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# Best proximity point theorems via measure of noncompactness and application to a system of Caputo fractional differential equations

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**Abstract.** In this article, we establish the best proximity point (pair) theorems for condensing operators defined on some new class of functions using the measure of noncompactness. Our theorems extend the Darbo type fixed point theorem and some recent investigations in this direction. We apply our results to the study of optimum solutions for a system of Caputo fractional differential equations.

## 1. Introduction

Many physical problems have been discussed using fractional calculus in the last three decades [1, 14, 17]. Due to science's adaptability, precision, and usefulness, it becomes very attractive for researchers [24]. Furthermore, depending on the nature of the problem, it becomes a method of reformulation and reconstruction. Fractional calculus has straightforward applications in various fields. Solvability for fractional differential equations has been extensively studied by many researchers [13, 15, 20] using the technique of measure of noncompactness and fixed point theory. Many fractional differential equations exist in the literature, but Caputo sense fractional differential equations are the simplest to solve. The main feature of the Caputo derivative is that it has a zero derivative of the constant function, whereas most other fractional derivatives do not. This characteristic helps the application of fixed point theorems to initial value problems.

The problem of determining points that minimize distances to a particular point or subset is one of the central problems in approximation theory. Analysts have always been drawn to the best approximation because it can be extended, mainly when using the functional analytic technique in nonlinear analysis. It was found in the middle of the 20th century that the existence of fixed points can be used to demonstrate the existence of the best approximation [4]. In the case of self-mapping, the best approximation is known as an invariant approximation. The measure of noncompactness and best proximity theory have various applications in solving different types of differential and integral equations, see for instance [7, 11, 22, 23]. Gabeleh et al. [12] established the best proximity point results for noncyclic $\varphi$-condensing operators in strictly convex Banach spaces by using a measure of noncompactness. Nashine et al. [21] discussed the best proximity results for cyclic and noncyclic FG-contractive operators in strictly convex Banach spaces via measure of noncompactness.

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In this article, we discuss the best proximity point results for cyclic and noncyclic mappings using measure of noncompactness and class of functions defined in [3, 16, 19]. The results of the paper generalize various results of the articles in [6, 12].

## 2. Preliminaries

Let $(Y, \|\cdot\|)$ be a Banach space and $\mathbb{R}_+ := [0, \infty)$. Suppose $S \subset Y$, then by $\overline{S}$ and conv$(S)$, we denote the closure and the closed convex hull of $S$, respectively. Let $\mathcal{M}_Y$ be the family of nonempty bounded subsets of $Y$ and $\mathcal{N}_Y$ its subfamily consisting of all relatively compact subsets of $Y$.

### *2.1. Best proximity points*

A pair $(C, D)$ of subsets of Banach space $Y$ satisfies a property if both $C$ and $D$ satisfy that property. For example, $(C, D) \subseteq (P, Q) \Leftrightarrow C \subseteq P, D \subseteq Q$; $(C, D)$ is convex if and only if both $C$ and $D$ are convex. For the pair $(C, D)$, we define

$$\begin{aligned} \text{dist}(C, D) &= \inf\{\|x - y\| : x \in C \text{ and } y \in D\},\\ C_0 &= \{x \in C : \exists\, y' \in D \,|\, \|x - y'\| = \text{dist}(C, D)\},\\ D_0 &= \{y \in D : \exists\, x' \in C \,|\, \|x' - y\| = \text{dist}(C, D)\}. \end{aligned}$$

We discuss that if $(C, D)$ is a pair of nonempty, convex and weakly compact pair in Banach space $Y$, then the pair $(C_0, D_0)$ is of same kind.

**Definition 2.1.** *A nonempty pair $(C, D)$ in a normed linear space $Y$ is said to be proximinal if $C = C_0$ and $D = D_0$.*

**Example 2.2.** *Let $Y$ be the Euclidean space $\mathbb{R}^2$ with the standard norm, and let*

$$\begin{aligned} C &= \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y = 0\},\\ D &= \{(x, y) \in \mathbb{R}^2 : x = 0,\ y \geq 0\}. \end{aligned}$$

*Here, $C$ and $D$ are nonempty subsets of $Y$ and $C_0$ includes all points in $C$ with $y = 0$, which is the entire nonnegative x-axis, and $D_0$ includes all points in $D$ with $x = 0$, which is the entire nonnegative y-axis. So, $C = C_0$ and $D = D_0$, hence $(C, D)$ is a proximinal pair.*

A mapping $L : C \cup D \to C \cup D$ with $L(C) \subseteq D,\ L(D) \subseteq C$ and $\|Lx - Ly\| \leq \|x - y\|$ for all $x \in C$ and $y \in D$ is known as cyclic relatively nonexpansive. If $C = D$, then $L$ is a nonexpansive self-mapping.
A point $x^* \in C \cup D$ is said to be the best proximity point of the mapping $L$ if

$$\|x^* - Lx^*\| = \text{dist}(C, D) =: \inf\{\|x - y\| : x \in C, y \in D\}.$$

In fact, best proximity point theorems have been studied to find the necessary conditions for the minimization problem

$$\min_{x \in C \cup D} \|x - Lx\|, \tag{1}$$

to have at least one solution.

**Theorem 2.3.** *[9] Consider $(C, D)$ is nonempty, compact and convex pair in a Banach space $Y$. If $L : C \cup D \to C \cup D$ is cyclic relatively nonexpansive mapping, then $L$ admits a best proximity point.*

A function $L : C \cup D \to C \cup D$ with $L(C) \subseteq C,\ L(D) \subseteq D$ and $\|Lx - Ly\| \leq \|x - y\|$ for all $x \in C$ and $y \in D$ is known as noncyclic relatively nonexpansive. Clearly, the class of noncyclic relatively nonexpansive functions contains all the nonexpansive functions. A noncyclic relatively nonexpansive functions need not be continuous. A point $(u, v) \in C \times D$ is called a best proximity pair if this is a solution of the following minimization problem:

$$\min_{x \in C} \|x - Lx\|,\ \min_{y \in D} \|y - Ly\|,\ \text{and} \min_{(x,y) \in C \times D} \|x - y\|. \tag{2}$$

Clearly, $(x, y) \in C \times D$ is a solution of the problem (2) if and only if $u = Lu$, $v = Lv$ and $\|u - v\| = \text{dist}(C, D)$.

