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On a new family of composite models with generalized log-Moyal tail for modeling insurance loss data

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Abstract. In this paper, four new families of composite models are developed for modeling claim severity of mixed sizes. These models are developed, considering generalized log-Moyal distribution for the tail using Mode-Matching techniques. The proposed models are applied to fit well-known Danish fire insurance data set. The comprehensive study of model selection suggests that the proposed composite models with generalized log-Moyal distribution for the tail are an appropriate choice as compared to other heavy-tailed distributions. In addition to this, the behavior of risk measures viz. Value-at-Risk and Limited expected value, for proposed models, are assessed and compared with other existing models.

1. The first section

In real-life situations, often the random variable under study may come from either a single probability distribution or from a set of distributions. Such a situation arises in modeling financial losses and claim severity in insurance. Modeling such a data set becomes easy, if the underlying distributions come from the same family, say Lognormal, Pareto, etc., but if they come from different families, the problem becomes complex. Moreover, it is important to calculate risk measures such as Value-at-Risk (VaR) and Limited Expectation Value (LEV) which are required in the computation of future price of the insurance product. One key feature of such data sets is that they possess small and moderate values with high frequency and have inevitably few large values with low frequency. Besides this, these data are unimodal and highly positively skewed. Hence, fitting appropriate probability distribution to the loss data becomes crucial in such modeling when the data spreads over a wide range of magnitude. This attracted researchers to develop methods such as a mixture of two or more families (See,[13]) and method of composition of two or more distributions and tested the adequacy of the developed models for their data sets.

For such cases the composition of the distributions is proposed at the point of composition, commonly known as the threshold point, which could be fixed or random. [8] proposed the first generation of

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composite models using Lognormal distribution to model small and moderate claims that occur with high frequency up to a fixed threshold and thereafter Pareto distribution to the large claims with low frequency in the tail. As the parameters increase while the composition of distributions, [8] introduced a common normalization constant and imposed additional constraints to ensure continuity and differentiability at the threshold to obtain a smooth composite density. [20] suggested an improvement in the composite Lognormal-Pareto model by taking unrestricted mixing weights and suggested mixing weights depending on the distribution parameters give a better fit as compared to constant coefficients which as in [8]. [7], [11], [21] considered Weibull distribution for the head instead of Lognormal distribution. [17] presented a new R package `CompLognormal` for computing the probability density function (pdf), cumulative density function (cdf), quantile function and for generating random numbers of any composite model based on the Lognormal distribution. The use of the package is illustrated using a real dataset. Further, adopting the methodology of [20], [16] proposed various composite models by considering Burr, Loglogistic, Paralogistic, and Generalized Pareto distribution for the tail of the data. The R package `gendist`, which calculates pdf, cdf, and quantile functions, and generates random values for numerous probability distribution models, such as the mixture model, composite model, folded model, skewed symmetric model, and the arc tan model is introduced by [3]. These models are commonly employed in the literature, and the R functions included in this package offer flexibility in handling various univariate distributions available in other R packages.

[6] introduced another composite model by matching the two families of distribution at modal value and proposed Lognormal-Stoppa and Weibull-Stoppa models. This approach replaces the differentiability conditions at the threshold value by matching the modal value of densities. Further, in the Mode-Matching (MM) method, reduction in parameters can be easier as compared to obtaining second-order derivative required to make the composite density smoother.

[10] undertook a comprehensive study of 256 different composite models by combining 16 commonly used parametric distributions as head and tail distributions. Their findings suggest that for the celebrated Danish fire loss data, distributions such as Weibull, Paralogistic, and Inverse Burr as the head distribution is found to be superior for small and moderate claims while Inverse Weibull, Inverse Paralogistic, Loglogistic, Burr, Inverse Gamma seem to be a better choice for modeling the tail. Surprisingly distributions like Pareto and Generalized Pareto are not included among the top 20 best models based on the Bayesian Information Criterion (BIC) value.

Recently, [4] analyzed 25 composite models generated from a foundation of 5 folded distributions for modeling `Danish` fire insurance loss data set. The distributions encompass the folded Cauchy (FC), the folded t-distribution (FT) with a scale parameter, the folded normal (FN) distribution, the folded logistic (FL) distribution, and the folded Gumbel (FG) distribution. To date, these models represent the most effective model in achieving a high level of goodness-of-fit according to Bayesian Information Criteria (BIC) for `Danish` fire insurance loss data set.

[5] proposed a two-parameter heavy-tailed distribution namely "generalized log-Moyal (GlogM) distribution", which gives a better fit than many classical two-parameter heavy-tailed distributions for modeling insurance claim/loss datasets. Moreover, actuarial measures, viz. Value-at-Risk (VaR), conditional tail expectation (CTE), and Limited Expected Value (LEV), obtained for GlogM distribution can also be expressed in closed form which encourages the practitioner to use it for insurance losses. [14] proposed a gamma mixture of the GlogM distribution and used it in analyzing the Chinese earthquake loss data set and Norwegian fire claim data set. Its multivariate generalization with dependence function is discussed in [15]. The fact that GlogM gives a fairly good fit at the tail when compared with the other two parameter heavy-tailed models motivates us to explore a new class of composite distributions with GlogM distribution as a tail distribution. Further, to make our study more exhaustive we consider Lognormal, Weibull, Paralogistic, and Inverse Burr distributions for the head as suggested in [10]. Hence in this omnibus study of composite models, we propose a new family of composite models using the "*Mode-Matching*(MM)" technique.

The rest of the paper is structured as follows. In section 2, a brief outline of composite models generated by CC and MM techniques and a summary of GlogM distribution are presented. Section 3 comprises of detailed discussion of the Lognormal-GlogM, Weibull-GlogM, Inverse Burr-GlogM and Paralogistic-GlogM composite models. Analytical expressions for actuarial risk measures of these composite distributions are presented in section 4. The effectiveness of the fitting of the proposed models to the insurance loss dataset and its comparison with existing composite distributions are demonstrated in Section 5. Finally, some concluding remarks along with future possible extensions, are drawn in section 6.

2. A general frame work of Composite Models

A general framework of composite models proposed by [2] and [6] are as follows:

2.1. Composite Model-1: CC Technique

The composite models proposed by [2] consider the right truncated and left truncated density at threshold θ as Head and Tail distributions respectively. They also use unrestricted mixing weight to make the resultant density smooth and differentiable. The probability density function (pdf) of the composite model can be written as

$$f_{H,T}(x) = \begin{cases} r^{CC}_{H,T} f^*_H(x|\Xi_1, \theta) & \text{for} \quad 0 < x \leq \theta \\ (1 - r^{CC}_{H,T}) f^*_T(x|\Xi_2, \theta) & \text{for} \quad \theta < x < \infty \end{cases} \tag{1}$$

where H and T stand for the head and tail part of the composite modeling framework. CC represents the composite models generated using the Classical Composition technique. Ξ_1 and Ξ_2 are the parameter space of the head and tail part density of the composite model 1. $r^{CC}_{H,T} \in [0, 1]$ and θ are mixing weight and threshold parameters of the composite model developed using the CC technique having H and T part density respectively. The function $f^*_H(x|\Xi_1, \theta) = \frac{f_H(x|\Xi_1)}{F_H(\theta|\Xi_1)}$ and $f^*_T(x|\Xi_2, \theta) = \frac{f_T(x|\Xi_2)}{1-F_T(\theta|\Xi_2)}$ are the adequate truncation of the pdfs f_H and f_T upto and after an unknown threshold value θ respectively.

