Filomat 38:14 (2024), 5127–5140
https://doi.org/10.2298/FIL2414127B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some general quantum integral inequalities for convex functions

Abdul Wakil Baidar[a,b], Mehmet Kunt[b]

[a] *Kabul University, Department of Mathematics, Kabul, Afghanistan*
[b] *Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Türkiye*

Abstract. In this paper, based on a new quantum analog of Hermite-Hadamard inequality, the quantum analog of some trapezoid, midpoint, Bullen and Simpson type inequalities are established. Moreover, by taking the special selection of λ and μ and taking limit $q \to 1^{-}$, some new results are obtained, and some old results are verified.

1. Introduction

Let I be an interval of real numbers. If $f : I \to \mathbb{R}$ is a convex function, then the following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{1}$$

holds for all $a, b \in I$ with $a < b$, and the inequality (1) is known as the Hermite-Hadamard inequality for convex functions .

Let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$, then the following inequality

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right], \tag{2}$$

is known as Bullen's inequality. In addition, if $f : [a, b] \to \mathbb{R}$ is four times continuously differentiable function on $[a, b]$ and $\left\|f^{(4)}\right\|_\infty = \sup_{t\in[a,b]}\left|f^{(4)}(t)\right| < \infty$, then the following inequality

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4, \tag{3}$$

is known as Simpson's inequality.

The process of obtaining a certain kind of inequality by providing an identity is one that is frequently employed in inequality theory. In this regard, in [15], Dragomir and Agarval derived several inequalities

2020 *Mathematics Subject Classification*. Primary 05C38, 15A15, ; Secondary 05A15, 15A18
Keywords. Simpson's inequality, Convex function, q-Integral, Bullen's inequality
Received: 12 April 2023; Revised: 18 February 2024; Accepted: 28 February 2024
Communicated by Miodrag Spalević
Email addresses: baidarmath87@ku.edu.af (Abdul Wakil Baidar), mkunt@ktu.edu.tr (Mehmet Kunt)

of the trapezoid type by identifying the difference between the right and middle of (1). Similarly, in [19], Kırmacı obtained some inequalities of the midpoint kind by identifying the difference between the left and middle of (1). Similar to this, Xi and Qi [34] and. Alomari et al. [6], obtained some Bullen and Simpson type inequalities respectively.

In recent years, generalizing integral inequalities to their quantum analogues has attracted the attention of many researchers. In this regard, for quantum analogs of the above-mentioned works and other works, the reader can see [1]-[5], [7]-[13], [16], [20]-[28], [31, 32] and [35].

Xi and Qi give the following general identity, in which trapezoid, midpoint, Bullen and Simpson type inequalities can all be obtained in particular choices of λ and μ.

Lemma 1.1. *[33, Lemma 2.1] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$. If $f' \in L[a, b]$ and $\lambda, \mu \in \mathbb{R}$, then*

$$\begin{aligned} & \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \qquad (4)\\ = & \frac{b-a}{4}\int_0^1 \left[\begin{array}{l} (1-\lambda-t) f'\left(ta + (1-t)\frac{a+b}{2}\right) \\ +(\mu - t) f'\left(t\left(\frac{a+b}{2}\right) + (1-t) b\right)\end{array}\right] dt. \end{aligned}$$

It is clear, if one takes $\lambda = \mu = 1, 0, 1/2$ and $1/3$ in identity (4), then one has trapezoid, midpoint, Bullen, and Simpson type identity respectively.

To the best of our knowledge, quantum analogs of the identity (4) and the resulting inequalities in [33] have not been studied so far. The aim of this paper is to establish quantum analogue's of the main identity and inequalities in [33], based on a new version of the quantum Hermite-Hadamard type inequality which has been obtained in [13] (see also [4]). The consequences of trapezoid, midpoint, Bullen and Simpson type inequalities are deduced as specific cases when $q \to 1$.

2. Preliminaries

In this section, we give some basic concepts of quantum derivative and quantum integral on an arbitrary finite interval $[a, b]$. From here on, $0 < q < 1$ will be considered constant throughout the paper.

Definition 2.1. *[28]Let $f : [a, b] \to \mathbb{R}$ be an arbitrary function, then the $q-$derivative of f on $[a, b]$ is defined by the following expression*

$$\begin{aligned} {}_aD_q f(t) &= \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)}, \quad t \neq a, \qquad (5)\\ {}_aD_q f(a) &= \lim_{t\to a^+} {}_aD_q f(t)\ . \end{aligned}$$

Clearly, if $a = 0$ in (5), then ${}_0D_q f(t) = D_q f(t)$ where $D_q f(t)$ is well known quantum derivative of the function f defined by

$$\begin{aligned} D_q f(t) &= \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \qquad (6)\\ D_q f(0) &= \lim_{t\to 0} D_q f(t). \end{aligned}$$

Definition 2.2. *[28]Let $f : [a, b] \to \mathbb{R}$ be an arbitrary function, then the quantum integral of f on $[a, b]$ is defined as follows:*

$$\int_a^b f(t)\ {}_ad_qt = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a). \qquad (7)$$

If the series in the right-hand side of (7) converges, then f is said to be quantum integrable on $[a,b]$. Also, for any number $c \in (a,b)$, it is defined in this wise:

$$\int_c^b f(t)\ {}_a d_q t = \int_a^b f(t)\ {}_a d_q t - \int_a^c f(t)\ {}_a d_q t\ . \tag{8}$$

Clearly, if $a = 0$ in (8), then

$$\int_0^b f(t)\ {}_0 d_q t = \int_0^b f(t)\ d_q t\ ,$$

where, $\int_0^b f(t)\ d_q t$ is well-known Jackson integral of f on $[0,b]$. For more details, see [18].

In [20], the authors have denoted (5) and (7) respectively as left quantum derivative and definite left quantum integral and it has been written in this wise :

$$\begin{aligned} {}_a D_q f(t) &= {}_{a^+} D_q f(t)\ , \\ \int_a^b f(t)\ {}_a d_q t &= \int_a^b f(t)\ {}_{a^+} d_q t\ . \end{aligned}$$

We use these notations in the rest of the paper.