**Theorem 2.4.** *[9] Consider $(C, D)$ is nonempty, compact and convex pair in a strictly convex Banach space $Y$. If $L : C \cup D \to C \cup D$ is noncyclic relatively nonexpansive mapping, then $L$ admits a best proximity point.*

**Definition 2.5.** *[10] Let $C$ and $D$ be nonempty subsets of a Banach space $Y$. A function $L : C \cup D \to C \cup D$ is said to be a relatively u-continuous function if it satisfies:*
*For each $\epsilon > 0$, there exists a $\delta > 0$ such that $\|Lx - Ly\| < \epsilon + dist(C, D)$, whenever $\|x - y\| < \delta + dist(C, D)$, $\forall\, x \in C, y \in D$.*
*Note that every relatively nonexpansive mapping is a relatively u-continuous mapping. The following example shows that the converse is not true.*

***Example 2.6.*** *Let us take $(\mathbb{R}^2, \|.\|_2)$ and $C = \{(0, t) : 0 \le t \le 1\}$ and $D = \{(1, t) : 0 \le t \le 1\}$. Define $L : C \cup D \to C \cup D$ by*

$$L(x, y) = \begin{cases} (1, \sqrt{y}) \text{ if } x = 0, \\ (0, \sqrt{y}) \text{ if } x = 1. \end{cases}$$

*Then $L$ is a relatively u-continuous mapping but not a relatively nonexpansive mapping.*

**Theorem 2.7.** *[12] Suppose $(C, D)$ is nonempty, bounded, closed and convex pair in a strictly convex Banach space $Y$ such that $C_0$ is nonempty. If $L : C \cup D \to C \cup D$ is noncyclic relatively u-continuous and compact mapping, then $L$ admits a best proximity point.*

**Theorem 2.8.** *[12] Let $(C, D)$ be nonempty, bounded, closed and convex pair in a strictly convex Banach space $Y$ such that $C_0$ is nonempty. If $L : C \cup D \to C \cup D$ is cyclic relatively nonexpansive and compact mapping, then $L$ admits a best proximity point.*

### 2.2. *Measure of noncompactness*

For a bounded subset $B$ of a metric space $Y$, Kuratowski [18] measure of noncompactness is defined as:

$$\alpha(B) = \inf\{\epsilon > 0 : B = \bigcup_{k=1}^{n} B_k,\ \operatorname{diam}(B_k) \le \epsilon,\ 1 \le k \le n < \infty\},$$

where $\operatorname{diam}(B_k)$ denotes the diameter of the set $B_k \subset Y$.

**Definition 2.9.** *[2] Let $\mathcal{M}_Y$ be a family of bounded subsets of $Y$. A mapping $\nu : \mathcal{M}_Y \to [0, \infty)$ is called a measure of noncompactness in $Y$ if*

- $(A_1)$ $\nu(S) = 0$ *if and only if $S$ is relatively compact;*
- $(A_2)$ $\nu(\overline{S}) = \nu(S)$*;*
- $(A_3)$ $\nu(S_1 \cup S_2) = \max\{\nu(S_1), \nu(S_2)\}$*, for all $S_1, S_2 \in \mathcal{M}_Y$.*

The Hausdorff (or ball) measure of noncompactness for $B$ is defined as follows:

$$\chi(B) = \inf\{\epsilon > 0 : B \text{ has a finite } \epsilon\text{-net in } Y\}.$$

Darbo gave the following result using the concept of measure of noncompactness.

**Theorem 2.10.** *[2, Darbo] Suppose $D$ is a nonempty, bounded, closed, and convex subset of a Banach space $Y$. Let $L : D \to D$ be a continuous function, and there is a constant $m \in [0, 1)$ such that $\chi(LD) \le m\chi(D)$. Then $L$ admits at least one fixed point in the set $D$.*

Gabeleh et al. established the following best proximity theorems:

**Theorem 2.11.** *[12] Let $Y$ be a strictly convex Banach space, and $(C, D)$ be a nonempty, bounded, closed, and convex pair in $Y$ such that $C_0$ is nonempty. Assume $L : C \cup D \to C \cup D$ is noncyclic, relatively u-continuous such that for any nonempty, convex, closed, proximinal, and L-invariant pair $(Q_1, Q_2) \subseteq (C, D)$ with $dist(Q_1, Q_2) = dist(C, D)$,*

$$\nu(L(Q_1) \cup L(Q_2)) \leq \theta(\nu(Q_1 \cup Q_2))\nu(Q_1 \cup Q_2), \tag{3}$$

*where $\nu$ is an arbitrary measure of noncompactness on $Y$, $\theta \in \Delta$, and $\Delta$ is a class of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following condition:*

*For each $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$. Then $L$ admits a best proximity point.*

**Theorem 2.12.** *[12] Let $Y$ be a strictly convex Banach space, and $(C, D)$ be a nonempty, bounded, closed, and convex pair in $Y$ such that $C_0$ is nonempty. Assume $L : C \cup D \to C \cup D$ is a cyclic relatively nonexpansive mapping such that for any nonempty, convex, closed, proximinal, and L-invariant pair $(Q_1, Q_2) \subseteq (C, D)$ with $dist(Q_1, Q_2) = dist(C, D)$,*

$$\nu(L(Q_1) \cup L(Q_1)) \leq \theta(\nu(Q_1 \cup Q_2))\nu(Q_1 \cup Q_2), \tag{4}$$

*where $\nu$ is an arbitrary measure of noncompactness on $Y$, $\theta \in \Delta$, and $\Delta$ is defined in the above theorem. Then $L$ has the best proximity point.*

**Definition 2.13.** *[16] Let $\mathcal{F}$ be the family of the functions $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:*

($\mathcal{F}_1$) $\max\{\tau, \mu\} \leq F(\tau, \mu)$ *for* $\tau, \mu \geq 0$;
($\mathcal{F}_2$) *$F$ is continuous and nondecreasing;*
($\mathcal{F}_3$) $F(\tau_1 + \mu_1, \tau_2 + \mu_2) \leq F(\tau_1, \tau_2) + F(\mu_1, \mu_2)$.