- The value of weight parameter $r^{CC}_{H,T}$ is obtained by continuity condition imposed at threshold θ i.e. $r^{CC}_{H,T} f^*_H(\theta|\Xi_1, \theta) = (1 - r^{CC}_{H,T}) f^*_T(\theta|\Xi_2, \theta)$. Hence, we get

$$r^{CC}_{H,T} = r^{CC}_{H,T}(\theta, \Xi_1, \Xi_2) = \frac{f_T(\theta|\Xi_2).F_H(\theta|\Xi_1)}{f_T(\theta|\Xi_2).F_H(\theta|\Xi_1) + f_H(\theta|\Xi_1).(1 - F_T(\theta|\Xi_2))} \tag{2}$$

- Further, imposing the differentiability condition at θ makes the resulting density smooth as

$$r^{CC}_{H,T} f^{*'}_H(\theta|\Xi_1, \theta) = (1 - r^{CC}_{H,T}) f^{*'}_T(\theta|\Xi_2, \theta) \tag{3}$$

These above conditions reduce the number of parameters and make the resulting density continuous and differentiable. Henceforth we refer to this technique of generating composite model as CC Technique.

2.2. Composite Model-2: MM Technique

[6] gave another viewpoint on the composition of two densities. Depending on how fast the probability decreases from the mode value, the densities at both sides of the modal value be considered. Hence, the resulting distribution supports the data in a more balanced way. The head part density of the composite model is used up to the modal value and thereafter the density of the tail part is considered. We refer to this technique as the "MM technique". The pdf of the composite distribution obtained from the MM technique is given as,

$$f_{H,T}(x) = \begin{cases} r^{MM}_{H,T} f^*_H(x) & \text{for} \quad 0 < x \leq x_{mo} \\ (1 - r^{MM}_{H,T}) f^*_T(x) & \text{for} \quad x_{mo} < x < \infty \end{cases} \tag{4}$$

where $r_{H,T}^{MM} \in [0,1]$ and x_{mo} are the mixing weight and threshold parameter of the composite model developed using MM technique having H and T part density respectively, $f_H^*(x) = \frac{f_H(x)}{F_H(x_{mo})}$ and $f_T^*(x) = \frac{f_T(x)}{1-F_T(x_{mo})}$. Instead of using traditional differentiability conditions at threshold value, an MM technique is used to ensure the differentiability of the resulting density. Denoting the mode of the head and the tail distribution by x_{mo}^H and x_{mo}^T respectively, the MM conditions are

$$x_{m0}^H = x_{m0}^T \tag{5}$$

$$r_{H,T}^{MM} f_H^*(x_{m0}^H) = (1 - r_{H,T}^{MM}) f_T^*(x_{m0}^T). \tag{6}$$

Equality in (5) allows us to drop the $'H'$ and $'T'$ labels. Equation (6) implies that the continuity condition is satisfied and gives the weight parameter $r_{H,T}^{MM}$ as

$$r_{H,T}^{MM} = \frac{f_T(x_{mo}).F_H(x_{mo})}{f_T(x_{mo}).F_H(x_{mo}) + f_H(x_{mo}).(1 - F_T(x_{mo}))} \tag{7}$$

Condition (5) surpasses the differentiability condition, as for uni-model distribution, the derivative of density at the modal value is zero. The easy implementation of the MM condition 5 as compared to the CC technique enables us to (i) extend the literature on the MM technique to generate composite models to analyze the data of mixed sizes (ii) reduce the computational burden of the estimation procedure. Similar to the traditional CC technique, this methodology includes unrestricted mixing weights. However, it offers a more straightforward model derivation compared to the traditional method, particularly when the distribution's mode exhibits a closed-form expression.

2.3. The generalized log-Moyal distribution

The generalized log-Moyal (GlogM) distribution was proposed by [5] as a two-parameter heavy-tailed distribution with parameters α and β. The pdf and cdf of GlogM distribution are given by

$$f_{GlogM}(x) = \frac{1}{\sqrt{2\pi}x\beta}\left(\frac{\alpha}{x}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}\left(\frac{\alpha}{x}\right)^{\frac{1}{\beta}}} \quad \text{for} \quad x > 0, \alpha > 0, \beta > 0 \tag{8}$$

$$F_{GlogM}(x) = \text{erfc}\left[\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x}\right)^{\frac{1}{2\beta}}\right] \tag{9}$$

where erfc(.) is the complementary of the error function given by erfc(x)=$\int_x^\infty e^{-t^2}dt$.
The closed-form expressions for r^{th} moment about origin, variance, Mode, Value-at-Risk (VaR), and Limited Expected Value (LEV) of GlogM(α,β) are

$$E(X^r) = \int_0^\infty x^r f(x)\,dx = \frac{\alpha^r}{2^{r\beta}\sqrt{\pi}}.\gamma_\beta(r), \quad r < \frac{1}{2\beta}$$

where $\gamma_\beta(r) = \Gamma(\frac{1}{2} - r\beta)$. By using the above equation one can easily find that

$$E(X) = \frac{\alpha}{2^\beta\sqrt{\pi}}.\gamma_\beta(1), \quad \beta < \frac{1}{2}. \tag{10}$$

and

$$V(X) = \frac{\alpha^2}{2^{2\beta}\sqrt{\pi}}.\left[\gamma_\beta(2) - \frac{1}{\sqrt{\pi}}\gamma_\beta^2(2)\right], \quad \beta < \frac{1}{4}. \tag{11}$$

Mode of the GlogM(α,β) is given by

$$x_{mo} = \frac{\alpha}{(1+2\beta)^\beta}. \tag{12}$$

Value-at-Risk for the GlogM(α,β) is

$$VaR_{\gamma}(X) = \alpha 2^{-\beta}\mathrm{erfc}^{\leftarrow}(q)^{-2\beta}. \tag{13}$$

Limited Expected Value (LEV) of GlogM(α,β) is given as

$$\mathtt{LEV}_u(X) = u.\mathrm{erf}\left(\frac{\left(\frac{\alpha}{u}\right)^{\frac{1}{2\beta}}}{\sqrt{2}}\right) + \frac{\alpha 2^{-\beta}\Gamma\left(\frac{1}{2}-\beta, \frac{1}{2}\left(\frac{\alpha}{u}\right)^{1/\beta}\right)}{\sqrt{\pi}}, \quad \beta < \frac{1}{2}. \tag{14}$$

