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Solving fractional differential equations using cubic Hermit spline functions

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Abstract. In this article, we solve fractional differential equations in the Caputo fractional derivative sense using cubic Hermite spline functions. We first construct the operational matrix to the fractional derivative of the cubic Hermite spline functions. Then using this matrix and some properties of these functions, we convert a fractional differential equation into a system of algebraic equations that can be solved to find the approximate solution. Numerous examples show that the results obtained by this method are in full agreement with the results presented by some previous works.

1. Introduction

The concept of derivatives of non-integer order can be traced back to Leibniz's note in a letter to L'Hospital, where he discussed the meaning of derivatives of order half. This note paved the way for the development of the theory of derivatives and integrals of arbitrary order, which took shape by the end of the 19th century thanks to Liouville, Grundwold, Letnikov, and Riemann [17, 38].

Fractional differential equations (FDEs) are a relatively new topic in applied mathematics that has garnered significant attention from researchers in recent years due to their frequent use across various disciplines. Nature employs fractional derivatives to create complex models of the real world, earning fractional calculus the moniker "21st-century calculus" for its ability to describe such models effectively. A comprehensive history of fractional differential operators and their applications can be found in [12, 16, 17, 25, 28, 29, 31, 38], with [17] and [38] being two excellent references on FDEs. Numerous authors have investigated analytic results on the existence and uniqueness of solutions to FDEs (see [16, 25] for examples). Additionally, some types of FDEs have been subject to analytic solution investigations [24].

Recently, FDEs have been used in a wide range of scientific fields, including physics, mechanics, fluid flow, image processing, and more. For examples, see [6, 28, 30, 31]. Since in most cases one cannot find the explicit solution of FDEs, the development, validation, and program implementation of numerical methods for these types of equations are of fundamental relevance. Some of these numerical approaches are recalled as follows.

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Abdelkawy et al. [4] successfully solved non-linear variable-order fractional Fredholm integro-differential equations using a collocation method based on fractional-order Legendre functions. In [9], an operational matrix method based on Laguerre polynomials was utilized to solve FDEs on the half line. The authors in [22] tackled FDEs by solving ordinary differential equations, while Shahmorad et al. [42] employed a geometric approach for solving nonlinear fractional integro-differential equations. Bahmanpour et al. [7] utilized a Müntz wavelets collocation method to solve FDEs, and in [45], two ideas based on discretization of the fractional differential operator and integral form of the FDE were used to develop higher order numerical techniques for solving FDEs. In addition, a finite element approach with cubic Hermit element was employed to solve a time fractional gas dynamics equation in [37]. Furthermore, Pourbabaee et al. [39] solved distributed order FDEs using a collocation method based on Chebyshev polynomials. For further research works on this problem, we recommend interested readers to refer to [1, 3, 10, 19, 32, 36, 43].

In this paper, we would like to solve a fractional initial value problem (FIVP) in the form

$$\begin{cases} {}_*D_0^\alpha y(\mathrm{x}) = \mathsf{F}(\mathrm{x}, y(\mathrm{x})), \quad 0 < \alpha < 1,\ 0 \le \mathrm{x} \le L, \\ y(0) = y_0, \end{cases} \tag{1}$$

where ${}_*D_0^\alpha$ denotes the fractional derivative in the Caputo sense and y_0 is real number.

First we introduce and construct the operational matrix of fractional derivative for cubic Hermite spline functions (CHSFs). Then we expand the unknown function as a linear combination of CHSFs with unknown coefficients. Using the operational matrix of derivative and collocation method we reduce the problem (1) to system of algebraic equations which can be solved to find the unknown coefficients.

This paper is organized as follow: In section 2, we give some preliminaries and definitions needed for our work. In section 3, the numerical method is presented. A convergence analysis of the method is discussed in section 4, and we show that if the unknown function lies in $H^4(\Omega)$, where Ω is the domain of the problem, then the order of convergence will be $O(2^{-4J})$. Some numerical examples are presented to show the efficiency and validity of the method in section 5. We finish the paper with a conclusion and suggesting a future work.

2. Preliminaries and notations

2.1. The fractional derivative in the Caputo sense and some other definitions

In this section, we review the basic concepts and definitions of fractional calculus which will be used throughout the next sections.

Definition 2.1. *[17] Let α be a positive real number. The Riemann-Liouville fractional integral operator of order α, denoted by J_a^α, is defined on the interval $[a, b]$ and acts on functions $L^1[a, b]$ as*

$$J_a^\alpha f(\mathrm{x}) := \frac{1}{\Gamma(\alpha)} \int_a^{\mathrm{x}} (\mathrm{x} - t)^{\alpha-1} f(t)dt,\ a \le \mathrm{x} \le b.$$

Specially when $\alpha = 0$ we define $J_a^0 f(\mathrm{x}) = f(\mathrm{x})$.

Definition 2.2. *[17] Let $\alpha \ge 0$ and $n = \lceil \alpha \rceil$. Then the Caputo fractional derivative (CFD) of order α is defined as*

$${}_*D_a^\alpha f(\mathrm{x}) = J_a^{n-\alpha} D^n f(\mathrm{x}),$$

when $D^n f \in L^1[a, b]$. Using definition 2.1, it is then written in the following form

$${}_*D_a^\alpha f(\mathrm{x}) = \frac{1}{\Gamma(n - \alpha)} \int_a^{\mathrm{x}} \frac{f^{(n)}(t)}{(\mathrm{x} - t)^{\alpha-n+1}} dt.$$

In the case of the Caputo derivative we have

$${}_*D_a^n C = 0 \quad (C \text{ is constant}),$$

$$
{}_*D_a^n\ \mathbf{x}^\beta = \begin{cases} 0 & \textit{for } \beta \in \mathbb{N}_0 \textit{ and } \beta < \lceil n \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}\ \mathbf{x}^{\beta-n} & \textit{for } \beta \in \mathbb{N}_0 \textit{ and } \beta \geqslant \lceil n \rceil \textit{ or } \beta \notin \mathbb{N} \textit{ and } \beta > \lfloor n \rfloor. \end{cases}
$$

It is worth noting that the ceiling function $\lceil n \rceil$ is the smallest integer greater than or equal to n and the floor function $\lfloor n \rfloor$ is the largest integer less than or equal to n.