*For example, $F(\tau, \mu) = \tau + \mu$.*

**Definition 2.14.** *[19] Let $\Phi$ be the class of functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

($\Phi_1$) *$\phi$ is nondecreasing;*
($\Phi_2$) $\lim_{k\to\infty} \phi(t_k) = 0 \Longleftrightarrow \lim_{k\to\infty} t_k = 0$ *for all* $\{t_k\} \subset (0, \infty)$;
($\Phi_3$) *$\phi$ is continuous.*

**Definition 2.15.** *[3] Let $\Xi$ be the set of mappings $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying:*

($\Xi_1$) *$\xi$ is monotone increasing;*
($\Xi_2$) $\lim_{k\to\infty} \xi^k(t) = 0, \quad \forall\, t > 0.$

*Such a class of mappings is known as a comparison function.*
*Examples of comparison functions:*

(*i*) $\xi_1(t) = \lambda t, \quad \lambda \in (0, 1), \quad \forall\, t > 0;$
(*ii*) $\xi_2(t) = \begin{cases} \frac{t}{2}, & 0 < t < 1 \\ \frac{t}{3}, & t \geq 1; \end{cases}$
(*iii*) $\xi_3(t) = \frac{t}{t+1}, \quad$ *for all* $t > 0$.

## 3. Main Results

**Theorem 3.1.** *Let $Y$ be a strictly convex Banach space, and $(C, D)$ be a nonempty, bounded, closed, and convex pair in $Y$ such that $C_0$ is nonempty. Assume $L : C \cup D \to C \cup D$ is noncyclic relatively u-continuous such that for any nonempty, bounded, convex, closed, proximinal, and L-invariant pair $(P, Q) \subseteq (C, D)$ with $dist(P, Q) = dist(C, D)$,*

$$\phi\left[F\left(\nu((LP) \cup (LQ)), \beta(\nu((LP) \cup (LQ)))\right)\right] \leq \xi[\phi(F(\nu(P \cup Q), \beta(\nu(P \cup Q)))]\beta(\nu(P \cup Q)), \tag{5}$$

*where $\nu$ is an arbitrary measure of noncompactness, $\phi \in \Phi$, $\xi \in \Xi$, $F \in \mathcal{F}$, and $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Then $L$ admits the best proximity point.*

*Proof.* Let $(C_0, D_0)$ be nonempty, closed, convex, and proximinal, and let $x \in C_0$. Then there exists $y \in D_0$ such that $\|x-y\| = \mathrm{dist}(C,D)$. As $L$ is relatively u-continuous, $\|L(x)-L(y)\| = \mathrm{dist}(C,D)$, and so $Lx \in C_0$. Thus $L(C_0) \subseteq C_0$. Similarly, $L(D_0) \subseteq D_0$. Hence $(C_0, D_0)$ is $L$-invariant.
Let us start by assuming $P^0 = C_0$ and $Q^0 = D_0$ and, for all $k \in \mathbb{N}$, define

$$P^k = \overline{\mathrm{conv}}(L(P^{k-1})), \;\; Q^k = \overline{\mathrm{conv}}(L(Q^{k-1})),$$

we have

$$P^1 = \overline{\mathrm{conv}}(L(P^0)) = \overline{\mathrm{conv}}(L(C_0)) \subseteq C_0 = P^0.$$

Continuing in this pattern and by induction, we obtain $P^{k-1} \supseteq P^k$ for all $k \in \mathbb{N}$. Similarly, $Q^{k-1} \supseteq Q^k$ for all $k \in \mathbb{N}$. If there exists an integer $l \in \mathbb{N}$ such that $\max\{\nu(P^l), \nu(Q^l)\} = 0$, then $(P^l, Q^l)$ is a compact pair and $L(P^l) \subseteq \overline{\mathrm{conv}}(L(P^l)) = P^{l+1} \subseteq P^l$. Similarly, we can see that $L(Q^l) \subseteq Q^l$, and so $L$ is noncyclic relatively u-continuous on $P^l \cup Q^l$, where $(P^l, Q^l)$ is a compact and convex pair in a strictly convex Banach space $Y$. Hence, Theorem 2.7 yields that $L$ has a best proximity pair.
So, we suppose that $\max\{\nu(P^k), \nu(Q^k)\} > 0$ for all $k \in \mathbb{N}$. If there exist $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$ such that $\nu(P^{m_1}) = \nu(Q^{m_2}) = 0$, since the sequence $\{P^k\}_{k \in \mathbb{N}\cup\{0\}}$ is a decreasing sequence, we have $P^{m_2} \subseteq P^{m_1}$, and so $\nu(P^{m_2}) \leq \nu(P^{m_1})$, which leads to $\nu(P^{m_2}) = 0$. Hence, $\max\{\nu(P^{m_2}), \nu(Q^{m_2})\} = 0$, which is a contradiction. Therefore, $\min\{\nu(P^k), \nu(Q^k)\} > 0$ for all $k \in \mathbb{N} \cup \{0\}$.

As $L$ is noncyclic relatively u-continuous, hence for the pair $(x, y) \in P^0 \times Q^0$ with $\|x-y\| = \mathrm{dist}(C,D)$, we have $\|L^k(x) - L^k(y)\| = \mathrm{dist}(C,D)$ for all $k \in \mathbb{N}$. By the definition of the pair $(P^k, Q^k)$, we have $(L^k(x), L^k(y)) \in P^k \times Q^k$, which implies that

$$\mathrm{dist}(P^k, Q^k) = \mathrm{dist}(C,D), \quad \forall\, k \in \mathbb{N}.$$

Now assume that $u \in P^1 = \overline{\mathrm{conv}}(L(P^0))$, then $u = \sum_i^n c_i L(u_i)$, where $u_i \in P^0$ for all $1 \leq i \leq n$ such that $c_i \geq 0$ and $\sum_i^n c_i = 1$. Since $(P^0, Q^0)$ is proximinal, for all $1 \leq i \leq n$, there exist $v_i \in Q^0$ such that $\|u_i - v_i\| = \mathrm{dist}(P^0, Q^0) = \mathrm{dist}(C,D)$. Put $v = \sum_i^n c_i L(v_i)$. Then $v \in Q^1$ and

$$\begin{aligned}\|u-v\| &= \left\| \sum_i^n c_i L(u_i) - \sum_i^n c_i L(v_i) \right\| \leq \sum_i^n \|L(u_i) - L(v_i)\| \\ &= \mathrm{dist}(C,D).\end{aligned}$$

Hence, the pair $(P^1, Q^1)$ is proximinal. By a similar argument, we obtained that the pair $(P^k, Q^k)$ is proximinal for all $k \in \mathbb{N} \cup \{0\}$. Thus $(P^k, Q^k)$ is nonempty, bounded, convex, closed, proximinal pair, which is $L$-invariant.