3. Composite *H*-generalized log-Moyal models via MM Technique

Composite H-generalized log-Moyal models make use, of generalized log-Moyal distribution beyond the unknown threshold value and four distributions for the head (H) part of the proposed composite models. In this section we derive the H-Composite generalized log-Moyal models by imposing the MM technique discussed in section 2.2. We consider four different families of distributions for modeling the head part of the composite model, these distributions are as follows

1. Lognormal (L) distribution:

$$f_L(x) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left\{-\frac{1}{2}\left(\frac{\log(x)-\mu}{\sigma}\right)^2\right\}, \quad \text{and} \quad F_L(x) = \Phi\left(\frac{\log(x)-\mu}{\sigma}\right),$$

2. Weibull (W) distribution:

$$f_W(x) = \frac{\tau}{\phi}\exp\left\{-\left(\frac{x}{\phi}\right)^{\tau}\right\}\left(\frac{x}{\phi}\right)^{\tau-1}, \quad \text{and} \quad F_W(x) = 1-\exp\left\{-\left(\frac{x}{\phi}\right)^{\tau}\right\},$$

3. Inverse Burr (IB) distribution:

$$f_{IB}(x) = \frac{\mu\sigma(x\tau)^{\mu\sigma}}{x\left[(x\tau)^{\sigma}+1\right]^{\mu+1}}, \quad \text{and} \quad F_{IB}(x) = \left[((\tau x)^{\sigma}+1)^{-\mu}(\tau x)^{\mu\sigma}\right],$$

4. Paralogistic (P) distribution:

$$f_P(x) = \frac{\mu^2(x\tau)^{\mu}}{x\left[(x\tau)^{\mu}+1\right]^{\mu+1}}, \quad \text{and} \quad F_P(x) = \left[1-\left(\frac{1}{(\tau x)^{\mu}+1}\right)^{\mu}\right].$$

Proposed Composite Models with GlogM tail

Let X follow the composite density with GlogM distribution as the tail distribution developed using MM technique having density of the form

$$f^{MM}_{H,GlogM}(x) = \begin{cases} r^{MM}_{H,GlogM}\frac{f_H(x)}{F_H(x_{mo})} & \text{for} \quad 0 < x \leq x_{mo} \\ (1-r^{MM}_{H,GlogM})\frac{\alpha^{\frac{1}{2\beta}}\exp\{-\frac{1}{2}\left(\frac{\alpha}{x}\right)^{\frac{1}{\beta}}\}}{\sqrt{2\pi}\beta x^{\frac{1}{2\beta}+1}\left(1-\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x_{mo}}\right)^{\frac{1}{2\beta}}\right)\right)} & \text{for} \quad x_{mo} < x < \infty \end{cases} \tag{15}$$

with cdf given by

$$F^{MM}_{H,GlogM}(x) = \begin{cases} r^{MM}_{H,GlogM}\frac{F_H(x)}{F_H(x_{mo})} & \text{for} \quad 0 < x \leq x_{mo} \\ r^{MM}_{H,GlogM} + (1-r^{MM}_{H,GlogM})\frac{\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x}\right)^{\frac{1}{2\beta}}\right)-\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x_{mo}}\right)^{\frac{1}{2\beta}}\right)}{1-\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x_{mo}}\right)^{\frac{1}{2\beta}}\right)} & \text{for} \quad x_{mo} < x < \infty. \end{cases} \tag{16}$$

where $r^{MM}_{H,GlogM}$ is the mixing weight parameter of composite distribution with Head distribution H, and GlogM tail distribution developed using MM technique and x_{mo} is the threshold parameter. Considering L, W, IB, and P distributions for the head part of the composite model, the mixing weight parameter and the parametric relation obtained from the MM technique are given in Table 1

Table 1: Mixing weight ($r^{MM}_{H,T}$) and expression for reduced parameter obtained from (5) and (6) for different composite models developed using MM technique

Head	Tail	$r^{MM}_{H,T}$	Reduced Parameter
Lognormal	GlogM	$r^{MM}_{L,GlogM} = \frac{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_L(x_{m0})}{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_L(x_{m0})+f_L(x_{m0}).\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}}\right)\right]}$	$\sigma = \sqrt{\mu - \log\left(\frac{\alpha}{(1+2\beta)^{\beta}}\right)}$
Weibull	GlogM	$r^{MM}_{W,GLogM} = \frac{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_W(x_{m0})}{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_W(x_{m0})+f_W(x_{m0}).\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}}\right)\right]}$	$\phi = \frac{\alpha}{(1+2\beta)^{\beta}}\left(\frac{\tau}{\tau-1}\right)^{\frac{1}{\tau}}$
Inverse Burr	GlogM	$r^{MM}_{IB,GlogM} = \frac{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_{IB}(x_{m0})}{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_{IB}(x_{m0})+f_{IB}(x_{m0}).\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}}\right)\right]}$	$\tau = \frac{(2\beta+1)^{\beta}\left(\frac{\mu\sigma-1}{\sigma+1}\right)^{1/\sigma}}{\alpha}$
Paralogistic	GlogM	$r^{MM}_{P,GlogM} = \frac{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_P(x_{m0})}{\frac{1}{\sqrt{2\pi}x_{m0}\beta}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}} e^{-\frac{1}{2}(\frac{\alpha}{x_{m0}})^{\frac{1}{\beta}}} .F_P(x_{m0})+f_P(x_{m0}).\left[\mathrm{erf}\left(\frac{1}{\sqrt{2}}\left(\frac{\alpha}{x_{m0}}\right)^{\frac{1}{2\beta}}\right)\right]}$	$\tau = \frac{(2\beta+1)^{\beta}\left(\frac{\mu-1}{\mu^2+1}\right)^{1/\mu}}{\alpha}$

4. Actuarial Measures of composite H-GlogM models

Risk measures such as Value-at-Risk (VaR), tail value-at-risk (TVaR), and Limited Expected Value (LEV) are used to ensure the insolvency of the business with a specified degree of certainty. The VaR, TVaR, and LEV for the composite model are as follows:

4.1. Value-at-Risk

The Value-at-Risk ($VaR_\gamma(X)$) of a rv X at level γ is the $\gamma - th$ quantile of rv X and is defined as

$$VaR_\gamma(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq \gamma\}.$$

Hence for composite H-GlogM models the $VaR_\gamma(X)$ is given as

$$VaR_\gamma(X) = \begin{cases} F_H^{\leftarrow}\left(\frac{\gamma}{r^{MM}_{H,GlogM}} F_H(x_{mo})\right) & \text{if} \quad 0 < \gamma \leq r^{MM}_{H,GlogM}, \\ F_{GlogM}^{\leftarrow}\left(\frac{\gamma - r^{MM}_{H,GlogM}}{1 - r^{MM}_{H,GlogM}}(1 - F_{GlogM}(x_{mo})) + F_{GlogM}(x_{mo})\right) & \text{if} \quad r^{CC}_{H,GlogM} < \gamma < 1, \end{cases} \tag{17}$$

where $F_H^{\leftarrow}$ and $F_{GlogM}^{\leftarrow}$ are inverse cdf of Head and GlogM distribution respectively. The analytical expressions of Value-at-Risk for different families of proposed composite models are given in Table 2.