Lemma 2.3. *[20, 28]Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then we have*

$$\lim_{q\to 1^-} {}_{a^+} D_q f(t) = \frac{df(t)}{dt}. \tag{9}$$

Lemma 2.4. *[20, 28]Let $f : [a,b] \to \mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\,dt$ is exist, then the following equality*

$$\lim_{q\to 1^-} \int_a^b f(t)\ {}_{a^+} d_q t = \int_a^b f(t)\ dt, \tag{10}$$

is true.

Definition 2.5. *[14, 20]Let a function f be defined on $[a,b]$. Then the right quantum derivative of f on $[a,b]$ is defined by*

$$\begin{aligned} {}_{b^-} D_q f(t) &:= \frac{f(t) - f(qt + (1-q)b)}{(1-q)(t-b)}, \quad t \neq b, \\ {}_{b^-} D_q f(b) &= \lim_{t\to b^-} {}_{b^-} D_q f(t)\ . \end{aligned} \tag{11}$$

Definition 2.6. *[14, 20]Let a function f be defined on $[a,b]$. Then the right quantum integral of f on $[a,b]$ is defined by*

$$\int_a^b f(t)\ {}_{b^-} d_q t = (1-q)(b-a) \sum_{n=0}^{\infty} q^n f(q^n a + (1-q^n) b). \tag{12}$$

If the series in right hand side of (12) converges, then f is said to be right quantum integrable on $[a,b]$. Also, for any number $c \in (a,b)$,it is defined in this wise:

$$\int_a^c f(t)\ {}_{b^-} d_q t = \int_a^b f(t)\ {}_{b^-} d_q t - \int_c^b f(t)\ {}_{b^-} d_q t\ . \tag{13}$$

Lemma 2.7. *[20]Let $f:[a,b]\to\mathbb{R}$ be a differentiable function. Then the following equality holds:*

$$\lim_{q\to 1^-} {}_{b^-}D_q f(t) = \frac{df(t)}{dt}. \tag{14}$$

Lemma 2.8. *[20]Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function. If $\int_a^b f(t)\,dt$ is exist, then we have*

$$\lim_{q\to 1^-}\int_a^b f(t)\ {}_{b^-}d_q t = \int_a^b f(t)\ dt. \tag{15}$$

Recently in [13], the authors have obtained the following results

Theorem 2.9. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $0<q<1$. Then we have*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_q t \leq\frac{f(a)+f(b)}{2}, \tag{16}$$

where

$$\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_q t = \int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_q t + \int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_q t\ . \tag{17}$$

If f is left quantum integrable on $\left[a,\frac{a+b}{2}\right]$ and right quantum integrable on $\left[\frac{a+b}{2},b\right]$, then $\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_q t$ is exist.

Moreover, we have

$$\lim_{q\to 1^-}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_q t = \int_a^b f(t)\ dt. \tag{18}$$

If n is a positive integer, then

$$[n]_q=\frac{1-q^n}{1-q},$$

is referred to as the q-analogue of n.

The following two lemmas that will be usefull to prove the main identity in the next section

Lemma 2.10. *[25]For the continuous functions $g,f:[a,b]\to\mathbb{R}$, the following equality holds true:*

$$\begin{aligned}&\int_0^1 g(t)\ {}_{a^+}D_q f(tb+(1-t)a)\ \ {}_{0^+}d_q t \\ =\ & \left.\frac{g(t)f(tb+(1-t)a)}{b-a}\right|_0^1-\frac{1}{b-a}\int_0^1\ {}_{0^+}D_q g(t)\ f(qtb+(1-qt)a)\ {}_{0^+}d_q t\ .\end{aligned} \tag{19}$$

Lemma 2.11. *[26]For the continuous functions $g,f:[a,b]\to\mathbb{R}$, the following equality holds true:*

$$\begin{aligned}&\int_0^1 g(t)\ {}_{b^-}D_q f(ta+(1-t)b)\ \ {}_{0^+}d_q t \\ =\ & \frac{1}{b-a}\int_0^1\ {}_{0^+}D_q g(t)\ f(qta+(1-qt)b)\ {}_{0^+}d_q t\ -\left.\frac{g(t)f(ta+(1-t)b)}{b-a}\right|_0^1.\end{aligned} \tag{20}$$

3. Main Lemmas

In this section, we derive some identities that are used to obtain some generalized quantum integral inequalities.

Lemma 3.1. *If $\beta \in [0,1]$ and q is in $(0,1)$, then*

$$S(q,\beta) := \int_0^1 |q\tau-\beta| \; _{0^+}d_q\tau = \begin{cases} \frac{2\beta^2([2]_q-1)-q[2]_q\beta+q^2}{q[2]_q}, & 0 \le \frac{\beta}{q} < 1 \\ \frac{[2]_2\beta-q}{[2]_q}, & 1 \le \frac{\beta}{q} \end{cases}, \tag{21}$$

$$T(q,\beta) := \int_0^1 |q\tau-\beta| \frac{\tau}{2} \; _{0^+}d_q\tau = \begin{cases} \frac{\beta^3(2[3]_q-2[2]_q)-q^2\beta[3]_q+q^3[2]_q}{2q^2[2]_q[3]_q}, & 0 \le \frac{\beta}{q} < 1 \\ \frac{\beta[3]_q-q[2]_q}{2[2]_q[3]_q}, & 1 \le \frac{\beta}{q} \end{cases}, \tag{22}$$

and

$$\begin{aligned} \Omega(q,\beta) &= \int_0^1 |q\tau-\beta|\left(1-\frac{\tau}{2}\right) \; _{0^+}d_q\tau \\ &= \begin{cases} \frac{4q\beta^2[3]_q([2]_q-1)-2q^2\beta[2]_q[3]_q+2q^3[3]_q-\beta^3(2[3]_q-2[2]_q)+q^2\beta[3]_q-q^3[2]_q}{2q^2[2]_q[3]_q}, & 0 \le \frac{\beta}{q} < 1 \\ \frac{\beta[3]_q(2[2]_q-1)+q([2]-2[3]_q)}{2[2]_q[3]_q}, & 1 \le \frac{\beta}{q} \end{cases}. \end{aligned} \tag{23}$$

Proof. Using the definition of left quantum integral, the proof can be obtained straightforward calculation, so it is omitted. □

Lemma 3.2. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. If $_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and $_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, then the following identity holds:*

$$\begin{aligned} &\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t) \; _{\frac{a^++b^-}{2}}d_qt \\ = &\frac{b-a}{4}\int_0^1 \left[\begin{array}{c} (qt-\lambda) \; _{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ +(\mu-qt) \; _{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right) \end{array}\right] \; _{0^+}d_qt, \end{aligned} \tag{24}$$

for all $\lambda, \mu \in \mathbb{R}$.