By equation (5),

$$\begin{aligned}&\phi\left[F\left(\nu\left(P^{k+1} \cup Q^{k+1}\right), \beta\left(\nu\left(P^{k+1} \cup Q^{k+1}\right)\right)\right)\right] \\ &\leq \xi^{k+1}\left[\phi\left(F\left(\nu\left(P^0 \cup Q^0\right), \beta\left(\nu\left(P^0 \cup Q^0\right)\right)\right)\right)\right] \times \beta\left(\nu\left(P^0 \cup Q^0\right)\right) \cdots \beta\left(\nu\left(P^{k-1} \cup Q^{k-1}\right)\right) \beta\left(\nu\left(P^k \cup Q^k\right)\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\phi\left[F\left(\nu\left(P^{k+1} \cup Q^{k+1}\right), \beta\left(\nu\left(P^{k+1} \cup Q^{k+1}\right)\right)\right)\right] \\ &\leq \xi^{k+1}\left[\phi\left(F(\nu(P^0 \cup Q^0), \beta(\nu(P^0 \cup Q^0))\right)\right] \beta\left(\nu\left(P^0 \cup Q^0\right)\right) \cdots \beta\left(\nu\left(P^{k-1} \cup Q^{k-1}\right)\right) \beta\left(\nu\left(P^k \cup Q^k\right)\right).\end{aligned} \tag{6}$$

Letting $k \to \infty$ in (6) and applying $(\Xi_2)$, we have

$$\phi\left[F\left(\nu\left(P^{k+1} \cup Q^{k+1}\right), \beta\left(\nu\left(P^{k+1} \cup Q^{k+1}\right)\right)\right)\right] = 0$$

as $k \to \infty$, which implies from $(\Phi_2)$ that

$$\lim_{k\to\infty}\left[F\left(\nu\left(P^{k+1}\cup Q^{k+1}\right),\beta\left(\nu\left(P^{k+1}\cup Q^{k+1}\right)\right)\right)\right]=0.$$

It gives

$$\lim_{k\to\infty}\nu(P^{k+1}\cup Q^{k+1})=0=\lim_{k\to\infty}\beta(\nu(P^{k+1}\cup Q^{k+1})).$$

It follows from the condition of $(A_6)$ in Definition 2.9 that the pair $(P_\infty, Q_\infty)$ is nonempty, convex, and closed, which is $L$-invariant, where $P_\infty = \cap_{k=0}^{\infty} P^k$ and $Q_\infty = \cap_{k=0}^{\infty} Q^k$. Furthermore, $\text{dist}(P_\infty, Q_\infty) = \text{dist}(C, D)$, and it is easy to see that $(P_\infty, Q_\infty)$ is proximinal. On the other hand, $\max\{\nu(P_\infty), \nu(Q_\infty)\} = 0$, which ensures that the pair $(P_\infty, Q_\infty)$ is compact. Now from Theorem 2.7, $L$ has a best proximity point. □

**Remark 3.2.** *We get Theorem 2.11 if we take $F(x,y) = x$, $\phi(t) = t$, $\xi(t) = \lambda t$, and $\beta = \frac{\theta(t)}{\lambda}$, $\lambda \in (0,1)$ in Theorem 3.1.*

**Example 3.3.** *Consider $Y = \mathbb{R}^2$ with the Euclidean norm. Let $C$ and $D$ be defined as follows:*

$$C = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y = 0\},$$
$$D = \{(x,y) \in \mathbb{R}^2 : x = 0, y \geq 0\}.$$

*These sets are nonempty, bounded, closed, and convex. $C_0$ includes all points in $C$ with $y = 0$, which is the entire nonnegative x-axis, and $D_0$ includes all points in $D$ with $x = 0$, which is the entire nonnegative y-axis. So, $(C, D)$ is a proximinal pair. Let $L : C \cup D \to C \cup D$ be defined as*

$$L(x,y)=\begin{cases}(1,\sqrt{y}) & \text{if } (x,y)\in C,\\ (\sqrt{x},1) & \text{if } (x,y)\in D.\end{cases}$$

*This mapping $L$ is noncyclic relatively u-continuous. For functions $F(x,y) = x$, $\phi(t) = t$, $\xi(t) = \frac{1}{2}t$, and $\beta = \frac{t}{2}$, $t \in (0,\infty)$, it is easy to see that $L$ satisfies all the conditions of Theorem 3.1, ensuring that $L$ admits the best proximity point.*

*We can see that $L$ does not satisfy the equation (3), hence Theorem 2.11 does not apply to this example.*

**Theorem 3.4.** *Let $Y$ be a strictly convex Banach space and $(C, D)$ be a nonempty, bounded, closed, and convex pair in $Y$ such that $C_0$ is nonempty. Assume $L : C \cup D \to C \cup D$ is a cyclic relatively nonexpansive function such that for any nonempty, convex, closed, proximinal, and $L$-invariant pair $(P, Q) \subseteq (C, D)$ with $dist(P, Q) = dist(C, D)$,*

$$\phi\left[F\left(\nu(L(P)\cup(LQ)),\beta(\nu(L(P)\cup(LQ)))\right)\right]\leq\xi\left[\phi\left(F\left(\nu\left(P\cup Q\right),\beta\left(\nu\left(P\cup Q\right)\right)\right)\right)\right]\beta\left(\nu\left(P\cup Q\right)\right),\tag{7}$$

*where $\nu$ is an arbitrary measure of noncompactness, $\phi \in \Phi$, $\xi \in \Xi$, $F \in \mathcal{F}$, and $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous mapping. Then $L$ admits a best proximity point.*

*Proof.* In a similar way to Theorem 3.1, we obtain that $(C_0, D_0)$ is nonempty, closed, convex, proximinal, and $L$-invariant, that is $L(C_0) \subseteq D_0$ and $L(D_0) \subseteq C_0$. Set $P^0 = C_0$ and $Q^0 = D_0$ and, for all $k \in \mathbb{N}$, define

$$P^k = \overline{conv}(L(P^{k-1})),\ \ Q^k = \overline{conv}(L(Q^{k-1})),$$

then we have

$$P^1 = \overline{conv}(L(P^0)) = \overline{conv}(L(C_0)) \subseteq D_0 = Q^0,$$

and so, $L(P^1) \subseteq L(Q^0)$, which implies that

$$P^2 = \overline{conv}(L(P^1)) \subseteq L(Q^0) = Q^1.$$

Continuing this pattern, we obtain $P^{k+1} \subseteq Q^k$. Also, we have

$$Q^1 = \overline{conv}(L(Q^0)) = \overline{conv}(L(D_0)) \subseteq C_0 = P^0,$$

and hence, $L(Q^1) \subseteq L(P^0)$. Thus,

$$Q^2 = \overline{conv}(L(Q^1)) \subseteq L(P^0) = P^1.$$

By induction, we obtained $Q^k \subseteq P^{k-1}$ for all $k \in \mathbb{N}$. Hence,

$$P^{k+2} \subseteq Q^{k+1} \subseteq P^k \subseteq Q^{k-1} \ \forall \ k \in \mathbb{N}.$$