4.2. Limited Expected Value

In deciding the reinsurance premium, one of the prime measures is the "expected value on or below the threshold u", known as the limited expected value (LEV), given as

$$\mathtt{LEV}_u(X) := \mathbb{E}(X \wedge u) = \int_0^u x f(x)dx + u(1 - F(u)) \tag{18}$$

For proposed composite models with generalized log-Moyal tail, the $\texttt{LEV}_u(X)$ is

$$\texttt{LEV}_u(X) = \begin{cases} \int_0^u x\frac{f_H(x)}{F_H(x_{mo})}dx + u\left(1 - r^{MM}_{H,GlogM}\frac{F_H(u)}{F_H(x_{mo})}\right) & \text{if} \quad u < x_{mo}, \\ r^{MM}_{H,GlogM}\int_0^{x_{mo}} x\frac{f_H(x)}{F_H(x_{mo})}dx + (1 - r^{MM}_{H,GlogM})\int_{x_{mo}}^u x\frac{f_{GlogM}(x)}{\overline{F}_{GlogM}(x_{mo})}dx & \\ +u\left(1 - \left(r^{MM}_{H,GlogM} + (1 - r^{MM}_{H,GlogM})\frac{F_{GlogM}(u) - F_{GlogM}(x_{mo})}{\overline{F}_{GlogM}(x_{mo})}\right)\right) & \text{if} \quad u > x_{mo}. \end{cases} \tag{19}$$

see [19] for the proof of the above expression. The expressions for LEV obtained for proposed composite models having various head distributions and GlogM as tail distribution are given in Table 3.

Table 2: Analytical Expression for Value-at-Risk at security level γ for proposed composite models developed using MM technique

Head	Tail	$VaR_\gamma(X)$
Lognormal	GlogM	$\begin{cases} \exp\left\{\mu + \sigma\left(\Phi^{\leftarrow}((\frac{\gamma}{r^{MM}_{L,GlogM}})\Phi(\frac{\log x_{mo}-\mu}{\sigma}))\right)\right\} & \text{if} \quad 0 < \gamma \le r^{MM}_{L,GlogM} \\ \left(\frac{1}{A^{MM}_{L,GlogM}}\right)^{2\beta} & \text{if} \quad r^{MM}_{L,GlogM} < \gamma \le 1 \end{cases}$
Weibull	GlogM	$\begin{cases} \left(-\phi^\tau \log\left(1 - (\frac{\gamma}{r^{MM}_{W,GlogM}})(1 - e^{-\left(\frac{x_{mo}}{\phi}\right)^\tau})\right)\right)^{\frac{1}{\tau}} & \text{if} \quad 0 < \gamma \le r^{MM}_{W,GlogM} \\ \left(\frac{1}{A^{MM}_{W,GlogM}}\right)^{2\beta} & \text{if} \quad r^{MM}_{W,GlogM} < \gamma \le 1 \end{cases}$
Inverse Burr	GlogM	$\begin{cases} \left(\frac{\tau^{-\sigma}}{1-\left(\frac{\gamma}{r^{MM}_{IB,GlogM}}\right)^{-1/\mu}\left((x_{mo}\tau)^{\mu\sigma}((x_{mo}\tau)^\sigma+1)^{-\mu}\right)^{-1/\mu}}\right)^{1/\sigma} & \text{if} \quad 0 < \gamma \le r^{MM}_{IB,GlogM} \\ \left(\frac{1}{A^{MM}_{IB,GlogM}}\right)^{2\beta} & \text{if} \quad r^{MM}_{IB,GlogM} < \gamma \le 1 \end{cases}$
Paralogistic	GlogM	$\begin{cases} \frac{1}{\tau}\left[\frac{1}{\left(1-\frac{\gamma}{r^{MM}_{P,GlogM}}\left(1-\frac{1}{((x_{mo}\tau)^\mu+1)^\mu}\right)\right)^{\frac{1}{\mu}}} - 1\right]^{\frac{1}{\mu}} & \text{if} \quad 0 < \gamma \le r^{MM}_{P,GlogM} \\ \left(\frac{1}{A^{MM}_{P,GlogM}}\right)^{2\beta} & \text{if} \quad r^{MM}_{P,GlogM} < \gamma \le 1 \end{cases}$

where $A^{MM}_{H,GlogM} = \frac{\sqrt{2}}{\alpha^{1/2\beta}}\texttt{erfc}^{\leftarrow}\left((\frac{\gamma - r^{MM}_{H,GlogM}}{1 - r^{MM}_{H,GlogM}})\texttt{erf}(\frac{1}{\sqrt{2}}(\frac{\gamma}{x_{mo}})^{\frac{1}{2\beta}}) + \texttt{erfc}(\frac{1}{\sqrt{2}}(\frac{\gamma}{x_{mo}})^{\frac{1}{2\beta}})\right)$. The subscript H in r^{MM}_H replaces with L, W, IB or P for Lognormal, Weibull, Inverse Burr and Paralogistic distribution.

5. Numerical Application

We discuss the applicability of the proposed composite models on one real-world insurance data set. The first data set, namely "Danish Fire Losses", is popularly used for severity modeling using composite distribution. It contains fire insurance losses in millions of Danish kroner (Dkr) from the year 1980 to 1990 inclusively, adjusted to reflect 1985 values. The data set is available in the `SMPracticals` package in `R`

Table 3: Analytical Expression for Limited Expected Value of Proposed Composite Models