Proof. Since $_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and $_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, then by using the linearity of left quantum integral we have

$$\begin{aligned} &\frac{b-a}{4}\int_0^1 \left[\begin{array}{c} (qt-\lambda) \; _{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \\ +(\mu-qt) \; _{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right) \end{array}\right] \; _{0^+}d_qt \\ = &\frac{b-a}{4}\left[\int_0^1 (qt-\lambda) \; _{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \; _{0^+}d_qt \right. \\ &\left. +\int_0^1 (\mu-qt) \; _{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right) \; _{0^+}d_qt \right] \\ = &\frac{b-a}{4}[B_1+B_2]. \end{aligned} \tag{25}$$

Using (19), we achieve

$$B_1 = \int_0^1 (qt-\lambda) \; _{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right) \; _{0^+}d_qt \tag{26}$$

$$
\begin{aligned}
&= \left.\frac{(qt-\lambda) f\left(t\frac{a+b}{2}+(1-t)a\right)}{\frac{a+b}{2}-a}\right|_0^1 \\
&\quad -\frac{1}{\frac{a+b}{2}-a}\int_0^1 {}_{0^+}D_q(qt-\lambda)\ f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)a\right)\ {}_{0^+}d_qt \\
&= \frac{2(q-\lambda) f\left(\frac{a+b}{2}\right)}{b-a}-\frac{2(-\lambda) f(a)}{b-a}-\frac{2q}{b-a}\int_0^1 f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)a\right)\ {}_{0^+}d_qt \\
&= \frac{2\lambda f(a)}{b-a}+\frac{2(q-\lambda) f\left(\frac{a+b}{2}\right)}{b-a}-\frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt+\frac{2(1-q) f\left(\frac{a+b}{2}\right)}{(b-a)} \\
&= \frac{2\lambda f(a)}{b-a}+\frac{2(1-\lambda) f\left(\frac{a+b}{2}\right)}{b-a}-\frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} f(t)\ {}_{a^+}d_qt\ .
\end{aligned}
$$

Similarly, using (20), we get

$$
\begin{aligned}
B_2 &= \int_0^1 (\mu-qt)\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\ {}_{0^+}d_qt \qquad (27)\\
&= \frac{1}{b-\frac{a+b}{2}}\int_0^1 {}_{0^+}D_q(\mu-qt)\ f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)b\right)\ {}_{0^+}d_qt \\
&\quad -\left.\frac{(\mu-qt) f\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)}{b-\frac{a+b}{2}}\right|_0^1 \\
&= \frac{-2q}{b-a}\int_0^1 f\left(qt\left(\frac{a+b}{2}\right)+(1-qt)b\right)\ {}_{0^+}d_qt-\frac{2(\mu-q) f\left(\frac{a+b}{2}\right)}{b-a}+\frac{2\mu f(b)}{b-a} \\
&= \frac{-4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt+\frac{2(1-q) f\left(\frac{a+b}{2}\right)}{b-a}-\frac{2(\mu-q) f\left(\frac{a+b}{2}\right)}{b-a}+\frac{2\mu f(b)}{b-a} \\
&= \frac{2(1-\mu) f\left(\frac{a+b}{2}\right)}{b-a}+\frac{2\mu f(b)}{b-a}-\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b f(t)\ {}_{b^-}d_qt\ .
\end{aligned}
$$

Combining (25), (26) and (27), we obtain (25). This completes the proof. □

Remark 3.3. *If f is differentiable on $[a,b]$ and $q \to 1^-$, then the identity (24) reduce to (4).*

4. Some generalized quantum integral inequalities

Theorem 4.1. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a continuous function, ${}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, and $0\le\lambda,\mu\le 1$. If $\left|{}_{a^+}D_qf\right|$ and $\left|{}_{b^-}D_qf\right|$ are convex functions on $[a,b]$, then we have the inequality*

$$
\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\right| \qquad (28)\\
\le\ & \frac{b-a}{4}\left[\begin{array}{l} T(q,\lambda)\left|{}_{a^+}D_qf(b)\right|+\Omega(q,\lambda)\left|{}_{a^+}D_qf(a)\right| \\ +T(q,\mu)\left|{}_{b^-}D_qf(a)\right|+\Omega(q,\mu)\left|{}_{b^-}D_qf(b)\right|\end{array}\right],
\end{aligned}
$$

where $T(q,\lambda)$, $T(q,\mu)$, $\Omega(q,\lambda)$ and $\Omega(q,\mu)$ are given in Lemma 3.1.

Proof. From Lemma 3.2 and using the convexity of $\left|{}_{a^+}D_q f\right|$ and $\left|{}_{b^-}D_q f\right|$, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\right| \qquad (29)\\
=&\left|\frac{b-a}{4}\int_0^1\left[\begin{array}{c}(qt-\lambda)\ {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\\ +(\mu-qt)\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{array}\right]{}_{0^+}d_qt\right|\\
\leq&\frac{b-a}{4}\left[\int_0^1\left|qt-\lambda\right|\left|{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|{}_{0^+}d_qt\right.\\
&\left.+\int_0^1\left|\mu-qt\right|\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|{}_{0^+}d_qt\right]\\
=&\frac{b-a}{4}\left[\int_0^1\left|qt-\lambda\right|\left|{}_{a^+}D_qf\left(\frac{t}{2}b+(1-\frac{t}{2})a\right)\right|{}_{0^+}d_qt\right.\\
&\left.+\int_0^1\left|\mu-qt\right|\left|{}_{b^-}D_qf\left(\frac{t}{2}a+(1-\frac{t}{2})b\right)\right|{}_{0^+}d_qt\right]\\
\leq&\frac{b-a}{4}\left[\begin{array}{c}\left|{}_{a^+}D_qf(b)\right|\int_0^1\left|qt-\lambda\right|\frac{t}{2}\ {}_{0^+}d_qt\\ +\left|{}_{a^+}D_qf(a)\right|\int_0^1\left|qt-\lambda\right|\left(1-\frac{t}{2}\right){}_{0^+}d_qt\\ +\left|{}_{b^-}D_qf(a)\right|\int_0^1\left|\mu-qt\right|\frac{t}{2}\ {}_{0^+}d_qt\\ +\left|{}_{b^-}D_qf(b)\right|\int_0^1\left|\mu-qt\right|\left(1-\frac{t}{2}\right){}_{0^+}d_qt\end{array}\right].
\end{aligned}$$