Therefore, $\{(P^{2k}, Q^{2k})\}_{k\geq 0}$ is a decreasing sequence of nonempty, bounded, closed, and convex pairs in $C_0 \times D_0$. Furthermore, for all $k \in \mathbb{N} \cup \{0\}$, we have

$$L(Q^{2k}) \subseteq L(P^{2k-1}) \subseteq \overline{conv}(P^{2k-1}) = P^{2k},$$
$$L(P^{2k}) \subseteq L(Q^{2k-1}) \subseteq \overline{conv}(Q^{2k-1}) = Q^{2k}.$$

So, we deduce that $(P^{2k}, Q^{2k})$ is $L$-invariant. As $L$ is relatively nonexpansive, for $(w, y) \in P^0 \times Q^0$ with $\|w-y\| = \text{dist}(C, D)$, we have $\text{dist}(P^{2k}, Q^{2k}) \leq \|L^{2k}w - L^{2k}y\| \leq \|w-y\| = \text{dist}(C, D)$. Also, $(L^{2k}w, L^{2k}y) \in P^{2k} \times Q^{2k}$. Similar to Theorem 3.1, we can see that $(P^{2k}, Q^{2k})$ is also proximinal for all $k \in \mathbb{N}$. If $\max\{\nu(P^{2k}), \nu(Q^{2k})\} = 0$ for some $k \in \mathbb{N}$, then the result follows from Theorem 2.8. So we assume that $\max\{\nu(P^{2k}), \nu(Q^{2k})\} > 0$ for all $k \in \mathbb{N}$. Again using a similar discussion of Theorem 3.1, we conclude that $\min\{\nu(P^{2k}), \nu(Q^{2k})\} > 0$.
By using (7), we have

$$\begin{aligned}
&\phi\left[F(\nu(P^{2k+2} \cup Q^{2k+2}), \beta(\nu(P^{2k+2} \cup Q^{2k+2}))\right] \\
&= \phi\left[F(\max\{\nu(P^{2k+2}), \nu(Q^{2k+2})\}, \beta(\max\{\nu(P^{2k+2}), \nu(Q^{2k+2})\})\right] \\
&\leq \phi\left[F(\max\{\nu(Q^{2k+1}), \nu(P^{2k+1})\}, \beta(\max\{\nu(Q^{2k+1}), \nu(P^{2k+1})\})\right] \\
&= \phi\left[F(\max\{\nu(\overline{conv}(LP^{2k})), \nu(\overline{conv}(LQ^{2k}))\}, \ \beta(\max\{\nu(\overline{conv}(LP^{2k}), \nu(\overline{conv}(LQ^{2k}))\})\right] \\
&= \phi\left[F(\max\{(\nu(LP^{2k}), \nu(LQ^{2k})\}, \beta(\max\{(\nu(LP^{2k}), \nu(LQ^{2k})\})\right] \\
&= \phi\left[F(\nu((LP^{2k}) \cup (LQ^{2k})), \beta(\nu((LP^{2k}) \cup (LQ^{2k})))\right] \\
&\leq \xi\left[\phi\left(F\left(\nu\left(P^{2k} \cup Q^{2k}\right), \beta\left(\nu\left(P^{2k} \cup Q^{2k}\right)\right)\right)\right) \beta\left(\nu\left(P^{2k} \cup Q^{2k}\right)\right)\right] \\
&\leq \xi^2\left[\phi\left(F\left(\nu\left(P^{2k-2} \cup Q^{2k-2}\right), \beta\left(\nu\left(P^{2k-2} \cup Q^{2k-2}\right)\right)\right)\right) \beta(\nu(P^{2k-1} \cup Q^{2k-1}))\beta(\nu(P^{2k} \cup Q^{2k}))\right] \\
&\quad \vdots \\
&\leq \xi^{k+1}\left[\phi\left(F\left(\nu\left(P^0 \cup Q^0\right), \beta\left(\nu\left(P^0 \cup Q^0\right)\right)\right)\right) \beta\left(\nu\left(P^0 \cup Q^0\right)\right) \cdots \beta\left(\nu\left(P^{2k-1} \cup Q^{2k-1}\right)\right) \beta\left(\nu\left(P^{2k} \cup Q^{2k}\right)\right)\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\phi\left[F(\nu(P^{2k+2} \cup Q^{2k+2}), \beta(\nu(P^{2k+2} \cup Q^{2k+2}))\right] \\
&\leq \xi^{k+1}(\phi[F(\nu(P^0 \cup Q^0), \beta(\nu(P^0 \cup Q^0)))]\beta(\nu(P^0 \cup Q^0))...\beta(\nu(P^{2k-1} \cup Q^{2k-1}))\beta(\nu(P^{2k} \cup Q^{2k})). \qquad (8)
\end{aligned}$$

Letting $k \to \infty$ in (8) and applying $(\Xi_2)$, we have
$\phi\left[F\left(\nu\left(P^{2k+2} \cup Q^{2k+2}\right), \beta\left(\nu\left(P^{2k+2} \cup Q^{2k+2}\right)\right)\right)\right] = 0$ as $k \to \infty$,
which implies from $(\Phi_2)$ that
$\lim\limits_{k\to\infty}\left[F\left(\nu\left(P^{2k+2} \cup Q^{2k+2}\right), \beta\left(\nu\left(P^{2k+2} \cup Q^{2k+2}\right)\right)\right)\right] = 0.$
It gives $\lim\limits_{k\to\infty} \nu\left(P^{2k+2} \cup Q^{2k+2}\right) = 0 = \lim\limits_{k\to\infty} \beta\left(\nu\left(P^{2k+2} \cup Q^{2k+2}\right)\right)$. So, $\max\left\{\lim\limits_{k\to\infty} \nu\left(P^{2k+2}\right), \lim\limits_{k\to\infty} \nu\left(Q^{2k+2}\right)\right\} = 0.$