Head	Tail	$\mathtt{LEV}_u(X)$
Lognormal	GlogM	$\begin{cases} \frac{r^{MM}_{L,GlogM} e^{\left(\frac{1}{2}\sigma^2+\mu\right)} \mathrm{erfc}\left(\frac{\sigma^2+\mu-\log(u)}{\sqrt{2}\sigma}\right)}{\mathrm{erfc}\left(\frac{\log(x_{mo})-\mu}{\sqrt{2}\sigma}\right)} + u\left(1 - r^{MM}_{L,GlogM}\left(\frac{\mathrm{erfc}\left(\frac{\mu-\log(u)}{\sqrt{2}\sigma}\right)}{\mathrm{erfc}\left(\frac{\mu-\log(x_{mo})}{\sqrt{2}\sigma}\right)}\right)\right) & \text{if} \quad 0 < u \le x_{mo} \\ \frac{r^{MM}_{L,GlogM} e^{\left(\frac{1}{2}\sigma^2+\mu\right)} \mathrm{erfc}\left(\frac{-\log(x_{mo})+\sigma^2+\mu}{\sqrt{2}\sigma}\right)}{\mathrm{erfc}\left(\frac{\log(x_{mo})-\mu}{\sqrt{2}\sigma}\right)} + \frac{(1-r^{MM}_{L,GlogM})\alpha 2^{\beta}\left(\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{u}{\alpha}\right)^{-1/\beta}\right)-\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{\alpha}{x_{mo}}\right)^{1/\beta}\right)\right)}{\sqrt{\pi}\,\mathrm{erf}\left(\frac{\left(\frac{\alpha}{x_{mo}}\right)^{\frac{1}{2\beta}}}{\sqrt{2}}\right)} & \\ +u\left[1-\left(r^{MM}_{L,GlogM}+(1-r^{MM}_{L,GlogM})\frac{F_{GlogM}(u)-F_{GlogM}(x_{mo})}{1-F_{GlogM}(x_{mo})}\right)\right] & \text{if} \quad x_{mo} < u \le \infty \end{cases}$
Weibull	GlogM	$\begin{cases} \frac{r^{MM}_{W,GlogM}\phi\left(\Gamma\left(\frac{1+\tau}{\tau}\right)-\Gamma\left(\frac{1+\tau}{\tau},\left(\frac{u}{\phi}\right)^{\tau}\right)\right)}{1-e^{-\left(\frac{u}{\phi}\right)^{\tau}}} + u\left(\frac{(1-e^{-\left(\frac{x_{mo}}{\phi}\right)^{\tau}})-r^{MM}_{W,GlogM}(1-e^{-\left(\frac{u}{\phi}\right)^{\tau}})}{(1-e^{-\left(\frac{x_{mo}}{\phi}\right)^{\tau}})}\right) & \text{if} \quad 0 < u \le x_{mo} \\ \frac{r^{MM}_{W,GlogM}\phi\left(\Gamma\left(\frac{1+\tau}{\tau}\right)-\Gamma\left(\frac{1+\tau}{\tau},\left(\frac{x_{mo}}{\phi}\right)^{\tau}\right)\right)}{1-e^{-\left(\frac{x_{mo}}{\phi}\right)^{\tau}}} + \frac{(1-r^{MM}_{W,GlogM})\alpha\left(\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{u}{\alpha}\right)^{-1/\beta}\right)-\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{\alpha}{x_{mo}}\right)^{1/\beta}\right)\right)}{\sqrt{\pi}2^{\beta}\mathrm{erf}\left(\frac{\left(\frac{\alpha}{x_{mo}}\right)^{\frac{1}{2\beta}}}{\sqrt{2}}\right)} & \\ +u\left[1-\left(r^{MM}_{W,GlogM}+(1-r^{MM}_{W,GlogM})\frac{F_{GlogM}(u)-F_{GlogM}(x_{mo})}{1-F_{GlogM}(x_{mo})}\right)\right] & \text{if} \quad x_{mo} < u \le \infty \end{cases}$
Inverse Burr	GlogM	$\begin{cases} \frac{\mu\sigma r^{MM}_{IB,GlogM} x_{mo}^{-\mu\sigma}((x_{mo}\tau)^{\sigma}+1)^{\mu} u_2^{\mu\sigma+1}F_1\left(\mu+1,\frac{1}{\sigma}+\mu,\frac{1}{\sigma}+\mu+1,-(\tau u)^{\sigma}\right)}{1+\mu\sigma} + u\left(\frac{((x_{mo}\tau)^{\mu\sigma}((x_{mo}\tau)^{\sigma}+1)^{-\mu})-r^{MM}_{IB,GlogM}((u\tau)^{\mu\sigma}((u\tau)^{\sigma}+1)^{-\mu})}{(1-e^{-\left(\frac{x_{mo}}{\phi}\right)^{\tau}})}\right) & \text{if} \quad 0 < u \le x_{mo} \\ \frac{\mu\sigma r^{MM}_{IB,GlogM}(x_{mo})_2F_1\left(1,\frac{1}{\sigma},\frac{1}{\sigma}+\mu+1,-(x_{mo}\tau)^{\sigma}\right)}{1+\mu\sigma} + \frac{(1-r^{MM}_{IB,GlogM})\alpha\left(\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{u}{\alpha}\right)^{-1/\beta}\right)-\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{\alpha}{x_{mo}}\right)^{1/\beta}\right)\right)}{\sqrt{\pi}2^{\beta}\mathrm{erf}\left(\frac{\left(\frac{\alpha}{x_{mo}}\right)^{\frac{1}{2\beta}}}{\sqrt{2}}\right)} & \\ +u\left[1-\left(r^{MM}_{IB,GlogM}+(1-r^{MM}_{IB,GlogM})\frac{F_{GlogM}(u)-F_{GlogM}(x_{mo})}{1-F_{GlogM}(x_{mo})}\right)\right] & \text{if} \quad x_{mo} < u \le \infty \end{cases}$
Paralogistic	GlogM	$\begin{cases} \frac{r^{MM}_{P,GlogM} u\Gamma\left(\frac{1}{\mu}\right)(\tau u)^{\mu}\,{}_2\tilde{F}_1\left(1+\frac{1}{\mu},\mu+1;2+\frac{1}{\mu};-(u\tau)^{\mu}\right)}{1-\left(\frac{1}{(x_{mo}\tau)^{\mu}+1}\right)^{\mu}} + u\left(1-\frac{r^{MM}_{P,GlogM}\left(1-\left(\frac{1}{(\tau u)^{\mu}+1}\right)^{\mu}\right)}{1-\left(\frac{1}{(x_{mo}\tau)^{\mu}+1}\right)^{\mu}}\right) & \text{if} \quad 0 < u \le x_{mo} \\ \frac{r^{MM}_{P,GlogM} x_{mo}\Gamma\left(\frac{1}{\mu}\right)(\tau x_{mo})^{\mu}\,{}_2\tilde{F}_1\left(1,\frac{1}{\mu}-\mu+1,2+\frac{1}{\mu},-(x_{mo}\tau)^{\mu}\right)}{1-\left(\frac{1}{(x_{mo}\tau)^{\mu}+1}\right)^{\mu}} + \frac{(1-r^{MM}_{P,GlogM})\alpha\left(\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{u}{\alpha}\right)^{-1/\beta}\right)-\Gamma\left(\frac{1}{2}-\beta,\frac{1}{2}\left(\frac{\alpha}{x_{mo}}\right)^{1/\beta}\right)\right)}{\sqrt{\pi}2^{\beta}\mathrm{erf}\left(\frac{\left(\frac{\alpha}{x_{mo}}\right)^{\frac{1}{2\beta}}}{\sqrt{2}}\right)} & \\ +u\left[1-\left(r^{MM}_{P,GlogM}+(1-r^{MM}_{P,GlogM})\frac{F_{GlogM}(u)-F_{GlogM}(x_{mo})}{1-F_{GlogM}(x_{mo})}\right)\right] & \text{if} \quad x_{mo} < u \le \infty \end{cases}$

where, ${}_2F_1\,(a,b,c,d)$ is Hypergeometric function and ${}_2\tilde{F}_1(a,b;c;d)$ is Hypergeometric ${}_2F_1$ Regularized functions.