Using Lemma 3.1 in (29), we get (28). This completes the proof. □

The following results are obtained from Theorem.4.1, by taking the specific values of λ,μ and the limit $q\to 1^-$.

Corollary 4.2. *If one takes $\lambda=\mu=1,0,\frac{1}{2}$ and $\frac{1}{3}$ respectively in Theorem.4.1, one has the following quantum trapezoid, midpoint, Bullen and Simpson type inequalities respectively:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\right| \qquad (30)\\
\leq&\frac{b-a}{4}\left[\begin{array}{c}T(q,1)\left|{}_{a^+}D_qf(b)\right|+\Omega(q,1)\left|{}_{a^+}D_qf(a)\right|\\ +T(q,1)\left|{}_{b^-}D_qf(a)\right|+\Omega(q,1)\left|{}_{b^-}D_qf(b)\right|\end{array}\right],
\end{aligned}$$

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\right| \qquad (31)\\
\leq&\frac{b-a}{4}\left[\begin{array}{c}T(q,0)\left|{}_{a^+}D_qf(b)\right|+\Omega(q,0)\left|{}_{a^+}D_qf(a)\right|\\ +T(q,0)\left|{}_{b^-}D_qf(a)\right|+\Omega(q,0)\left|{}_{b^-}D_qf(b)\right|\end{array}\right],
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\right| \qquad (32)\\
\leq&\frac{b-a}{4}\left[\begin{array}{c}T\left(q,\frac{1}{2}\right)\left|{}_{a^+}D_qf(b)\right|+\Omega\left(q,\frac{1}{2}\right)\left|{}_{a^+}D_qf(a)\right|\\ T\left(q,\frac{1}{2}\right)\left|{}_{b^-}D_qf(a)\right|+\Omega\left(q,\frac{1}{2}\right)\left|{}_{b^-}D_qf(b)\right|\end{array}\right],
\end{aligned}$$

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\right| \qquad (33)$$

$$\leq \frac{b-a}{4}\left[\begin{array}{c} T\left(q,\frac{1}{3}\right)\left|\ {}_{a^+}D_q f(b)\ \right| + \Omega\left(q,\frac{1}{3}\right)\left|\ {}_{a^+}D_q f(a)\ \right| \\ T\left(q,\frac{1}{3}\right)\left|\ {}_{b^-}D_q f(a)\ \right| + \Omega\left(q,\frac{1}{3}\right)\left|\ {}_{b^-}D_q f(b)\ \right| \end{array}\right].$$

Remark 4.3. *In inequalities* (30) *and* (31), *we recapture the inequalities [4, Theorem 5.1 and Theorem 4.1], respectively.*

Corollary 4.4. *In particular if f is differentiable on $[a,b]$ and $q \to 1^-$ respectively in* (30) – (33) *inequalities, one has the following trapezoid, midpoint, Bullen and Simpson type inequalities respectively:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right], \tag{34}$$

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right], \tag{35}$$

$$\begin{aligned} &\left|\frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{b-a}{16}\left[\left|f'(a)\right| + \left|f'(b)\right|\right], \end{aligned} \tag{36}$$

$$\begin{aligned} &\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ \leq\ & \frac{5(b-a)}{72}\left[\left|f'(b)\right| + \left|f'(a)\right|\right]. \end{aligned} \tag{37}$$

Remark 4.5. *In inequalities* (34) – (37), *we recapture the inequalities [15, Theorem 2.2], [19, Theorem 2.2], [33, Corollary 3.4, first inequality] and [33, Corollary 3.4, third inequality] (see also [29, Theorem 5]) respectively.*

Theorem 4.6. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function, ${}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)a\right)$ and ${}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)b\right)$ are left quantum integrable on $[0,1]$, and $0 \leq \lambda, \mu \leq 1$. If $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex functions on $[a,b]$, where $r \geq 1$, then we have*

$$\begin{aligned} &\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^+ + b^-}{2}}d_q t\ \right| \\ \leq\ & \frac{b-a}{4}\left[S^{1-\frac{1}{r}}(q,\lambda)\left(T(q,\lambda)\left|\ {}_{a^+}D_q f(b)\ \right|^r + \Omega(q,\lambda)\left|\ {}_{a^+}D_q f(a)\ \right|^r\right)^{\frac{1}{r}}\right. \\ & \left. + S^{1-\frac{1}{r}}(q,\mu)\left(T(q,\mu)\left|\ {}_{b^-}D_q f(a)\ \right|^r + \Omega(q,\mu)\left|\ {}_{b^-}D_q f(b)\ \right|^r\right)^{\frac{1}{r}}\right], \end{aligned} \tag{38}$$

where $S(q,\lambda)$, $S(q,\mu)$, $T(q,\lambda)$, $T(q,\mu)$, $\Omega(q,\lambda)$ and $\Omega(q,\mu)$ are given in Lemma 3.1.