The operator ${}_*D_a^n$ is linear, i.e.,

$$
{}_*D_a^n\Big(Af(\mathbf{x}) + Bg(\mathbf{x})\Big) = A\ {}_*D_a^n f(\mathbf{x}) + B\ {}_*D_a^n g(\mathbf{x}),
$$

where A and B are arbitrary constants.

Definition 2.3. *[17] For $n > 0$, one parameter Mittage-Leffler function is defined by*

$$
E_n(z) := \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\Gamma(\kappa n + 1)}.
$$

This function has been introduced by Mittage-Leffler [34]. It is evident that

$$
E_1(z) = \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\kappa!} = exp(z),
$$

is just the well known exponential function.

Definition 2.4. *For $n, m > 0$, two-parameter Mittage-Leffler function is defined by*

$$
E_{n,m}(z) := \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\Gamma(\kappa n + m)}.
$$

It is immediately apparent that

$$
E_{0,1}(z) := \sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \qquad |z| < 1.
$$

It is noteworthy that the one-parameter Mittage-Leffler function represents a specific instance of the two-parameter variant,i.e. $E_n(z) = E_{n,1}(z)$.

Definition 2.5. *Let Ω, be an open subset of $\mathbb{R}$. Then the space $H^s(\Omega)$ defined by*

$$
H^s(\Omega) = \{f \in L^2(\Omega);\ f^{(\alpha)} \in L^2(\Omega), \forall\alpha,\ 0 \leq \alpha \leq s\},
$$

is called Sobolev space of order s [27]. The norm for this space is defined by

$$
\|f\|_{s,\Omega} = \left(\sum_{\alpha=0}^{s} \|f^{(\alpha)}\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.
$$

For the non-integer value $s \in (0, 1)$, the fractional Sobolev space is defined as [8]

$$
W^{s,2}(a, b) = \left\{ f \in L^2(a, b) : \frac{f(\mathbf{x}) - f(y)}{|\mathbf{x} - y|^{\frac{1}{2}+s}} \in L^2([a, b] \times [a, b]) \right\},
$$

with the related norm

$$
\|f\|_{W^{s,2}(a,b)} = \left[\int_a^b |f(\mathbf{x})|^2 d\mathbf{x} + \int_a^b \int_a^b \frac{|f(\mathbf{x}) - f(y)|^2}{|\mathbf{x} - y|^{1+2s}} d\mathbf{x} dy \right]^{\frac{1}{2}}.
$$

If $s = n + \mu > 1$ with $n \in \mathbb{N}$ and $\mu \in \mathbb{N}$, then the fractional Sobolev space is defined as

$$
W^{s,2}(a, b) = \left\{ f \in H^n(a, b) : D^n f \in W^{\mu,2}(a, b) \right\}.
$$

2.2. *Cubic Hermite spline functions on* $[0, L]$

A cubic Hermite spline or cubic Hermite interpolator is a spline function such that each piece is a polynomial of third-degree specified in Hermite form. In other words, cubic Hermite spline functions are defined by [14, 35]

$$\phi_1(\mathrm{x}) = \begin{cases} (\mathrm{x}+1)^2(-2\mathrm{x}+1), & \mathrm{x} \in [-1, 0], \\ (1-\mathrm{x})^2(2\mathrm{x}+1), & \mathrm{x} \in [0, 1], \\ 0, & o.w., \end{cases} \tag{2}$$

$$\phi_2(\mathrm{x}) = \begin{cases} (\mathrm{x}+1)^2\mathrm{x}, & \mathrm{x} \in [-1, 0], \\ (1-\mathrm{x})^2\mathrm{x}, & \mathrm{x} \in [0, 1], \\ 0, & o.w., \end{cases} \tag{3}$$

and satisfy the following interpolation properties

$$\phi_1(k) = \delta_{0,k}, \quad \phi_1'(k) = 0, \quad \phi_2(k) = 0, \quad \phi_2'(k) = \delta_{0,k} \ (k \in \mathbb{Z}),$$

where

$$\delta_{i,k} = \begin{cases} 1, & i = k, \\ 0, & o.w., \end{cases}$$

The integer transformations of ϕ_1 and ϕ_2 form a basis for the space of C^1-continuous piecewise cubic functions on $\mathbb{R}$ that interpolate function values and their first derivatives at $k \in \mathbb{Z}$. The dilation and translation parameters are denoted by 2^j and k, respectively where $j, k \in \mathbb{Z}$. For $l = 1, 2$, we can express any function in this space as a linear combination of $\phi_l(2^j\mathrm{x} - k)$.

Suppose

$$B_{j,k}(\mathrm{x}) = supp\Big[\phi_l^{j,k}(\mathrm{x})\Big] = clos\Big\{\mathrm{x} : \phi_l^{j,k}(\mathrm{x}) \neq 0\Big\},$$

with simple calculations, it can be shown that

$$B_{j,k}(\mathrm{x}) = \Big[\frac{k-1}{2^j}, \frac{k+1}{2^j}\Big], \quad \text{for } \ j, k \in \mathbb{Z}.$$

Define a set of indices as follows

$$S_j = \Big\{B_{j,k} \cap (0, L) \neq \emptyset\Big\}, \quad j, k \in \mathbb{Z}.$$

It is evident that

$$S_j = \Big\{0, 1, 2, ..., L \times 2^j\Big\}, \quad j \in \mathbb{Z}.$$

We require the cubic Hermite functions to be defined on the interval $[0, L]$. Therefore we introduce

$$\phi_l^{j,k}(\mathrm{x}) = \phi_l^{j,k}(\mathrm{x}).\chi_{[0,L]}(\mathrm{x}), \quad j \in \mathbb{Z}, \ k \in S_j, \ l = 1, 2.$$

2.3. *Function approximation*

Suppose $\Phi_j(.)$ is $2(L \times 2^j + 1)$ vector as

$$\Phi_j(.) = \Big[\phi_1^{j,0}(.), \phi_2^{j,0}(.), ..., \phi_1^{j,L\times 2^j}(.), \phi_2^{j,L\times 2^j}(.)\Big]^T, \quad j \in \mathbb{Z}. \tag{4}$$

Due to the interpolatory nature of the functions ϕ_1 and ϕ_2, when $j = J$ is fixed, it is possible to approximate a function $f \in H^4[0, L]$ with cubic Hermite functions as

$$f(\mathtt{x}) \simeq \sum_{\kappa=0}^{L\times 2^J} \Big(c_{1,\kappa}\ \phi_1^{J,\kappa}(\mathtt{x}) + c_{2,\kappa}\ \phi_2^{J,\kappa}(\mathtt{x}) \Big) = C^T\ \Phi_J(\mathtt{x}), \tag{5}$$

where

$$c_{1,\kappa} = f(\frac{\kappa}{2^J}),\ \ c_{2,\kappa} = 2^{-J} f'(\frac{\kappa}{2^J}), \quad \kappa = 0, 1, \dots, L \times 2^J,$$

and C is a N vector as

$$C = \Big[c_{1,0},\ c_{2,0}, \dots, c_{1,L\times 2^J},\ c_{2,L\times 2^J} \Big]^T,$$

in which $N = 2(L \times 2^J + 1)$.