It follows from the condition of $(A_6)$ in Definition 2.9 that the pair $(P_\infty, Q_\infty)$ is nonempty, closed, and convex, which is $L$-invariant, where $P_\infty = \cap_{k=0}^{\infty} P^{2k}$ and $Q_\infty = \cap_{k=0}^{\infty} Q^{2k}$. Furthermore, $\mathrm{dist}(P_\infty, Q_\infty) = \mathrm{dist}(C, D)$, and it is easy to see that $(P_\infty, Q_\infty)$ is proximinal. On the other hand, $\max\{\nu(P_\infty), \nu(Q_\infty)\} = 0$, which ensures that the pair $(P_\infty, Q_\infty)$ is compact. Now from Theorem 2.8, $L$ has a best proximity point. □

**Remark 3.5.** *We get Theorem 2.12 if we take* $F(w, y) = w$, $\phi(t) = t$, $\xi(t) = \lambda t$, *and* $\beta = \frac{\theta(t)}{\lambda}$, $\lambda \in (0, 1)$ *in Theorem 3.4.*

**Remark 3.6.** *We get Theorem 2.10 if we take* $F(w, y) = w$, $\phi(t) = t$, $\xi(t) = \lambda t$, $\lambda \in (0, 1)$ *and* $\beta = 1$, *and* $P = Q$ *in Theorem 3.4.*

**Example 3.7.** *Consider* $Y = \mathbb{R}^2$ *with the Euclidean norm. Let C and D be defined as follows:*

$$C = \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}$$
$$D = \{(0, y) \in \mathbb{R}^2 : y \geq 0\}.$$

*These sets are nonempty, bounded, closed, and convex.* $C_0$ *includes all points in C with* $y = 0$, *which is the entire nonnegative x-axis, and* $D_0$ *includes all points in D with* $x = 0$, *which is the entire nonnegative y-axis. So,* $(C, D)$ *is a proximinal pair. Let* $L : C \cup D \to C \cup D$ *be defined as*

$$L(x, y) = \begin{cases} (0, \sqrt{x}) & \text{if } x > 0, \\ (\sqrt{y}, 0) & \text{if } y > 0. \end{cases}$$

*This mapping L is cyclic and relatively nonexpansive. For the functions* $\phi(t) = t$, $\xi(t) = \lambda t$ *for some* $\lambda \in (0, 1)$, $F(\tau, \mu) = \tau + \mu$, *and* $\beta(t) = t$, *it is easy to see that L satisfies all the conditions of Theorem 3.4 , ensuring that L admits the best proximity point.*

*We can see that L does not satisfy the equation (4), and conditions of Theorem 2.10, hence Theorem 2.10 and Theorem 2.12 do not apply to this example.*

## 4. Application to optimal solution of Caputo's fractional differential equations

In this section, we apply our main results to study the existence of optimum solutions for a system of Caputo fractional differential equations with initial conditions featuring integral order derivatives.
Let $J = [0, \tau], \tau > 0$ and $(Y, \|\cdot\|)$ be a Banach space. Let $V_1 = B(u_0, \kappa)$ and $V_2 = B(v_0, \kappa)$ be closed balls in $Y$, where $\kappa$ is real number and $u_0, v_0 \in Y$. Assume that $f : J \times V_1 \to Y$, $g : J \times V_2 \to Y$ are continuous functions. Let us take the following system of Caputo fractional differential equations of arbitrary order featuring initial conditions at integer derivatives [8]:

$$\begin{cases} {}^c\mathrm{D}_0^\gamma u(t) = f(t, u(t)), \\ u(0) = u_0^{(0)},\ u'(0) = u_0^{(1)}, ..., u^{m-1}(0) = u_0^{(m-1)}; \end{cases} \tag{9}$$

$$\begin{cases} {}^c\mathrm{D}_0^\gamma v(t) = g(t, v(t)), \\ v(0) = v_0^{0},\ v'(0) = v_0^{(1)}, ..., v^{m-1}(0) = v_0^{(m-1)}, \end{cases} \tag{10}$$

where $m = \lceil \gamma \rceil$ denotes the integer such that $\gamma - 1 < m \leq \gamma$; ${}^c\mathrm{D}_0^\gamma$ is the Caputo fractional differential operator of order $\gamma > 0$ with ${}^c\mathrm{D}_0^\gamma h := \mathrm{D}_0^\gamma[h - T_{m-1}[h; 0]]$; $\mathrm{D}_0^\gamma$ is Riemann-Liouville fractional differential operator of order $\gamma > 0$; $T_{m-1}[h; 0]$ denotes the Taylor polynomial of degree $m - 1$ for the function $h$, centred at 0.

**Lemma 4.1.** *[8, Lemma 6.2] The initial value problem (9) is equivalent to the following integral equation:*

$$u(t) = \sum_{i=0}^{m-1} \frac{t^i u_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int_0^t (t - r)^{\gamma - 1} f(r, u(r)) dr,\ t \in I.$$

Let $I \subseteq J$ and $S = C(I, Y)$ be Banach space of continuous functions from $I$ into $Y$ with supremum norm. Let

$$S_1 := C(I, V_1) = \{u \in C(I, Y) : u(0) = u_0^{(0)},\ u'(0) = u_0^{(1)}, ..., u^{m-1}(0) = u_0^{(m-1)}\},$$
$$S_2 := C(I, V_2) = \{v \in C(I, Y) : v(0) = v_{(0)}^{0},\ v'(0) = v_0^{(1)}, ..., v^{m-1}(0) = v_0^{(m-1)}\}.$$

So, $(S_1, S_2)$ is nonempty, bounded, closed and convex pair in $S \times S$. Now for every $p \in S_1$ and $q \in S_2$, we have $\|p - q\| = \sup \|p(r) - q(r)\| \geq \|u_0^{(0)} - v_0^{(0)}\|$, so $\mathrm{dist}(S_1, S_2) = \|u_0^{(0)} - v_0^{(0)}\|$.
Now, let us define the operator $L : S_1 \cup S_2 \to S$ as follows:

$$Lu(t) = \begin{cases} \sum\limits_{i=0}^{m-1} \frac{t^i v_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int\limits_0^t (t-r)^{\gamma-1} g(r, u(r))dr, \ u \in S_1, \\ \sum\limits_{i=0}^{m-1} \frac{t^i u_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int\limits_0^t (t-r)^{\gamma-1} f(r, u(r))dr, \ u \in S_2. \end{cases} \tag{11}$$