Proof. Since $\left|{}_{a^+}D_q f\right|^r$ and $\left|{}_{b^-}D_q f\right|^r$ are convex functions, from Lemma 3.2 and using power mean inequality, we get

$$\begin{aligned} &\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^+ + b^-}{2}}d_q t\ \right| \\ =\ & \left|\frac{b-a}{4}\int_0^1 \left[\begin{array}{c} (qt-\lambda)\ {}_{a^+}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)a\right) \\ +(\mu - qt)\ {}_{b^-}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)b\right) \end{array}\right]\ {}_{0^+}d_q t\ \right| \end{aligned} \tag{39}$$

$$
\begin{aligned}
&\leq \frac{b-a}{4}\Bigg[\int_0^1 \left|qt-\lambda\right|\left|\ {}_{a^+}D_qf\left(t\left(\tfrac{a+b}{2}\right)+(1-t)a\right)\ \right|\ {}_{0^+}d_qt \\
&\quad +\int_0^1 \left|\mu-qt\right|\left|\ {}_{b^-}D_qf\left(t\left(\tfrac{a+b}{2}\right)+(1-t)b\right)\ \right|\ {}_{0^+}d_qt\ \Bigg] \\
&= \frac{b-a}{4}\Bigg[\int_0^1 \left|qt-\lambda\right|\left|\ {}_{a^+}D_qf\left(\tfrac{t}{2}b+(1-\tfrac{t}{2})a\right)\ \right|\ {}_{0^+}d_qt \\
&\quad +\int_0^1 \left|\mu-qt\right|\left|\ {}_{b^-}D_qf\left(\tfrac{t}{2}a+(1-\tfrac{t}{2})b\right)\ \right|\ {}_{0^+}d_qt\ \Bigg] \\
&\leq \frac{b-a}{4}\Bigg[\left(\int_0^1 \left|qt-\lambda\right|\ {}_{0^+}d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1 \left|qt-\lambda\right|\left|\ {}_{a^+}D_qf\left(\tfrac{t}{2}b+(1-\tfrac{t}{2})a\right)\ \right|^r\ {}_{0^+}d_qt\right)^{\frac{1}{r}} \\
&\quad +\left(\int_0^1 \left|\mu-qt\right|\ {}_{0^+}d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1 \left|\mu-qt\right|\left|\ {}_{b^-}D_qf\left(\tfrac{t}{2}a+(1-\tfrac{t}{2})b\right)\ \right|^r\ {}_{0^+}d_qt\right)^{\frac{1}{r}}\Bigg] \\
&\leq \frac{b-a}{4}\Bigg[\left(\int_0^1 \left|qt-\lambda\right|\ {}_{0^+}d_qt\right)^{1-\frac{1}{r}}\left(\begin{array}{c}\left|\ {}_{a^+}D_qf(b)\ \right|^r\int_0^1\left|qt-\lambda\right|\tfrac{t}{2}\ {}_{0^+}d_qt \\ +\left|\ {}_{a^+}D_qf(a)\ \right|^r\int_0^1\left|qt-\lambda\right|\left(1-\tfrac{t}{2}\right)\ {}_{0^+}d_qt\end{array}\right)^{\frac{1}{r}} \\
&\quad +\left(\int_0^1 \left|\mu-qt\right|\ {}_{0^+}d_qt\right)^{1-\frac{1}{r}}\left(\begin{array}{c}\left|\ {}_{b^-}D_qf(a)\ \right|^r\int_0^1\left|\mu-qt\right|\tfrac{t}{2}\ {}_{0^+}d_qt \\ +\left|\ {}_{b^-}D_qf(b)\ \right|^r\int_0^1\left|\mu-qt\right|\left(1-\tfrac{t}{2}\right)\ {}_{0^+}d_qt\end{array}\right)^{\frac{1}{r}}\Bigg].
\end{aligned}
$$

Using Lemma 3.1 in (39), we have (38). This completes the proof. □

The following results are obtained from Theorem.4.6, by taking the specific values of λ,μ and the limit $q \to 1^-$.

Corollary 4.7. *If one takes $\lambda = \mu = 1, 0, \frac{1}{2}$ and $\frac{1}{3}$ respectively in Theorem.4.6, one has the following quantum trapezoid, midpoint, Bullen and Simpson type inequalities respectively:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\ \right| \qquad (40)\\
&\leq \frac{b-a}{4}S^{1-\frac{1}{r}}(q,1)\left[\begin{array}{c}\left(T(q,1)\left|\ {}_{a^+}D_qf(b)\ \right|^r+\Omega(q,1)\left|\ {}_{a^+}D_qf(a)\ \right|^r\right)^{\frac{1}{r}} \\ +\left(T(q,1)\left|\ {}_{b^-}D_qf(a)\ \right|^r+\Omega(q,1)\left|\ {}_{b^-}D_qf(b)\ \right|^r\right)^{\frac{1}{r}}\end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\ \right| \qquad (41)\\
&\leq \frac{b-a}{4}S^{1-\frac{1}{r}}(q,0)\left[\begin{array}{c}\left(T(q,0)\left|\ {}_{a^+}D_qf(b)\ \right|^r+\Omega(q,0)\left|\ {}_{a^+}D_qf(a)\ \right|^r\right)^{\frac{1}{r}} \\ +\left(T(q,0)\left|\ {}_{b^-}D_qf(a)\ \right|^r+\Omega(q,0)\left|\ {}_{b^-}D_qf(b)\ \right|^r\right)^{\frac{1}{r}}\end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\ \right| \qquad (42)\\
&\leq \frac{b-a}{4}S^{1-\frac{1}{r}}\left(q,\frac{1}{2}\right)\left[\begin{array}{c}\left(T\left(q,\tfrac{1}{2}\right)\left|\ {}_{a^+}D_qf(b)\ \right|^r+\Omega\left(q,\tfrac{1}{2}\right)\left|\ {}_{a^+}D_qf(a)\ \right|^r\right)^{\frac{1}{r}} \\ +\left(T\left(q,\tfrac{1}{2}\right)\left|\ {}_{b^-}D_qf(a)\ \right|^r+\Omega\left(q,\tfrac{1}{2}\right)\left|\ {}_{b^-}D_qf(b)\ \right|^r\right)^{\frac{1}{r}}\end{array}\right],
\end{aligned}
$$

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^{+}+b^{-}}{2}}d_qt\ \right| \tag{43}$$
$$\leq \frac{b-a}{4}S^{1-\frac{1}{r}}\left(q,\frac{1}{3}\right)\left[\begin{array}{c}\left(T\left(q,\frac{1}{3}\right)\left|\ {}_{a^{+}}D_qf(b)\ \right|^r+\Omega\left(q,\frac{1}{3}\right)\left|\ {}_{a^{+}}D_qf(a)\ \right|^r\right)^{\frac{1}{r}}\\+\left(T\left(q,\frac{1}{3}\right)\left|\ {}_{b^{-}}D_qf(a)\ \right|^r+\Omega\left(q,\frac{1}{3}\right)\left|\ {}_{b^{-}}D_qf(b)\ \right|^r\right)^{\frac{1}{r}}\end{array}\right].$$