2.4. Fractional Derivative Operational Matrix

The CFD of the functions $\phi_i^{J,k}(.), k = 0, 1, 2, \dots, L \times 2^J, i = 1, 2$ of order $n, (0 \le n < 1)$ can be approximated as

$$\begin{aligned} {}_*D_0^\alpha \phi_i^{J,\ell}(\mathtt{x}) &= {}_*D_0^\alpha \phi_i(2^J \mathtt{x} - \ell) \\ &\simeq \textstyle\sum_{k=0}^{L\times 2^J} \Big\{ {}_*D_0^\alpha \phi_i(k-\ell)\phi_1(2^J\mathtt{x} - k) + \frac{1}{2^J} D\Big({}_*D_0^\alpha \phi_i(\mathtt{x}) \Big)\Big|_{\mathtt{x}=k-\ell}\ \phi_2(2^J\mathtt{x} - k) \Big\}, \end{aligned} \tag{6}$$

where D denotes the first order classical derivative and $\ell = 0, 1, \dots, L \times 2^J$. Using the relation (6) we can find the CFD of order α for the vector function Φ_J in form

$${}_*D_0^\alpha \Phi_J(\mathtt{x}) \simeq \mathcal{D}_\alpha\ \Phi_J(\mathtt{x}), \tag{7}$$

where $\mathcal{D}_\alpha$ is the $N \times N$ operational matrix of fractional derivative. For the special cases $J = 0$, $J = 1$ and $\alpha = \frac{1}{10}$, the matrix $\mathcal{D}_\alpha$ is as follows, respectively:

$$\mathcal{D}_\alpha = \begin{pmatrix} 0 & 0 & \frac{-600}{551\Gamma(\frac{9}{10})} & \frac{20}{57\Gamma(\frac{9}{10})} \\ 0 & 0 & \frac{-10}{551\Gamma(\frac{9}{10})} & \frac{11}{171\Gamma(\frac{9}{10})} \\ 0 & 0 & \frac{600}{551\Gamma(\frac{9}{10})} & \frac{-20}{57\Gamma(\frac{9}{10})} \\ 0 & 0 & \frac{200}{4959\Gamma(\frac{9}{10})} & \frac{220}{171\Gamma(\frac{9}{10})} \end{pmatrix},$$

and

$$\mathcal{D}_\alpha = \begin{pmatrix} 0 & 0 & \frac{-600}{551} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{40}{57} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{16000}{1653} \frac{(\frac{11}{20} - \frac{49}{80} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} & \frac{3200}{57} \frac{(\frac{21}{40} - \frac{39}{80} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} \\ 0 & 0 & \frac{-10}{551} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{22}{171} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{16000}{1653} \frac{(\frac{631}{2400} - \frac{59}{240} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} & \frac{1600}{57} \frac{(\frac{1051}{2400} - \frac{49}{120} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} \\ 0 & 0 & \frac{600}{551} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{-40}{57} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{-16000}{1653} \frac{(\frac{11}{20} - \frac{1}{2} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} & \frac{-3200}{57} \frac{(\frac{21}{40} - \frac{1}{2} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} \\ 0 & 0 & \frac{200}{4959} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{440}{171} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{8000}{1653} \frac{(\frac{31}{30} - \frac{29}{30} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} & \frac{1600}{57} \frac{(\frac{41}{60} - \frac{19}{30} 2^{\frac{1}{10}})}{\Gamma(\frac{9}{10})} \\ 0 & 0 & 0 & 0 & \frac{600}{551} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{-40}{57} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} \\ 0 & 0 & 0 & 0 & \frac{200}{4959} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} & \frac{440}{171} \frac{2^{\frac{1}{10}}}{\Gamma(\frac{9}{10})} \end{pmatrix}.$$

3. Methodology Description

In this section we solve the FIVP (1) by using cubic Hermite spline functions. The unknown function y in the problem (1), using relation (5), can be approximated as

$$y(\mathtt{x}) \simeq y_J(\mathtt{x}) = \sum_{k=0}^{L\times 2^J} \left\{ Y_{1,k}\ \phi_1^{J,k}(\mathtt{x}) + Y_{2,k}\ \phi_2^{J,k}(\mathtt{x}) \right\} = Y^T \Phi_J(\mathtt{x}), \tag{8}$$

where Y is an unknown vector as

$$Y = \left[Y_{1,0}, Y_{2,0}, Y_{1,1}, Y_{2,1}, \ldots, Y_{1,L\times 2^J}, Y_{2,L\times 2^J} \right]^T.$$

Also using the operational matrix in relation (7), we can approximate the value ${}_*D_0^\alpha y(\mathtt{x})$ as

$${}_*D_0^\alpha y(\mathtt{x}) \simeq {}_*D_0^\alpha Y^T \Phi_J(\mathtt{x}) \simeq Y^T \mathcal{D}_\alpha \Phi_J(\mathtt{x}). \tag{9}$$

By replacing the relations (8) and (9) in Eq. (1), we get

$$Y^T \mathcal{D}_\alpha \Phi_J(\mathtt{x}) \simeq \mathsf{F}\Big(\mathtt{x}, Y^T \Phi_J(\mathtt{x})\Big), \quad 0 < \mathtt{x} \le L, \tag{10}$$

also by replacing (8) in the initial condition of problem (1), we obtain

$$Y^T \Phi_J(0) = y_0. \tag{11}$$

By collocating Eq. (10) at the $N-1$ equally spaced nodes $\mathtt{x}_i \in [0, L]$, for $i = 1, 2, ..., N-1$, we have

$$Y^T \mathcal{D}_\alpha \Phi_J(\mathtt{x}_i) \simeq \mathsf{F}\Big(\mathtt{x}_i, Y^T \Phi_J(\mathtt{x}_i)\Big), \quad i = 1, 2, 3, \ldots, N-1. \tag{12}$$

Equations (12) together with equation (11) give a system of algebraic equations with N equations and N unknowns, which can be solve to obtain the unknown vector Y. So, the solution y of problem (1) can be found.