**Lemma 4.2.** *[22] The operator $L : S_1 \cup S_2 \to S$ defined by (11) is cyclic if*

$$\max\left\{\sum_{i=0}^{m-1} \frac{t^i v_0^{(i)}}{i!} + \frac{M_1 \tau^\gamma}{\Gamma(\gamma+1)}, \sum_{i=0}^{m-1} \frac{t^i u_0^{(i)}}{i!} + \frac{M_2 \tau^\gamma}{\Gamma(\gamma+1)}\right\} \leq \eta,$$

*where $M_1 = \sup\{|f(r, u(r))| : r \in I, u \in S_1\}$ and $M_2 = \sup\{|g(r, u(r))| : r \in I, u \in S_2\}$.*

**Assumptions:**

($N_1$) For any bounded pair $(P_1, P_2) \subseteq (V_1, V_2)$,

$$\phi\left[F\left(\nu(f(I \times P_2)) \cup \nu(g(I \times P_1)), \beta(\nu(f(I \times P_2) \cup (\nu(g(I \times P_1)))))\right)\right]$$
$$\leq \xi(\phi[F(\nu(P_1 \cup P_2), \beta(\nu(P_1 \cup P_2)))]\beta(\nu(P_1 \cup P_2))).$$

($N_2$) There exists $\rho \geq 0$ such that

$$\left|\sum_{i=0}^{m-1} \frac{t^i \left(u_0^{(i)} - v_0^{(i)}\right)}{i!}\right| \leq \rho,\ \forall\ t \in I,$$

and $\|f(t, u(t)) - g(t, v(t))\| \leq \frac{(\Gamma(\gamma+1))}{\tau^\gamma}(\|u(t) - v(t)\| - \rho)$, for all $(u, v) \in S_1 \times S_2$.

For fractional differential, we recall the Mean-Value Theorem.

**Theorem 4.3.** *[5] Let $f$ be integrable on $I$ and let $M$ and $m$ be the supremum and infimum of $f$ respectively on $I$. Then there exists a point $\xi$ in $I$ such that*

$$\frac{1}{\Gamma(\gamma)} \int\limits_0^t (t-r)^{\gamma-1} f(r, u(r))dr = \frac{t^{\gamma-1}}{\Gamma(\gamma)} f(\zeta, u(\zeta)). \tag{12}$$

**Theorem 4.4.** *Under the hypothesis of Lemma 4.2, hypotheses ($N_1$) and ($N_2$), the system in (9)-(10) has an optimum solution whenever $\frac{t^{\gamma-1}}{\Gamma(\gamma)} \leq 1$.*

*Proof.* By using Lemma 4.2, $L$ is a cyclic operator. It is easy to see that $L(S_1)$ is bounded subset of $S_2$. We prove that $L(S_1)$ is also an equicontinuous subset of $S_2$.

For $\gamma \in [0,1]$ and for $t_1, t_2 \in I$ with $t_1 < t_2$ and $u \in S_1$, we have

$$\begin{aligned}
\|Lu(t_1) - Lu(t_2)\| &= \left\| \frac{1}{\Gamma(\gamma)} \int_0^{t_1} (t_1 - r)^{\gamma-1} g(r, u(r))\, dr - \frac{1}{\Gamma(\gamma)} \int_0^{t_2} (t_2 - r)^{\gamma-1} g(r, u(r))\, dr \right\| \\
&= \left\| \frac{1}{\Gamma(\gamma)} \int_0^{t_1} ((t_1 - r)^{\gamma-1} - (t_2 - r)^{\gamma-1}) g(r, u(r))\, dr - \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} (t_2 - r)^{\gamma-1} g(r, u(r))\, dr \right\| \\
&\leq [2(t_2 - t_1)^{\gamma} + t_1^{\gamma} - t_2^{\gamma}] \frac{M_1}{\Gamma(\gamma+1)} \\
&\leq (t_2 - t_1)^{\gamma} \frac{M_1}{\Gamma(\gamma+1)}.
\end{aligned}$$

If $\gamma > 1$, then the polynomial $\sum_{i=0}^{m-1} \frac{t^i v_0^{(i)}}{i!}$ has uniformly bounded and continuous derivatives on $I$, and there exists $K > 0$ such that

$$\left| \sum_{i=0}^{m-1} \frac{t_1^i v_0^{(i)}}{i!} - \sum_{i=0}^{m-1} \frac{t_2^i v_0^{(i)}}{i!} \right| \leq K |t_1 - t_2|.$$

Thus, for $\gamma > 1$ and for any $t_1, t_2 \in I$ with $t_1 \leq t_2$ and $(t_2 - t_1) \leq 1$, we have

$$\begin{aligned}
\|Lu(t_1) - Lu(t_2)\| &= \left\| \sum_{i=0}^{m-1} \frac{t_1^i v_0^{(i)}}{i!} - \sum_{i=0}^{m-1} \frac{t_2^i v_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int_0^{t_1} (t_1 - r)^{\gamma-1} g(r, u(r)) dr - \frac{1}{\Gamma(\gamma)} \int_0^{t_2} (t_2 - r)^{\gamma-1} g(r, u(r)) dr \right\| \\
&\leq \left\| \sum_{i=0}^{m-1} \frac{t_1^i v_0^{(i)}}{i!} - \sum_{i=0}^{m-1} \frac{t_2^i v_0^{(i)}}{i!} \right\| + \left\| \frac{1}{\Gamma(\gamma)} \int_0^{t_1} (t_1 - r)^{\gamma-1} g(r, u(r)) dr - \frac{1}{\Gamma(\gamma)} \int_0^{t_2} (t_2 - r)^{\gamma-1} g(r, u(r)) dr \right\| \\
&= \sup \left| \sum_{i=0}^{m-1} \frac{t_1^i v_0^{(i)}}{i!} - \sum_{i=0}^{m-1} \frac{t_2^i v_0^{(i)}}{i!} \right| + \\
&\quad \sup \left| \frac{1}{\Gamma(\gamma)} \int_0^{t_1} (t_1 - r)^{\gamma-1} g(r, u(r)) dr - \frac{1}{\Gamma(\gamma)} \int_0^{t_2} (t_2 - r)^{\gamma-1} g(r, u(r)) dr \right| \\
&\leq \mathrm{K} \sup |t_1 - t_2| + \sup [2(t_2 - t_1)^{\gamma} + t_1^{\gamma} - t_2^{\gamma}] \frac{M_1}{\Gamma(\gamma+1)} \\
&\leq \sup |t_1 - t_2| \left( \mathrm{K} + \frac{2M_1}{\Gamma(\gamma+1)} \right) \\
&= \left( \mathrm{K} + \frac{2M_1}{\Gamma(\gamma+1)} \right) \|t_2 - t_1\|.
\end{aligned}$$