Remark 4.8. *In inequalities* (40) *and* (41), *we recapture the inequalities [4, Theorem 5.2 and Theorem 4.2], respectively.*

Corollary 4.9. *In particular if f is differentiable on $[a,b]$ and $q\to 1^-$ respectively in* (40) – (43) *inequalities, one has the following trapezoid, midpoint, Bullen and Simpson type inequalities respectively:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{44}$$
$$\leq \frac{b-a}{8}\left(\frac{1}{6}\right)^{\frac{1}{r}}\left[\left(\left|f'(b)\right|^r+5\left|f'(a)\right|^r\right)^{\frac{1}{r}}+\left(\left|f'(a)\right|^r+5\left|f'(b)\right|^r\right)^{\frac{1}{r}}\right],$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{45}$$
$$\leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{r}}\left[\left(\left|f'(b)\right|^r+2\left|f'(a)\right|^r\right)^{\frac{1}{r}}+\left(\left|f'(b)\right|^r+2\left|f'(a)\right|^r\right)^{\frac{1}{r}}\right],$$

$$\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{46}$$
$$\leq \frac{b-a}{16}\left(\frac{1}{12}\right)^{\frac{1}{r}}\left[\left(3\left|f'(b)\right|^r+9\left|f'(a)\right|^r\right)^{\frac{1}{r}}+\left(3\left|f'(a)\right|^r+9\left|f'(b)\right|^r\right)^{\frac{1}{r}}\right],$$

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \tag{47}$$
$$\leq \frac{5(b-a)}{72}\left(\frac{1}{90}\right)^{\frac{1}{r}}\left[\left(29\left|f'(b)\right|^r+61\left|f'(a)\right|^r\right)^{\frac{1}{r}}+\left(29\left|f'(a)\right|^r+61\left|f'(b)\right|^r\right)^{\frac{1}{r}}\right].$$

Remark 4.10. *In inequalities* (46) *and* (47), *we recapture the inequalities [33, Corollary 3.3, first inequality] and [33, Corollary 3.3, third inequality] (see also [29, Theorem 7]) respectively. Also, we consider that the inequalities* (44) *and* (45) *are new.*

Theorem 4.11. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a continuous function, ${}_{a^{+}}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)$ and ${}_{b^{-}}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)$ are left quantum integrable on $[0,1]$, and $0\leq\lambda,\mu\leq 1$. If $\left|{}_{a^{+}}D_qf\right|^r$ and $\left|{}_{b^{-}}D_qf\right|^r$ are convex functions on $[a,b]$, where $\frac{1}{r}+\frac{1}{p}=1$ and $r>1$, then we have*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^{+}+b^{-}}{2}}d_qt\ \right| \tag{48}$$
$$\leq \frac{b-a}{4}\left[\begin{array}{c}M^{\frac{1}{p}}(q,\lambda)\left(\frac{\left|\ {}_{a^{+}}D_qf(b)\ \right|^r+3\left|\ {}_{a^{+}}D_qf(a)\ \right|^r}{4}\right)^{\frac{1}{r}}\\+M^{\frac{1}{p}}(q,\mu)\left(\frac{\left|\ {}_{b^{-}}D_qf(a)\ \right|^r+3\left|\ {}_{b^{-}}D_qf(b)\ \right|^r}{4}\right)^{\frac{1}{r}}\end{array}\right],$$

where

$$M(q,\lambda)=\int_0^1 \left|qt-\lambda\right|^p \ {}_{0^+}d_qt \ ,$$

and,

$$M(q,\mu)=\int_0^1 \left|\mu-qt\right|^p \ {}_{0^+}d_qt \ .$$

Proof. Since $\left|{}_{a^+}D_qf\right|^r$ and $\left|{}_{b^-}D_qf\right|^r$ are convex functions, from Lemma 3.2 and using Hölder inequality, we get

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\ {}_{\frac{a^++b^-}{2}}d_qt\right| \qquad (49)\\
=&\left|\frac{b-a}{4}\int_0^1\left[\begin{array}{c}(qt-\lambda)\ {}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\\ +(\mu-qt)\ {}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\end{array}\right]{}_{0^+}d_qt\right|\\
\leq&\ \frac{b-a}{4}\left[\int_0^1\left|qt-\lambda\right|\left|{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|{}_{0^+}d_qt\right.\\
&\left.+\int_0^1\left|\mu-qt\right|\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|{}_{0^+}d_qt\right]\\
\leq&\ \frac{b-a}{4}\left[\left(\int_0^1\left|qt-\lambda\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{a^+}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)a\right)\right|^r{}_{0^+}d_qt\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\int_0^1\left|\mu-qt\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{b^-}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)b\right)\right|^r{}_{0^+}d_qt\right)^{\frac{1}{r}}\right]\\
=&\ \frac{b-a}{4}\left[\left(\int_0^1\left|qt-\lambda\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{a^+}D_qf\left(\tfrac{t}{2}b+(1-\tfrac{t}{2})a\right)\right|^r{}_{0^+}d_qt\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\int_0^1\left|\mu-qt\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}_{b^-}D_qf\left(\tfrac{t}{2}a+(1-\tfrac{t}{2})b\right)\right|^r{}_{0^+}d_qt\right)^{\frac{1}{r}}\right]\\
\leq&\ \frac{b-a}{4}\left[\left(\int_0^1\left|qt-\lambda\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\left|{}_{a^+}D_qf(b)\right|^r\int_0^1\frac{t}{2}\,{}_{0^+}d_qt+\left|{}_{a^+}D_qf(a)\right|^r\int_0^1\left(1-\frac{t}{2}\right){}_{0^+}d_qt\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\int_0^1\left|\mu-qt\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\left|{}_{b^-}D_qf(a)\right|^r\int_0^1\frac{t}{2}\,{}_{0^+}d_qt+\left|{}_{b^-}D_qf(b)\right|^r\int_0^1\left(1-\frac{t}{2}\right){}_{0^+}d_qt\right)^{\frac{1}{r}}\right]\\
\leq&\ \frac{b-a}{4}\left[\left(\int_0^1\left|qt-\lambda\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}_{a^+}D_qf(b)\right|^r+(1+2q)\left|{}_{a^+}D_qf(a)\right|^r}{2(1+q)}\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\int_0^1\left|\mu-qt\right|^p{}_{0^+}d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}_{b^-}D_qf(a)\right|^r+(1+2q)\left|{}_{b^-}D_qf(b)\right|^r}{2(1+q)}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

This completes the proof. □

The following results are obtained from Theorem 4.11, by taking the specific values of λ,μ and the limit $q\to1^-$.