4. Convergence Analysis

Theorem 4.1. *[13, 14] Let $f \in H^4[0, L]$ and f_J be the projection of f on the space V_J. Then we have*

$$\inf \|f(x) - f_J(x)\|_{L^2([0,L])} \le 2^{-4J} \|f\|_{4,[0,L]} = O(2^{-4J}). \tag{13}$$

Let y and y_J denote the exact and approximate solutions, respectively, of the FIVP (1). Then we have

$$\begin{cases} {}_*D_0^\alpha y_J(\mathtt{x}) = \mathsf{F}(\mathtt{x}, y_J(\mathtt{x})) + r_J(\mathtt{x}), & 0 < \mathtt{x} \le L, \\ y_J(0) = y_0, \end{cases} \tag{14}$$

where $r_J(\mathtt{x})$ is the residual term of the approximation. In the following theorem we find an upper bound for $r_J(\mathtt{x})$.

Theorem 4.2. *Let $y(t) \in H^4[0, L]$ and $y_J(t)$ are the exact and approximate solutions of the FIVP (1) , respectively. Let $\tilde{D}^\alpha$ and $\tilde{y}_J$ are approximated values of ${}_*D_0^\alpha$ and y_J respectively. Also let the function F be a Lipschitz continuous function with respect to its second variable, i.e.,*

$$F(x, t_1) - F(x, t_2) \le L_p |t_1 - t_2|,$$

with the lipschitz constant L_p. Then we have

$$|r_J(\mathtt{x})| = |\tilde{D}^\alpha \tilde{y}_J(\mathtt{x}) - F\Big(\mathtt{x}, \tilde{y}_J(\mathtt{x})\Big)| = O(2^{-4J}). \tag{15}$$

Proof: Using the definition of finction $r_J(\mathrm{x})$, we can conclude that:

$$\begin{aligned}|r_J(\mathrm{x})| &= \left|\tilde{D}^{\alpha}\tilde{y}_J(\mathrm{x}) - \mathsf{F}\left(\mathrm{x},\tilde{y}_J(\mathrm{x})\right)\right| \\ &= \left|\left\{{}_{*}D_0^{\alpha}y(\mathrm{x}) - \mathsf{F}(\mathrm{x},y(\mathrm{x}))\right\} - \left\{\tilde{D}^{\alpha}\tilde{y}_J(\mathrm{x}) - \mathsf{F}\left(\mathrm{x},\tilde{y}_J(\mathrm{x})\right)\right\}\right| \\ &= \left|\left\{{}_{*}D_0^{\alpha}y(\mathrm{x}) - \tilde{D}^{\alpha}\tilde{y}_J(\mathrm{x})\right\} - \left\{\mathsf{F}(\mathrm{x},y(\mathrm{x})) - \mathsf{F}\left(\mathrm{x},\tilde{y}_J(\mathrm{x})\right)\right\}\right| \\ &= \left|\left\{{}_{*}D_0^{\alpha}y(\mathrm{x}) - \tilde{D}^{\alpha}y(\mathrm{x}) + \tilde{D}^{\alpha}y(\mathrm{x}) - \tilde{D}^{\alpha}\tilde{y}_J(\mathrm{x})\right\} + \left\{\mathsf{F}(\mathrm{x},y(\mathrm{x})) - \mathsf{F}\left(\mathrm{x},\tilde{y}_J(\mathrm{x})\right)\right\}\right| \\ &\leq \left|{}_{*}D_0^{\alpha}y(\mathrm{x}) - \tilde{D}^{\alpha}y(\mathrm{x})\right| + \left|\tilde{D}^{\alpha}y(\mathrm{x}) - \tilde{D}^{\alpha}\tilde{y}_J(\mathrm{x})\right| \\ &\quad + \left|\mathsf{F}(\mathrm{x},y(\mathrm{x})) - \mathsf{F}\left(\mathrm{x},\tilde{y}_J(\mathrm{x})\right)\right|. \end{aligned} \tag{16}$$

By applying (13), we can obtain expressions for the first and second terms on the right-hand side of the inequality (16):

$$\left|{}_{*}D_0^{\alpha}y(\mathrm{x}) - \tilde{D}^{\alpha}y(\mathrm{x})\right| = O(2^{-4J}), \tag{17}$$

$$\left|\tilde{D}^{\alpha}y(\mathrm{x}) - \tilde{D}^{\alpha}\tilde{y}_J(\mathrm{x})\right| = O(2^{-4J}). \tag{18}$$

To evaluate the last term on the right-hand side of inequality (16), we utilize the Lipschitz condition to obtain:

$$\left|\mathsf{F}(\mathrm{x},y(\mathrm{x})) - \mathsf{F}\left(\mathrm{x},\tilde{y}_J(\mathrm{x})\right)\right| \leq L_p|y(\mathrm{x}) - \tilde{y}_J(\mathrm{x})|.$$

Using (13) for the last relation we get

$$\left|\mathsf{F}(\mathrm{x},y(\mathrm{x})) - \mathsf{F}\left(\mathrm{x},\tilde{y}_J(\mathrm{x})\right)\right| = O(2^{-4J}). \tag{19}$$

From (17),(18) and (19) we conclude the proof.