repository. `Danish` data set is being used extensively to demonstrate the applicability of different composite models for loss severity modeling. The descriptive statistics for the `Danish` data set are given in Table 4. and it can be observed that the data set exhibits characteristics like high skewness value indicating right-skewed data, a huge difference between Maximum to Q_3 i.e. 260 for `Danish` data set indicating that long tail behavior and unimodality. We use a method of maximum likelihood for estimating the parameters of all the proposed models. As the likelihood equation for composite models is not in closed form, we use the numerical optimization tool "`nlm()`" in `R` programming language to optimize the likelihood and obtain estimated parameters. The initial values of parameters are selected using the random selection method proposed in [10]. We compare the fitting results for proposed composite distributions obtained using the MM technique with distributions proposed in [6]. The estimated values of parameters obtained for different composite models using the MM technique for `Danish` data set are given in Table 5. These results of fitting using MM techniques for `Danish` data set are shown in Table 6.

We also compute Negative loglikelihood value (NLL), Akaike information criterion (AIC) given as $2\times$NLL$+2k$, where k is a number of parameters and Bayesian information criterion (BIC) computed as $2\times$NLL$+k \times \log n$ for the model assessment. Note that for all these above information criteria, smaller values indicate a better fit of the model to the data. The models are presented in order of BIC value as it accommodates both the number of observations (n) as well as a number of parameters (k).

Table 4: Descriptive statistics of `Danish` Dataset

Measures	`Danish`
n	2492
Mean	3.06
Std.Dev	7.98
Min	0.31
Mode	0.89
Q1	1.16
Median	1.63
Q3	2.65
CV	2.60
Max.	263.25
Skewness	19.88

Table 5: Parameter estimates obtained for the top 15 best fitting composite models developed using MM techniques for `Danish` dataset (arranged according to the ascending values BIC).

Head	Tail	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\tau}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\psi}$	$\hat{x_0}$	$r_{H,T}^{\hat{MM}}$	$\hat{x_{mo}}$
Weibull	GlogM	–	–	16.314	1.121	0.338	-	-	0.944	-	0.076	0.940
Inverse Burr	GlogM	0.24	60.000	1.056	1.103	0.342	-	-	-	-	0.064	0.923
Paralogistic	GlogM	16.401	-	0.892	1.121	0.338	-	-	-	-	0.077	0.941
Weibull	Burr	0.330	4.320	16.188	–	1.141	-	-	0.946	-	0.077	0.943
Inverse Burr	Burr	0.116	127.266	1.080	0.366		3.916	1.150		-	0.055	0.910
Paralogistic	Burr	0.329	4.3283	16.268	–	1.141	-	-	0.889	-	0.078	0.943
Weibull	Stoppa	-	-	16.171	-	-	1.730	1.495	0.942	0.741	0.081	0.945
Weibull	Lomax	0.969	–	15.345	0.561	0.971		–	–	–	0.139	0.991
Inverse Burr	Inverse Weibull	1.52×10^{-2}	1.04×10^{3}	1.118	1.715	-	1.163	–	–	-	0.043	0.8903
Weibull	Inverse Weibull	-	1.198	17.150	1.751	-	-	-	0.929	-	0.065	0.9259
Paralogistic	Inverse Weibull	17.206	1.199	0.911	1.752	–	-	-	-	-	0.065	0.926
Inverse Burr	Stoppa	0.238	64.999	1.078	–	–	54.998	1.722	–	0.114	0.053	0.905
Paralogistic	Stoppa	17.133	–	0.909	–	–	54.996	1.744	–	0.120	0.066	0.927
Inverse Burr	Inverse Paralogistic	4.36×10^{-4}	3.53×10^{4}	1.130	1.952	1.099	-	-	-	-	0.039	0.887
Weibull	Inverse Paralogistic	1.974	1.082	18.154	-	-	-	-	0.914	-	0.056	0.911

Table 6 shows that the top three ranked models developed using the MM technique for `Danish` dataset

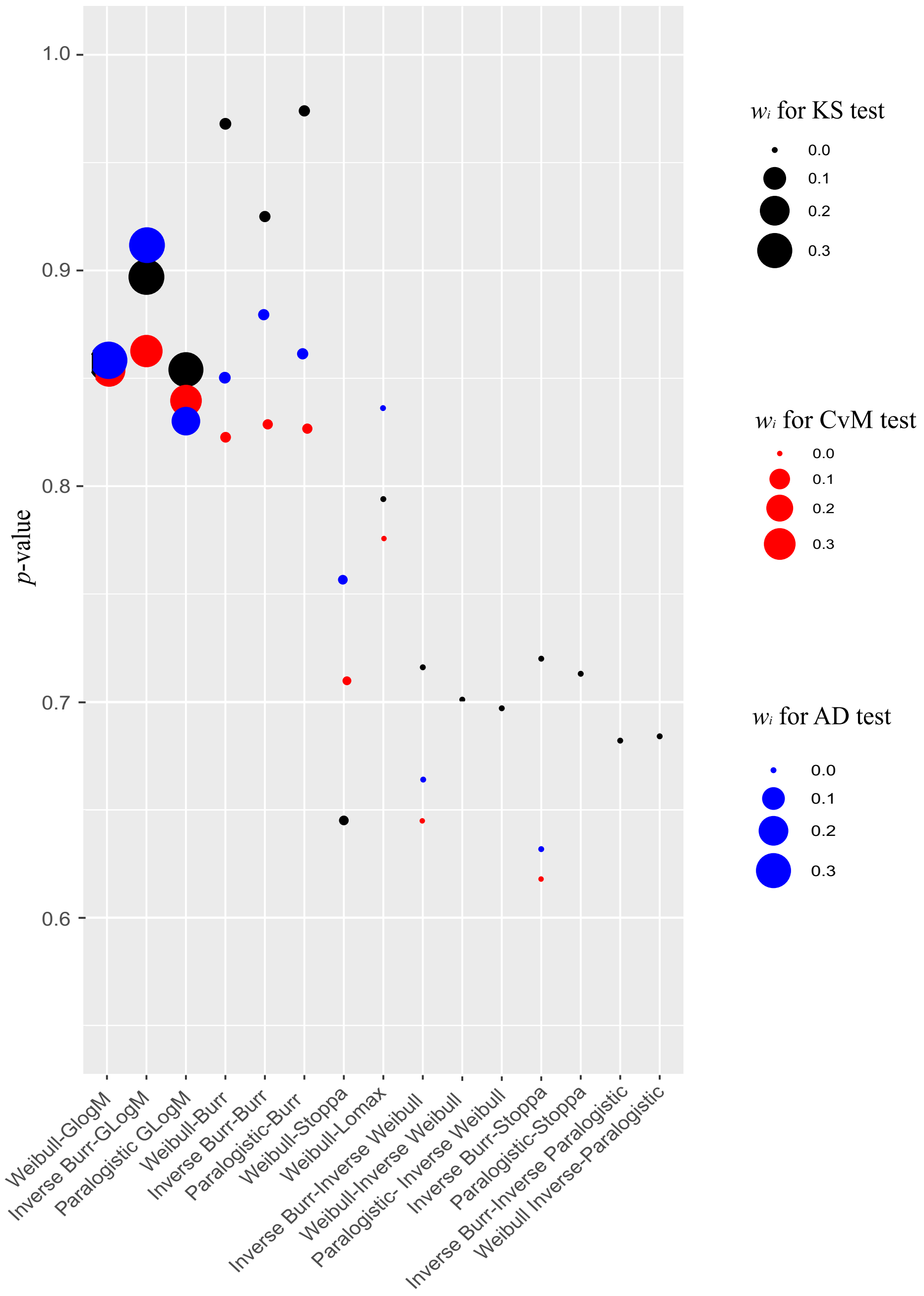
Table 6: Results for the top 15 best fitting composite models developed using MM technique for Danish dataset (arranged in the ascending order of BIC).