Corollary 4.12. *If one takes $\lambda = \mu = 1, 0, \frac{1}{2}$ and $\frac{1}{3}$ respectively in Theorem.4.11, one has the following quantum trapezoid, midpoint, Bullen and Simpson type inequalities respectively:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \ {}_{\frac{a^+ + b^-}{2}} d_q t \right| \tag{50}$$
$$\leq \frac{b-a}{4} M^{\frac{1}{p}}(q,1) \left[\begin{array}{c} \left(\frac{\left| {}_{a^+}D_q f(b) \right|^r + (1+2q) \left| {}_{a^+}D_q f(a) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \\ + \left(\frac{\left| {}_{b^-}D_q f(a) \right|^r + (1+2q) \left| {}_{b^-}D_q f(b) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \end{array} \right],$$

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) \ {}_{\frac{a^+ + b^-}{2}} d_q t \right| \tag{51}$$
$$\leq \frac{b-a}{4} M^{\frac{1}{p}}(q,0) \left[\begin{array}{c} \left(\frac{\left| {}_{a^+}D_q f(b) \right|^r + (1+2q) \left| {}_{a^+}D_q f(a) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \\ + \left(\frac{\left| {}_{b^-}D_q f(a) \right|^r + (1+2q) \left| {}_{b^-}D_q f(b) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \end{array} \right],$$

$$\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t) \ {}_{\frac{a^+ + b^-}{2}} d_q t \right| \tag{52}$$
$$\leq \frac{b-a}{4} M^{\frac{1}{p}}\left(q,\frac{1}{2}\right) \left[\begin{array}{c} \left(\frac{\left| {}_{a^+}D_q f(b) \right|^r + (1+2q) \left| {}_{a^+}D_q f(a) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \\ + \left(\frac{\left| {}_{b^-}D_q f(a) \right|^r + (1+2q) \left| {}_{b^-}D_q f(b) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \end{array} \right],$$

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(t) \ {}_{\frac{a^+ + b^-}{2}} d_q t \right| \tag{53}$$
$$\leq \frac{b-a}{4} M^{\frac{1}{p}}\left(q,\frac{1}{3}\right) \left[\begin{array}{c} \left(\frac{\left| {}_{a^+}D_q f(b) \right|^r + (1+2q) \left| {}_{a^+}D_q f(a) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \\ + \left(\frac{\left| {}_{b^-}D_q f(a) \right|^r + (1+2q) \left| {}_{b^-}D_q f(b) \right|^r}{2(1+q)} \right)^{\frac{1}{r}} \end{array} \right].$$

Remark 4.13. *In inequalities* (50) *and* (51), *we recapture the inequalities [4, Theorem 5.3 and Theorem 4.3], respectively.*

Corollary 4.14. *In particular if f is differentiable on $[a, b]$ and $q \to 1^-$ respectively in* (50) − (53) *inequalities, one has the following trapezoid, midpoint, Bullen and Simpson type inequalities respectively:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{54}$$
$$\leq \frac{b-a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(\frac{\left| f'(b) \right|^r + 3 \left| f'(a) \right|^r}{4} \right)^{\frac{1}{r}} + \left(\frac{\left| f'(a) \right|^r + 3 \left| f'(b) \right|^r}{4} \right)^{\frac{1}{r}} \right],$$

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \tag{55}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(b)\right|^{r}+3\left|f'(a)\right|^{r}}{4}\right)^{\frac{1}{r}}+\left(\frac{\left|f'(a)\right|^{r}+3\left|f'(b)\right|^{r}}{4}\right)^{\frac{1}{r}}\right],$$

$$\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right| \tag{56}$$
$$\leq \frac{b-a}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(b)\right|^{r}+3\left|f'(a)\right|^{r}}{4}\right)^{\frac{1}{r}}+\left(\frac{\left|f'(a)\right|^{r}+3\left|f'(b)\right|^{r}}{4}\right)^{\frac{1}{r}}\right],$$

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right| \tag{57}$$
$$\leq \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(b)\right|^{r}+3\left|f'(a)\right|^{r}}{4}\right)^{\frac{1}{r}}+\left(\frac{\left|f'(a)\right|^{r}+3\left|f'(b)\right|^{r}}{4}\right)^{\frac{1}{r}}\right].$$

Remark 4.15. *In inequality* (57)*, we recapture the inequality [29, Theorem 6]. Also, we consider that the inequalities* (54) – (56) *are new.*

5. Conclusion

In this study, some of the obtained results are new and some are the generalization of the results in [4, 15, 19, 29, 33], to their quantum forms. Moreover, the method for establishing the quantum forms of the integral inequalities is different from the methods which have been used in the literature. We hope that applying the method and techniques of this paper in post-quantum integral and as well as for obtaining the quantum form of integral inequalities in different classes of functions will help the researchers for further study in these areas.

References

[1] Ali, M. A., Abbas, M., Budak, H., Agarwal, P., Murtaza, G., & Chu, Y. M. (2021). New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. Advances in Difference Equations, 2021(1), 1-21.

[2] Ali, M. A., Budak, H., Abbas, M., & Chu, Y. M. (2021). Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives. Advances in Difference Equations, 2021(1), 1-12.

[3] Ali, M. A., Budak, H., Zhang, Z., & Yildirim, H. (2021). Some new Simpson's type inequalities for coordinated convex functions in quantum calculus. Mathematical Methods in the Applied Sciences, 44(6), 4515-4540.