Theorem 4.3. *Let $y(t) \in H^4[0,L]$ and $y_J(t)$ are the exact and approximate solution of the FIVP (1), respectively, and the function F be a Lipschitz continuous function with respect to its second variable. Then, we have*

$$e_J(\mathrm{x}) = |y(\mathrm{x}) - y_J(\mathrm{x})| = O(2^{-4J}). \tag{20}$$

Proof: By using Riemann-Liouville fractional integral on both sides of FDEs in problems (1), (14) and the related initial values, we get

$$y(x) = y(0) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F(t,y(t))dt, \tag{21}$$

and

$$y_J(x) = y(0) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left(F(t,y_J(t)) + r(t)\right). \tag{22}$$

By subtrcting Eq. (22) from (21) we get

$$|y(x) - y_J(x)| = |\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left(F(t,y(t) - F(t,y_J(t))\right)dt - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}r(t)dt|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^x |x-t|^{\alpha-1} dt \left(L_p \max_{0\leq x\leq L} |F(x,y(x)) - F(x,y_J(x))| + \max_{0\leq x\leq L} |r(x)| \right)$$
$$\leq \frac{L_p}{\alpha\Gamma(\alpha)} \max_{0\leq x\leq L} x^{\alpha} \left(\max_{0\leq x\leq L} |F(x,y(x)) - F(x,y_J(x))| + \max_{0\leq x\leq L} |r(x)| \right).$$

and so

$$|y(x) - y_J(x)| \leq \frac{L_p L^{\alpha}}{\alpha\Gamma(\alpha)} \left(\max_{0\leq x\leq L} |F(x,y(x)) - F(x,y_J(x))| + \max_{0\leq x\leq L} |r(x)| \right). \tag{23}$$

Applying the relations (15) *and* (19) *in* (23) *implies*

$$e_J(x) = |y(x) - y_J(x)| \leq O(2^{-4J}),$$

which completes the proof.

5. Numerical examples

In this section, we present the computational results from numerical experiments using the method described in the previous sections to support our theoretical analysis.

We tabulate L_∞ and L_2 errors for various levels of J. To determine the numerical convergence order using the relation (20) we assume $e_J(x) = O(2^{-pJ})$, and solve for p accordingly

$$p = \log_2 \frac{\|e_{J-1}(x)\|_{L^{\infty}([0,L])}}{\left\|e_J(x)\right\|_{L^{\infty}([0,L])}}. \tag{24}$$

Example 5.1. *As the first example, for $\alpha \in (0,1)$ we consider the following linear initial value problem [41]*

$$\begin{cases} {}_*D_0^{\alpha} y(x) = \frac{24}{\Gamma(5-\alpha)} x^{4-\alpha} - \frac{3}{\Gamma(4-\alpha)} x^{3-\alpha} - \frac{1}{2}x^3 + x^4 - y(x), \\ y(0) = 0. \end{cases}$$

The exact solution of this problem is $y(x) = x^4 - \frac{1}{2}x^3$.

We report the L_∞ and L_2 errors for different values of J in Table 1. Also Table 1 shows the numerical convergence order for different values of α and J, which confirm our theoretical results. Table 2 presents a comparison between the results obtained in this study and those obtained using the method proposed in [41].

Example 5.2. *In this example, for $\alpha \in (0,1)$ and $\gamma > 0$ we consider the following linear initial value problem [41]*

$$\begin{cases} {}_*D_0^{\alpha} y(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha} + x^{\gamma} - y(x), \\ y(0) = 0. \end{cases}$$

The exact solution of this problem is $y(x) = x^{\gamma}$.

Here we assume the cases $\gamma = 1.8, 2.3, 2.7, 3.4$ and 4.7, where only for the last case, the exact solution lies in $H^4[0,1]$. But for the other cases the exact solutions belongs to $W^{\gamma,2}[0,1]$. The obtained results show that the numerical order of convergence (24) for the case $\gamma = 4.7$ is as $O(2^{-pJ})$, where p is close to 4, which shows the good agreements with our theoretical results (see Table 7.). But for the other cases it reduces as $O(2^{-\gamma J})$ (see Tables 3-6).

The presented method has been used to obtain the L_∞ and L_2 errors for these two cases for various values of J in Table 3 and Table 4 , respectively.

The graph of absolute error function $|y(x) - y_J(x)|$ for $\alpha = 0.5$, $J = 7$ and $\gamma = 1.8, 2.7, 3.4$ and 4.7 are given in Figure 1 and Figure 2 respectively.

Table 1: L_∞ and L_2 errors for $y(\mathtt{x})$ using presented method for Example 5.1.

	J	L_∞ error	L_2 error	p
$\alpha = 0.25$	1	4.31×10^{-3}	2.05×10^{-3}	
	2	3.79×10^{-4}	1.83×10^{-4}	3.51
	3	2.67×10^{-5}	1.40×10^{-5}	3.83
	4	1.67×10^{-6}	9.42×10^{-7}	4.00
	5	1.02×10^{-7}	5.97×10^{-8}	4.02
	6	6.33×10^{-9}	3.71×10^{-9}	4.02
	7	3.91×10^{-10}	2.29×10^{-10}	4.01
$\alpha = 0.5$	1	3.42×10^{-3}	1.70×10^{-3}	
	2	2.37×10^{-4}	1.19×10^{-4}	3.85
	3	1.54×10^{-5}	7.87×10^{-6}	3.94
	4	1.02×10^{-6}	5.14×10^{-7}	3.91
	5	6.94×10^{-8}	3.45×10^{-8}	3.88
	6	4.84×10^{-9}	2.42×10^{-9}	3.84
	7	3.49×10^{-10}	1.80×10^{-10}	3.80
$\alpha = 0.75$	1	2.77×10^{-2}	1.51×10^{-3}	
	2	1.95×10^{-4}	9.80×10^{-5}	3.83
	3	1.43×10^{-5}	6.74×10^{-6}	3.77
	4	1.11×10^{-6}	5.23×10^{-7}	3.68
	5	9.23×10^{-8}	4.64×10^{-8}	3.60
	6	8.21×10^{-9}	3.53×10^{-9}	3.49
	7	7.67×10^{-10}	4.62×10^{-10}	3.42

Table 2: The Computational results for Example 5.1.