Head	Tail	Parameter	NLL	AIC	BIC	Estimated Mode
Weibull	GlogM	3	3818.42	7642.84	7660.31	0.9407
Inverse Burr	GlogM	4	3814.62	7637.23	7660.52	0.9230
Paralogistic	GlogM	3	3818.58	7643.16	7660.63	0.9413
Weibull	Burr	4	3817.89	7643.78	7667.06	0.9422
Inverse Burr	Burr	5	3814.12	7638.24	7667.34	0.9100
Paralogistic	Burr	4	3818.04	7644.08	7667.36	0.9429
Weibull	Stoppa	4	3818.82	7645.64	7668.92	0.9647
Weibull	Lomax	4	3823.70	7655.40	7676.68	0.9913
Inverse Burr	Inverse Weibull	4	3823.96	7655.92	7679.20	0.8902
Weibull	Inverse Weibull	3	3832.77	7671.54	7689.00	0.9258
Paralogistic	Inverse Weibull	3	3832.99	7671.98	7689.44	0.9267
Inverse Burr	Stoppa	5	3825.44	7660.89	7689.99	0.9055
Paralogistic	Stoppa	4	3832.24	7672.49	7695.77	0.9276
Inverse Burr	Inverse Paralogistic	4	3847.08	7702.16	7725.44	0.8873
Weibull	Inverse Paralogistic	3	3858.65	7723.30	7740.76	0.9118

based on the BIC values are having GlogM distribution at the tail and Weibull, Inverse Burr and Paralogistic at the head with BIC values 7660.31, 7660.52 and 7660.63 respectively. It is worth highlighting that the difference between the BIC value of third rank model and fourth rank model is significant, which suggests that modeling the tail part of Danish data with GlogM distribution is a good option to reduce the penalty associated with number of parameters of the model. It is noted that, for the Danish data set, the three composite models reported in [4] namely FC-FC, FC-FT, FL-FT have the smallest BIC values i.e. 7642.491, 7620.623, and 7651.295 respectively which are lower than the BIC values of the proposed models. However, the estimates of risk measures for the Danish data set of the models proposed by [4] are far away from their empirical counterparts as can be seen in 5.2.

5.1. Goodness-of-fit test

We use the following three Goodness-of-fit (GoF) tests based on the empirical df, which measures the 'distance' between the distribution function of the fitted model and empirical df obtained from the data. Denote the cdf of the fitted model by $\hat{F}$, and arrange the data in increasing order of magnitude by $x_{(1)}, \cdots, x_{(n)}$. Then we have: (i) Kolmogorov–Smirnov test statistic ($\mathtt{KS}_n$), $\mathtt{KS}_n = \max\{D^+, D^-\}$, where $D^+ = \max_{j=1,\cdots,n}\left(j/n - F(X_{(j)}; \hat{\sigma}_n)\right)$ and $D^- = \max_{j=1,\cdots,n}\left(F(X_{(j)}; \hat{\sigma}_n) - (j-1)/n\right)$, (ii) the Cramér-von Mises test statistic ($\mathtt{CvM}_n$), $\mathtt{CvM}_n = \frac{1}{12n} + \sum_{j=1}^{n}\left(F(X_{(j)}; \hat{\sigma}_n) - \frac{2j-1}{2n}\right)^2$, and the (iii) Anderson–Darling test ($\mathtt{AD}_n$) $\mathtt{AD}_n = -n - \frac{1}{n}\sum_{j=1}^{n}\log\left(F^{(2j-1)}(X_{(j)}; \hat{\sigma}_n)\bar{F}^{(2n-2j+1)}(X_{(j)}; \hat{\sigma}_n)\right)$. For all the GoF measures mentioned above, smaller values indicate a better fit of the model to the data. These values are indicated in Table 7. To perform the goodness-of-fit tests, it is required that the proposed models are specified completely. However, in the case where parameters are estimated from data, the critical values of GoF tests produced using the standard procedure are no longer valid (see, [1]). Hence we obtain bootstrap p-value (see [6]) for acceptance and rejection of GoF of the models. In Figure 5.1, we organize the top 15 composite models on the x-axis and p-values on the y-axis. Each bubble on this plot represents the size of the individual model weight, w_i. The model weights are computed based on the approximate calculation of the posterior probability of each model, assuming equal prior model weights. Specifically, the formula $w_i = e^{-\frac{\Delta BIC_i}{2}} / \sum_{i=1}^{15} e^{-\frac{\Delta BIC_i}{2}}$ where $\Delta BIC_i = BIC_i - min_{1\leq i\leq 15}(BIC_i)$, is used in determining the posterior model weights. Readers may refer [12] for how to determine posterior model weights. The bubble with a large diameter indicates the minimum value of BIC.



1.0
0.9
0.8
0.7
0.6
p-value
Weibull-GlogM
Inverse Burr-GLogM
Paralogistic GLogM
Weibull-Burr
Inverse Burr-Burr
Paralogistic-Burr
Weibull-Stoppa
Weibull-Lomax
Inverse Burr-Inverse Weibull
Weibull-Inverse Weibull
Paralogistic- Inverse Weibull
Inverse Burr-Stoppa
Paralogistic-Stoppa
Inverse Burr-Inverse Paralogistic
Weibull Inverse-Paralogistic
w_i for KS test
0.0
0.1
0.2
0.3
w_i for CvM test
0.0
0.1
0.2
0.3
w_i for AD test
0.0
0.1
0.2
0.3

Table 7: KS Test, CvM Test, and AD test statistics values for Danish data set for different composite models obtained using MM techniques.