[4] Ali, M. A., Budak, H., Fečkan, M., & Khan, S. (2023). A new version of q-Hermite-Hadamard's midpoint and trapezoid type inequalities for convex functions. Mathematica Slovaca, 73(2), 369-386.

[5] Ali, M. A., Chu, Y. M., Budak, H., Akkurt, A., Yıldırım, H., & Zahid, M. A. (2021). Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. Advances in Difference Equations, 2021(1), 1-26.

[6] Alomari, M., Darus, M., & Dragomir, S. S. (2009). New inequalities of Simpson's type for s-convex functions with applications. Research report collection, 12(4).

[7] Alp, N., Sarıkaya, M. Z., Kunt, M., & İşcan, İ. (2018). q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. Journal of King Saud University-Science, 30(2), 193-203.

[8] Awan, M. U., Talib, S., Kashuri, A., Noor, M. A., & Chu, Y. M. (2020). Estimates of quantum bounds pertaining to new q-integral identity with applications. Advances in Difference Equations, 2020(1), 1-15.

[9] Awan, M. U., Talib, S., Kashuri, A., Noor, M. A., Noor, K. I., & Chu, Y. M. (2020). A new q-integral identity and estimation of its bounds involving generalized exponentially μ-preinvex functions. Advances in Difference Equations, 2020(1), 1-12.

[10] Budak, H., Ali, M. A., & Tarhanaci, M. (2020). Some new quantum Hermite–Hadamard-like inequalities for coordinated convex functions. Journal of Optimization Theory and Applications, 186(3), 899-910.

[11] Budak, H., Ali, M. A., & Tunç, T. (2020). Quantum Ostrowski-type integral inequalities for functions of two variables. Mathematical Methods in the Applied Sciences.

[12] Budak, H., Erden, S., & Ali, M. A. (2021). Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. Mathematical Methods in the Applied Sciences, 44(1), 378-390.

[13] Baidar, A., Kunt, M., (2022). Quantum analogue of some trapezoid and midpoint type inequalities for convex functions, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 71(2), 456-480.

[14] On q-Hermite–Hadamard inequalities for general convex functions. Acta mathematica hungarica, 162, 364-374.

[15] Dragomir, S. S., & Agarwal, R. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Applied mathematics letters, 11(5), 91-95.

[16] Erden, S., Iftikhar, S., Delavar, M. R., Kumam, P., Thounthong, P., & Kumam, W. (2020). On generalizations of some inequalities for convex functions via quantum integrals. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 114(3), 1-15.

[17] Fedotov, I., & Dragomir, S. S. (1999). An inequality of Ostrowski type and its applications for Simpson's rule and special means. RGMIA research report collection, 2(1).

[18] Kac, V., and Cheung, P., Quantum calculus. Springer Science & Business Media, 2001.

[19] Kirmaci, Uğur. S. (2004). Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Applied mathematics and computation, 147(1), 137-146.

[20] Kunt, M., Baidar, A., & Şanlı, Z. (2022). Left-Right quantum derivatives and definite integrals, Turkish Journal of Science and Technology, 17(2), 343-356.

[21] Khan, M. A., Mohammad, N., Nwaeze, E. R., & Chu, Y. M. (2020). Quantum Hermite–Hadamard inequality by means of a Green function. Advances in Difference Equations, 2020(1), 1-20.

[22] Noor, M. A., Noor, K. I., & Awan, M. U. (2015). Some quantum estimates for Hermite–Hadamard inequalities. Applied Mathematics and Computation, 251, 675-679.

[23] Prabseang, J., Nonlaopon, K., & Ntouyas, S. K. (2020). On the refinement of quantum Hermite-Hadamard inequalities for continuous convex functions J. Math. Inequal,, 14(3), 875-885.

[24] Rashid, S., Butt, S. I., Kanwal, S., Ahmad, H., & Wang, M. K. (2021). Quantum Integral Inequalities with Respect to Raina's Function via Coordinated Generalized-Convex Functions with Applications. Journal of Function Spaces, 2021.

[25] Soontharanon, J., Ali, M. A., Budak, H., Nonlaopon, K., & Abdullah, Z. (2022). Simpson's and Newton's Type Inequalities for (α, m)-Convex Functions via Quantum Calculus. Symmetry, 14(4), 736

[26] Sial, I. B., Mei, S., Ali, M. A., & Nonlaopon, K. (2021). On some generalized Simpson's and Newton's inequalities for (α, m)-convex functions in q-calculus. Mathematics, 9(24), 3266

[27] Sudsutad, W., Ntouyas, S. K., & Tariboon, J. (2015). Quantum integral inequalities for convex functions. J. Math. Inequal, 9(3), 781-793.

[28] Tariboon, J., & Ntouyas, S. K. (2013). Quantum calculus on finite intervals and applications to impulsive difference equations. Advances in Difference Equations, 2013(1), 1-19..

[29] Sarikaya, M. Z., Set, E., & Özdemir, M. E. (2010). On new inequalities of Simpson's type for convex functions, RGMIA Res. Rep. Coll, 13(2).

[30] Ujević, N. (2007). New error bounds for the Simpson's quadrature rule and applications. Computers & Mathematics with Applications, 53(1), 64-72.

[31] Vivas-Cortez, M., Aamir Ali, M., Kashuri, A., Bashir Sial, I., & Zhang, Z. (2020). Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus. Symmetry, 12(9), 1476.

[32] Vivas-Cortez, M., Kashuri, A., Liko, R., & Hernández, J. E. H. (2020). Some new q-integral inequalities using generalized quantum Montgomery identity via preinvex functions. Symmetry, 12(4), 553.

[33] Xi, B. Y., & Qi, F. (2012). Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means. Journal of Function Spaces and Applications, 2012.

[34] Xi, B. Y., & Qi, F. (2013). Some Hermite-Hadamard Type Inequalities for Differentiable Convex Functions and Applications. Hacettepe Journal of Mathematics and Statistics, 42(3), 243-257.

[35] You, X., Kara, H., Budak, H., & Kalsoom, H. (2021). Quantum Inequalities of Hermite–Hadamard Type for-Convex Functions. Journal of Mathematics, 2021.