	Method [41]	Absolute error [41]	Present Method	L_∞ error
$\alpha = 0.1$	$h = 1/15$	3.79×10^{-6}	$J = 4$	3.77×10^{-6}
	$h = 1/30$	2.68×10^{-7}	$J = 5$	2.65×10^{-7}
	$h = 1/60$	1.88×10^{-8}	$J = 6$	1.57×10^{-8}
	$h = 1/120$	1.31×10^{-9}	$J = 7$	9.51×10^{-10}
$\alpha = 0.2$	$h = 1/15$	1.08×10^{-5}	$J = 4$	2.01×10^{-6}
	$h = 1/30$	8.08×10^{-7}	$J = 5$	1.26×10^{-7}
	$h = 1/60$	5.98×10^{-8}	$J = 6$	7.53×10^{-9}
	$h = 1/120$	4.40×10^{-9}	$J = 7$	4.56×10^{-10}
$\alpha = 0.4$	$h = 1/15$	4.51×10^{-5}	$J = 4$	9.91×10^{-7}
	$h = 1/30$	3.83×10^{-6}	$J = 5$	6.45×10^{-8}
	$h = 1/60$	3.21×10^{-7}	$J = 6$	4.12×10^{-9}
	$h = 1/120$	2.67×10^{-8}	$J = 7$	2.70×10^{-10}
$\alpha = 0.6$	$h = 1/15$	1.43×10^{-4}	$J = 4$	8.91×10^{-7}
	$h = 1/30$	1.39×10^{-5}	$J = 5$	6.71×10^{-8}
	$h = 1/60$	1.34×10^{-6}	$J = 6$	5.27×10^{-9}
	$h = 1/120$	1.28×10^{-7}	$J = 7$	4.31×10^{-10}
$\alpha = 0.8$	$h = 1/15$	4.00×10^{-4}	$J = 4$	1.10×10^{-6}
	$h = 1/30$	4.52×10^{-5}	$J = 5$	9.61×10^{-8}
	$h = 1/60$	5.01×10^{-6}	$J = 6$	8.94×10^{-9}
	$h = 1/120$	5.49×10^{-7}	$J = 7$	8.77×10^{-10}
$\alpha = 0.9$	$h = 1/15$	6.47×10^{-4}	$J = 4$	6.31×10^{-6}
	$h = 1/30$	7.89×10^{-5}	$J = 5$	1.01×10^{-6}
	$h = 1/60$	9.38×10^{-6}	$J = 6$	2.06×10^{-7}
	$h = 1/120$	1.10×10^{-6}	$J = 7$	6.96×10^{-8}

Example 5.3. *As an other example, for $\alpha \in (0, 1)$, we examine a linear initial value problem as follows [15, 18, 33, 40]:*

$$\begin{cases} {}_*D_0^\alpha y(x) = -y(x), \\ y(0) = 1. \end{cases}$$

The solution to this problem is given by [18]:

$$y_(\mathrm{x}) = \sum_{k=0}^{\infty} \frac{(-\mathrm{x}^\alpha)^k}{\Gamma(\alpha k + 1)}. \tag{25}$$

We provide the results for L_∞ and L_2 errors for various values of J and $\alpha = 0.85$ in Table 8. To compare our method with existing approaches, we include Table 9, which compares our results with those obtained from the methods presented in [33, 40]. Furthermore, we plot the numerical solutions for $y(\mathrm{x})$ with $J = 6$ and $\alpha = 0.5, 0.75, 0.95$, and 1 in Fig3. It is worth noting that as α approaches to 1, our numerical solution converges to the analytical solution given by $y(\mathrm{x}) = exp(-\mathrm{x})$. Overall, these results support our theoretical discussion and demonstrate the effectiveness of our proposed method for solving this problem.

Table 3: L_∞ and L_2 errors for $\gamma = 1.8$ and $y(\mathbf{x})$ using presented method for Example 5.2.

	J	L_∞ errors	L_2 errors	p
$\alpha = 0.5$	1	2.10×10^{-3}	1.01×10^{-3}	
	2	6.22×10^{-4}	2.07×10^{-4}	1.76
	3	1.83×10^{-4}	4.20×10^{-5}	1.76
	4	5.36×10^{-5}	8.59×10^{-6}	1.77
	5	1.56×10^{-5}	1.76×10^{-6}	1.78
	6	d 4.52×10^{-6}	3.60×10^{-7}	1.78
	7	1.31×10^{-6}	7.37×10^{-8}	1.79

Table 4: L_∞ and L_2 errors for $\gamma = 2.3$ and $y(\mathbf{x})$ using presented method for Example 5.2.

	J	L_∞ errors	L_2 errors	p
$\alpha = 0.25$	1	1.21×10^{-3}	6.98×10^{-4}	
	2	2.45×10^{-4}	1.11×10^{-4}	2.31
	3	4.95×10^{-5}	1.59×10^{-5}	2.31
	4	1.00×10^{-5}	2.22×10^{-6}	2.31
	5	2.02×10^{-6}	3.13×10^{-7}	2.31
	6	4.08×10^{-8}	4.42×10^{-8}	2.31
	7	8.22×10^{-8}	6.26×10^{-9}	2.31
$\alpha = 0.5$	1	1.05×10^{-3}	5.68×10^{-4}	
	2	2.21×10^{-4}	8.19×10^{-5}	2.25
	3	4.63×10^{-5}	1.17×10^{-5}	2.26
	4	9.60×10^{-6}	1.67×10^{-6}	2.27
	5	1.98×10^{-6}	2.40×10^{-7}	2.28
	6	d 4.07×10^{-7}	3.45×10^{-8}	2.28
	7	8.34×10^{-8}	4.96×10^{-9}	2.29
$\alpha = 0.75$	1	1.42×10^{-3}	6.21×10^{-4}	
	2	3.05×10^{-4}	9.42×10^{-5}	2.22
	3	6.43×10^{-5}	1.43×10^{-5}	2.25
	4	1.33×10^{-5}	2.19×10^{-6}	2.27
	5	2.75×10^{-6}	3.38×10^{-7}	2.28
	6	d 5.64×10^{-7}	5.27×10^{-8}	2.29
	7	1.15×10^{-7}	8.33×10^{-9}	2.29

Table 5: L_∞ and L_2 errors for $\gamma = 2.7$ and $y(\mathbf{x})$ using presented method for Example 5.2.

	J	L_∞ errors	L_2 errors	p
$\alpha = 0.25$	3	2.01×10^{-5}	6.99×10^{-6}	
	4	3.08×10^{-6}	5.18×10^{-7}	2.70
	5	4.72×10^{-7}	7.89×10^{-8}	2.71
	6	7.25×10^{-8}	8.43×10^{-9}	2.71
	7	1.11×10^{-8}	9.03×10^{-10}	2.71
$\alpha = 0.5$	3	1.67×10^{-5}	4.78×10^{-6}	
	4	2.61×10^{-6}	5.18×10^{-7}	2.67
	5	4.09×10^{-7}	5.63×10^{-8}	2.68
	6	d 6.38×10^{-8}	6.12×10^{-9}	2.68
	7	9.91×10^{-9}	6.66×10^{-10}	2.69
$\alpha = 0.75$	3	2.38×10^{-5}	5.27×10^{-6}	
	4	3.75×10^{-6}	5.89×10^{-7}	2.66
	5	5.86×10^{-7}	6.57×10^{-8}	2.68
	6	9.10×10^{-8}	7.31×10^{-9}	2.69
	7	1.40×10^{-8}	8.13×10^{-10}	2.69

Table 6: L_∞ and L_2 errors for $\gamma = 3.4$ and $y(\mathbf{x})$ using presented method for Example 5.2.