Now, for given $\epsilon > 0$, we may choose $\delta = \frac{\epsilon \Gamma(\gamma+1)}{\mathrm{K}\Gamma(\gamma+1) + 2M_1}$, then we have $\|Lu(t_1) - Lu(t_2)\| < \epsilon$ whenever $\|t_2 - t_1\| < \delta$. Thus, $L(S_1)$ is equicontinuous. Similarly, we can prove that $L(S_2)$ is bounded and equicontinuous subset of $S_1$. Thus, by using Arzela-Ascoli theorem, $(S_1, S_2)$ is relatively compact. Now we prove that $L$ is relatively nonexpansive. For any $(u,v) \in S_1 \times S_2$, we have

$$\begin{aligned}
\|Lu(t) - Lv(t)\| &= \left\| \left( \sum_{i=0}^{m-1} \frac{t^i v_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} g(r,u(r))dr \right) - \left( \sum_{i=0}^{m-1} \frac{t^i u_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} f(r,v(r))dr \right) \right\| \\
&\leq \left\| \sum_{i=0}^{m-1} \frac{t^i v_0^{(i)}}{i!} - \sum_{i=0}^{m-1} \frac{t^i u_0^{(i)}}{i!} \right\| + \left\| \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} g(r,u(r))dr - \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} f(r,v(r))dr \right\| \\
&\leq \rho + \frac{1}{\Gamma(\gamma)} \frac{(\Gamma(\gamma+1))}{\tau^\gamma} \int_0^t (t-r)^{\gamma-1} (\|u(r)-v(r)\| - \rho) ds \text{ (by assumption (N}_2)) \\
&\leq \rho + (\|u(t)-v(t)\| - \rho) = \|u(t)-v(t)\|.
\end{aligned}$$

Hence, $\|Lu(t) - Lv(t)\| \leq \|u(t) - v(t)\|$. Therefore, $L$ is relatively nonexpansive.

Now, we shall show $L$ satisfies condition (5). Suppose that $(R_1, R_2) \subseteq (S_1, S_2)$ is nonempty, bounded, convex, closed, proximinal pair which is $L$-invariant and such that $\text{dist}(R_1, R_2) = \text{dist}(S_1, S_2) = \|x_0 - y_0\|$. Now

$$\phi\left[F\left(\nu((LR_1) \cup (LR_2)), \beta(\nu(LR_1) \cup (LR_2))\right)\right] = \phi\left[F\left(\max\{\nu(LR_1), \nu(LR_2)\}\right), \beta(\max\{\nu(LR_1), \nu(LR_2)\})\right].$$

Using theorem 2.11 of [25] and assumption $(N_1)$, we have

$$\begin{aligned}
\max\{\nu(LR_1), \nu(LR_2)\} &= \max\left\{ \sup_{t\in I}\{\nu(Lu(t) : u \in R_1)\}, \sup_{t\in I}\{\nu(Lv(t) : v \in R_2)\} \right\} \\
&= \max\left\{ \sup_{t\in I}\left\{ \nu\left( \sum_{i=0}^{m-1} \frac{t^i v_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} g(r,u(r))ds : u \in R_1 \right) \right\}, \right. \\
&\quad \left. \sup_{t\in I}\left\{ \nu\left( \sum_{i=0}^{m-1} \frac{t^i u_0^{(i)}}{i!} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-r)^{\gamma-1} f(r,v(r))dr : v \in R_2 \right) \right\} \right\}.
\end{aligned}$$

From Theorem 4.3, it follows that

$$\begin{aligned}
&\max\{\nu(LR_1), \nu(LR_2)\} \\
&\quad \leq \max\left\{ \sup_{t\in I}\left\{ \nu\left( \sum_{i=0}^{m-1} \frac{t^i v_0^{(i)}}{i!} + \frac{s^{\gamma-1}}{\Gamma(\gamma)} g(\zeta, u(\zeta)) : u \in R_1,\ \zeta \in [0,t] \right) \right\}, \right. \\
&\quad \left. \sup_{t\in I}\left\{ \nu\left( \sum_{i=0}^{m-1} \frac{t^i u_0^{(i)}}{i!} + \frac{s^{\gamma-1}}{\Gamma(\gamma)} f(\zeta, v(\zeta)) : v \in R_2,\ \zeta \in [0,s] \right) \right\} \right\} \\
&\quad = \max\left\{ \frac{s^{\gamma-1}}{\Gamma(\gamma)} \nu\left(g(I \times R_1)\right), \frac{s^{\gamma-1}}{\Gamma(\gamma)} \nu\left(f(I \times R_2)\right) \right\} \\
&\quad = \frac{s^{\gamma-1}}{\Gamma(\gamma)} \left\{ \nu\left(g(I \times R_1) \cup f(I \times R_2)\right) \right\} \\
&\quad \leq \left\{ \nu\left(g(I \times R_1) \cup f(I \times R_2)\right) \right\}.
\end{aligned}$$

Hence,

$$\phi\left[F\left(\nu((LR_1)\cup(LR_2)),\beta(\nu(LR_1)\cup(LR_2))\right)\right]$$
$$\leq \phi\left[F\left(\{\nu\left(g(I\times R_1)\cup f(I\times R_2)\right)\}\right),\beta(\{\nu\left(g(I\times R_1)\cup f(I\times R_2)\right)\})\right]$$
$$\leq \xi\left[\phi\left(F\left(\nu\left(R_1\cup R_2\right),\beta\left(\nu\left(R_1\cup R_2\right)\right)\right)\right)\right]\beta\left(\nu\left(R_1\cup R_2\right)\right) \text{ (by assumption } N_1).$$

Therefore, $L$ satisfies all the conditions of Theorem 3.4. Hence $L$ admits a best proximity point $y \in S_1 \cup S_2$ which is an optimal solution for the system (9) and (10).

□

## 5. Conclusion

In this work, we introduced a new condensing operator and extended the results given by Gabeleh and Markin that every cyclic (noncyclic) relatively nonexpansive mapping, which is condensing, has the best proximity point (pair) (Theorems 2.11, 2.12). We have extended these theorems by considering some appropriate class of functions defined in [3, 16, 19] and applying it to find the optimum solution for systems of Caputo fractional differential equations.

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