3*S.No.			Danish Data		
	Head	Tail	MM-Technique		
			KS	CvM	AD
1	Weibull	GlogM	0.017	0.104	0.848
2	Paralogistic	GlogM	0.017	0.104	0.852
3	Inverse Burr	GlogM	0.016	0.095	0.736
4	Weibull	Inverse Weibull	0.034	0.798	4.700
5	Paralogistic	Inverse Weibull	0.034	0.806	4.747
6	Inverse Burr	Inverse Weibull	0.032	0.537	3.207
7	Weibull	Inverse Paralogistic	0.046	1.822	10.835
8	Inverse Burr	Inverse Paralogistic	0.044	1.594	9.524
9	Weibull	Burr	0.013	0.084	0.720
10	Paralogistic	Burr	0.014	0.084	0.721
11	Inverse Burr	Burr	0.014	0.082	0.649
12	Weibull	Stoppa	0.017	0.126	0.882
13	Weibull	Lomax	0.026	0.338	1.910
14	Inverse Burr	Stoppa	0.032	0.592	3.517
15	Paralogistic	Stoppa	0.034	0.775	4.575

5.2. Estimation of VaR and LEV

Table 8 provides a summary of VaR at 0.95 and 0.99 security levels for the top 15 best fitting models developed using MM technique for Danish data set. Using the values from Table 8, the percentage difference between estimated VaR and empirical VaR can be calculated as $\frac{|\widehat{VaR_\gamma}(X)-VaR_\gamma(X)|}{VaR_\gamma(X)} \times 100$. Where $\widehat{VaR_\gamma}(X)$ is the estimated VaR of the various composite models and $VaR_\gamma(X)$ is the empirical VaR of the Danish data set at security level γ. One can observe that, for the proposed model, the value of VaR for Weibull-Burr, Inverse-Burr, and Paralogistic-Burr have VaR closer to empirical VaR at security level 0.95 as compared to composite models with GlogM tail. The estimated VaR of the GlogM distribution for Danish data set at security level 0.95 and 0.99 are 8.55 and 21.82 respectively (see, [5]) which in turn shows that the single density of the GlogM distribution for the Danish data set underestimates the extreme quantile (security level 0.99). Among the proposed composite models, the composite Inverse-Burr model with GlogM tail gives the closest estimates of the VaR for security level 0.95 & 0.99 for the Danish data set i.e. the percentage difference between the estimated VaR based on composite Inverse Burr-GlogM model and the empirical VaR at security level 0.95 & 0.99 are 2.7% and 2.8% respectively. For the composite models, FC-FC, FC-FT, and FL-FT proposed by [4], the estimated VaR at 0.95 security level is 8.45, 7.52, and 7.53 respectively, and 42.30, 34.16, and 34.23 for the 0.99 security level. The percentage difference between estimated VaR and empirical VaR for the models given in [4] is 0.5%, 10.6% and 10.5% for security level 0.95 and 71.9%, 38.8% and 39.1% for the security level 0.99. The LEV for various composite models with GlogM tail have been obtained using MM technique at different levels u and the results are shown in Table 9 for Danish data set. Table 9 shows that the GlogM distribution underestimates the limited expected value when the losses exceed 110 million Danish Kroner. It can be seen that the proposed composite models demonstrate a better fit to the data in the high quantiles, likewise suggesting that the proposed composite models are favorable models for the Danish data set as compared to fitting a single GlogM density to the Danish data set. Composite Inverse-Burr distribution with GlogM tail provides the nearest estimates of the LEV for the extreme values of u that is $u = 110, 170, 260$.

Table 8: Results of VaR at the 0.95 and 0.99 security level for the top 15 best fitting composite models developed using MM technique for Danish data set (arranged in the ascending order of BIC).

		$\mathtt{VaR}_{0.95}$	$\mathtt{VaR}_{0.99}$
Empirical Estimates		8.41	24.61
Head	Tail		
Weibull	GlogM	8.07	23.35
Inverse Burr	GlogM	8.18	23.93
Paralogistic	GlogM	8.07	23.34
Weibull	Burr	8.26	25.54
Inverse Burr	Burr	8.23	25.24
Paralogistic	Burr	8.26	25.55
Weibull	Stoppa	8.21	24.22
Weibull	Lomax	8.20	22.65
Inverse Burr	Inverse Weibull	7.34	18.93
Weibull	Inverse Weibull	7.20	18.22
Paralogistic	Inverse Weibull	7.19	18.20
Inverse Burr	Stoppa	7.31	18.78
Paralogistic	Stoppa	7.22	18.33
Inverse Burr	Inverse Paralogistic	6.62	15.29
Weibull	Inverse Paralogistic	6.57	15.04

Table 9: Results of LEV for the proposed composite models developed using MMtechnique for Danish data set.

u	Lognormal-GlogM	Weibull-GlogM	Paralogistic-GlogM	Inverse Burr-GlogM	GlogM	Empirical
1	0.928	0.988	0.988	0.988	0.983	0.935
2	1.547	1.569	1.569	1.566	1.585	1.551
3	1.830	1.851	1.851	1.848	1.877	1.839
5	2.114	2.141	2.141	2.140	2.166	2.155
8	2.315	2.353	2.353	2.355	2.374	2.387
10	2.394	2.438	2.438	2.442	2.454	2.483
15	2.516	2.572	2.572	2.578	2.576	2.653
21	2.599	2.665	2.665	2.674	2.658	2.762
40	2.722	2.806	2.806	2.820	2.780	2.919
70	2.798	2.899	2.898	2.916	2.854	2.997
110	2.846	2.957	2.957	**2.977**	2.900	3.045
170	2.882	3.003	3.002	**3.025**	2.934	3.093
260	2.909	3.039	3.038	**3.064**	2.940	3.111

6. Conclusions

In this paper, new families of composite models are proposed considering the MM technique having generalized log-Moyal distribution in the tail and Lognormal, Weibull, Paralogistic, and Inverse Burr as head distributions. The simple form of generalized log-Moyal distribution allows one to get closed-form expressions for actuarial measures such as VaR and LEV. These measures have been calculated for the proposed models. The exhaustive study of the proposed four families of composite distributions was conducted using one real-world insurance data set namely `Danish` fire insurance data set. Further, to examine the risk behavior, two actuarial risk measures such as VaR, TVaR and LEV were studied and compared. [10] suggested that the Paralogistic and Inverse Burr are also suitable distributions for modeling the head part of the composite model. Our findings lie on the line of [10]. Weibull, Paralogistic, and Inverse Burr distributions in the head with generalized log-Moyal in the tail are found to be ideal for modeling the small and moderate-size claims. A comparison of the proposed models with the models given in [3] shows that the proposed composite models provide a good fit for the tail of the `Danish` fire insurance data set. Along similar lines to [9], one can use the proposed composite models for fitting the insurance data having a heavy-tailed response in the presence of covariate information for future investigation.

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