	J	L_∞ errors	L_2 errors	p
$\alpha = 0.5$	1	1.15×10^{-3}	6.34×10^{-4}	
	2	1.08×10^{-4}	4.92×10^{-5}	3.42
	3	1.00×10^{-5}	3.57×10^{-6}	3.42
	4	9.33×10^{-7}	2.57×10^{-7}	3.42
	5	8.72×10^{-8}	1.86×10^{-8}	3.42
	6	d 8.17×10^{-9}	1.37×10^{-9}	3.42
	7	1.04×10^{-10}	7.64×10^{-10}	3.42

Table 7: L_∞ and L_2 errors for $\gamma = 4.7$ and $y(\mathtt{x})$ using presented method for Example 5.2.

	J	L_∞ errors	L_2 errors	p
	3	7.23×10^{-5}	2.76×10^{-5}	
	4	5.02×10^{-6}	1.95×10^{-6}	3.85
$\alpha = 0.25$	5	3.24×10^{-7}	1.28×10^{-7}	3.95
	6	2.05×10^{-8}	8.02×10^{-9}	3.99
	7	1.27×10^{-9}	4.97×10^{-10}	4.01
	3	4.57×10^{-5}	1.64×10^{-5}	
	4	2.61×10^{-6}	1.08×10^{-6}	4.12
$\alpha = 0.5$	5	2.13×10^{-7}	7.20×10^{-8}	3.61
	6	1.46×10^{-8}	4.99×10^{-9}	3.87
	7	1.04×10^{-9}	3.64×10^{-10}	3.80
	3	4.02×10^{-5}	1.35×10^{-5}	
	4	3.10×10^{-6}	1.01×10^{-6}	3.70
$\alpha = 0.75$	5	2.45×10^{-7}	8.74×10^{-8}	3.66
	6	2.15×10^{-8}	8.42×10^{-9}	3.51
	7	1.98×10^{-9}	8.55×10^{-10}	3.44

Table 8: L_∞ and L_2 errors for $y(\mathtt{x})$ using presented method for Example 5.3.

J	L_∞ errors	L_2 errors
3	2.94×10^{-3}	1.99×10^{-3}
4	1.58×10^{-3}	1.03×10^{-3}
5	8.12×10^{-4}	5.28×10^{-4}
6	4.09×10^{-4}	2.67×10^{-4}

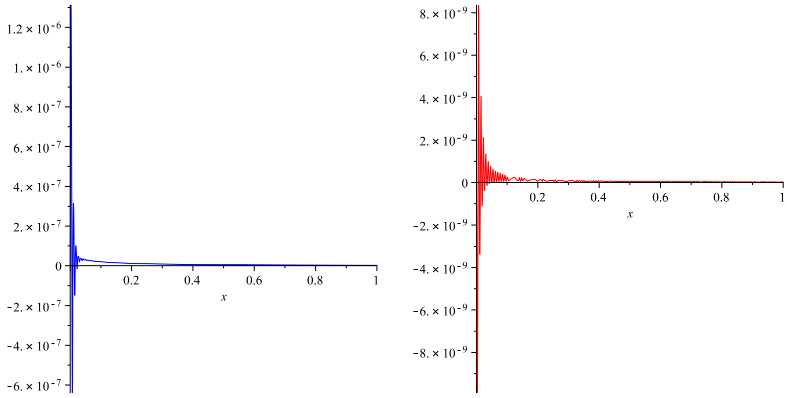

Figure 1: Plots of absolute errors $|y(\mathrm{x}) - y_J(\mathrm{x})|$ for $\alpha = 0.5$, $J = 7$ and $\gamma = 1.8, 2.7$, for Example 5.2 .

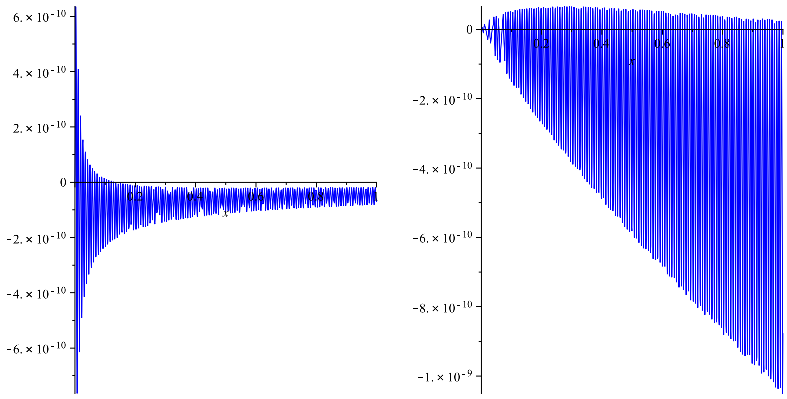

Figure 2: Plots of absolute errors $|y(\mathrm{x}) - y_J(\mathrm{x})|$ for $\alpha = 0.5$, $J = 7$ and $\gamma = 3.4, 4.7$, for Example 5.2 .

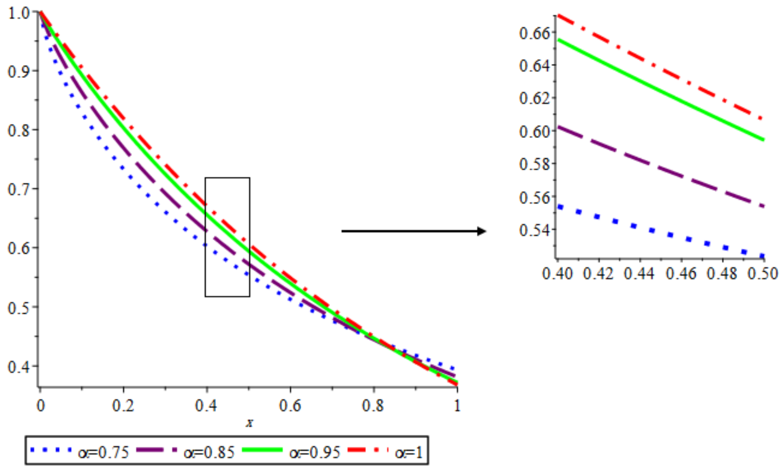

Figure 3: Comparison of $y(\mathrm{x})$ for $J = 6$ and $\alpha = 0.5, 0.75, 0.95$ and the exact solution for $\alpha = 1$, Example 5.3.

