rangle g$ is followed. The proof is completed. □

Corollary 2.6. Let $A = f \otimes g$, where f, g are nonzero vectors in $\mathcal{H}$. If $\langle Xf, g\rangle \neq 0$ for an operator $X \in \mathcal{B}(\mathcal{H})$, then X is a solution of the equation $AXA = XAX$ if and only if X and X^* have eigenpairs $(\langle f, g\rangle, f)$ and $(\langle g, f\rangle, g)$, respectively.

Remark 2.7. In fact, in Theorem 2.5, the solution is commuting solution of the equation $AXA = XAX$ since

$$AX = (f \otimes g)X = f \otimes (X^* g) = \langle f, g\rangle f \otimes g = (Xf) \otimes g = X(f \otimes g) = XA.$$

Theorem 2.1 and Theorem 2.5 give the necessary and sufficient condition for existence of the nontrivial solution of the equation $AXA = XAX$.

Theorem 2.8. Let $A = f \otimes g$, where f, g are nonzero vectors in $\mathcal{H}$. Then the equation $AXA = XAX$ has a nontrivial solution $X \in \mathcal{B}(\mathcal{H})$ if and only if one of the following conditions holds for $X \neq 0, A$:

(1) $0 \in \sigma_p(X)$ and f is the eigenvector corresponding to 0, or $0 \in \sigma_p(X^*)$ and g is the eigenvector corresponding to 0;

(2) $\langle f, g\rangle \in \sigma_p(X) \cap \overline{\sigma_p(X^*)}$, and f, g are eigenvectors of X and X^* corresponding to $\langle f, g\rangle$, respectively.

One interesting question arises: is X possibly a rank-one, and A is not? Note that the characteristic of the operator equation $AXA = XAX$, we give the following result on rank-one solutions by Theorem 2.8.

Theorem 2.9. Let $A \in \mathcal{B}(\mathcal{H})$. Then the equation $AXA = XAX$ has a rank-one solution if and only if one of the following conditions holds:

(1) $0 \in \sigma_p(A)$, or $0 \in \sigma_p(A^*)$;

(2) there exist nonzero vectors $f, g \in \mathcal{H}$ such that $\langle f, g\rangle \in \sigma_p(A) \cap \overline{\sigma_p(A^*)}$, and f, g are eigenvectors of A and A^* corresponding to $\langle f, g\rangle$, respectively.

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