Table 9: The Computational results for Example 5.3.

Methods	L_∞ error
Legendre polynomials Method [40]	
$x = 0.7, m = 2$	2.4×10^{-2}
$x = 0.4, m = 5$	2.9×10^{-3}
$x = 0.2, m = 8$ 1.2×10^{-3}	
B-spline Method [33]	
$J = 5$	9.1×10^{-3}
$J = 6$	4.6×10^{-3}
$J = 7$	2.5×10^{-3}
$J = 8$	1.4×10^{-3}
Present Method	
$J = 1$	1.26×10^{-2}
$J = 2$	6.24×10^{-3}
$J = 3$	2.94×10^{-3}
$J = 4$	1.58×10^{-3}
$J = 5$	8.12×10^{-4}
$J = 6$	4.09×10^{-4}

Example 5.4. *For* $\alpha \in (0,1)$ *consider the following fractional Riccati equation*

$$\begin{cases} {}_*D_0^\alpha y(x) = 1 - y^2(x), \\ y(0) = 0. \end{cases} \tag{26}$$

The exact solution of the initial value problem (26) for $\alpha = 1$ is

$$y(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

The exact solutions for the values of $\alpha \neq 1$ are not exist. The approximate solutions obtained by the present method for various values of $\alpha = 0.25, 0.5, 0.75.0.85, 0.95$ are plotted in Figure 4 , which shows that as α tends to 1, the approximate solutions approaches to the plot of $y(x) = e^{2x} - 1/e^{2x} + 1$.

Example 5.5. *As the last example, for* $\alpha \in (0,1)$ *we consider the following nolinear initial value problem*

$$\begin{cases} {}_*D_0^\alpha y(x) = 2e^{2x} + e^{\frac{4}{3}x} - y^{2/3}(x) \\ y(0) = 1 \end{cases} \tag{27}$$

The exact solution of the initial value problem (27) with $\alpha = 1$ is $y(x) = e^{2x}$. The present method yields approximate solutions at various values of $\alpha(0.5, 0.75, 0.85, 0.95)$ which are plotted in Figure 5. As α approaches 1, we observe that the solution of the fractional differential equation converges towards that of the integer-order differential equation.

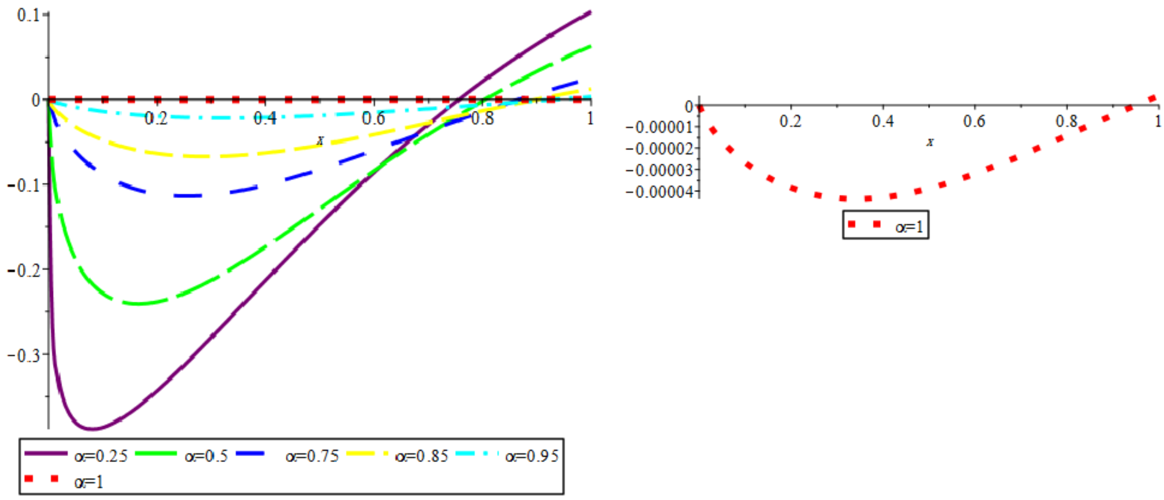

Figure 4: Plot of approximate solutions for $\alpha = 0.25, 0.5, 0.75, 0.85, 0.95$, for Example 5.4.

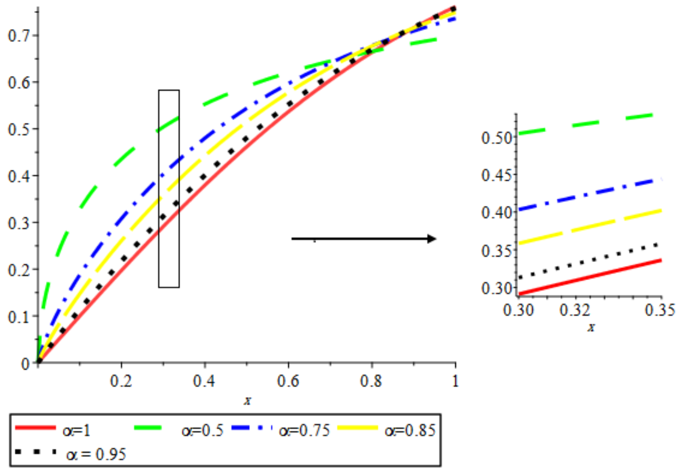

Figure 5: Plot of approximate solutions for $\alpha = 0.25, 0.5, 0.75, 0.85, 0.95$, for Example 5.5.

6. Conclusion

In this paper, we utilize cubic Hermite spline functions to solve an initial value fractional differential equation. We first construct the operational matrix of the Caputo-type fractional derivative and then use the collocation method to obtain the solution. We demonstrate that our method exhibits good agreement with the numerical order of convergence when the exact solution lies within $H^4[0, L]$. However, when the exact solution lies within the fractional Sobolev space $W^{s,2}(0, L)$, where $1 \leq s < 4$, we observe a reduction in convergence order to $O(2^{-sJ})$. Future work could focus on theoretically determining the convergence order of our method in fractional Sobolev spaces.

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